Formalising Real Numbers in Homotopy Type Theory

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Abstract

Cauchy reals can be defined as a quotient of Cauchy sequences of rationals. In this case, the limit of a Cauchy sequence of Cauchy reals is defined through lifting it to a sequence of Cauchy sequences of rationals.

This lifting requires the axiom of countable choice or excluded middle, neither of which is available in homotopy type theory. To address this, the Univalent Foundations Program uses a higher inductive-inductive type to define the Cauchy reals as the free Cauchy complete metric space generated by the rationals.

We generalize this construction to define the free Cauchy complete metric space generated by an arbitrary metric space. This forms a monad in the category of metric spaces with Lipschitz functions. When applied to the rationals it defines the Cauchy reals. Finally, we can use Altenkirch and Danielson (2016)'s partiality monad to define a semi-decision procedure comparing a real number and a rational number.

The entire construction has been formalized in the Coq proof assistant. It is available at `https://github.com/SkySkimmer/HoTTClasses/tree/CPP2017`.

Categories and Subject Descriptors F.4.1 [*Mathematical Logic and Formal Languages*]: Mathematical Logic

Keywords homotopy type theory, inductive inductive types, higher inductive types, real numbers, metric space, premetric space, monads, Coq

1. Introduction

The usual process of defining the set of Cauchy real numbers proceeds in three stages: first define Cauchy sequences of rationals, then define an equivalence between Cauchy sequences, and finally quotient Cauchy sequences by the equivalence. However, proving that the so-defined Cauchy reals are Cauchy complete, i.e. that Cauchy sequences of Cauchy reals have Cauchy real limits requires the axiom of countable choice.

Alternatively, the quotient step can be replaced by working with Cauchy sequences as a setoid: this approach is used e.g. in OConnor07 (2007) which defines the completion of arbitrary metric spaces. This requires many results, e.g. in abstract algebra, to be generalized to setoids and prevents us from using the many properties of the identity type.

The Higher Inductive Inductive types (HIIT) from Homotopy Type Theory (HoTT 2013) provide another construction, in only one step and without the need for an axiom of choice to prove completeness. The construction and the proof that it produces an Archimedean ordered field were outlined in the HoTT book, however formalization in the Coq proof assistant would have required workarounds for the lack of inductive-inductive types until an experimental branch by M. Sozeau started in 2015. Such workarounds can be seen in M. Shulman's implementation of the surreal numbers based on the HoTT book (`https://github.com/HoTT/HoTT/blob/master/theories/Spaces/No.v`).

We start by explaining the context of the development in section 2, specifying the theory we work in, how much is presumed known, and general use notations.

In section 3 we define a notion of *premetric space*, which on the meta level is a generalization of a metric space. From this we can define basic notions such as Lipschitz functions and limits of Cauchy sequences (or rather the equivalence but easier to work with Cauchy approximations).

Section 4 generalizes the construction of the Cauchy completion of rationals from the HoTT book to arbitrary premetric spaces. This generalization shows that Cauchy completion is a monadic operator on premetric spaces (where the arrows are Lipschitz functions).

Lemmas relating to the specific structure of Cauchy reals (such as lemmas about the order on reals) are retained as shown in section 5. The monadic structure also provides a more natural way to define multiplication than that used in HoTT (2013).

In section 6 we investigate how partial functions as per Partiality, Revisited (2016) can be defined on our definition of Cauchy reals through the example of a semi-decision procedure for the property $0 < x$.

CPP'17, January 16–17, 2017, Paris, France
ACM. 978-1-4503-4705-1/17/01...$15.00
http://dx.doi.org/10.1145/3018610.3018614

2. Context

We work in the type theory of HoTT (2013). This gives us a Martin-Löf style type theory with the powerful (higher) inductive types.

We assume understanding of HoTT (2013) sections 1.1 (the type theory), 1.3 (homotopic "mere" propositions and sets, propositional truncation) and 1.5 and 1.6 (induction and higher induction). The following concepts are particularly used:

- $Prop$ is a universe of mere propositions, i.e. $\Sigma_{T:Type} IsHProp\, T$.
- $\exists x : A, P$ ("there merely exists $x : A$ such that P", with x bound in P) is the propositional truncation of the sigma type $\Sigma_{x:A} P$.
- $\forall x : A, P$ is the dependent function type $\Pi x : A, P$.
- $A \wedge B$ is the product type $A * B$.
- $A \vee B$ is the truncated disjunction $\|A + B\|$.
- **0** is the empty type, **1** is the type with unique element $\star$ and **2** is the type with 2 elements $true$ and $false$.

Additionally we use the notation $\cup$ (resp. $\cap$) for the binary join (resp. meet) of a lattice. If the lattice is a total order, such as $\mathbb{Q}$, it is the binary maximum (resp. minimum).

3. Premetric Spaces

We follow OConnor07 (2007) in defining distance as a relation expressing when two elements are sufficiently close. For O'Connor a metric space is a space with a relation $B_\varepsilon(x, y)$ where x and y are elements of the space and $\varepsilon : \mathbb{Q}^+$, which is interpreted as $d(x, y) \leq \varepsilon$.

In contrast, HoTT (2013) defines a relation $x \approx_\varepsilon y$ for x and y Cauchy reals which is interpreted as $d(x, y) < \varepsilon$. We follow HoTT in using the strict order $<$. This choice informs for instance the roundedness property in the following definition, or definition 3.15.

Definition 3.1 (Premetric space)**.** *A* premetric space *is a type A together with a parametric mere relation $\_ \approx_{\_} \_ : \mathbb{Q}^+ \to A \to A \to Prop$ verifying the following properties:*

- *reflexivity:* $\forall (\varepsilon : \mathbb{Q}^+)(x : A), x \approx_\varepsilon x$
- *symmetry:* $\forall (\varepsilon : \mathbb{Q}^+)(x\ y : A), x \approx_\varepsilon y \to y \approx_\varepsilon x$
- *separatedness:* $\forall x\ y : A, (\forall \varepsilon : \mathbb{Q}^+, x \approx_\varepsilon y) \to x =_A y$
- *triangularity:* $\forall (x\ y\ z : A)(\varepsilon\ \delta : \mathbb{Q}^+), x \approx_\varepsilon y \to y \approx_\delta z \to x \approx_{\varepsilon+\delta} z$
- *roundedness:* $\forall (\varepsilon : \mathbb{Q}^+)(x\ y : A), x \approx_\varepsilon y \leftrightarrow \exists \delta : \mathbb{Q}^+, \delta < \varepsilon \wedge x \approx_\delta y$

$\approx$ is called the closeness relation of A, with $x \approx_\varepsilon y$ read as "x and y are ε-close" or "the distance between x and y is less than ε".

Throughout this paper, for every type there is at most one obvious closeness relation upon it, either introduced with it or constructed according to its shape (e.g. for function types after definition 3.15).

We do not use the fact that any set can be made into a premetric space using the closeness relation $x \approx_\varepsilon y := x = y$.

Lemma 3.2. *If A is a type with a closeness relation $\_ \approx_{\_} \_ : \mathbb{Q}^+ \to A \to A \to Prop$ which has the separatedness property then A is a set. Consequently premetric spaces are sets.*

Proof. By HoTT (2013) theorem 7.2.2 and separatedness. □

Remark 3.3. *Classically, we can take $d(x, y) = \inf\{\varepsilon : \mathbb{Q}^+, x \approx_\varepsilon y\}$ with values in $\mathbb{R} + \{\infty\}$ to turn a premetric space into a metric space.*

If we remain constructive, we expect a need for a locatedness property such as $\forall (x\ y : A)\ (q\ r : \mathbb{Q}^+), q < r \to x \approx_r y \vee x \not\approx_q y$.

We have not carried out the constructions due to lack of time, so these may not be the exact properties required. For instance without countable choice the position of the truncation may need to be different: this can be seen in HoTT (2013) lemma 11.4.1.

We now work in an arbitrary premetric space A.

Definition 3.4 (Cauchy approximation)**.**

$$\text{Approximation}\, A := \Sigma_{x:\mathbb{Q}^+ \to A} \forall \varepsilon\ \delta : \mathbb{Q}^+, x_\varepsilon \approx_{\varepsilon+\delta} x_\delta$$

A Cauchy approximation $x : \text{Approximation}\, A$ can be seen as a function which given ε produces a value at distance up to ε of an hypothetical limit.

By abuse of notation when $x : \text{Approximation}\, A$ we use x for its first projection.

Definition 3.5 (Limit)**.** *$l : A$ is a limit of the approximation x when*

$$\forall \varepsilon, \delta : \mathbb{Q}^+, x_\varepsilon \approx_{\varepsilon+\delta} l$$

Since we want to express $d(x_\varepsilon, l) \leq \varepsilon$ but closeness is interpreted as $<$ we introduce an additional δ.

Lemma 3.6. *Limits are unique: if l_1 and l_2 are limits of $x : \text{Approximation}\, A$ then $l_1 = l_2$.*

We may then talk about the *limit of an approximation.*

Proof. By separatedness and triangularity. □

Definition 3.7 (Cauchy completeness)**.** *A is Cauchy complete when every Cauchy approximation has a limit. Since the limit is unique, this is equivalent to having a function*

$$lim : \text{Approximation}\, A \to A$$

producing the limit for every approximation.

Theorem 3.8. *Rationals form a premetric space with the closeness relation $q \approx_\varepsilon r := |q - r| < \varepsilon$.* □

The following lemmas make working with limits easier.

