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Abstract

In this paper, we study the hedging problem based on the CVaR in incomplete markets. As the superhedging is quite expensive in terms of initial capital, we construct a self-financing strategy that minimizes the CVaR of hedging risk under a budget constraint on the initial capital. In incomplete markets, no explicit solution can be provided. To approximate the problem, we apply the Neyman-Pearson lemma approach with a specific equivalent martingale measure. Afterwards, we explicit the solution for call options hedging under the exponential-Lévy class of price models. This approach leads to an efficient and easy to implement method using the fast Fourier transform. We illustrate numerical results for the Merton model.

Keywords: Conditional Value-at-Risk; Exponential-Lévy models; Incomplete Market; Neyman Pearson Lemma; Esscher Martingale Measure; Fast Fourier Transform.

1 Introduction

A classical problem in mathematical finance is to construct optimal strategies to hedge option risk by trading only on the underlying asset. On one hand, this question is well understood in frictionless complete markets and completely solved by Harrison and Kreps [22]. In this case, it suffices to buy the replicating portfolio in order to completely offset the risk. This approach works well in the standard Black-Scholes or Cox-Ross-Rubinstein setup, but not much beyond.

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On the other hand, it has often been observed that real market data exchange perform heavy tails and volatility clustering. Common ways to account for such phenomena are the use of stochastic volatility or jump processes, or combination of both. For models with stochastic volatility, there are usually more uncertainties than traded securities. In models with jumps, problems become more involved. When the jump sizes are predictable, the market can be completed by introducing other securities so that the risky securities span the market. But once the securities have jumps with unpredictable sizes, the market becomes incomplete. This occurs as soon as the security can have more than a single jump size. Such markets cannot be completed by any number of traded securities.

In this paper we study models with jumps and assume that the logarithmic stock price follows a Lévy process. Processes of this type play by now an important role in the modeling of financial data [2, 11, 12, 29].

In incomplete markets, under the assumption of no arbitrage, there are infinitely many martingale measures. This induces an interval of arbitrage-free prices. Replicating portfolios typically do not exist in such framework. One approach is then to study the range of arbitrage-free prices, we obtain thus the concept of super-hedging [14, 24, 13, 26]. The idea is to use the smallest amount to construct a self-financing portfolio whose payoff is greater than the contingent claim for all possible scenarios. By this strategy, the hedger is ensured to have sufficient fund to cover all his future obligations. The downside is that the strategy is too costly to be of practical interest [21]. Since that, other approaches have appeared and consist in introducing subjective criteria which lead to more reasonable hedging strategies.

One idea is to relax the constraint on the strategies to be self-financing, while still asking that the terminal portfolio value equals the value of the contingent claim. As the strategies are no longer self-financing, there is a cost process. The local risk-minimization hedging aims to minimize this risk in a sequential way, locally. This problem was first studied by [18] in the martingale setting, and extended to semimartingale setting by [17]. Interestingly, locally risk-minimizing strategies are shown to be *mean-self-financing* which means that the cost process is a martingale. From here it is possible to derive necessary and sufficient conditions for existence and uniqueness of locally risk-minimization strategies. These conditions essentially involve a decomposition of the contingent claim H into orthogonal components. This is somewhat similar to the Kunita-Watanabe decomposition [27].

Alternatively, other studies suggest to keep the condition on the trading strategies to be self-financing, and try to minimize, in some sense, the terminal hedging risk

$$L(V_0, \xi) = H - V_0 - \int_0^T \xi_t dS_t, \tag{1}$$

where H is the option payoff, $(S_t)_{0\leqslant t\leqslant T}$ is the asset price and (V_0, ξ) is the initial capital and the hedging strategy process. One of the most studied method is the mean-variance hedging. It suggests to minimize the second moment of the hedging risk $\mathbb{E}[L^2]$, where the expectation refers to the historical probability measure [28, 46, 20, 23].

Despite the well-developing of the mean-variance hedging and the existence of explicit solutions for complex model [23], this method has obvious disadvantages. One of the drawbacks is that the mean-variance hedging does not distinguish between loss and profit and equally penalizes both. Thus, other criteria which focus only on the losses L_+, were developed. Optimization under these criteria on all admissible strategies (V_0, ξ) leads naturally to the super-hedging in which L is negative for all possible scenarios. Thus, these criteria were studied in literature under a budget constraint by fixing a maximum amount $\tilde{V}_0$ for initial capital. This cost is typically much smaller than the corresponding perfect hedging and the super-hedging strategies. The first proposed criterion, in this context of "partial hedging", is the quantile hedging [15, 10] which ensures that, for a given budget constraint, the probability to have a successful hedge (i.e., a negative terminal hedging risk) $\mathbb{P}(L \leqslant 0)$ is optimally maximized. Föllmer and Leurkert solved this problem in complete market for $\tilde{V}_0$ smaller than the unique risk-neutral price [15]. They applied the Neyman-Pearson lemma to translate the problem to a hypothesis testing one and give a semi-explicit solution. Their article also analyzes the problem in the difficult context of incomplete markets.

Other variants of optimal partial hedging strategies are proposed in the literature. For example, the approach of quantile hedging was generalized in [16] to the problem of minimizing the expected loss $\mathbb{E}[L_+]$ and more generally, $\mathbb{E}[\ell(L_+)]$ for some loss function $\ell(\cdot)$. Nakado [33] considers the problem in the context of minimizing some coherent risk measures, which essentially can be represented as the expected value of the shortfall under a particular probability measure. These results were extended after to some convex risk measures in [37]. More recently, Melnikov ans Smirnov [31] adopt a criterion of minimizing the conditional Value-at-Risk (CVaR) of the hedger's portfolio in a complete market, and they derive a semi-explicit solution to the optimal partial hedging problem by applying results from [16] and [34]. By minimizing the Value-at-Risk (VaR) of the hedger's total risk exposure, [6] derives analytically that the optimal partial hedging strategy is either a knock-out call strategy or a bull call spread strategy, depending on the admissible classes of hedging strategies. Then the results of [6] were extended to the CVaR context in [7]. Other related results on optimal partial hedging can be found in [8, 32, 44] and references therein. Also, the quantile hedging has been successfully applied to many specific financial and insurance contracts ; see, for example, [25, 30, 43, 47].

