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Abstract: In this paper, we present an effective flexible formulation for the capacitated multi-item lot-sizing problem with setup carryovers and setup times. The formulation can accommodate setup times, single or multi-period setup carry-overs, backorders with limits on the number of backorder periods, and shelf-life restrictions without the need for any additional variables and constraints. We provide empirical evidence of the superiority of our model over conventional formulations by comparing LP lower bounds generated on a number of randomly generated test problems. Our flexible formulation dominated the results in 100% of the problem cases.

Keywords: Production Lot Sizing, Setup Times, Setup Carryover, Backorders, Generalized Formulation

1 Introduction

The single-level capacitated multi-item lot-sizing problem (CMLSP) without setup times is well known. It consists of scheduling N items over a planning horizon of T periods in a single-level production facility. Demands are time varying and are known, and are to be satisfied without backlogging. The objective is to minimize the sum of setup and carrying costs over the planning horizon subject to limited capacity in each time period.

The CMLSP continues to be of importance in batch manufacturing environments. Over the years, formulations of the problem have expanded from earlier versions to incorporate various problem features such as backorders, setup times, and setup carryovers. In order to accommodate these additional features, several of the formulations introduce new variables and constraints that often led to greater model complexity. The added complexity has often meant that the problems became harder to solve optimally, largely because the LP lower bounds were not particularly strong.

In this paper, we propose a very flexible formulation that accommodates without the need for any additional variables and constraints, problem features such as setup times, setup carry-overs, backorders, shelf-life restrictions, and preventive maintenance. The formulation is a generalization of a formulation proposed by [1]. We demonstrate that the model is very effective in providing strong lower bounds from the LP relaxation of the formulation, hence providing the opportunity for reasonable sized problems to be solved to optimality, or for large problems to yield good feasible solutions from heuristic approaches that require good lower bounds.

In the next section we provide a brief literature summary. In section 3 we discuss the various model formulations. Section 4 provides empirical evidence of the dominance of the proposed model over the conventional formulation. The paper concludes in section 5 with discussion and conclusions.

2 Literature review

The capacitated multi-item lot-sizing problem without setup times is known to be NP-hard [2]. When positive setup times are incorporated in the model the feasibility problem becomes NP-complete [3].

Over the last 20-30 years there have been numerous approaches to the problem and its variants. There are several math programming approaches that use an LP relaxation of the model to generate a lower bound and subsequently employ a solution scheme such as branch and bound or Lagrangian relaxation to close the gap between the lower

bound and a feasible primal solution. Lagrangian-based algorithms such as those of [1, 4-10] are all examples of approaches that require an efficient lower bound.
Many of the solution approaches to the capacitated lot-sizing problem that employ a lower bounding scheme often begin with the mathematical formulation found in [11]. However [1,7] presents a formulation that produces superior lower bounds to the formulation found in [11]. The model formulation in found in [11] often has to be modified by introducing additional variables and constraints whenever variants of the model must include features such as: backorders, setup times, and setup carryovers. The modifications often make the resulting problem cumbersome and difficult to solve [8, 12, 13].

3 Formulations of the CMLSP and its Variants

Much of the early work on solving the CMLSP was predicated on the formulation of the model presented in [11] and [2]. The model parameters and variables are as follows:

x_{it} - the quantity of product i produced in period t;

$$y_{it} = \begin{cases} 1 & \text{if } x_{it} > 0 \\ 0 & \text{Otherwise} \end{cases}$$

s_i - setup cost for item i;

h_{it} - the direct variable cost (holding cost) associated with one unit of item i produced in period t;

N - the set of items as well as its cardinality;

a_i - the capacity absorption rate in time per unit for item i;

C_t - the capacity available in period t in time units;

T - the set of periods as well as its cardinality;

d_{it} - the demand in units for item i in period t;

I_{it} - the inventory in units for item i in period t;

3.1 The Capacitated Multi-Item Lot Sizing Problem Without Backlogging

This model was proposed by Dixon and Silver [2] as a 2-index model. The model is as follows:

$$M1: Min \sum_{i \in \mathrm{N}} \sum_{t \in \mathrm{T}} h_{it} I_{it} + \sum_{i \in \mathrm{N}} \sum_{t \in \mathrm{T}} s_i y_{it} \tag{1}$$