Lemma 3.9. *Let y : Approximation A, l_y and $x : A$ and ε and $\delta : \mathbb{Q}^+$ such that l_y is the limit of y and $x \approx_\varepsilon y_\delta$. Then $x \approx_{\varepsilon+\delta} l_y$.*

Proof. First strengthen the hypothesis $x \approx_\varepsilon y_\delta$ by roundedness, then finish with triangularity. □

Lemma 3.10. *Let x and y : Approximation A, and $\varepsilon\ \delta\ \kappa : \mathbb{Q}^+$ such that $x_\delta \approx_\varepsilon y_\kappa$, then if l_x is the limit of x and l_y is the limit of y, $l_x \approx_{\varepsilon+\delta+\kappa} l_y$.*

Proof. By two applications of lemma 3.9. □

Lemma 3.11. *If $x\ y$: Approximation A and $\varepsilon : \mathbb{Q}^+$ are such that $\forall \delta\ \kappa : \mathbb{Q}^+, x_\kappa \approx_{\varepsilon+\delta} y_\kappa$, then for l_x limit of x and l_y limit of y, $\forall \delta : \mathbb{Q}^+, l_x \approx_{\varepsilon+\delta} l_y$.*

Proof. Using lemma 3.10, since $\varepsilon + \delta = (\varepsilon + \frac{\delta}{3}) + \frac{\delta}{3} + \frac{\delta}{3}$. □

3.1 Continuity Notions

We will be interested in certain properties of functions between premetric spaces A and B.

Definition 3.12 (Lipschitz function)**.** *A function $f : A \to B$ is* Lipschitz *with constant $L : \mathbb{Q}^+$ when*

$$\forall (\varepsilon : \mathbb{Q}^+)(x, y : A), x \approx_\varepsilon y \to f\ x \approx_{L*\varepsilon} f\ y$$

If L is 1 we say that f is non-expanding.

Definition 3.13 (Continuous function)**.** *A function $f : A \to B$ is continuous when*

$$\forall (\varepsilon : \mathbb{Q}^+)(x : A), \exists \delta : \mathbb{Q}^+, \forall y : A, x \approx_\delta y \to f\ x \approx_\varepsilon f\ y$$

Lemma 3.14. *Lipschitz functions are continuous.*

Proof. Using $\delta := \frac{\varepsilon}{L}$. □

Premetric spaces with continuous functions form a category.

Premetric spaces with Lipschitz functions also form a category.

3.2 The Premetric Space of Functions

Let A a type and B a premetric space.

Definition 3.15 (Closeness of functions)**.**

$$f \approx_\varepsilon g := \exists \delta : \mathbb{Q}^+, \delta < \varepsilon \wedge \forall x : A, f\ x \approx_\delta f\ y$$

This expresses that $d(f, g) = \sup\{d(f\ x, g\ x) | x : A\}$.

Lemma 3.16. *For $\varepsilon : \mathbb{Q}^+$ and $f\ g : A \to B$, if $f \approx_\varepsilon g$ then $\forall x : A, f\ x \approx_\varepsilon g\ x$.*

Proof. By roundedness. □

Theorem 3.17. *$A \to B$ forms a premetric space. If B is Cauchy complete then so is $A \to B$, and the limit of s : Approximation$(A \to B)$ is $\lambda y, lim\ (\lambda\varepsilon, s\ \varepsilon\ y)$.* □

Lemma 3.18 (Limit of Lipschitz functions)**.** *Suppose A is a premetric space and B is Cauchy complete.*

If s : Approximation$(A \to B)$ is such that $\forall \varepsilon : \mathbb{Q}^+$, $s\ \varepsilon$ is Lipschitz with constant L, then $lim\ s$ is Lipschitz with constant L.

Proof. Let $\varepsilon : \mathbb{Q}^+$ and $x\ y : A$ such that $x \approx_\varepsilon y$. By roundedness there merely is $\delta\ \kappa : \mathbb{Q}^+$ such that $\varepsilon = \delta + \kappa$ and $x \approx_\delta y$.

By hypothesis $\forall \eta : \mathbb{Q}^+, s_\eta\ x \approx_{L*\delta} y$, then by roundedness $\forall \eta\ \eta' : \mathbb{Q}^+, s_\eta\ x \approx_{L*\delta+\eta'} s_\eta\ y$.

By lemma 3.11 and unfolding the definition of $\lim s$ we have $\forall \eta : \mathbb{Q}^+, \lim s\ x \approx_{L*\delta+\eta} \lim s\ y$, then since $L * \varepsilon = L * \delta + L * \kappa$ we have $\lim s\ x \approx_{L*\varepsilon} \lim s\ y$. □

4. Cauchy Completion

4.1 Definition and Eliminators

In classical logic, we define the completion of a metric space T as the quotient of the Cauchy sequences (or equivalently of Cauchy approximations) in T by the equivalence $\lim f = \lim g$ (or rather an equivalent statement which doesn't assume the limit is defined). The axiom of countable choice is then used to prove that Cauchy approximations in the quotient have limits in the quotient.

Using higher inductive types, we can instead define $\mathcal{C}\,T$ the free complete premetric space generated by the premetric space T. By unfolding this statement we can see what constructors it needs:

- generated by T: so there is a constructor of type $T \to \mathcal{C}\,T$.
- premetric space: so we need to construct the closeness relation, and truncate $\mathcal{C}\,T$ to make it separated.
- Cauchy complete: there is a constructor of type Approximation$(\mathcal{C}\,T) \to \mathcal{C}\,T$.

Definition 4.1. *$\mathcal{C}\,T$ has the following constructors*

$$\eta : T \to \mathcal{C}\,T$$
$$\lim : \text{Approximation}(\mathcal{C}\,T) \to \mathcal{C}\,T$$

The constructors of the closeness relation and the path constructors for $\mathcal{C}\,T$ and its closeness construct proof-irrelevant values. As such, we do not name them but instead give them as inference rules:

$$\frac{\forall \varepsilon : \mathbb{Q}^+, x \approx_\varepsilon y}{x = y} \qquad \frac{p, q : x \approx_\varepsilon y}{p = q}$$

$$\frac{q \approx_\varepsilon r}{\eta\,q \approx_\varepsilon \eta\,r} \qquad \frac{x_\delta \approx_{\varepsilon-\delta-\kappa} y_\kappa}{\lim x \approx_\varepsilon \lim y}$$

$$\frac{\eta\,q \approx_{\varepsilon-\delta} y_\delta}{\eta\,q \approx_\varepsilon \lim y} \qquad \frac{x_\delta \approx_{\varepsilon-\delta} \eta\,r}{\lim x \approx_\varepsilon \eta\,r}$$

The fully general induction principle produces values in the following predicates A and B when applied to functions corresponding to each constructor and path constructor of

$\mathcal{C}\,T$ and its closeness.

$$A : \mathcal{C}\,T \to Type$$
$$B : \forall x\ y\ \varepsilon, A\ x \to A\ y \to x \approx_\varepsilon y \to Type$$

At this time Coq cannot yet infer such fully general principles (with B depending on A) for inductive-inductive types. It can however be checked by Coq.

The path constructor cases must be guessed by the programmer for higher inductive types. For this paper, we use the obvious generalization of the definition in the HoTT book.

In practice we only use the following specializations:

Definition 4.2 (Simple $\mathcal{C}$ −induction)**.** *Given a mere predicate $A : \mathcal{C}\,T \to Prop$, we have $\forall x : \mathcal{C}\,T, A\ x$ so long as the following hypotheses are verified:*

$$\frac{}{A\ (\eta\, q)} \qquad \frac{\forall \varepsilon : \mathbb{Q}^+, A\ x_\varepsilon}{A\ (\lim x)}$$

Definition 4.3 (Simple $\approx$ −induction)**.** *Given a mere predicate*

$$P : \mathbb{Q}^+ \to \mathcal{C}\,T \to \mathcal{C}\,T \to Prop$$

and the following hypotheses:

$$\frac{q \approx_\varepsilon r}{P\ \varepsilon\ (\eta\, q)\ (\eta\, r)}$$

$$\frac{x_\delta \approx_{\varepsilon-\delta-\kappa} y_\kappa \qquad P\ (\varepsilon - \delta - \kappa)\ x_\delta\ y_\kappa}{P\ \varepsilon\ (\lim x)\ (\lim y)}$$

$$\frac{\eta\, q \approx_{\varepsilon-\delta} y_\delta \qquad P\ (\varepsilon - \delta)\ (\eta\, q)\ y_\delta}{P\ \varepsilon\ (\eta\, q)\ (\lim y)}$$

$$\frac{x_\delta \approx_{\varepsilon-\delta} \eta\, r \qquad P\ (\varepsilon - \delta)\ x_\delta\ (\eta\, r)}{P\ \varepsilon\ (\lim x)\ (\eta\, r)}$$

then

$$\forall \varepsilon\ x\ y, x \approx_\varepsilon y \to P\ \varepsilon\ x\ y$$

Definition 4.4 (Mutual $\mathcal{C}$ −recursion)**.** *Let $A : Type$, a mere predicate $\sim : \mathbb{Q}^+ \to A \to A \to Prop$ and functions*

$$f_\eta : T \to A$$
$$f_{lim} : \forall (x : \text{Approximation}(\mathcal{C}\,T))(f_{rec} : \mathbb{Q}^+ \to A),$$
$$(\forall (\varepsilon, \delta : \mathbb{Q}^+), f_{rec}\ \varepsilon \sim_{\varepsilon+\delta} f_{rec}\ \delta) \to A$$

which verify the following inference rules:

$$\frac{\forall \varepsilon : \mathbb{Q}^+, x \sim_\varepsilon y}{x = y}$$

$$\frac{q \approx_\varepsilon r}{f_\eta\ q \sim_\varepsilon f_\eta\ r} \qquad \frac{f_x\ \delta \sim_{\varepsilon-\delta-\kappa} f_y\ \kappa}{f_{lim}\ x\ f_x\ H_x \sim_\varepsilon f_{lim}\ y\ f_y\ H_y}$$

$$\frac{f_\eta\ q \sim_{\varepsilon-\delta} f_y\ \delta}{f_\eta\ q \sim_\varepsilon f_{lim}\ y\ f_y\ H_y} \qquad \frac{f_x\ \delta \sim_{\varepsilon-\delta} f_\eta\ r}{f_{lim}\ x\ f_x\ H_x \sim_\varepsilon f_\eta\ r}$$

then we have the following functions

$$f : \mathcal{C}\,T \to A$$
$$f_\approx : \forall (x, y : \mathcal{C}\,T)(\varepsilon : \mathbb{Q}^+), x \approx_\varepsilon y \to f\ x \sim_\varepsilon f\ y$$

and the following computation rules

$$f\ (\eta\, q) := f_\eta\ q$$
$$f\ (\lim x) := f_{lim}\ x\ (f \circ x)\ (\lambda \varepsilon\ \delta, f_\approx\ (\varepsilon + \delta)\ x_\varepsilon\ y_\delta)$$

4.2 Properties of the Completion

We now seek to

- show that $\mathcal{C}\,T$ is indeed a premetric space, and that $\lim$ constructs limits.
- characterize the closeness relation: for instance $\eta\, q \approx_\varepsilon \eta\, r$ should be equivalent to $q \approx_\varepsilon r$.