Our paper follows a similar framework as the one of [31], which considers the complete market case with applications for a call option in the Black-Scholes model and for an embedded call option in an equity-linked life insurance contract. The novelty of our paper is to consider the incomplete market where the asset price is driven by a Lévy process with jumps. In this context, we give an approximate semi-explicit solution for the minimal CVaR by relaxing the constraint in the dual problem and constructing a pricing rule among the set of equivalent martingale measures.

Outline. The paper is organized as follows. In Section 2, we formulate the CVaR optimization problem and introduce the different needed definitions. The optimization problem are treated in two steps: an Expected Shortfall minimization problem and a one-dimensional optimization problem. In Section 3, we transfer the Expected Shortfall minimization problem to a hypothesis testing one and we provide an approximation of the optimal solution using the Esscher martingale measure. In section 5, we study the case of a call option. Finally, in Section 5.2, we test our method and give some numerical results for an example of exponential Lévy models: the Merton model.

2 Statement of the problem

In this section, we define the exponential Lévy model for the stock price and the CVaR optimization problem. We then introduce the approach used to solve this latter.

Definition 2.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. The discounted price process $(S_t)_{t\in[0,T]}$, of the stock asset, is defined by:*

$$S_t = S_0 \; e^{X_t}, \tag{2}$$

where $S_0 > 0$ is a constant and $(X_t)_{t\in[0,T]}$ is a one-dimensional Lévy process [1, 3, 41] with a characteristic triplet (b, σ^2, ν).

The law of X_t at any time t is determined by the triplet (b, σ^2, ν). In particular, the Lévy-Khintchine representation gives the characteristic function of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}$

$$\Phi_t(u) := \mathbb{E}[e^{iuX_t}] = e^{t\Psi(u)}, \quad u \in \mathbb{R}, \tag{3}$$

where Ψ, called the characteristic exponent, is given by

$$\Psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x). \tag{4}$$

The measure ν on $\mathbb{R}$, called the Lévy measure, determines the intensity of jumps of different sizes: $\nu([a_1, a_2])$ is the expected number of jumps on the time interval $[0, 1]$, whose sizes fall in $[a_1, a_2]$. The Lévy measure satisfies the integrability condition

$$\int_{\mathbb{R}} 1 \wedge |x|^2 \nu(dx) < \infty.$$

Let $\mathcal{P}$ denote the set of equivalent martingale measures. The no-arbitrage assumption can be expressed by the existence of at least one equivalent martingale measure. This holds as soon as $\sigma \neq 0$ or $\nu \neq 0$. Under exponential Lévy models, we are mostly in the incomplete market case which means there are infinitely many equivalent martingale measures. Throughout this article, we assume that we are in such a case. In what follows, equations and inequalities between random variables should always be understood as $\mathbb{P}$-a.s.

A self-financing strategy (V_0, ξ) is given by an initial capital $V_0 \geqslant 0$ and by a predictable process ξ such that the resulting discounted value process

$$V_t = V_0 + \int_0^t \xi_s dS_s, \quad \forall t \in [0, T] \tag{5}$$

is well defined. A self-financing strategy (V_0, ξ) is called *admissible* if the corresponding discounted value process V satisfies

$$V_t \geqslant 0, \quad \text{for all } t \in [0, T]. \tag{6}$$

Let $\mathcal{A}$ denote the set of all admissible self-financing strategies.

$$\mathcal{A} = \{(V_0, \xi) \mid (V_0, \xi) \text{ self-financing and admissible}\}. \tag{7}$$

Consider a contingent claim. Its discounted payoff at maturity T is given by a $\mathcal{F}_T$-measurable, non-negative random variable $H \in L^1$. We assume that

$$U_0 = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H] < \infty, \tag{8}$$

where $\mathbb{E}^*$ denotes expectation with respect to $\mathbb{P}^*$.

The value U_0 is the smallest amount V_0 such that there exists an admissible strategy (V_0, ξ) whose discounted value process satisfies $V_T \geqslant H$.

In a complete market, the equivalent martingale measure $\mathbb{P}^*$ is unique. Therefore $U_0 = \mathbb{E}^*[H]$ is the unique arbitrage-free price of the contingent claim H. In an incomplete market, there is more than one equivalent martingale measure and we have a range of arbitrage-free prices. The super-hedging consists in the optional decomposition [14, 26] of the supermartingale $U_t = \operatorname*{esssup}_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H|\mathcal{F}_t]$ in

$$U_t = U_0 + \int_0^t \xi_s \mathrm{d}S_s - C_t, \quad 0 \leqslant t \leqslant T,$$

where $(C_t)_{t\in[0,T]}$ is an adapted increasing process. The strategy (U_0, ξ) ensures that $V_T \geqslant H$.

Since the super-hedging can be quite expensive in incomplete markets (see e.g. [21]), we are looking for the best hedge an investor can achieve with a smaller amount $\tilde{V}_0 < U_0$, where $\tilde{V}_0$ is a budget constraint of the investor. In this paper, we focus on the CVaR, since it is a coherent risk measure and it is very popular among financial institutions and insurance companies. The definition of the CVaR is given as follows:

Definition 2.2. *For a given strategy (V_0, ξ) and a fixed confidence level $a \in (0, 1)$ (which in applications would be a value fairly close to 1), the CVaR at a% level is defined as the expected return of the portfolio in the worst a% cases. In short,*

$$\mathrm{CVaR}_a(L) = \frac{1}{1-a} \int_a^1 \mathrm{VaR}_x(L) dx, \tag{9}$$

where L is the terminal hedging risk given by (1) *and the Value-at-Risk (VaR) is the a-quantile q_a^L of L:*

$$\mathrm{VaR}_a(L) = q_a^L = \inf\{x \in \mathbb{R} : \mathbb{P}(L \leqslant x) > a\}.$$