Subject to:

$$\sum_{i \in N} a_i x_{it} \le c_t \qquad \forall_t \tag{2}$$

$$I_{it-1} + x_{it} - I_{it} - d_{it} = 0 \qquad \forall_{it} \tag{3}$$

$$x_{it} - m_{it} y_{it} \le 0 \qquad \forall_{it} \tag{4}$$

$$x_{it}, I_{it} \ge 0 \qquad \forall_{it} \tag{5}$$

$$y_{it} \in \{0, 1\} \qquad \forall_{it} \tag{6}$$

The objective of model M1 is to minimize the sum of set up and holding costs. Constraint (2) limits the capacity consumed in each period. Constraint (3) is the inventory balance constraint. Constrain (4) is a logical constraint that triggers a setup when production of item i takes place in period t. The constant m_{it} must be $\ge x_{it}$. It is the minimum of the capacity in period t or the sum of the demands that can be satisfied by production in that period. The constant m_{it} is computed as follows:

$$m_{it} = \min\left\{c_t, \sum_{r=t,T} d_{ir}\right\} \qquad \forall_{it} \qquad (7)$$

Constrain (5) specifies x_{it} as a continuous variable, while constraint (6) specifies y_{it} as a binary variable.

3.2 The CMLSP with Backlogging: The Conventional Formulation

This model allows unmet demand to be satisfied in the future. It requires the addition of a backorder variable B_{it}, and a unit backorder cost, p_{it}.

$$M2: Min \sum_{i\in N}\sum_{t\in T} s_i y_{it} + \sum_{i\in N}\sum_{t\in T} h_{it} I_{it} + \sum_{i\in N}\sum_{t\in T} p_{it} B_{it} \qquad (8)$$

Subject to:

(2), (4), (5), (6)

$$I_{it-1} + x_{it} - I_{it} - B_{it-1} + B_{it} - d_{it} = 0 \qquad \forall_{it} \qquad (9)$$

We note that constraint (9) is the inventory balance constraint incorporating backorders. The constant in the logic constraint is computed as follows:

$$m_{it} = \min\left\{c_{it}, \sum_{r=1,T} d_{ir}\right\} \qquad \forall_{it} \qquad (10)$$

We note that for a given problem setting, the constant would be much larger in the backlogging case compared to the case with no backlogging. Consequently, the logical constraint in the LP relaxation of M2 will likely have significant slack in the optimal solution.

3.3 A Three-Index Model

A 3-index model for the production lot-sizing problem was proposed in [1]. The proposed model introduces a 3-index production variable x_{itk}.

x_{itk} - the quantity of product i produced in period t for use in period k;

The model referred to as M3 requires an additional summation over periods k, k=1, T.

$$M3: Min \sum_{i\in N}\sum_{t\in T}\sum_{k\in T} h_{itk} x_{itk} + \sum_{i\in N}\sum_{t\in T} s_i y_{it}$$

Subject to:

$$\sum_{i\in N}\sum_{t\in T} a_i x_{itk} \le c_t \qquad \forall_t$$

$$\sum_{t\in T} x_{itk} \ge d_{ik} \qquad \forall_{ik}$$

$$x_{itk} - d_{ik} y_{it} \le 0 \qquad \forall_{itk} \qquad (11)$$

$$y_{it} \in \{0, 1\} \qquad \forall_{it}$$

$$x_{itk} \ge 0 \qquad \forall_{itk}$$

Apart from the difference between the 2-index vs. the 3-index production variable, a key difference between the two models shows up in the logical constraint, constraint (11). In the case of M1, the constant m_{it} can be much larger than the production quantity x_{it}. As such, in an LP relaxation, constraint (4) could have significant levels of slack leading to poor lower bounds. In constraint (11), it is highly likely that the production is equal to the demand, i.e., $x_{itk} = d_{ik}$. Hence in an LP relaxation, many of the logical constraints will have zero slack producing tighter lower bounds and very likely the optimal integer solution to the problem.

3.4 The CMLSP with Backlogging: The 3-Index Model

Accommodating backorders in the Millar-Yang model is rather simple. We allow k to take on values less than t. Values of k < t signal backorders. No additional variables or constraints are needed. This model was tested and presented in [7].