Constructors of $\approx$ give us separatedness and proof irrelevance.

Lemma 4.5 (Reflexivity)**.**

$$\forall (u : \mathcal{C}\,T)(\varepsilon : \mathbb{Q}^+), u \approx_\varepsilon u$$

Proof. By simple induction on u:

- Let $u : T$ and $\varepsilon : \mathbb{Q}^+$. T is a premetric space so $u \approx_\varepsilon u$, then $\eta\, u \approx_\varepsilon \eta\, u$.
- Let $x : \text{Approximation}(\mathcal{C}\,T)$ such that

$$\forall (\varepsilon, \delta : \mathbb{Q}^+), x_\varepsilon \approx_\delta x_\varepsilon$$

Let $\varepsilon : \mathbb{Q}^+$. Then $x_{\varepsilon/3} \approx_{\varepsilon/3} x_{\varepsilon/3}$, so $\lim x \approx_\varepsilon \lim x$. □

Lemma 4.6. *$\mathcal{C}\,T$ is a set.*

Proof. By lemma 3.2. □

Lemma 4.7 (Symmetry)**.**

$$\forall (\varepsilon : \mathbb{Q}^+)(xy : \mathcal{C}\,T), x \approx_\varepsilon y \to y \approx_\varepsilon x$$

Proof. By simple $\approx$ −induction, since T has a symmetric closeness relation. □

To go further we need a way to deconstruct proofs of closeness. This is done by defining a function $B_\_(\_,\_) : \mathbb{Q}^+ \to \mathcal{C}\,T \to \mathcal{C}\,T \to Prop$ recursively on the two $\mathcal{C}\,T$ arguments which is equivalent to $\approx$. The entire definition is a generalization of HoTT (2013) theorem 11.3.32.

B will be defined by mutual $\mathcal{C}$ −recursion as it is proof-relevant. In order to be able to prove the side conditions we will first inhabit a subtype then obtain B by projection.

For each $x : \mathcal{C}\,T$ we will (re)define $x \approx_\_ \_$ which can be seen as the concentric open balls around x. This is an inhabitant of the following type:

Definition 4.8 (Concentric balls)**.** *A set of concentric balls is a value of type*

$$Balls := \Sigma_{B : \mathcal{C}\,T \to \mathbb{Q}^+ \to Prop}\ (\forall y\ \varepsilon, B_\varepsilon\ y \leftrightarrow \exists \delta < \varepsilon, B_\delta\ y) \wedge (\forall \varepsilon\ \delta\ y\ z, y \approx_\varepsilon z \to B_\delta\ y \to B_{\delta+\varepsilon}\ z)$$

We call the first property ball roundedness*, and the second* ball triangularity.

For $\varepsilon : \mathbb{Q}^+$ and B_1, B_2 : $Balls$, let $B_1 \approx_\varepsilon B_2$ when for $\{i, j\} = \{1, 2\}$

$$\forall y\ \delta, B_{i\,\delta}\ y \to B_{j\,\delta+\varepsilon}\ y$$

In a second level of induction we will define $x \approx_\_ y$ for some x and y. This can be seen as an upper cut of positive rationals:

Definition 4.9 (Upper cut)**.** *An upper cut is a predicate on $\mathbb{Q}^+$ which is upward rounded, i.e.*

$$Upper := \Sigma_{U:\mathbb{Q}^+ \to Prop}\,(\forall \varepsilon, U_\varepsilon \leftrightarrow \exists \delta < \varepsilon, U_\delta)$$

For $\varepsilon : \mathbb{Q}^+$ and U_1, U_2 : $Upper$, let $U_1 \approx_\varepsilon U_2$ when for $\{i, j\} = \{1, 2\}$

$$\forall \delta, U_{i\,\delta} \to U_{j\,\delta+\varepsilon}$$

Lemma 4.10. *The closeness on $Balls$ is separated.*

Proof. Let $B^{(1)}, B^{(2)}$: $Balls$ such that $B^{(1)}$ and $B^{(2)}$ are $\varepsilon-$close for all ε. Let ε and y, we need $B^{(1)}_\varepsilon\ y = B^{(2)}_\varepsilon\ y$. By univalence this is $B^{(1)}_\varepsilon\ y \leftrightarrow B^{(2)}_\varepsilon\ y$.

Suppose $B^{(1)}_\varepsilon\ y$, by ball roundedness there merely is $\delta < \varepsilon$ such that $B^{(1)}_\delta\ y$. $B^{(1)}$ and $B^{(2)}$ are $(\varepsilon - \delta)-$close, so we have $B^{(2)}_\varepsilon\ y$.

The second direction is the same by symmetry. ☐

Lemma 4.11. *The closeness on $Upper$ is separated.*

Proof. Like with lemma 4.10 we use first roundedness then the definition of upper cut closeness at the appropriate $\varepsilon - \delta$. ☐

Lemma 4.12 (Concentric balls from upper cuts)**.** *Suppose $B : \mathcal{C}\,T \to Upper$ is non-expanding, then the underlying $\mathcal{C}\,T \to \mathbb{Q}^+ \to Prop$ is a set of concentric balls.*

Proof. The ball roundedness property is exactly upper cut roundedness.

B verifies ball triangularity because it is non-expanding. ☐

Definition 4.13 (Balls around a base element)**.** *Let $q : T$. The set of concentric balls around q is $B\_(\eta\,q, \_)$ defined by mutual $\mathcal{C}-$recursion as a non-expanding function of type $\mathcal{C}\,T \to Upper$ suitable for lemma 4.12.*

The proof relevant values are as follows:

- *base case: $B_\varepsilon(\eta\,q, \eta\,r) := q \approx_\varepsilon r$. This produces an upper cut by roundedness of T.*
- *limit case: $B_\varepsilon(\eta\,q, \lim x) := \exists \delta < \varepsilon, B_{\varepsilon-\delta}(\eta\,q, x_\delta)$. This produces an upper cut by the induction hypothesis and roundedness at the recursive call.*

The remaining hypotheses expressing that the construction is non-expanding are hard to see through on paper. In Coq however reduction makes how to proceed obvious. Let us consider the $\eta - lim$ case.

Let $q\ \ r$: T, $\varepsilon\ \ \delta : \mathbb{Q}^+$ such that $\delta < \varepsilon$, and y : $\mathrm{Approximation}(\mathcal{C}\,T)$ such that we have $(\lambda \kappa\ \xi, B_\xi(\eta\,q, y_\kappa))$. This later function is an approximation on upper cuts. Finally the induction hypothesis is that $(\lambda \kappa, q \approx_\kappa r) \approx_{\varepsilon-\delta} (\lambda \kappa, B_\kappa(\eta\,q, x_\delta))$ as upper cuts.

In that context, we need to prove that $(\lambda \kappa, q \approx_\kappa r) \approx_\varepsilon (\lambda \kappa, B_\kappa(\eta\,q, \lim x))$ as upper cuts. Let $\kappa : \mathbb{Q}^+$, we have two goals:

- *If $q \approx_\kappa r$ then $B_{\kappa+\varepsilon}(\eta\,q, \lim x)$ i.e.*

  $$\exists \delta < \kappa + \varepsilon, B_{\kappa+\varepsilon-\delta}(\eta\,q, x_\delta)$$

  By the induction hypothesis and $q \approx_\kappa r$ we have

  $$B_{\varepsilon-\delta+\kappa}(\eta\,q, x_\delta)$$

  with $\delta < \varepsilon < \varepsilon + \kappa$.
- *If $\exists \xi < \kappa, B_{\kappa-\xi}(\eta\,q, x_\xi)$ then $q \approx_{\kappa+\varepsilon} r$.*
  Because $(\lambda \kappa\ \xi, B_\xi(\eta\,q, y_\kappa))$ is a cut approximation we have $B_{\kappa-\xi+\delta+\xi}(\eta\,q, x_\delta) = B_{\kappa+\delta}(\eta\,q, x_\delta)$. Then by induction hypothesis $q \approx_{\kappa+\varepsilon} r$. ☐

We then similarly define the concentric balls around a limit point, and show that this definition and definition 4.13 respect $\approx$ using simple $\mathcal{C}-$induction. In order to have space for more interesting proofs we shall simply recap what results we obtain from this process.

Theorem 4.14. *We have for all $(\varepsilon : \mathbb{Q}^+)$ and $x\ \ y : \mathcal{C}\,T$, $B_\varepsilon(x, y)$: $Prop$ such that $\lambda x\ \ y\ \ \varepsilon, B_\varepsilon(x, y)$ is a non-expanding function from $\mathcal{C}\,T$ to $Balls$. Additionally we have the following definitional identities:*

$$B_\varepsilon(\eta\,q, \eta\,r) \equiv q \approx_\varepsilon r$$
$$B_\varepsilon(\eta\,q, \lim y) \equiv \exists \delta < \varepsilon, B_{\varepsilon-\delta}(\eta\,q, y_\delta)$$
$$B_\varepsilon(\lim x, \eta\,r) \equiv \exists \delta < \varepsilon, B_{\varepsilon-\delta}(x_\delta, \eta\,r)$$
$$B_\varepsilon(\lim x, \lim y) \equiv \exists \delta\ \kappa : \mathbb{Q}^+, \delta + \kappa < \varepsilon, B_{\varepsilon-\delta-\kappa}(x_\delta, y_\kappa)$$

☐

Theorem 4.15. *$B_\varepsilon(x, y)$ and $x \approx_\varepsilon y$ are equivalent.*

Proof. We prove both sides of the equivalence separately:

- $\forall (u, v : \mathcal{C}\,T)(\varepsilon : \mathbb{Q}^+), B_\varepsilon(u, v) \to u \approx_\varepsilon v$
  By simple induction on u then v, then using the computation rules of B and the constructors of $\approx$.
- $\forall (\varepsilon : \mathbb{Q}^+)(u, v : \mathcal{C}\,T), u \approx_\varepsilon v \to B_\varepsilon(u, v)$
  By simple $\approx -$induction, with each case being trivial. ☐

We can now use the computation rules in theorem 4.14 as computation rules for $\approx$.