We denote by $\mathcal{A}_{\tilde{V}_0}$ the set of all admissible strategies satisfying the wealth constraint:

$$\mathcal{A}_{\tilde{V}_0} = \{(V_0, \xi) \mid (V_0, \xi) \in \mathcal{A},\ V_0 \leqslant \tilde{V}_0\}. \tag{10}$$

The aim of this work is to solve the following optimization problem

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}(L(V_0, \xi)). \tag{11}$$

Problem (11) is complex as the objective function is defined via the quantile function and the corresponding numerical methods involve ordering of the position values. Rockafellar and Uryasev [34, 35] found an equivalent formula for CVaR as a convex function, thus opening the door for convex programming methods. They proved that the CVaR is equivalent to

$$\mathrm{CVaR}_a(L(V_0,\xi)) = \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-a}\mathbb{E}[(L(V_0,\xi)-z)_+] \right\}.$$

By introducing the function

$$f(z) = z + \frac{1}{1-a} \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathbb{E}[(L(V_0,\xi)-z)_+], \tag{12}$$

we rewrite the optimization problem (11) as

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}_\alpha(L(V_0,\xi)) = \min_{z\in\mathbb{R}} f(z). \tag{13}$$

To succeed, we proceed as follows:

1. *Compute $f(z)$:* For each $z \in \mathbb{R}$, we explicit the expression of $f(z)$ by finding $(V_0^*(z), \xi^*(z))$, such that

$$(V_0^*(z), \xi^*(z)) \in \underset{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}}{\operatorname{argmin}} \mathbb{E}[(L(V_0,\xi)-z)_+]. \tag{14}$$

2. *Minimize the CVaR:* We solve the one-dimensional optimization problem

$$z^* \in \underset{z\in\mathbb{R}}{\operatorname{argmin}}\, f(z). \tag{15}$$

To conclude, we observe that the strategy $(V_0^*(z^*), \xi^*(z^*))$ is the optimal one:

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}_a(L(V_0,\xi)) = \mathrm{CVaR}_a(L(V_0^*(z^*), \xi^*(z^*))).$$

3 Dual problem

In this section, we focus on the minimization of $\mathbb{E}[(L(V_0,\xi)-z)_+]$ in $\mathcal{A}_{\tilde{V}_0}$, for all z, in order to obtain a numerical valuation of the function $f(z)$. To deal with the constraint $V_0 \leqslant \tilde{V}_0$, we rewrite Problem (14) in the form of hypothesis testing one that can be treated using the generalized Neyman Pearson approach. We introduce $\mathcal{R}$ a class of "randomized tests" defined by

$$\mathcal{R} = \{\varphi : \Omega \to [0,1] \mid \varphi \text{ is } \mathcal{F}_T\text{-measurable}\}.$$

The application of Proposition 3.1 of [16] to the discounted payoff $(H-z)_+$ gives the following result.

Theorem 3.1. *For all $z \in \mathbb{R}$, there exists a solution $\varphi^*(z) \in \mathcal{R}$ to the optimization problem*

$$\begin{aligned} \varphi^*(z) \in & \underset{\varphi(z)\in\mathcal{R}}{\operatorname{argmin}}\, \mathbb{E}[(1-\varphi(z))(H-z)_+] \\ & \text{s.t.} \quad \sup_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[\varphi(z)(H-z)_+] \leqslant \tilde{V}_0. \end{aligned} \tag{16}$$

Moreoover, we have the following equality

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathbb{E}[(L(V_0,\xi)-z)_+] = \begin{array}{l} \displaystyle\min_{\varphi(z)\in\mathcal{R}} \mathbb{E}[(1-\varphi(z))(H-z)_+] \\ \text{s. t.} \displaystyle\sup_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[\varphi(z)(H-z)_+] \leqslant \tilde{V}_0. \end{array} \tag{17}$$

The existence of the solution to (16) permits the transfer of the minimization problem from the space $\mathcal{A}_{\tilde{V}_0}$ to the space of randomized tests $\mathcal{R}$.

Corollary 3.1. *For each $z \in \mathbb{R}$, the argmins $(V_0^*(z), \xi^*(z))$ of (14) and $\varphi^*(z)$ of (16) are connected in the following way*

- *Given the optimal solution $(V_0^*(z), \xi^*(z))$, its success ratio, defined by*

$$\varphi(V_0^*(z), \xi^*(z)) = \mathbf{1}_{\{V_T^*(z)\geqslant(H-z)_+\}} + \frac{V_T^*(z)}{(H-z)_+}\mathbf{1}_{\{V_T^*(z)<(H-z)_+\}}, \tag{18}$$

  is equal to $\varphi^*(z)$.

- *Conversely, given $\varphi^*(z)$, the following optional decomposition*

$$\operatorname*{esssup}_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[\varphi^*(z)(H-z)_+|\mathcal{F}_t] = V_0^*(z) + \int_0^t \xi^*(z)_s dS_s - C_t, \quad 0 \leqslant t \leqslant T,$$

  where $(C_t)_{t\in[0,T]}$ is an adapted increasing process, gives the optimal strategy $(V_0^*(z), \xi^*(z))$.

Idea of proof. As H and V_T are positive, we have

$$(L(V_0,\xi)-z)_+ = (H - V_T - z)_+ = ((H-z)_+ - V_T)_+.$$

Thus, the minimization of $\mathbb{E}[(L(V_0,\xi)-z)_+]$ can be treated as a problem of expected loss minimization with payoff $(H-z)_+$ that depends on a real-valued parameter z.