3.5 The Capacitated Multi-Item Lot Sizing Problem with Setup Times

To incorporate setup times, the capacity constraint must be modified in both the two-index and three-index models. The modifications are as follows:

$$\sum_{i\in N} a_i x_{it} + \sum_{i\in \mathrm{N}} v_i y_{it} \le c_t \qquad \forall_{t\ge 2} \qquad (12)$$

In the case of Millar and Yang's model, the modification is similar.

$$\sum_{i\in \mathrm{N}} \sum_{t\in \mathrm{T}} a_i x_{itk} + \sum_{i\in \mathrm{N}} v_i y_{it} \le c_t \qquad \forall_{t\ge 2} \qquad (13)$$

The total of production time plus setup time in any given period must not exceed the capacity of that period.

3.5 The Capacitated Multi-Item Lot Sizing Problem with Setup Carryover: A Two-Index Model

Setup carryovers involve the case where a setup for one of the items in a given period is carried over to the next period. The item with the setup carryover would be produced last in the current period and first in the next. [13] and [8] proposed two-index formulations that introduce a new variable z_{it}, a binary variable set to 1 if the setup carryover for item I is initiated in period t. Hasse's model [8] will carry a setup for at most one period.

Sox and Gao [13] propose a two-index model that allows setup carryovers over multiple periods. The model modifies the logical constraint and introduces four new constraints (14)-(18) to capture the logic of multi-period setup carryovers.

$$M5: Min \sum_{i\in \mathrm{N}} \sum_{t\in \mathrm{T}} s_i y_{it} + \sum_{i\in \mathrm{N}} \sum_{t\in \mathrm{T}} h_{it} I_{it}$$

Subject to:

(2), (3), (5)

$$x_{it} \le m_{it}(y_{it} + z_{it}) \qquad \forall_{t\ge 2} \qquad (14)$$

$$\sum_{i\in I} z_{it} = 1 \qquad \forall_{t\ge 2} \qquad (15)$$

$$z_{it} - y_{it-1} - z_{it-1} \le 0 \qquad \forall_{i,t} \qquad (17)$$

$$z_{it} + z_{it-1} - y_{it-1} + z_{jt-1} \qquad \forall_{t\ge 2} \qquad (18)$$

$$y_{it}, z_{it} \in \{0,1\} \qquad \forall_{i,t} \qquad (19)$$

We note that the models use the form of the logical constraint present in the conventional formulation of the capacitated lot-sizing problem. Given the additional setup carryover constraints, it is highly likely that in the optimal LP relaxations, the logical constraints will possess significant levels of slack.

3.8 A Flexible 3-Index Model

We propose the following mathematical formulation that incorporates backorders, setup times and setup carryovers with two sets of variables (X, Y). The more conventional model of [8] would require a total of four different variables (X, Y, B, Z) to do the same, where B represents the backorder variables. The proposed model considers setup carryovers by converting the set up variable y_{it}, into y_{itq}, a binary variable set to 1 if item i is setup in period t and carried over to period q, where $q \geq t$. Further single or multi-period carryovers are accommodated by simply allowing $q > t+1$. There is no need for additional variables or constraints. The model handles limits on the number of periods of backorders allowed or the number of periods of inventory carried (shelf-life limits) by simply limiting k in the variable x_{itk}. For example, if a maximum of 3 periods of backorders and/or inventory are allowed, then k is allowed to vary such that $t-3 \leq k \leq t+3$. The proposed model is formulated as follows:

$$\Psi = Min \sum_{i \in N} \sum_{t \in T} \sum_{k \in T} h_{itk} x_{itk} + \sum_{i \in N} \sum_{t \in T} \sum_{q=t,T} s_i y_{itq} + \sum_{j \in J} \sum_{t \in T} g_j r_{jt}$$

subject to:

$$\sum_{i \in N} \sum_{k \in T} a_i x_{itk} + \sum_{i \in N} \sum_{q=t,T} v_i y_{itq} - \sum_{j \in J} r_{jt} = 0 \qquad \forall t \qquad (20)$$

$$\sum_{t \in T} x_{itk} \geq d_{ik} \qquad \forall ik \qquad (21)$$

$$x_{itk} - d_{ik} \sum_{p=1,t} \sum_{q=t,T} y_{ipq} \leq 0 \qquad \forall itk \qquad (22)$$