Theorem 4.16. *$\mathcal{C}\,T$ forms a premetric space.*

Proof. Roundedness of B as a closeness relation is obtained from roundedness as a function into $Balls$, then we use that B equals $\approx$ to have roundedness of $\approx$.

The triangularity property of B as a function into balls together with theorem 4.15 shows that $\approx$ is triangular.

Separatedness comes by definition of $\mathcal{C}\,T$, and the other properties of a premetric space are already proven in lemmas 4.5 and 4.7. □

Corollary 4.17. *η is injective.*

Proof. By separatedness. □

Theorem 4.18. *$\mathcal{C}\,T$ is Cauchy complete, i.e. for all x : $\text{Approximation}(\mathcal{C}\,T)$, $\lim x$ is the limit of x.*

Proof. Lemma 3.9 also holds for $\mathcal{C}\,T$:

$$\forall (u : \mathcal{C}\,T)(y : \text{Approximation}(\mathcal{C}\,T))(\varepsilon, \delta : \mathbb{Q}^+),\\ u \approx_\varepsilon y_\delta \to u \approx_{\varepsilon+\delta} \lim y$$

By simple induction on u:

- Let $v : T, y : \text{Approximation}(\mathcal{C}\,T)$ and $\varepsilon, \delta : \mathbb{Q}^+$ such that $\eta\, v \approx_\varepsilon y_\delta$.
  Then $\eta\, v \approx_{\varepsilon+\delta} \lim y$ (by definition of $\approx$).
- Let x : $\text{Approximation}(\mathcal{C}\,T)$ such that (induction hypothesis)

  $$\forall (\varepsilon_0, \varepsilon, \delta : \mathbb{Q}^+)(y : \text{Approximation}(\mathcal{C}\,T)),\\ x_{\varepsilon_0} \approx_\varepsilon y_\delta \to x_{\varepsilon_0} \approx_{\varepsilon+\delta} \lim y$$

  and let y, ε, δ such that $\lim x \approx_\varepsilon y_\delta$.
  By roundedness, there merely exist $\kappa, \theta : \mathbb{Q}^+$ such that $\varepsilon = \kappa + \theta$ and $\lim x \approx_\kappa y_\delta$.
  The induction hypothesis used with $y := x$ and reflexivity of $\approx$ gives that $\forall (\varepsilon, \delta : \mathbb{Q}^+), x_\varepsilon \approx_{\varepsilon+\delta} \lim x$ (i.e. $\lim x$ is the limit of x). Specifically, $x_{\theta/4} \approx_{3\theta/4} \lim x$.
  By triangularity, $x_{\theta/4} \approx_{3\theta/4+\kappa} y_\delta$.
  By constructor $\lim x \approx_{\theta+\kappa+\delta} \lim y$.
  Then $\lim x \approx_{\varepsilon+\delta} \lim y$.

Then using this result and lemma 4.7 shows that $\lim x$ is the limit of x. □

4.3 Monadic Structure of the Completion

Continuity lets us characterize functions on $\mathcal{C}\,T$ based on their behaviour on the base elements $\eta\, x$. If a function is sufficiently continuous, i.e. Lipschitz, we can even define its value on $\mathcal{C}\,T$ from its value on T: this turns the completion into a monad.

Theorem 4.19. *Let A a premetric space and $f, g : \mathcal{C}\,T \to A$ continuous functions such that*

$$\forall u : T, f\ (\eta\, u) = g\ (\eta\, u)$$

Then

$$\forall x : \mathcal{C}\,T, f\ x = g\ x$$

Proof. By simple induction on x (the desired property is a mere proposition because premetric spaces are sets). The base case is trivial.

Let $x : \text{Approximation}(\mathcal{C}\,T)$ with the induction hypothesis

$$\forall \varepsilon : \mathbb{Q}^+, f\ x_\varepsilon = g\ x_\varepsilon$$

By separatedness it suffices to prove that

$$\forall \varepsilon : \mathbb{Q}^+, f(\lim x) \approx_\varepsilon g(\lim x)$$

Let $\varepsilon : \mathbb{Q}^+$. Continuity of f and g at $\lim x$ and $\varepsilon/2$ shows that there merely exist δ_f and $\delta_g : \mathbb{Q}^+$ such that

$$\forall y : \mathcal{C}\,T, \lim x \approx_{\delta_f} y \to f(\lim x) \approx_{\varepsilon/2} f\ y$$

$$\forall y : \mathcal{C}\,T, \lim x \approx_{\delta_g} y \to g(\lim x) \approx_{\varepsilon/2} g\ y$$

Let δ : $\mathbb{Q}^+$ such that $\delta < \delta_f$ and δ_g. By roundedness and because $\lim x$ is the limit of x, $\lim x \approx_{\delta_f} x_\delta$ and $\lim x \approx_{\delta_g} x_\delta$.

Then $f\ (\lim x) \approx_{\varepsilon/2} f\ x_\delta = g\ x_\delta$ and $g\ (\lim x) \approx_{\varepsilon/2} g\ x_\delta$.

By triangularity $f\ (\lim x) \approx_\varepsilon g\ (\lim x)$. □

Repeated application of theorem 4.19 lets us deal with multiple variables. For instance, if f and $g : \mathcal{C}\,T_1 \to \mathcal{C}\,T_2 \to A$ are continuous in both arguments (i.e. for all x, $f\ x$ and $g\ x$ are continuous, and for all y, $\lambda x, f\ x\ y$ and $\lambda x, g\ x\ y$ are continuous) and they coincide on T_1 and T_2 then they are equal.

Theorem 4.20. *Let A a Cauchy complete premetric space and f : $T \to A$ Lipschitz with constant L. There exists $\overline{f} : \mathcal{C}\,T \to A$ Lipschitz with constant L such that*

$$\forall x : T, \overline{f}(\eta\, x) = f\ x$$

Proof. We define $\overline{f} : \mathcal{C}\,T \to A$ by mutual recursion, guaranteeing that the images of ε-close values are $L * \varepsilon$-close. This condition is exactly that $\overline{f}$ is Lipschitz with constant L.

In the base case we simply use f.

In the limit case, the induction hypothesis is $\overline{f_x} : \mathbb{Q}^+ \to A$ such that

$$\forall \varepsilon, \delta : \mathbb{Q}^+, \overline{f_x}\ \varepsilon \approx_{L*(\varepsilon+\delta)} \overline{f_x}\ \delta$$

Then $\lambda \varepsilon, \overline{f_x}\ (\varepsilon/L)$ is a Cauchy approximation and we take its limit.

The coherence properties necessary for mutual recursion are easy given lemmas 3.9 and 3.10. □

Theorem 4.21. *If T is Cauchy complete then $\mathcal{C}\,T = T$.*

Proof. The identity of T is non-expanding, so it can be extended into $\overline{id_T} : \mathcal{C}\,T \to T$.

$\overline{id_T} \circ \eta_T$ is convertible to id_T.

$\eta_T \circ \overline{id_T} = id_{\mathcal{C}\,T}$ by continuity.

Then $\overline{id_T}$ is an equivalence from $\mathcal{C}\,T$ to T, and by univalence they are equal. □

Aside from the obvious use in the above theorem, univalence is only used through 4.14 to show that $Balls$ and $Upper$ have the separatedness property (since by univalence equivalent propositions are equal).

Theorem 4.22. *The Cauchy completion is an idempotent monad on the category of premetric spaces with Lipschitz functions.*

Proof. Given $f : A \to B$ a Lipschitz function with constant L, $\eta \circ f : A \to \mathcal{C}\,B$ and $\overline{\eta \circ f} : \mathcal{C}\,A \to \mathcal{C}\,B$ are Lipschitz functions with constant L.

The identities about extension of identity and extension of composition are verified by continuity.

Then completion is a functor, and the previous theorem shows it is an idempotent monad. □

Remark 4.23. *OConnor07 (2007) defines Cauchy completion as a monad on the category of metric spaces with uniformly continuous functions (with setoid identities). Whether we can extend uniformly continuous functions with our definition remains to be investigated.*

Repeated Lipschitz extension can be applied to functions taking multiple arguments: if $f : A \to B \to T$ is Lipschitz in both arguments, the function $f_1 : A \to \mathcal{C}\,B \to T$ obtained by pointwise Lipschitz extension is itself a Lipschitz function into the Cauchy complete space $\mathcal{C}\,B \to T$.