The equivalent problem (16) is then given by applying the approach of Föllmer and Leukert [16] to a contingent claim with payoff $(H-z)+$. The existence of such a solution $\varphi^*(z)$ is ensured by Proposition 3.1. □

Under convenable assumptions, many generalizations of Neyman Pearson lemma were developed to solve problems of the form (16). We cite for example [9, 36, 37, 38, 39, 42]. These studies prove that the solution is not unique and give only the structure of the solution. More precisely, under the assumption that the set of densities $Z_{\mathbb{P}^*} := \{\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} : \mathbb{P}^* \in \mathcal{P}\}$ is compact, the results of [39] give the structure of the optimal randomized test $\varphi^*(z)$ of Problem (16) as

$$\varphi^*(z) = \begin{cases} 1 \text{ if } \quad (H-z)_+\dfrac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} > (H-z)_+\displaystyle\int_{\mathcal{P}} \frac{d\mathbb{P}^*}{d\mathbb{P}} d\tilde{\lambda}(\mathbb{P}^*) \\ 0 \text{ if } \quad (H-z)_+\dfrac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} < (H-z)_+\displaystyle\int_{\mathcal{P}} \frac{d\mathbb{P}^*}{d\mathbb{P}} d\tilde{\lambda}(\mathbb{P}^*). \end{cases} \tag{19}$$

Besides

$$\mathbb{E}^*[\varphi^*(z)(H-z)_+] = \tilde{V}_0, \quad \tilde{\lambda} - a.s., \tag{20}$$

where $(\tilde{Q},\tilde{\lambda})$ is a solution to some dual problem. Delivering numerical solutions from this result is still not possible. Furthermore, [45] shows that the set $Z_{\mathbb{P}^*}$ is never compact, except when $\mathcal{P}$ is a singleton, which corresponds to the complete market case. In the following, we prove that this hypothesis of compactness is not verified by exponential-Lévy models. We then suggest in Section 4 to relax the constraint in (16) in order to get a semi-explicit approximation for the solution.

Proposition 3.1. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process under $\mathbb{P}$ with characteristic triplet (b,σ^2,ν). The set of densities $Z_{\mathbb{P}^*} = \{\frac{d\mathbb{P}^*}{d\mathbb{P}} \; : \; \mathbb{P}^* \in \mathcal{P}\}$ is given by*

$$Z_{\mathbb{P}^*} = \left\{ e^{U_T^{(\theta,\phi)}} \; : \; \theta \in \mathbb{R} \text{ and } \phi : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}} \left(e^{\phi(x)/2}-1\right)^2 \nu(dx) < \infty \text{ and } \right.$$
$$\left. b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{\phi(x)}(e^x-1) - x\mathbf{1}_{|x|\leqslant 1}\right)\nu(dx) = 0 \right\},$$

where $\left(U_t^{(\theta,\phi)}\right)_{t\in[0,T]}$ is a Lévy process under $\mathbb{P}$ with characteristic triplet

$$\left(-\frac{\sigma^2\theta^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{|x|\leqslant 1})\nu\circ\phi^{-1}(dx), \sigma^2\theta^2, \nu\circ\phi^{-1}\right).$$

Proof. Using [41, Theorem 33.1 and Theorem 33.2], that gives the conditions of equivalence of measures, we can describe the set of measures equivalent to $\mathbb{P}$ by

$$\left\{ \mathbb{P}^*_{(\theta,\phi)} \; : \; \theta \in \mathbb{R} \text{ and } \phi : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}} \left(e^{\phi(x)/2}-1\right)^2 \nu(dx) < \infty \right\},$$

with $\frac{\mathrm{d}\mathbb{P}^*_{(\theta,\phi)}}{\mathrm{d}\mathbb{P}} = e^{U_T^{(\theta,\phi)}}$ and $\left(U^{(\theta,\phi)}\right)_{t\in[0,T]}$ a Lévy process under $\mathbb{P}$ with characteristic triplet

$$\left(-\frac{\sigma^2\theta^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{|x|\leqslant 1})\nu\circ\phi^{-1}(dx), \sigma^2\theta^2, \nu\circ\phi^{-1}\right).$$

Under $\mathbb{P}^*_{(\theta,\phi)}$, $(X_t)_{t\in[0,T]}$ is a Lévy process with characteristic triplet $\left(b^*_{(\theta,\phi)}, \sigma^2, \nu^*_{(\theta,\phi)}\right)$ such that

$$\nu^*_{(\theta,\phi)}(dx) = e^{\phi(x)}\nu(dx)$$

and

$$b^*_{(\theta,\phi)} = b + \sigma^2\theta + \int_{|x|\leqslant 1} x\nu^*_{(\theta,\phi)}(dx) - \int_{|x|\leqslant 1} x\nu(dx).$$

On the other hand, using [41, Theorems 25.3 and 25.17], $(S_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}^*$ if and only if

$$b^*_{(\theta,\phi)} + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} \left(e^x - 1 - x\mathbf{1}_{|x|\leqslant 1}\right)\nu^*_{(\theta,\phi)}(dx) = 0,$$

which gives

$$b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{\phi(x)}(e^x - 1) - x\mathbf{1}_{|x|\leqslant 1}\right) \nu(dx) = 0.$$

We conclude that

$$Z_{\mathbb{P}*} = \Bigg\{ e^{U_T^{(\theta,\phi)}} \;:\; \theta \in \mathbb{R} \text{ and } \phi : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}} \left(e^{\phi(x)/2} - 1\right)^2 \nu(dx) < \infty \text{ and }$$
$$b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{\phi(x)}(e^x - 1) - x\mathbf{1}_{|x|\leqslant 1}\right) \nu(dx) = 0 \Bigg\}.$$

Thus, we finish the proof. □

We see from this proposition that if $\nu \neq 0$ then the set $Z_{\mathbb{P}*}$ is infinite because there are infinitely many functions ϕ that satisfy the conditions. Thus, $Z_{\mathbb{P}*}$ cannot be compact unless $\nu = 0$. Conversely, if $\nu = 0$ then $Z_{\mathbb{P}*}$ is reduced to a singleton, so it is compact. This is summarized in the following corollary.

Corollary 3.2. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process under $\mathbb{P}$ with characteristic triplet (b, σ^2, ν). The set of densities $Z_{\mathbb{P}*} = \{\frac{d\mathbb{P}^*}{d\mathbb{P}} \;:\; \mathbb{P}^* \in \mathcal{P}\}$ is compact if and only if $\nu = 0$. In such case, $Z_{\mathbb{P}*}$ is reduced to a singleton*

$$Z_{\mathbb{P}*} = \left\{ \exp\left(-\frac{\sigma^2\theta^2}{2} T + \sigma\theta B_T \right) \right\},$$

with $\theta = (-b - \frac{\sigma^2}{2})/\sigma^2$ and $(B_t)_{t\in[0,T]}$ a standard Brownian motion.