$$\sum_{i \in I} \sum_{p=1,t} \sum_{q=t+1,T} y_{ipq} \leq 1 \qquad \forall t \leq T-1 \qquad (23)$$

$$\sum_{q=t,T} y_{itq} \leq 1 \qquad \forall i,t \qquad (24)$$

$$\sum_{i \in I} \sum_{p=1,t-1} \sum_{q=t+1,T} y_{ipq} + y_{itt} \leq 1 \qquad \forall it \qquad (25)$$

$$0 \leq r_{jt} \leq c_{jt} \qquad \forall jt \qquad (26)$$

$$y_{itq} \in \{0,1\}$$

$$x_{itk} \geq 0$$

Where

$y_{itq} \in \{0,1\}$ - is a binary variable which is set to 1 if a setup for item i is initiated in period t and carried until period q.

r_{jt} - is the consumption of capacity resource type j in period t

C_{jt} - is the capacity available for resource type j in period t

The variable r_{jt} allows multiple capacity sources to be considered, such as overtime and subcontracting.

Constraints (20) – (22) are standard capacity, demand, and logical constraints modified to reflect the 3-index setup/setup carryover variable. Constraints (23) – (25) address the logic of the carryovers.

4 Computational results

In this section we present empirical evidence of the effectiveness of the formulation. First we randomly generate 36 small test problems and solve them to optimality using the conventional Dixon-Silver model [2, 11], and the Millar-

Yang model [1,7]. No backorders, setup times, or setup carryovers were considered. A branch and bound algorithm in CPLEX was used to solve the problems optimally. Table 1 compares computational effort for the two models. It shows the average number of simplex iterations (SI) and branch and bound (BB) nodes searched during the solution procedure. BAK represents the conventional model [2, 11], and MY the Millar-Yang model [1,7].

Table 1: Comparison of Computational Effort for the 2-index vs. the 3-Index Models

SI-BAK	*BB-BAK*	*SI-MY*	*BB-MY*
463	368	169	27

Tables 2 show the results for 72 randomly generated test problems of size *i=8* products and *t=8* periods. The first 36 problems exclude setup times while the second 36 include setup times. The LP gaps are calculated as follows:

$$Gap(\%) = \frac{(MIP\ optimal - LP)}{MIP\ Optimal}\ x\ 100$$

The table columns are as follows:

BAK Gap – LP gap based on the Baker et al. formulation [11]
MY Gap – LP gap based on the 3-index formulation in Millar and Yang [1,7]
SG Gap – LP gap based on the 2-index formulation of Sox and Gao [8]
FLX Gap – LP gap based on the newly proposed model

5 Conclusion

We have presented a flexible and efficient formulation for the multi-item capacitated lot-sizing problem with several features that include setup times, setup carryover, backorders restrictions, and shelf-life limits. Our model is a 3-index model that allows for the definition of constraints that typically have smaller slack values than their 2-index equivalents. This feature is the principal factor driving the superiority of the LP lower bounds. We provide empirical evidence of the effectiveness of our formulation by solving to optimality 72 randomly generated problems, and obtaining LP relaxations for the 2-index and 3-index variants of the problem.

The results in Table 2 show our 3-index formulation dominated the performance of the 2-index formulations in 100% of all cases. Figure 1 provides a visual of the difference between the results for Sox and Gao [11] and our proposed model. Though not presented in the table, the proposed model required significantly less computational effort in all cases compared to the two-index models.

In conclusion, researchers wishing to explore solution approaches to the lot sizing problem and who are in need of good lower bounds as a starting point now have at their disposal a highly flexible and efficient model.

