# Adaptive observer for age-structured population with spatial diffusion

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**Abstract**

We investigate the inverse problem of simultaneously estimating the state and the spatial diffusion coefficient for an age-structured population model. The time evolution of the population is supposed to be known on a subdomain in space and age. We generalize to the infinite dimensional setting an adaptive observer originally proposed for finite dimensional systems.

## 1 Introduction

We consider the following system modelling the evolution of an age-structured population with spatial diffusion:

$$\begin{cases} \partial_t p(a,x,t) + \partial_a p(a,x,t) & a \in (0,a^*),\ x \in \Omega,\ t > 0, \\ \qquad = -\mu(a)p(a,x,t) + k\Delta p(a,x,t), & \\ p(a,x,t) = 0, & a \in (0,a^*),\ x \in \partial\Omega,\ t > 0, \\ p(a,x,0) = p_0(a,x), & a \in (0,a^*),\ x \in \Omega, \\ p(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,x,t)\,\mathrm{d}a, & x \in \Omega,\ t > 0. \end{cases} \tag{1.1}$$

In the above equations:

- $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, denotes a smooth bounded domain, $k$ is a positive constant diffusion coefficient and $\Delta$ the laplacian with respect to the space variable $x$.
- $p(a,x,t)$ denotes the distribution density of the population of age $a$ at spatial position $x \in \Omega$ at time $t$;
- $p_0$ denotes the initial distribution;
- $a^* \in (0,+\infty)$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates, which are supposed to be independent of $x$ and satisfy

$$\beta \in L^\infty(0,a^*), \quad \beta \geqslant 0 \text{ a.e. in } (0,a^*),$$
$$\mu \in L^1_{\mathrm{loc}}(0,a^*), \quad \mu \geqslant 0 \text{ a.e. in } (0,a^*),$$

and

$$\lim_{a\to a^*} \int_0^a \mu(s)\,\mathrm{d}s = +\infty. \tag{1.2}$$

The last equation in (1.1) describing the birth process is the so-called *renewal equation*. We assume here homogeneous Dirichlet boundary conditions (in space) which model a hostile habitat at the boundary $\partial\Omega$.

We assume here that the diffusion coefficient $k$ is not well known. To be more specific we shall assume that $k \in [k_0 - r_k, k_0 + r_k]$ where $k_0$ (an approximate value of $k$) and $r_k$ (the uncertainty on $k$) are known.

Inverse problems for population dynamics models have been studied in several papers. Rundell *et al.* [14, 12, 5] studied the determination of the death rate for an age-structured population dynamics from the knowledge population profiles at two distinct times. Gyllenberg *et al.* [6] investigate the identifiability of birth and death rates in a linear age-structured population model from data on total population size and cumulative number of births (more realistic data than those used by Rundell *et al.* [14, 12, 5]). Perthame and Zubelli [11] considered the problem of determining the division (birth) rate coefficient for a size-structured model for cell division from measured stable size distribution of the population. More recently, Perasso and Razafison [10] studied the identifiability of the age-dependent mortality rate for Mc Kendrick model. Let us emphasize that all these works do not take into account the effect of spatial diffusion. On the contrary, Di Blasio and Lorenzi [3, 4] investigated such models from the point of view of identifiability (existence, uniqueness and continuous dependence). Traore investigated estimation problems for population dynamics with spatial diffusion to recover the state from distributed observation [16] or boundary observation [15] in space and full observation in age.

In this paper, we investigate the following inverse problem: *Assuming the initial age distribution $p_0$ to be unknown, but knowing the age distribution*

$$y(a,x,t) := p(a,x,t), \qquad t \in (0,T),\ a \in (a_1,a_2),\ x \in \mathcal{O},$$

*where $\mathcal{O}$ is some given subset of $\Omega$ and $0 \leqslant a_1 < a_2 \leqslant a^*$, estimate simultaneously:*

- *the age distribution $p(a,x,T)$ when $T \to +\infty$ , for $x \in \Omega$ and $a \in (0,a^*)$*
- *and the diffusion coefficient $k$.*

In [13], the authors answered the above question in the case where the diffusion coefficient $k$ is known. To do so, they constructed an observer for system (1.1), *i.e.* a new evolution system using the available measurements as inputs and whose dynamics is suitably chosen to make its state $\hat{p}(t)$ converge (asymptotically in time) to the state of the initial system $p(t)$. The design of this observer crucially uses the fact that the initial system has a finite number of unstable modes (corresponding to eigenvalues with non negative real parts), and the infinite dimensional observer is then constructed by designing a Luenberger observer for the finite dimensional unstable part of the system. The main contribution of this work is to extend this approach to the case where the diffusion coefficient $k$ is unknown. This is far from being obvious as the eigenvalues of the infinite dimensional system are then unknown. However, we can take advantage from the fact that eigenfunctions are known and this allows us to design a new observer following an idea proposed by Kreisselmeier in a finite dimensional setting in [9] (see also [8]). Let us emphasize that this observer requires more measurements than the observer proposed in [13], as it uses the projected outputs not only on the unstable modes but also on a finite number of stable modes (see (2.4)).