Thus, under exponential-Lévy models, the set of densities $Z_{\mathbb{P}*}$ is not compact except for the case of diffusion models, like the Black and Scholes model. These models correspond to complete markets. Then, we have a unique equivalent martingale measure and the minimization problem can be solved using the simple Neyman-Pearson lemma. We conclude that the proposed generalizations for this lemma, that deal with the case of infinite equivalent martingale measures, are not applicable in the financial framework where the compactness hypothesis required by these generalizations are not fulfilled for incomplete markets.

4 Approximation of dual problem

In this section, we propose to relax the constraint $\sup_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[\varphi(z)(H - z)_+] \leqslant \tilde{V}_0$ in (16). Instead of considering the supremum over all the equivalent martingale measures, we fix an equivalent martingale measure $\mathbb{P}^\dagger$ and we optimize under the constraint $\mathbb{E}^\dagger[\varphi(z)(H - z)_+] \leqslant \tilde{V}_0$, where $\mathbb{E}^\dagger$ is the expectation under the chosen measure $\mathbb{P}^\dagger$. We opt for the choice of the Esscher martingale measure [4, 19]. This measure is defined by the following density process

$$\left.\frac{\mathrm{d}\mathbb{P}^\dagger}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{\mathbb{E}\left[e^{\theta X_t}\right]}, \tag{21}$$

where $\theta \in \mathbb{R}$ is solution to

$$b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{\theta x}(e^x - 1) - x\mathbf{1}_{|x|\leqslant 1}\right) \nu(dx) = 0.$$

The advantage of using this equivalent martingale measure is that the density process only depends on the current stock price, which permits more explicit computing. See [40] for more details. In terms of Proposition 3.2, the Esscher martingale measure corresponds to the choice $\phi(x) = \theta x$. An approximation of the problem (16) is given by

$$\begin{aligned} \hat{\varphi}(z) =& \underset{\varphi(z)\in\mathcal{R}}{\operatorname{argmin}}\ \mathbb{E}[(1-\varphi(z))(H-z)_+] \\ & \text{s. t.}\quad \mathbb{E}^\dagger[\varphi(z)(H-z)_+] \leqslant \tilde{V}_0, \end{aligned} \tag{22}$$

where $\mathbb{E}^\dagger$ is the expectation under the Esscher martingale measure $\mathbb{P}^\dagger$.

Hypothesis 4.1. We suppose that $\tilde{V}_0 < \hat{U}_0 = \mathbb{E}^\dagger[H]$.

Under this hypothesis, we do not consider the trivial case where $\hat{\varphi}(z) = 1$ for all $z \leqslant 0$ and $\widehat{\mathrm{CVaR}}_a$ is equal to zero.

The solution to Problem (22) can be constructed by using the Neyman-Pearson lemma.

Theorem 4.1. *The solution to Problem* (22) *is given by*

$$\hat{\varphi}(z) = \mathbf{1}_{\left\{e^{-\theta X_T} > c(z)\right\}}$$

with

$$c(z) = \inf\left\{u \geqslant 0 \ :\ \mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\left\{e^{-\theta X_T} > u\right\}}\right] \leqslant \tilde{V}_0\right\}.$$

Proof. Let $\mathbb{Q}$ et $\mathbb{Q}^\dagger$ be the measures given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{(H-z)_+}{\mathbb{E}[(H-z)_+]} \quad \text{and} \quad \frac{\mathrm{d}\mathbb{Q}^\dagger}{\mathrm{d}\mathbb{P}^\dagger} = \frac{(H-z)_+}{\mathbb{E}^\dagger[(H-z)_+]}.$$

The optimization problem

$$\begin{aligned} &\min_{\varphi(z)\in\mathcal{R}} \mathbb{E}[(1-\varphi(z))(H-z)_+] = \max_{\varphi(z)\in\mathcal{R}} \mathbb{E}[\varphi(z)(H-z)_+] \\ &\text{s. t.}\quad \mathbb{E}^\dagger[\varphi(z)(H-z)_+] \leqslant \tilde{V}_0, \end{aligned}$$

takes the form

$$\begin{aligned} &\max_{\varphi(z)\in\mathcal{R}} \int \varphi(z)\mathrm{d}\mathbb{Q} \\ &\text{s. t.} \int \varphi(z)\mathrm{d}\mathbb{Q}^\dagger \leqslant \alpha(z) := \frac{\tilde{V}_0}{\mathbb{E}^\dagger[(H-z)_+]}. \end{aligned}$$

Thus the solution of this last problem is identified as the optimal randomized test in a problem of testing the simple hypothesis $\mathbb{Q}^\dagger$ against the simple alternative $\mathbb{Q}$. The Neyman-Pearson [48, 9] provides an explicit solution, given by

$$\hat{\varphi}(z) = \mathbf{1}_{\left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > c''(z)\right\}} + \gamma(z) \cdot \mathbf{1}_{\left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} = c''(z)\right\}}, \tag{23}$$

where

$$c''(z) = \inf\left\{u \geqslant 0 \ :\ \mathbb{Q}^\dagger\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > u\right) \leqslant \alpha(z)\right\}$$

and

$$\gamma(z) = \frac{\alpha(z) - \mathbb{Q}^\dagger\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > c''(z)\right)}{\mathbb{Q}^\dagger\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} = c''(z)\right)},$$

with the convention $0/0 = 0$.