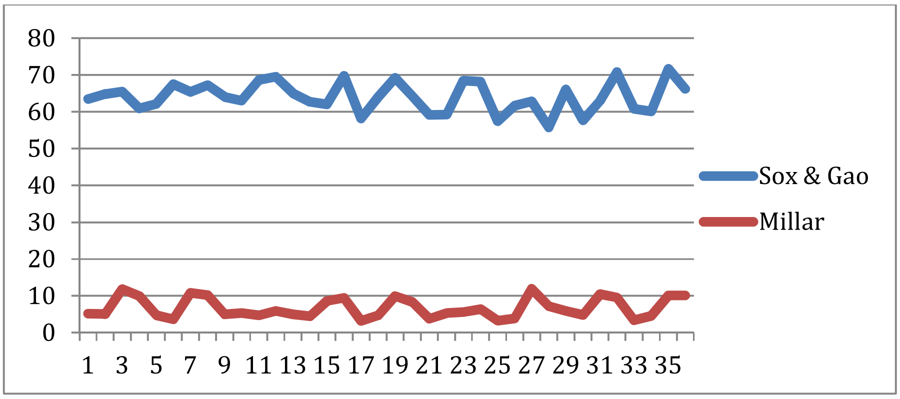

Fig. 1. A graphical comparison of the performance of the proposed formulation with that of Sox and Gao [13]

Table 2: Comparison of LP Gaps (%) for the 2-Index vs. the 3-Index Models

Prob. #	BAK	MY	SG	FLX	Prob. #	BAK	MY	SG	FLX
1	26.4	0.1	63.5	5.2	37	24.79	0.18	62.12	5.22
2	28.4	0.5	64.8	5.1	38	18.52	0.28	60.28	3.70
3	9.5	0.4	65.5	11.8	39	12.58	1.38	68.19	10.91
4	11.2	3.1	60.9	10.0	40	11.28	0.78	59.66	9.84
5	27.1	0.3	62.2	4.7	41	27.47	0.77	59.31	2.19
6	32.0	0.0	67.5	3.6	42	25.94	0.71	57.80	4.67
7	17.3	2.8	65.4	10.8	43	10.65	2.37	67.24	11.81
8	22.2	2.7	67.2	10.2	44	23.68	6.12	66.98	9.33
9	18.4	1.6	64.0	5.0	45	31.73	0.92	60.63	7.93
10	32.5	2.9	63.0	5.3	46	18.20	0.88	56.83	4.45
11	19.0	1.5	68.6	4.7	47	11.85	3.97	60.35	12.00
12	24.6	2.3	69.5	5.9	48	21.45	1.82	67.11	8.92
13	25.3	0.2	65.0	5.0	49	15.25	0.63	61.78	3.82
14	24.5	0.6	62.7	4.5	50	18.14	0.00	62.09	4.92
15	11.7	1.5	62.0	8.6	51	6.67	0.94	66.23	13.49
16	12.0	2.7	69.8	9.5	52	4.27	1.44	62.45	9.55
17	26.0	0.7	58.2	3.2	53	24.02	0.78	54.03	8.04
18	28.2	0.5	64.0	4.7	54	19.24	0.34	59.74	3.69
19	8.4	1.7	69.2	10.0	55	8.28	2.57	72.05	8.60
20	17.5	1.8	64.3	8.4	56	1.43	0.18	61.26	12.82
21	27.3	0.9	59.1	3.8	57	14.54	0.01	53.30	7.53
22	33.6	2.9	59.2	5.3	58	25.58	0.70	56.65	3.83
23	22.6	5.4	68.4	5.6	59	11.41	2.46	72.88	8.76
24	17.3	3.5	68.2	6.4	60	8.07	2.97	65.62	8.44
25	21.33	0.56	57.45	3.31	61	9.31	1.10	67.52	6.27
26	23.11	0.22	61.67	3.85	62	21.89	0.09	52.68	5.75
27	8.30	1.85	62.83	11.94	63	4.44	2.29	74.05	16.11
28	8.96	0.60	55.75	7.16	64	8.83	2.13	67.75	13.73
29	25.68	2.25	66.02	5.88	65	11.66	0.45	60.06	5.95
30	22.79	1.12	57.71	4.83	66	15.10	0.03	52.93	3.36
31	15.15	3.35	62.97	10.51	67	3.45	0.76	64.32	13.34
32	27.32	5.33	70.82	9.58	68	5.59	0.08	69.05	18.22
33	26.53	0.95	60.85	3.34	69	12.98	0.12	64.52	7.15
34	26.93	2.83	60.14	4.58	70	15.49	1.08	59.66	8.22
35	14.10	2.43	71.65	10.13	71	13.76	2.93	66.31	10.31
36	22.29	6.60	66.20	10.12	72	17.55	2.13	77.95	13.24

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