For the sake of clarity, the main results are stated in Section 2, and their proofs are given in Section 3.

## 2 Statement of the main results

Using a semigroup formulation, we first rewrite problem (1.1) in the abstract form (throughout the paper, the dot denotes the derivative with respect to time)

$$\begin{cases} \dot{p}(t) = Ap(t), & t \in (0,T), \\ p(0) = p_0, \end{cases} \tag{2.1}$$

where $A : \mathcal{D}(A) \to X$ is the generator of a $C_0$-semigroup on a Hilbert space $X := L^2((0,a^*) \times \Omega)$ defined by

$$\mathcal{D}(A) = \Big\{ \varphi \in X \cap L^2\left((0,a^*), H_0^1(\Omega)\right) \Big| -\frac{\partial \varphi}{\partial a} - \mu\varphi + k\Delta\varphi \in X; \\ \varphi(a,\cdot)|_{\partial\Omega} = 0 \text{ for almost all } a \in (0,a^*); \\ \varphi(0,x) = \int_0^{a^*} \beta(a)\varphi(a,x)\,\mathrm{d}a \text{ for almost all } x \in \Omega \Big\}$$

$$A\varphi = -\partial_a\varphi - \mu\varphi + k\Delta\varphi, \qquad \forall \varphi \in \mathcal{D}(A),$$

(see Chan and Guo [2], for more details). Similarly, the available observation can also be reformulated using a bounded observation operator $C \in \mathcal{L}(X,Y)$, where $Y := L^2((a_1,a_2) \times \mathcal{O})$, defined by $C\varphi = \varphi\big|_{(a_1,a_2)\times\mathcal{O}}$ $(\varphi \in X)$:

$$y(t) = Cp(t), \qquad t \in (0,T).$$

We recall here some results about the spectrum of $A$, we refer to [2, 13] for more details:

- The operator $A$ has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity.
- The eigenvalues of $A$ are given by

$$\sigma(A) = \left\{ \lambda_i^0 - k\lambda_j^D \,|i,j \in \mathbb{N}^* \right\}, \tag{2.2}$$

  where $(\lambda_n^D)_{n\geqslant 1}$ denotes the increasing positive sequence of eigenvalues of the Dirichlet Laplacian and $(\lambda_n^0)_{n\geqslant 1}$ denotes the sequence of eigenvalues of the free diffusion operator ($k = 0$), which are the solutions of the characteristic equation

$$F(\lambda) := \int_0^{a^*} \beta(a) e^{-\lambda a - \int_0^a \mu(s)\,\mathrm{d}s}\,\mathrm{d}a = 1. \tag{2.3}$$

- $A$ has a real dominant eigenvalue $\lambda_1$:

$$\lambda_1 = \lambda_1^0 - k\lambda_1^D > \mathrm{Re}\,(\lambda), \qquad \forall\, \lambda \in \sigma(A),\ \lambda \neq \lambda_1.$$

- The eigenspace associated to an eigenvalue $\lambda$ of $A$ is given by

$$\mathrm{Span}\left\{ e^{-\lambda_i^0 a - \int_0^a \mu(s)\,\mathrm{d}s} \varphi_j^D(x) \mid \lambda_i^0 - \lambda_j^D = \lambda \right\}$$

  where $(\varphi_n^D)_{n\geqslant 1}$ denotes an orthonormal basis of $L^2(\Omega)$ constituted of eigenfunctions of the Dirichlet Laplacian.
- Every vertical strip of the complex plane contains a finite number of eigenvalues of $A$.
- The semigroup $e^{tA}$ generated on $X$ by $A$ is compact for $t \geqslant a^*$, which implies in particular that the exponential stability of $e^{tA}$ is equivalent to the condition $\omega_0(A) := \sup\{\mathrm{Re}\,\lambda \,|\lambda \in \sigma(A)\} < 0$ (see Zabczyk [17], Section 2).