Setting $c'(z) = c''(z)\frac{\mathbb{E}[(H-z)_+]}{\mathbb{E}^\dagger[(H-z)_+]}$ and $c(z) = \frac{c'(z)}{\mathbb{E}[e^{\theta X_T}]}$, we have the following equalities

$$\left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > c''(z)\right\} = \left\{\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^\dagger} > c'(z)\right\} = \left\{e^{-\theta X_T} > c(z)\right\}$$

and

$$\begin{aligned}
\mathbb{Q}^\dagger\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > c''(z)\right) &= \int \mathbf{1}_{\left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > c''(z)\right\}}\mathrm{d}\mathbb{Q}^\dagger \\
&= \int \mathbf{1}_{\{e^{-\theta X_T} > c(z)\}}\frac{(H-z)_+}{\mathbb{E}^\dagger[(H-z)_+]}\mathrm{d}\mathbb{P}^\dagger \\
&= \frac{\mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\{e^{-\theta X_T} > c(z)\}}\right]}{\mathbb{E}^\dagger[(H-z)_+]}.
\end{aligned}$$

We can easily deduce that

$$c(z) = \inf\left\{u \geqslant 0 \ : \ \mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\left\{e^{-\theta X_T} > u\right\}}\right] \leqslant \tilde{V}_0\right\}.$$

On the other hand, the stochastic continuity of the Lévy process insures that the probability of a fixed value is zero. Thus

$$\mathbb{Q}^\dagger\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} = c''(z)\right) = \frac{\mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\{e^{-\theta X_T} = c(z)\}}\right]}{\mathbb{E}^\dagger[(H-z)_+]} = 0.$$

Hence, $\mathbb{Q}^\dagger\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^\dagger} > c''(z)\right) = \alpha(z)$ and therefore $\gamma(z) = 0$.

We have then shown that $\hat{\varphi}(z) = \mathbf{1}_{\left\{e^{-\theta X_T} > c(z)\right\}}$ and we conclude the proof. □

We introduce $\hat{f}(z)$, an approximation of $f(z)$, by

$$\hat{f}(z) = z + \frac{1}{1-a}\mathbb{E}\left[(1-\hat{\varphi}(z))(H-z)_+\right], \tag{24}$$

where $\hat{\varphi}(z)$ is given by Theorem 4.1. We study the approximated problem of the CVaR minimization

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(L(V_0,\xi)) = \min_{z\in\mathbb{R}}\hat{f}(z). \tag{25}$$

We develop results for call options in the next section.

5 Hedging a call option

In this section, we focus on hedging a call option with strike K. Let r be the free-risk interest rate. The discounted contingent claim has a payoff equal to

$$(S_T - Ke^{-rT})_+.$$

5.1 Solution of the approximated problem

The approximated problem of CVaR minimization (25) is solved in the next theorem. The strength of this theorem is that the objective function is only expressed with quantities of the form $\mathbb{E}[(S_T - x)_+]$, $\mathbb{P}(S_T > x)$, $\mathbb{E}^\dagger[(S_T - x)_+]$ and $\mathbb{P}^\dagger(S_T > x)$. These quatities can always be numerically computed by using the fast Fourier transform (FFT). The only required condition is the existence of explicit expressions of the characteristic functions Φ_T and $\Phi_T^\dagger$ of X_T under $\mathbb{P}$ and $\mathbb{P}^\dagger$ respectively. This condition is verified by the most known Lévy processes.

Theorem 5.1. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process under $\mathbb{P}$ with characteristic triplet (b, σ^2, ν) and let $\mathbb{P}^\dagger$ be the Esscher martingale measure with parameter θ. The solution to the approximated problem of the CVaR minimization (25) is given, with respect to the sign of θ, by*

- *If $\theta < 0$,*

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(H - V_T) = \min_{z\in(0,z_c)} \left\{ z + \frac{1}{1-a} \left(\mathbb{E}[(S_T - K(z))_+] \right.\right.$$
$$\left.\left. -\mathbb{E}[(S_T - c_1(z))_+] - (c_1(z) - K(z))\mathbb{P}(S_T > c_1(z))\right)\right\}, \quad (26)$$

where $K(z) = Ke^{-rT} + z$, $z_c > 0$ such that $\mathbb{E}^\dagger[(S_T - K(z_c))_+] = \tilde{V}_0$ and

$$c_1(z) = \inf\left\{u \geqslant K(z),\ \mathbb{E}^\dagger[(S_T - u)_+] + (u - K(z))\mathbb{P}^\dagger(S_T > u) \leqslant \tilde{V}_0\right\}. \quad (27)$$

- *If $\theta > 0$,*

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(H - V_T) = \min_{z\in(0,z_c)} \left\{ z + \frac{1}{1-a} \left(\mathbb{E}[(S_T - c_1(z))_+] \right.\right.$$
$$\left.\left. +(c_1(z) - K(z))\mathbb{P}(S_T > c_1(z))\right)\right\}, \quad (28)$$

where $K(z) = Ke^{-rT} + z$, $z_c > 0$ such that $\mathbb{E}^\dagger[(S_T - K(z_c))_+] = \tilde{V}_0$ and

$$c_1(z) = \sup\left\{u \geqslant K(z),\ \mathbb{E}^\dagger[(S_T - K(z))_+] - \mathbb{E}^\dagger[(S_T - u)_+] \right.$$
$$\left. -(u - K(z))\mathbb{P}^\dagger(S_T > u) \leqslant \tilde{V}_0\right\}. \quad (29)$$

Proof. Preliminaries. The function $\mathbb{E}^\dagger[(H - z)_+]$ is continuous and decreasing in z, equals to $\mathbb{E}^\dagger[H]$, which is strictly greater than $\tilde{V}_0$, for $z = 0$ and goes to $-\infty$ as $z \to +\infty$. Thus, there exists $z_c > 0$ such that $\mathbb{E}^\dagger[(H - z)_+] = \tilde{V}_0$.

For $z \geqslant z_c$, $\mathbb{E}^\dagger[(H - z)_+] > \tilde{V}_0$, so $\hat{\varphi}(z) = 1$ and $\hat{f}(z) = z$. Thus, $\min_{z\in\mathbb{R}} \hat{f}(z)$ may not be reached in $(z_c, +\infty)$ since the function $\hat{f}$ is increasing on that interval. Furthermore, we have seen that the approximated VaR is the smallest $\operatorname{argmin}_{z\in\mathbb{R}} \hat{f}(z)$. The VaR is always positive, thus the set $\operatorname{argmin}_{z\in\mathbb{R}} \hat{f}(z)$ is included in $(0, z_c)$.