Lemma 4.24. *If A is Cauchy complete and $f, g : T \to A$ are Lipschitz functions with constant L and $\varepsilon : \mathbb{Q}^+$ is such that $\forall (u : T)(\delta : \mathbb{Q}^+), f\ u \approx_{\varepsilon+\delta} g\ u$, then*

$$\forall (u : \mathcal{C}\,T)(\delta : \mathbb{Q}^+), \overline{f}\ u \approx_{\varepsilon+\delta} \overline{g}\ u$$

Proof. By simple induction on u, using lemma 3.11 in the limit case. □

Theorem 4.25 (Binary Lipschitz extension)**.** *If T is Cauchy complete and $f : A \to B \to T$ is such that for all $x : A$, $f\ x\ \_$ is Lipschitz with constant L_1 and for all $y : B$, $f\ \_\ y$ is Lipschitz with constant L_2, then f can be extended into $\overline{\overline{f}} : \mathcal{C}\,A \to \mathcal{C}\,B \to T$ with the same Lipschitz properties and coinciding with f on η values.*

Proof. Unary Lipschitz extension gives us $f_1 := \lambda x, \overline{f\ x} : A \to \mathcal{C}\,B \to T$ such that for all $x : A$, $f_1\ x\ \_$ is Lipschitz with constant $L1$.

f_1 is Lipschitz with constant L_2: let $\varepsilon : \mathbb{Q}^+$ and $x, y : A$ such that $x \approx_\varepsilon y$. We need to show that $f_1\ x \approx_{L_2 * \varepsilon} f_1\ y$, i.e. there merely exist $\delta_1, \kappa_1 : \mathbb{Q}^+$ such that $L_2 * \varepsilon = \delta_1 + \kappa_1$ and $\forall z : B, \overline{f\ x}\ z \approx_{\delta_1} \overline{f\ y}\ z$.

By roundedness there merely exist $\delta, \kappa : \mathbb{Q}^+$ such that $\varepsilon = \delta + \kappa$ and $x \approx_\delta y$. Use $\delta_1 := L_2 * \delta$ and $\kappa_1 := L_2 * \kappa$.

By roundedness there merely exist $\delta', \kappa' : \mathbb{Q}^+$ such that $\delta = \delta' + \kappa'$ and $x \approx_{\delta'} y$. By lemma 4.24 it suffices to prove

$$\forall (z : B)(\theta : \mathbb{Q}^+), f\ x\ z \approx_{L_2 * \delta' + \theta} f\ y\ z$$

Since $f\ \_\ z$ is Lipschitz with constant L_2 we have

$$f\ x\ z \approx_{L_2 * \delta'} f\ y\ z$$

then by roundedness the desired property.

$\mathcal{C}\,B \to T$ is Cauchy complete, so we have $\overline{\overline{f}} := \overline{f_1} : \mathcal{C}\,A \to \mathcal{C}\,B \to T$ Lipschitz with constant L_2.

By lemma 3.16 we have that for all $y : \mathcal{C}\,B$, $\overline{\overline{f}}\ \_\ y$ is Lipschitz with constant L_2.

By $\mathcal{C}$-induction and lemma 3.18 we have that for all $x : \mathcal{C}\,A$, $\overline{\overline{f}}\ x\ \_$ is Lipschitz with constant L_1. □

5. Cauchy Reals

We now have enough to define concepts specific to the Cauchy completion of the rationals, i.e. the Cauchy reals. Our goal is to show that they form an archimedean ordered field, a lattice, and that the closeness relation has the intended meaning $x \approx_\varepsilon y \leftrightarrow |x - y| < \varepsilon$ (with absolute value of x being the sup of x and $-x$).

Note that we use the constructive sense of ordered field, such that we have an apartness relation $x \# y$ expressing $0 < |x - y|$ and multiplicative inverse can only be applied on values apart from 0.

5.1 Addition and Order Relations

The Cauchy reals $\mathbb{R}_c$ are the Cauchy completion of the rationals $\mathcal{C}\,\mathbb{Q}$. Let $\mathrm{rat} : \mathbb{Q} \to \mathbb{R}_c$ be an alias for η.

We follow HoTT (2013) for the additive and order structure of $\mathbb{R}_c$: $0_{\mathbb{R}_c}$ is $\mathrm{rat}\ 0_{\mathbb{Q}}$, $1_{\mathbb{R}_c}$ is $\mathrm{rat}\ 1_{\mathbb{Q}}$, and $+, -, \cup, \cap$ and $|\_|$ are defined by Lipschitz extension. Then $x \leq y := x \cup y = y$ and $x < y := \exists q\ r : \mathbb{Q}, x \leq \mathrm{rat}\, q \wedge q < r \wedge \mathrm{rat}\, r \leq y$.

The HoTT book states:

> Furthermore, the extension is unique as long as we require it to be non-expanding in each variable, and just as in the univariate case, identities on rationals extend to identities on reals. Since composition of non-expanding maps is again non-expanding, we may conclude that addition satisfies the usual properties, such as commutativity and associativity.

This is a simple application of theorem 4.19. More complex uses require a little more attention to two issues:

- Consider transitivity of $\leq$:

  $$\forall x\ y\ z : \mathbb{R}_c, x \cup y = y \to y \cup z = z \to x \cup z = z$$

  This cannot be directly proven by continuity as the statement of theorem 4.19 does not allow for hypotheses which depend on the universally quantified variables.

  We can however strengthen this specific statement into one that can be solved by theorem 4.19: $\forall x\ y\ z : \mathbb{R}_c, x \cup ((x \cup y) \cup z) = (x \cup y) \cup z$. We will point out if this strengthening cannot be easily done.

- When showing that $\mathbb{R}_c$ is a group we need to prove $\forall x : \mathbb{R}_c, x + (-x) = 0$.

The issue is that for a binary function $f : A \to B \to C$, knowing that for all x and y $\lambda y, f\ x\ y$ and $\lambda x, f\ x\ y$ are continuous is not sufficient to show that $\lambda x, f\ x\ x$ is continuous. The hypothesis we really want is that f as the uncurried function from $A \times B$ to C is continuous.

If $\lambda y, f\ x\ y$ and $\lambda x, f\ x\ y$ are both Lipschitz with respective constant L and K then f is Lipschitz with constant $L + K$, so this is not a problem when dealing with functions defined through Lipschitz extension like addition. However, showing that multiplication is continuous as an uncurried function deserves an explicit proof.

Except for those which have to do with multiplication, the proofs from HoTT (2013) can be adapted with at most minor adjustments aside from the above remarks. Then $\mathbb{R}_c$ is a group, a lattice, $x \approx_\varepsilon y$ is equivalent to $|x - y| < \varepsilon$, etc.

The book lacks the proof that $\lambda y, x + y$ preserves $<$. We show this by proving that $x < y$ if and only if there merely is $\varepsilon : \mathbb{Q}^+$ such that $x + \operatorname{rat} \varepsilon \le y$, which then allows us to use properties proven by continuity.

Lemma 5.1. *Let $x, y : \mathbb{R}_c$ such that $x < y$. Then $\exists \varepsilon : \mathbb{Q}^+, x + \operatorname{rat} \varepsilon \le y$.*

Proof. By definition of $<$ there merely are $q, r : \mathbb{Q}$ such that $x \le \operatorname{rat} q < \operatorname{rat} r \le y$. We take $\varepsilon \coloneqq r - q$.
$x \le \operatorname{rat} q$ so

$$x + \operatorname{rat} \varepsilon = \operatorname{rat}(r - q) + x \le \operatorname{rat}(r - q) + \operatorname{rat} q = \operatorname{rat} r \le y$$

□

For the second direction, it is enough to show that $x < x + \operatorname{rat} \varepsilon$. We need a helper lemma first.

Lemma 5.2. *Let $\varepsilon : \mathbb{Q}^+$ and $x, y : \mathbb{R}_c$ such that $x \approx_\varepsilon y$. Then $y \le x + \operatorname{rat} \varepsilon$.*

Proof. $y - x \le |x - y| < \operatorname{rat} \varepsilon$ so $y \le x + \operatorname{rat} \varepsilon$. □

We can generalize HoTT (2013) lemma 11.3.43:

Lemma 5.3. *Let $x\ y\ z : \mathbb{R}_c$ and $\varepsilon : \mathbb{Q}^+$ such that $x < y$ and $x \approx_\varepsilon z$. Then $z < y + \operatorname{rat} \varepsilon$.*

Proof. There merely is $q : \mathbb{Q}$ between x and y. By HoTT (2013) lemma 11.3.43, $z < \operatorname{rat}(q + \varepsilon) \le y + \operatorname{rat} \varepsilon$.

Note here that we cannot prove $\operatorname{rat}(q + \varepsilon) < y + \operatorname{rat} \varepsilon$ since we prove that $\lambda u, u + \operatorname{rat} \varepsilon$ preserves $<$ using this lemma. □

Lemma 5.4. *$<_{\mathbb{R}_c}$ is cotransitive:*

$$\forall x, y, z : \mathbb{R}_c, x < y \to x < z \lor z < y$$

Note that $\lor$ is the truncated disjunction, i.e. the case distinction can only be made when proving a mere proposition.

Proof. By definition of $<$ we can reduce to the case where $x \coloneqq \operatorname{rat} q$ and $y \coloneqq \operatorname{rat} r$ for some $q, r : \mathbb{Q}$. Then we use simple $\mathcal{C}$ −induction on z.

In the base case, we inherit the property from $\mathbb{Q}$.

In the limit case, we have $x : \operatorname{Approximation} \mathbb{R}_c$ such that (induction hypothesis)

$$\forall (\varepsilon : \mathbb{Q}^+)(q, r : \mathbb{Q}), q < r \to \operatorname{rat} q < x_\varepsilon \lor x_\varepsilon < \operatorname{rat} r$$

Let $q, r : \mathbb{Q}$ such that $q < r$. There are $q_1, r_1 : \mathbb{Q}$ such that $q < q_1 < r_1 < r$, and $\delta : \mathbb{Q}^+$ such that $\delta < q_1 - q$ and $\delta < r - r_1$.
Using the induction hypothesis with δ and $q_1 < r_1$ we can do a case distinction:

- if $\operatorname{rat} q_1 < x_\delta$, we have $-x_\delta < \operatorname{rat}(-q_1)$ and since $x_\delta \approx_{q_1 - q} \lim x$ and $-$ is non-expanding we have using lemma 5.3 that $-\lim x < \operatorname{rat}(-q_1 + (q_1 - q)) = \operatorname{rat}(-q)$.
- if $x_\delta < \operatorname{rat} r_1$ using lemma 5.3 we have $\lim x < \operatorname{rat}(r_1 + (r - r_1)) = \operatorname{rat} r$. □

Lemma 5.5. *For all $x : \mathbb{R}_c$ and $\varepsilon : \mathbb{Q}^+$, $x < x + \operatorname{rat} \varepsilon$.*

Proof. By simple $\mathcal{C}$ −induction on x.