We denote by $M$ the number of eigenvalues of $A$ (counted without multiplicities) with positive real part and we assume that $A$ has no eigenvalue of real part equal to zero:

$$\cdots \leqslant \operatorname{Re}\lambda_{M+1} < 0 < \operatorname{Re}\lambda_M \leqslant \cdots \leqslant \operatorname{Re}\lambda_2 < \lambda_1.$$

To solve our estimation problem, we shall first construct an observer for the finite dimensional system in $\mathbb{C}^M$ corresponding to the unstable eigenvalues. To design this observer, we need to use an observation coming not only from the $M$ unstable modes, but also from some additional stable ones. More precisely let us choose $N \in \mathbb{N}^*$ such that

$$\operatorname{Re}\lambda_{N+1} < -3\lambda_1 \leqslant \operatorname{Re}\lambda_N < 0. \tag{2.4}$$

In the sequel, we also need to define $\alpha > 0$ such that

$$\operatorname{Re}\lambda_{N+1} < -\alpha < -3\lambda_1. \tag{2.5}$$

According to (2.2), the eigenvalues of $A$ depend linearly on the diffusion coefficient $k$ and, hence, $N$ also depends *a priori* on $k$. We will assume that the (finite) number of eigenvalues of $A$ of real part greater than $-3\lambda_1$ is constant when $k$ varies in $[k_0 - r_k, k_0 + r_k]$. This assumption is not crucial but is made for the sake of simplicity. Then, let us consider a curve $\Gamma^N$ in the complex plane enclosing the set of eigenvalues $\Sigma^N := \{\lambda_1, \ldots, \lambda_N\}$ but no other elements of the spectrum of $A$. We denote by $P^N$ the projection operator defined by

$$P^N := -\frac{1}{2\pi i}\int_{\Gamma^N} (\xi - A)^{-1}\mathrm{d}\xi\,.$$

We set $X^N = P^N(X)$ and $X_-^N = (\mathrm{Id} - P^N)(X)$, and then $P^N$ provides the following decomposition of $X$

$$X = X^N \oplus X_-^N.$$

Moreover $X^N$ and $X_-^N$ are invariant subspaces under $A$ (since $A$ and $P^N$ commute) and the spectra of the restricted operators $A_{|X^N}$ and $A_{|X_-^N}$ are respectively $\Sigma^N$ and $\sigma(A)\setminus\Sigma^N$ (see [7]). We also set

$$A^N := A_{|\mathcal{D}(A)\cap X^N} : \mathcal{D}(A)\cap X^N \to X^N \quad \text{and} \quad A_-^N := A_{|\mathcal{D}(A)\cap X_-^N} : \mathcal{D}(A)\cap X_-^N \to X_-^N.$$

Throughout the paper, we make the following assumption about $A^N$:

**Assumption 2.1.** *The restriction $A^N$ of operator $A$ to $X^N$ is diagonalizable.*

Let us emphasize that under this assumption, we have

$$X^N = \bigoplus_{n=1}^{N} \operatorname{Ker}(A - \lambda_n) = \operatorname{Span}\{\varphi_n,\ 1 \leqslant n \leqslant N\}\,,$$

where $(\varphi_n)_{1\leqslant n\leqslant N}$ denotes a basis of $X^N$ constituted of eigenfunctions of $A^N$ associated to the eigenvalues $(\lambda_k)_{1\leqslant n\leqslant N}$. We also denote by $\psi_n$ an eigenfunction of $A^*$ corresponding to the eigenvalue $\overline{\lambda_n}$, $1 \leqslant n \leqslant N$. It can be shown (see [1, p. 1453]) that the family $(\psi_n)_{1\leqslant n\leqslant N}$ can be chosen such that $(\varphi_n)_{1\leqslant n\leqslant N}$ and $(\psi_n)_{1\leqslant n\leqslant N}$ form bi-orthogonal sequences: $(\varphi_n, \psi_m)_X = \delta_{nm}$. It follows then that the projection operator $P^N \in \mathcal{L}(X, X^N)$ can be expressed as

$$P^N z = \sum_{n=1}^{N} \langle z, \psi_n\rangle \varphi_n, \qquad \forall z \in X.$$

With these notation, any solution $p$ of (2.1) admits the decomposition

$$p(t) = p^N(t) + p_-^N(t) \tag{2.6}$$

with

$$\begin{cases} p^N(t) := P^N(p(t)) = \displaystyle\sum_{n=1}^{N} e^{\lambda_n t} p_n^0 \varphi_n, \qquad p_n^0 = \langle p(0), \psi_n \rangle, \\ p_-^N(t) := (\mathrm{Id} - P^N)(p(t)) = e^{tA} p_-^N(0). \end{cases} \tag{2.7}$$