Let $z \in (0, z_c)$, then

- setting $K(z) = Ke^{-rT} + z$, we have

$$(H-z)_+ = ((S_T - Ke^{-rT})_+ - z)_+ = (S_T - (Ke^{-rT} + z))_+ = (S_T - K(z))_+.$$

- For all $u \geqslant 0$, we have

$$\mathbb{E}\left[(H-z)_+ \mathbf{1}_{\{S_T > u\}}\right] = \begin{cases} \mathbb{E}[(S_T - K(z))_+] & \text{if } u < K(z), \\ \mathbb{E}[(S_T - u)_+] + (u - K(z))\mathbb{P}(S_T > u) & \text{if } u \geqslant K(z). \end{cases} \tag{30}$$

- For all $u \geqslant 0$, we have

$$\mathbb{E}\left[(H-z)_+ \mathbf{1}_{\{S_T < u\}}\right] = \begin{cases} 0 & \text{if } u < K(z), \\ \mathbb{E}[(S_T - K(z))_+] - \mathbb{E}[(S_T - u)_+] - (u - K(z))\mathbb{P}(S_T > u) & \text{if } u \geqslant K(z). \end{cases} \tag{31}$$

Note that this does not depend on the measure $\mathbb{P}$ and we have the same expressions for $\mathbb{P}^\dagger$.

Main part of the proof. Let $z \in (0, z_c)$. The solution $\hat{\varphi}(z) = \mathbf{1}_{\left\{e^{-\theta X_T} > c(z)\right\}}$ to the approached problem (16), where $c(z) = \inf\left\{u \geqslant 0 \ : \ \mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\left\{e^{-\theta X_T} > u\right\}}\right] \leqslant \tilde{V}_0\right\}$, depends on the sign of the Esscher parameters θ. We must treat both cases $\theta < 0$ and $\theta > 0$ separately.

1[**st**] **case :** $\theta < 0$.

We can easily prove that $\hat{\varphi}(z)$ can be written in terms of S_T as follows

$$\hat{\varphi}(z) = \mathbf{1}_{\{S_T > c_1(z)\}},$$

where

$$c_1(z) = S_0(c(z))^{-1/\theta} = \inf\left\{u \geqslant 0 \ : \ \mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\{S_T > u\}}\right] \leqslant \tilde{V}_0\right\}.$$

The expectation $\mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\{S_T > u\}}\right]$ takes the form of (30). Moreover, $\mathbb{E}^\dagger[(S_T - K(z))_+] = \mathbb{E}^\dagger[(H-z)_+] > \tilde{V}_0$ for $z \in (0, z_c)$ since this expectation is decreasing on z and equals to $\tilde{V}_0$ for z_c. Thus,

$$c_1(z) = \inf\left\{u \geqslant K(z) \ : \ \mathbb{E}^\dagger[(S_T - u)_+] + (u - K(z))\mathbb{P}^\dagger(S_T > u) \leqslant \tilde{V}_0\right\}.$$

The function $\hat{f}(z)$ is then equal to

$$\hat{f}(z) = z + \frac{1}{1-a}\mathbb{E}\left[\mathbf{1}_{\{S_T < c_1(z)\}} \cdot (H-z)_+\right].$$

Since $c_1(z) \geqslant K(z)$, then by using (31), we have

$$\hat{f}(z) = z + \frac{1}{1-a}\left(\mathbb{E}[(S_T - K(z))_+] - \mathbb{E}[(S_T - c_1(z))_+] - (c_1(z) - K(z))\mathbb{P}(S_T > c_1(z))\right).$$

Since $\min\limits_{(V_0,\xi) \in \mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(H - V_T) = \min\limits_{z \in (0, z_c)} \hat{f}(z)$, we obtain the result (26) for $\theta < 0$.

2nd case : $\theta > 0$.

We can easily prove that $\hat{\varphi}(z)$ can be written in terms of S_T as follows

$$\hat{\varphi}(z) = \mathbf{1}_{\{S_T < c_1(z)\}},$$

where

$$c_1(z) = S_0(c(z))^{-1/\theta} = \sup\left\{u \geqslant 0 \ : \ \mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\{S_T<u\}}\right] \leqslant \tilde{V}_0\right\}.$$

The expectation $\mathbb{E}^\dagger\left[(H-z)_+ \cdot \mathbf{1}_{\{S_T<u\}}\right]$ takes the form of (31). Thus,

$$c_1(z) = \sup\left\{u \geqslant K(z) \ : \ \mathbb{E}^\dagger[(S_T - K(z))_+] - \mathbb{E}^\dagger[(S_T-u)_+] - (u-K(z))\mathbb{P}^\dagger(S_T > u) \leqslant \tilde{V}_0\right\}.$$

The function $\hat{f}(z)$ is then equal to

$$\hat{f}(z) = z + \frac{1}{1-a}\mathbb{E}\left[\mathbf{1}_{\{S_T > c_1(z)\}} \cdot (H-z)_+\right].$$

Since $c_1(z) \geqslant K(z)$, then by using (30), we have

$$\hat{f}(z) = z + \frac{1}{1-a}\left(\mathbb{E}[(S_T - c_1(z))_+] + (c_1(z) - K(z))\mathbb{P}(S_T > c_1(z))\right).$$

Since $\min\limits_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(H - V_T) = \min\limits_{z\in(0,z_c)} \hat{f}(z)$, we prove the result (28) for $\theta > 0$. $\square$

5.2 Example : the Merton model

In this section, we give numerical results in the context of the Merton model. In this model, the discounted price at time t is given by $S_t = S_0 e^{X_t}$ with

$$X_t = (\gamma - r)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i,$$

where $\gamma \in \mathbb{R}$, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion, $(N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ and $(Y_i)_{i\geqslant 1}$ are i.i.d. Gaussian random variables with parameters m and δ^2. For each $0 \leqslant t \leqslant T$, the characteristic function of X is given by

$$\Phi_t(u) = \mathbb{E}[e^{iuX_t}] = e^{t\Psi(u)},$$

and

$$\Psi(u) = i(\gamma - r)u - \frac{\sigma^2}{2}u^2 + \lambda\left(\exp(imu - \frac{\delta^2}{2}u^2) - 1\right).$$

The existence of the Esscher martingale measure $\mathbb{P}^\star$ depends on the existence of a real θ solution to

$$(\gamma - r) + \sigma^2\theta + \frac{\sigma^2}{2} + \lambda\left(e^{m(\theta+1)+\frac{\delta^2}{2}(\theta+1)^2} - e^{m\theta + \frac{\delta^2}{2}\theta^2}\right) = 0.$$

A numerical application with parameters of Table 1 gives the result $\theta = -0.352$. The equation is solved by Newton-Raphson method.