In the base case we inherit the result from $\mathbb{Q}$.

In the limit case, let $x : \operatorname{Approximation} \mathbb{R}_c$ such that (induction hypothesis)

$$\forall \varepsilon, \delta : \mathbb{Q}^+, x_\varepsilon < x_\varepsilon + \operatorname{rat} \delta$$

Let $\varepsilon : \mathbb{Q}^+$. By lemma 5.3 and the induction hypothesis we have $\forall \delta, \kappa : \mathbb{Q}^+, \lim x < x_\delta + \operatorname{rat}(\delta + \kappa)$.
Using $\delta \coloneqq \varepsilon/3$ and $\kappa \coloneqq 2\varepsilon/9$, by cotransitivity of $<$ (HoTT (2013) lemma) for $\lim x + \operatorname{rat} \varepsilon$ we have either

- $\lim x < \lim x + \operatorname{rat} \varepsilon$ as desired
- $\lim x + \operatorname{rat} \varepsilon < x_\delta + \operatorname{rat}(\delta + \kappa)$, but this is absurd:
  By lemma 5.2 $x_\delta \le \lim x + \operatorname{rat}(\delta + \varepsilon/9)$, then by adding $\delta + \kappa = 11\varepsilon/9$ to both sides $x_\delta + \operatorname{rat}(\delta + \kappa) \le \lim x + \operatorname{rat} \varepsilon < x_\delta + \operatorname{rat}(\delta + \kappa)$. □

We also need to prove $x \le y$ from $\neg y < x$.

Lemma 5.6. *Real numbers can be approximated from below: let $x : \mathbb{R}_c$, then $\lambda \varepsilon : \mathbb{Q}^+, x - \operatorname{rat} \varepsilon$ is an approximation with limit x.*

Proof. HoTT (2013) theorem 11.3.44 (expressing $x \approx_\varepsilon y$ as $|x - y| < \operatorname{rat} \varepsilon$) lets us reduce this to bureaucratic work. □

The following lemma is easy:

Lemma 5.7. *Let $f : \mathbb{R}_c \to \mathbb{R}_c$ Lipschitz with constant L and $x : \operatorname{Approximation} \mathbb{R}_c$. Then $\lambda \varepsilon, f\ x_{\varepsilon/L}$ is an approximation with limit $f\ (\lim x)$.* □

Lemma 5.8. *Given $x, y : \mathbb{R}_c$, if $x < y$ is false then $y \le x$.*

Proof. Let $x, y : \mathbb{R}_c$ such that $x < y$ is false. Let $z := x - y$.

First note that $\forall \varepsilon : \mathbb{Q}^+, -\operatorname{rat}\varepsilon < z$: let $\varepsilon : \mathbb{Q}^+$. Since $y - \operatorname{rat}\varepsilon < y$ by cotransitivity either $y - \operatorname{rat}\varepsilon < x$ as desired, or $x < y$ which is absurd.

$y \leq x$ is equivalent to $0 \leq z$ i.e. $0 \cup z = z$. By lemma 5.6 $0 = \lim(\lambda\varepsilon, -\operatorname{rat}\varepsilon)$ so by lemma 5.7 $0 \cup z = \lim(\lambda\varepsilon, -\operatorname{rat}\varepsilon \cup z) = \lim(\lambda\varepsilon, z) = z$. □

We still need to define multiplication, prove that it is continuous and behaves well regarding $<$, and show that reals apart from 0 are invertible.

5.2 Multiplication

We cannot use binary Lipschitz extension (theorem 4.25) to define multiplication as the Lipschitz constant of a partially applied multiplication $(\lambda r : \mathbb{Q}, q * r)$ depends on q. The definition in HoTT (2013) first defines squaring and uses the identity $u * v = \frac{(u+v)^2 - u^2 - v^2}{2}$ to define multiplication from it. We stay closer to simple Lipschitz extension by defining multiplication on bounded intervals then joining these to cover $\mathbb{R}_c$.

Definition 5.9 (Definition by surjection)**.** *Let A B and C sets, and $f : A \to C$ and $g : A \to B$ functions such that g is a surjection and f respects $\sim_g$ the equivalence relation on A induced by g.*

Then B is equivalent to $A/\sim_g$ the quotient of A by $\sim_g$ and there is a function $f_{\sim_g} : A/\sim_g \to C$ acting like f.

Composing $f_{\sim_g}$ with the equivalence defines the function $\overline{f}_{\sim_g} : B \to C$ such that $\forall x : A, \overline{f}_{\sim_g}\ (g\ x) = f\ x$.

Definition 5.10 (Intervals)**.** *For $a, b : \mathbb{Q}$ (resp. $a, b : \mathbb{R}_c$), the interval space $[a, b] := \Sigma_x a \leq x \leq b$ inheriting the closeness relation from the first projection forms a premetric space.*

For $x : \mathbb{Q}$ (resp. $x : \mathbb{R}_c$), $a \leq a \cup (x \cap b) \leq b$ so we can define $[x]_{a,b} : [a, b]$. If $a \leq x \leq b$ then $[x]_{a,b}$ has its first projection equal to x.

Definition 5.11 (Left multiplication by a rational)**.** *For any $q : \mathbb{Q}$, $\lambda r : \mathbb{Q}, q * r$ is Lipschitz with constant $|q| + 1$, so we define $\lambda(q : \mathbb{Q})(y : \mathbb{R}_c), q * y$ by Lipschitz extension.*

Definition 5.12 (Bounded multiplication)**.** *For $a : \mathbb{Q}^+$ and $y : [-\operatorname{rat} a, \operatorname{rat} a]$ we define $\lambda x : \mathbb{R}_c, x *_a y$ by Lipschitz extension.*

Proof. We need to check that $\lambda q : \mathbb{Q}, q * y$ is Lipschitz with constant a. Using HoTT (2013) theorem 11.3.44 it suffices to show that for $x : \mathbb{R}_c$ such that $|x| \leq \operatorname{rat} a$ we have $\forall q\ r : \mathbb{Q}, |q * x - r * x| \leq \operatorname{rat}(|q - r| * a)$. This is obtained by continuity. □

Lemma 5.13. *Cauchy reals are bounded by rationals, i.e. for all $x : \mathbb{R}_c$ there merely is $q : \mathbb{Q}^+$ such that $|x| < \operatorname{rat} q$.*

Proof. By simple $\mathcal{C}$ −induction on x.

In the base case we take $q := |x| + 1$.

In the limit case, where x is $\lim f$, by the induction hypothesis there merely is $q : \mathbb{Q}^+$ such that $|f\ 1| < \operatorname{rat} q$. $|f\ 1| \approx_2 |x|$ so $x < \operatorname{rat}(q + 2)$. □

Lemma 5.14. *Let the following function be defined by the obvious projections:*

$$\{\_\} : \Sigma_{a:\mathbb{Q}^+}[-\operatorname{rat} a, \operatorname{rat} a] \to \mathbb{R}_c$$

It is surjective and respects bounded multiplication, i.e.

$$\forall x\ y\ z, \{x\} = \{y\} \to z *_{x_1} x_2 = z *_{y_1} y_2$$

Proof. The function is surjective because reals are bounded by rationals. It respects bounded multiplication by continuity. □

Definition 5.15 (Multiplication)**.** *For $x : \mathbb{R}_c$ we define $\lambda y : \mathbb{R}_c, x * y$ from*

$$\lambda y : \Sigma_{a:\mathbb{Q}^+}[-\operatorname{rat} a, \operatorname{rat} a], x *_{y_1} y_2$$

and surjectivity of $\{\_\}$.

Multiplication is now defined, with the following properties by definition:

Lemma 5.16. *For $x : \mathbb{R}_c$ and $a : \mathbb{Q}^+$ and $y : [-\operatorname{rat} a, \operatorname{rat} a]$*

$$x * y_1 = x *_a y$$

Proof. By unfolding definition 5.9. □

Lemma 5.17. *Multiplication computes on rationals:*

$$\forall q, r : \mathbb{Q}, \operatorname{rat} q * \operatorname{rat} r \equiv \operatorname{rat}(q * r)$$

Proof. Checking a conversion is decidable so this proof is left as an exercise to the reader. □

We now need to show that multiplication is continuous as an uncurried function.

Lemma 5.18. *For $a : \mathbb{Q}^+$ and $y : \mathbb{R}_c$ such that $|y| \leq \operatorname{rat} a$, $\lambda x : \mathbb{R}_c, x * y$ is Lipschitz with constant a.*

Proof. Using lemma 5.16 and definition 5.12. □

Lemma 5.19. *For all $y : \mathbb{R}_c$, $\lambda x : \mathbb{R}_c, x * y$ is continuous.*

Proof. Let $y : \mathbb{R}_c$, there merely is $a : \mathbb{Q}^+$ such that $|y| \leq \operatorname{rat} a$. By lemma 5.18 $\lambda x : \mathbb{R}_c, x*y$ is Lipschitz with constant a and therefore continuous. □

Lemma 5.20. *For $q : \mathbb{Q}$ and $x : \mathbb{R}_c$, $\operatorname{rat} q * x = q * x$.*

Proof. Using lemma 5.16 for some a bounding x. □

Lemma 5.21. *Multiplication and negation distribute inside the absolute value:*

$$\forall a, b, c : \mathbb{R}_c, |a * b - a * c| = |a| * |b - c|$$

Proof. We can reduce to the case where a is rational by continuity, then use lemma 5.20 to replace real to real multiplication with rational to real multiplication and finish by continuity. $\square$

Lemma 5.22. *Multiplication is compatible with $\leq$ under absolute value: for $a, b, c, d : \mathbb{R}_c$, if $|a| \leq |c|$ and $|b| \leq |d|$ then $|a| * |b| \leq |c| * |d|$.*

Proof. Again we use continuity to reduce a and c (the variables appearing to the left of the multiplications) to their rational case, then rewrite the desired property to use multiplication of a rational and a real and finish with continuity. $\square$

Theorem 5.23. *Multiplication is continuous as a function of 2 variables, i.e. given u_1 and $v_1 : \mathbb{R}_c$ and $\varepsilon : \mathbb{Q}^+$ there merely exists $\delta : \mathbb{Q}^+$ such that for all u_2 and $v_2 : \mathbb{R}_c$, if $u_1 \approx_\delta u_2$ and $v_1 \approx_\delta v_2$ then $u_1 * v_1 \approx_\varepsilon u_2 * v_2$.*

Proof. Let $u_1, v_1 : \mathbb{R}_c$ and $\varepsilon : \mathbb{Q}^+$. There merely is $\delta : \mathbb{Q}^+$ such that $|u_1| < \operatorname{rat}\delta$ and $|v_1| < \operatorname{rat}\delta$. Let $\kappa := \delta + 1$, then in the lemma's statement we take $\delta := 1 \cap \frac{\varepsilon}{2(\kappa+1)}$.

Let $u_2, v_2 : \mathbb{R}_c$ such that

- $u_1 \approx_{1 \cap \frac{\varepsilon}{2(\kappa+1)}} u_2$
- $v_1 \approx_{1 \cap \frac{\varepsilon}{2(\kappa+1)}} v_2$

Then:

- $u_1 * v_1 \approx_{\varepsilon/2} u_2 * v_1$:
  $|v_1| \leq \operatorname{rat}\delta$ so $\lambda y : \mathbb{R}_c, y * v_1$ is Lipschitz with constant δ and it suffices to prove $u_1 \approx_{\varepsilon/2\delta} u_2$.
  This is true from roundedness and the first $\approx$ hypothesis since $1 \cap \frac{\varepsilon}{2(\kappa+1)} \leq \varepsilon/2\delta$.
- $u_2 * v_1 \approx_{\varepsilon/2} u_2 * v_2$:
  By HoTT (2013) theorem 11.3.44 we look to prove $|u_2 * v_1 - u_2 * v_2| = |u_2| * |v_1 - v_2| < \varepsilon/2$.
  In fact we have
  - $|u_2| \leq |\kappa| = \kappa$ since $|u_1| \leq \kappa$ and $u_1 \approx_1 u_2$.
  - $|v_1 - v_2| \leq |\frac{\varepsilon}{2(\kappa+1)}| = \frac{\varepsilon}{2(\kappa+1)}$ since $|v_1 - v_2| < 1 \cap \frac{\varepsilon}{2(\kappa+1)}$.

  Then by lemma 5.22 we have $|u_2| * |v_1 - v_2| \leq |\kappa| * |\frac{\varepsilon}{2(\kappa+1)}| = \varepsilon/2 * \frac{\kappa}{\kappa+1} < \varepsilon/2$.

By triangularity $u_1 * v_1 \approx_\varepsilon u_2 * v_2$. $\square$

This is enough to show that $\mathbb{R}_c$ forms a partially ordered ring, but we still need to link multiplication and $<$.

Lemma 5.24. *Multiplication of positive values produces a positive value: let $x, y : \mathbb{R}_c$ such that $0 < x$ and $0 < y$, then $0 < x * y$.*

Proof. Let $x, y : \mathbb{R}_c$ such that $0 < x$ and $0 < y$, then there merely are $\varepsilon, \delta : \mathbb{Q}^+$ such that $\varepsilon < x$ and $\delta < y$.

By continuity multiplication preserves $\leq$ for nonnegative values, so $0 < \operatorname{rat}(\varepsilon * \delta) \leq x * y$. $\square$

Lemma 5.25. *For $x, y : \mathbb{R}_c$, if $0 \leq x$ and $0 < x * y$ then $0 < y$.*

Proof. There merely is $\varepsilon : \mathbb{Q}^+$ such that $\operatorname{rat}\varepsilon < x * y$. By lemma 5.13 there merely is $\delta : \mathbb{Q}^+$ such that $|x| < \operatorname{rat}\delta$. Then it suffices to prove $0 < \varepsilon/\delta \leq y$.

We do this using lemma 5.8: suppose $y < \varepsilon/\delta$. Since $0 \leq y$ (if $y < 0$ then $x * y \leq 0$ which is absurd), $x * y \leq |x| * y \leq \varepsilon < x * y$ which is absurd. $\square$

5.3 Multiplicative Inverse

The multiplicative inverse for $\mathbb{Q}$ is Lipschitz on intervals $[\varepsilon, +\infty]$ for $\varepsilon : \mathbb{Q}^+$. We use this to extend it to positive reals, then to reals apart from 0 using negation.

Definition 5.26. *For $\varepsilon : \mathbb{Q}^+$ the function $\lambda q : \mathbb{Q}, \frac{1}{\varepsilon \cup q}$ is defined and Lipschitz with constant ε^{-2}.*

Then for $x : \Sigma_{\varepsilon:\mathbb{Q}^+, x:\mathbb{R}_c} \operatorname{rat}\varepsilon \leq x$ we define

$$/_\Sigma x := \overline{\left(\lambda q : \mathbb{Q}, \frac{1}{x_\varepsilon \cup q}\right)} x_x$$

Definition 5.27. *We define the inverse of positive reals by surjection (definition 5.9) using $/_\Sigma$ and the obvious surjection from $x : \Sigma_{\varepsilon:\mathbb{Q}^+, x:\mathbb{R}_c} \operatorname{rat}\varepsilon \leq x$ to $\Sigma_{x:\mathbb{R}_c} 0 < x$.*

For negative values we use the identity $\frac{1}{x} := -\frac{1}{-x}$.

This gives $\frac{1}{x}$ for any x such that $x \# 0$.

Lemma 5.28. $\forall q : \mathbb{Q}, \operatorname{rat} q \# 0 \to \frac{1}{\operatorname{rat} q} = \operatorname{rat}(\frac{1}{q})$

Proof. The negative case is easily reduced to the positive case.

In the positive case there merely are $r, s : \mathbb{Q}$ such that $0 \leq r < s \leq q$, then $\frac{1}{\operatorname{rat} q}$ reduces to $\operatorname{rat} \frac{1}{q \cup s}$ which is equal to $\operatorname{rat} \frac{1}{q}$ since $s \leq q$. $\square$

The following lemma is easily obtained:

Lemma 5.29. *For $x : \mathbb{R}_c$ and $\varepsilon : \mathbb{Q}^+$ such that $\operatorname{rat}\varepsilon \leq x$, $\frac{1}{x} = \overline{\left(\lambda q : \mathbb{Q}, \frac{1}{\varepsilon \cup q}\right)} x$.* $\square$

Lemma 5.30. *$\forall x : \mathbb{R}_c$, if $x \# 0$ then $x * \frac{1}{x} = 1$.*

Proof. We can reduce to the case where $0 < x$. Then there merely is $\varepsilon : \mathbb{Q}^+$ such that $\operatorname{rat}\varepsilon \leq x$.

By continuity $x * \overline{\left(\lambda q : \mathbb{Q}, \frac{1}{\varepsilon \cup q}\right)} x = 1$ for all x such that $\operatorname{rat}\varepsilon \leq x$, and by definition, $\frac{1}{x} = \overline{\left(\lambda q : \mathbb{Q}, \frac{1}{\varepsilon \cup q}\right)} x$. $\square$

Together with the results from HoTT (2013) section 11.3.3 we now have all results needed for $\mathbb{R}_c$ to form an Archimedean ordered field as desired.

6. A Partial Function on Cauchy Reals

Without additional axioms, we can't define any non-constant function from $\mathbb{R}_c$ to booleans $\mathbb{B}$. In other words, no non-trivial property on $\mathbb{R}_c$ is decidable.

However we can encode non-termination as an effect in the *partiality monad*, where the type of computations producing values of type A is denoted $A_\bot$. Then we can define a function $isPositive : \mathbb{R}_c \to 2_\bot$ which produces $true$ on positive reals, $false$ on negative reals and does not terminate on 0.

6.1 The Partiality Monad

In Partiality, Revisited (2016), Altenkirch and Danielsson define the type $A_\bot$ of computations producing values of type A as a HIIT. They implemented it in Agda and proved certain properties such as the existence of fixpoints and that it forms the free $\omega-$CPO on A.

Definition 6.1 (Increasing sequences)**.** *Let $A : Type$ with some relation $\leq$ on it.*

$$\text{IncreasingSequence}\, A := \Sigma_{f:\mathbb{N}\to A} \forall n, f_n \leq f_{n+1}$$

As with Cauchy approximations when f : IncreasingSequence A we write f for the underlying function.

In this paper we will only consider increasing sequences in the following type:

Definition 6.2. *Given A a type, the type $A_\bot$ is defined simultaneously with its order. It has the following constructors:*

- $\eta : A \to A_\bot$
- $\bot : A_\bot$
- sup : IncreasingSequence $A_\bot \to A_\bot$

with a path constructor of type $\forall x, y : A_\bot, x \leq y \to y \leq x \to x = y$.

The order has constructors of types

- $\forall x : A_\bot, x \leq x$
- $\forall x : A_\bot, \bot \leq x$
- $\forall f, x, sup\ f \leq x \to \forall n, f_n \leq x$
- $\forall f, x, (\forall n, f_n \leq x) \to sup\ f \leq x$

and is truncated to be propositional.

As with the Cauchy completion we have simple induction on values and simple induction on the auxiliary relation $\leq$ to prove inhabitedness of propositional types depending on computations, and non-dependent mutual recursion to define values from computations.

Altenkirch and Danielsson suggest a way of defining $isPositive : \mathbb{R}^q \to \mathbf{2}_\bot$, where $\mathbb{R}^q$ is the quotient of Cauchy sequences of $\mathbb{Q}$ by the appropriate equivalence (e.g. that the difference has limit 0).