Market	Merton	FFT
$S_0 = 100$	$\gamma = 0.1$	$N = 4096$
$r = 0.02$	$\sigma = 0.3$	$\Delta v = 0.25$
$T = 0.5$	$\lambda = 1$	
$K = 110$	$m = -0.1$	$\Delta k = \frac{2\pi}{N\Delta v}$
	$\delta = 0.2$	$k_1 = -\frac{N}{2}\Delta k$

Table 1: Models parameters for numerical applications.

The characteristic function of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}^\star$ is given by

$$\Phi_t^\dagger(u) = e^{t\Psi^\dagger(u)},$$

where

$$\Psi^\dagger(u) = i(\gamma - r + \sigma^2\theta)u - \frac{\sigma^2}{2}u^2 + \lambda\left(e^{m(\theta+iu)+\frac{\delta^2}{2}(\theta+iu)^2} - e^{m\theta+\frac{\delta^2}{2}\theta^2}\right).$$

So, with the Esscher martingale measure, we are able to provide analytic expression of the characteristic function under an equivalent martingale measure. This analytic form is inverted numerically with the fast Fourier transform (FFT) to compute the expectations and the probabilities in the expressions of Theorem 5.1. We use the technique developed by Carr and Madan [5] in the context of option pricing under exponential Lévy model.

Consider a European call option with strike price of K and the time to maturity T. For parameters of Table 1, we numerically compute all quantities of types $\mathbb{E}[(S_T - x)_+]$, $\mathbb{P}(S_T > x)$, $\mathbb{E}^\dagger[(S_T - x)_+]$ and $\mathbb{P}^\dagger(S_T > x)$ by using the fast Fourier transform. The optimization problem is then reduced to the numerical vector minimization. Figure 1 illustrates the approximate solution for the CVaR_a optimization problem with repect to the proportion $\tilde{V}_0/\hat{U}_0$, that represents the permitted proportion of the risk-neutral price under the Esscher martingale measure. It represents results for different confidence level a of CVaR.

We see in Figure 1 that the minimal CVaR becomes insensitive to the confidence level as we increase the inital capital. This is due to the fact that the minimum of $\hat{f}(z)$, for high values of $\tilde{V}_0$, is attained in z^*, thus $\min \hat{f}(z) = \hat{f}(z^*) = z^*$ which does not depend on a. In this case, the tail risk is concentrated entirely at the a-quantile of the loss distribution, and Value-at-Risk coincides with Conditional Value-at-Risk.

6 Conclusions and perspectives

In this paper, we studied the hedging problem in incomplete markets, while considering the CVaR as the criterion of optimization. No explicit solution can be provided to this problem. We propose a first simplification of the problem by introducing the variational representation of the CVaR due to Rockafellar and Uryasev. Then, we construct an approximation of the minimal CVaR by applying the Neyman-Pearson lemma to a specific equivalent martingale measure, called the Esscher martingale measure. In the case of European options and under exponential-Lévy models, the approximation is numerically derived by an iterative use of the fast Fourier transform. For future works, we

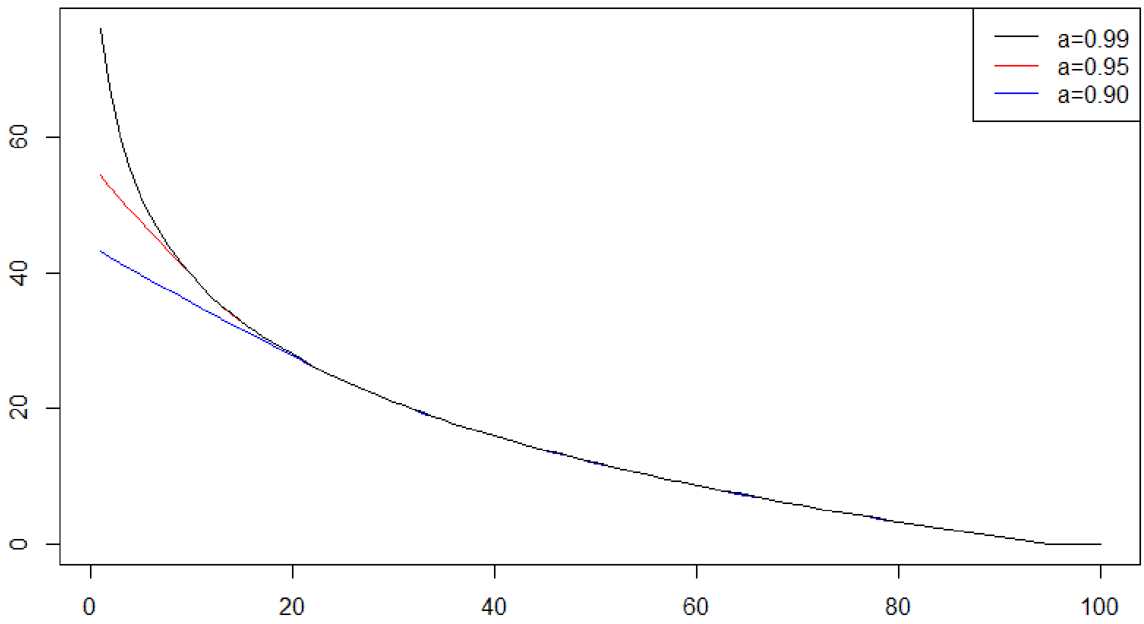

Figure 1: $\widehat{\mathrm{CVaR}}_a$, the approximated minimal CVaR, with respect to $\tilde{V}_0/\hat{U}_0$, the permitted proportion of the risk-neutral price under the Esscher martingale measure, under a Merton model.

wish to extend the approximation to other equivalent martingale measures or to a set of parameterized equivalent martingale measures, and to compare these results.

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