They first define it on Cauchy sequences of $\mathbb{Q}$ using the fixpoint operator provided by the partiality functor, then show that it respects the equivalence and extend it to the quotient $\mathbb{R}^q$.

We could not adapt that definition for the HIIT Cauchy real numbers. In the rest of this paper, we will use an alternate method:

- For $P : Prop$, $(\Sigma_{p:\mathbf{1}_\bot} p = \eta\star \leftrightarrow P)$ is propositional.

  We can use simple $\mathcal{C}-$induction to define for each $x : \mathbb{R}_c$ and $q : \mathbb{Q}$ an element of this $\Sigma-$type where P is $x < \operatorname{rat} q$.
- From p and $q : \mathbf{1}_\bot$ such that p and q are not both $\eta\star$, we define $interleave\ p\ q : \mathbf{2}_\bot$ indicating which if any is $\eta\star$.
- We interleave the values defined from $-x < 0$ and $x < 0$ to define $isPositive\ x$.

We assume the properties of $A_\bot$ for arbitrary A from Partiality, Revisited (2016). Let us then focus on the properties of $\mathbf{1}_\bot$.

6.2 The Sierpinski Space

If $A_\bot$ is the type of possibly non-terminating computations returning a value of type A, then $\mathbf{1}_\bot$ is the type of semi-decision procedures: $p : \mathbf{1}_\bot$ semi-decides all propositions equivalent to $p = \eta\star$.

Definition 6.3. *$\mathbf{1}_\bot$ has a greatest element $\top := \eta\star$.*

Proof. $\forall x : \mathbf{1}_\bot, x \leq \top$ by simple induction on x. ☐

We can interpret $p : \mathbf{1}_\bot$ as the proposition $p = \eta\star$ (equivalently, $\eta\star \leq p$).

Then trivially we always have $\top$ and never $\bot$.

Lemma 6.4. *For all $a\ b : \mathbf{1}_\bot$, $a \leq b$ if and only if $a \to b$.*

Proof.
if $a \leq b$ then $a \to b$: suppose a, i.e. $\top \leq a$. Then $\top \leq a \leq b$, i.e. b.
if $a \to b$ then $a \leq b$: by simple induction on a, each case being trivial. ☐

We can also interpret $\vee$ into $\mathbf{1}_\bot$ (and $\wedge$, but we do not need it for $isPositive$).

Definition 6.5 (Join on $\mathbf{1}_\bot$)**.** *Proof.* We define an auxiliary function by mutual recursion: for all $y : \mathbf{1}_\bot$ there is $\cup_y : \mathbf{1}_\bot \to \Sigma_{z:\mathbf{1}_\bot} y \leq z$, then $x \cup y := (\cup_y\ x)_1$. Then $x \cup y$ is the first projection of $\cup_y\ x$. It computes as follows:

- $\top \cup\ y := \top$
- $\bot \cup\ y := y$
- $(\sup f) \cup\ y := \sup(\lambda n, f_n \cup\ y)$

The proofs of the required properties are trivial. Note that we need the auxiliary function as we need a proof that $\forall x : \Sigma_{z:\mathbf{1}_\bot} y \leq z, \cup_y \bot = y \leq x_1$. ☐

Lemma 6.6. *$x \cup y$ is the least upper bound of x and y. Then $\cup$ is a monoid operator with identity element $\bot$.* ☐

Lemma 6.7. *For all $a\ b : \mathbf{1}_\bot$, $a \cup b$ if and only if $a \vee b$.* ☐

$\mathbf{1}_\bot$ has a countable join operator, but it is limited to increasing sequences. Thanks to the binary join we can remove this limit to define interpret properties $\exists n : \mathbb{N}, P_n$ and even $\exists x : A, P\ x$ when A is enumerable.

Definition 6.8. *For all* $f : \mathbb{N} \to \mathbf{1}_\perp$ *there is a least upper bound* $\sup f$ *of all the* f_n.

Proof. We have $\sup f$ for monotonous sequences, so for arbitrary $f : \mathbb{N} \to \mathbf{1}_\perp$ we define $f^{\leq} : \mathbb{N} \to \mathbf{1}_\perp$ by $f^{\leq}\ n := \bigcup_{m \leq n} f\ m$.

Then $f^{\leq}$ is monotonous and $\sup f := \sup f^{\leq}$ is the least upper bound of all the f_n. □

That $\sup f$ semi-decides $\exists x, f\ x$ is trivial.

6.3 Interleaving

Definition 6.9 (Disjoint)**.** *a and $b : \mathbf{1}_\perp$ are disjoint when they do not both hold, i.e.* $a \to b \to \mathbf{0}$.

Interleaving lets us define a value in $\mathbf{2}_\perp$ from two values in $\mathbf{1}_\perp$ which are not both $\top$. If we see $x\ y : \mathbf{1}_\perp$ as semi-decision procedures then the interleaving of x and y is $\eta\, true$ if x terminates (i.e. $x = \top$), $\eta\, false$ if y terminates and does not terminate if neither terminates. If computing on a Turing machine it would be obtained by interleaving simulated steps of x and y until one terminates, then returning a value depending on which one terminated.

We can only interleave disjoint values: a Turing machine could pick whichever one terminates first, but we have hidden those distinctions away using higher inductive types.

The following definition uses the fully general mutual induction principle of $\mathbf{1}_\perp$, making it fairly complex. Hopefully it can be simplified in the future.

Definition 6.10. *We define by mutual induction a function*

$$\begin{aligned} interleave\star : \forall a\ b : \mathbf{1}_\perp, disjoint\ a\ b \to \\ \Sigma_{c:\mathbf{2}_\perp}(map\ (\lambda\_, false)\ b) \leq c) \end{aligned}$$

where $map : \forall A\ B : Type, (A \to B) \to A_\perp \to B_\perp$ is the map of the partiality monad, and in parallel a proof that for all $a\ a' : \mathbf{1}_\perp$, if $a \leq a'$ then for all $b : \mathbf{1}_\perp$ disjoint with a and with a', $interleave_\star\ a\ b \leq interleave_\star\ a'\ b$.

Then the interleaving function $interleave$ is the first projection of $interleave_\star$. It computes as follows:

- $interleave\ \top\ b := \eta\, true$
- $interleave\ \perp\ b := map\ (\lambda\_, false) b$
- $interleave\ (\sup f)\ b := \sup\,(\lambda n, interleave\ f_n\ b)$

Some attention must be taken to keep track of the disjointness proofs which are left implicit on paper.

Lemma 6.11. *If $a : \mathbf{1}_\perp$ is disjoint from $\top$ then*

$$interleave\ a\ \top = \eta\, false$$

Proof. a is disjoint from $\top$ so $a = \perp$ and $interleave\ a\ \top = map\ (\lambda\_, false)\ \top = \eta\, false$. □

Lemma 6.12. *For $a\ b : \mathbf{1}_\perp$ disjoint $interleave\ a\ b = \eta\, true$ (resp. $\eta\, false$) if and only if a holds (resp. b holds).*

Proof. By simple induction on a in the first direction, by computation in the second (note that if b then $a = \perp$ as they are disjoint). □

6.4 Partial Comparison of Real Numbers with Rational Numbers

Lemma 6.13. *For all $x : \mathbb{R}_c$ and $q : \mathbb{Q}$, $x < \operatorname{rat} q$ is semi-decidable, i.e. we define $s : \mathbf{1}_\perp$ such that $s \leftrightarrow x < \operatorname{rat} q$.*

Proof. By simple induction on x.

In the base case, for all $q\ r : \mathbb{Q}$, $\operatorname{rat} q < \operatorname{rat} r$ is decidable so we pick $s := \top$ or $\perp$ as appropriate.

In the limit case, if $x : \mathrm{Approximation}\,\mathbb{R}_c$ such that for all ε and q, $x_\varepsilon < \operatorname{rat} q$ is semi-decidable, let $q : \mathbb{Q}$, we take $s := \exists \varepsilon\ \delta : \mathbb{Q}^+, x_\varepsilon < \operatorname{rat}(q - \varepsilon - \delta)$ (interpreted as a value in $\mathbf{1}_\perp$). Then to show correctness:

- if $\exists \varepsilon\ \delta : \mathbb{Q}^+, x_\varepsilon < \operatorname{rat}(q - \varepsilon - \delta)$ then $\lim x < \operatorname{rat} q = \operatorname{rat}(q - \varepsilon - \delta + \varepsilon + \delta)$ by lemma 5.3.
- if $\lim x < \operatorname{rat} q$, there merely is $r : \mathbb{Q}$ such that $\lim x < \operatorname{rat} r$ and $r < q$. Let $\varepsilon := q - r$.
  Then $x_{\frac{\varepsilon}{4}} < \operatorname{rat}(q - \varepsilon - \varepsilon) = \operatorname{rat}(r + \varepsilon + \varepsilon)$ by lemma 5.3.

□

Definition 6.14. *For $x : \mathbb{R}_c$ let $isPositive\ x$ be the interleaving of the semi-decisions for $-x < 0$ and $x < 0$.*

Theorem 6.15. *Let $x : \mathbb{R}_c$.*

- $0 < x$ *if and only if* $isPositive\ x = \eta\, true$
- $x < 0$ *if and only if* $isPositive\ x = \eta\, false$
- $isPositive\ 0 = \perp$

Proof. By lemmas 6.11 and 6.12 and computation. □

7. Conclusion

We have defined a Cauchy completion operation which is a monad on the category of premetric spaces with an appropriate closeness relation and Lipschitz functions. When applied to the space of rational numbers it produces a Cauchy complete archimedean ordered field generated by rationals and limits of Cauchy approximations, i.e. the Cauchy reals. Finally we have defined and proven correct a semi-decision procedure (in the sense of Partiality, Revisited (2016)) for comparing a Cauchy real and a rational number.

Acknowledgements

This paper is the result of two internships under the direction of Bas Spitters, at Radbough University and at the University of Aarhus.

This work has been funded by the CoqHoTT ERC Grant 637339.

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