Throughout the paper, we will use bold letters/symbols to denote vectors and matrices. The above decomposition suggests to introduce the following finite dimensional state variable:

$$\mathbf{p}^N(t) = (p_1^N(t), \dots, p_N^N(t)) := (e^{\lambda_1 t} p_1^0, \dots, e^{\lambda_N t} p_N^0)^T \in \mathbb{C}^N$$

whose dynamics is simply given by

$$\dot{\mathbf{p}}^N(t) = \mathbf{\Lambda}^N \mathbf{p}^N(t), \tag{2.8}$$

where $\mathbf{\Lambda}^N := \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ (recall that the initial state $\mathbf{p}^N(0)$ is unknown and that the eigenvalues $\lambda_n$, $n = 1, \dots, N$, depend on $k$ and are thus also unknown). As mentioned above, we shall construct a finite dimensional observer for the $M$ unstable modes of the system, i.e. the $M$ first components of $\mathbf{p}^N(t)$, denoted $\mathbf{p}(t)$:

$$\mathbf{p}(t) := (e^{\lambda_1 t} p_1^0, \dots, e^{\lambda_M t} p_M^0)^T \in \mathbb{C}^M.$$

To do so, system (2.8) needs to be supplemented by finite dimensional observations, which can be easily obtained from those of the infinite dimensional system as follows. Defining the quantities

$$y_n(t) = \langle y(t), C\varphi_n \rangle_Y = \langle Cp(t), C\varphi_n \rangle_Y, \qquad n = 1, \dots, N,$$

and setting

$$\mathbf{y}^N(t) := (y_1(t), \dots, y_N(t))^T \in \mathbb{C}^N,$$

the decomposition (2.6) immediately shows that

$$\mathbf{y}^N(t) = \mathbf{C}^N \mathbf{p}^N(t) + \mathbf{y}_-^N(t), \tag{2.9}$$

where the matrix $\mathbf{C}^N := (C_{mn}^N)_{1 \leqslant n,m \leqslant N}$ is defined by $C_{mn}^N = \langle C\varphi_n, C\varphi_m \rangle_Y$ and

$$\mathbf{y}_-^N(t) := (\langle Cp_-^N(t), C\varphi_1 \rangle_Y, \dots, \langle Cp_-^N(t), C\varphi_N \rangle_Y) \in \mathbb{C}^N.$$

The family $(C\varphi_n)_{1 \leqslant n \leqslant N}$ being linearly independent in $X$ (see [1] or [13]), the matrix $\mathbf{C}^N$ is invertible. Consequently, equation (2.9) equivalently reads

$$\mathbf{q}^N(t) = \mathbf{p}^N(t) + \mathbf{q}_-^N(t), \tag{2.10}$$

where

$$\mathbf{q}^N(t) = (q_1^N(t), \dots, q_N^N(t))^T := (\mathbf{C}^N)^{-1} \mathbf{y}^N(t)$$

and

$$\mathbf{q}_-^N(t) = (q_{-,1}^N(t), \dots, q_{-,N}^N(t))^T := (\mathbf{C}^N)^{-1} \mathbf{y}_-^N(t).$$

Note that $\mathbf{q}^N(t)$ is an available measure since $\mathbf{C}^N$ and $\mathbf{y}^N(t)$ are known.

Following Kreisselmeier ([9, 8]), the proposed finite dimensional observer in $\mathbb{C}^M$ is defined by

$$\hat{\mathbf{p}}(t) = \mathbf{M}(t) \mathbf{E}\, \boldsymbol{\theta}(t) \tag{2.11}$$

with

$$\begin{cases} \dot{\mathbf{M}}(t) = \mathbf{F}\mathbf{M}(t) + [q_1^N(t)\mathbf{Id}, \dots, q_M^N(t)\mathbf{Id}] & (2.12a) \\ \mathbf{M}(0) = 0 & (2.12b) \\ \dot{\boldsymbol{\theta}}(t) = -\gamma \mathbf{E}^* \mathbf{M}^*(t)(\hat{\mathbf{p}}(t) - \Pi_M \mathbf{q}^N(t)), & (2.12c) \\ \boldsymbol{\theta}(0) = 0 & (2.12d) \end{cases}$$

where

- $\hat{\mathbf{p}}$ and $\boldsymbol{\theta}$ are vectors in $\mathbb{C}^M$ whose components are respectively $\hat{p}_i$ and $\theta_i$;
- $\mathbf{M}(t)$ denotes a $M \times M^2$ matrix;
- $\mathbf{E} = (\mathbf{E}_1, \dots, \mathbf{E}_M)^{\mathrm{T}}$ denotes a $M^2 \times M$ matrix, with $\mathbf{E}_n = \mathrm{diag}(0, \dots, 0, 1, 0, \dots, 0)$, the term 1 being at the $n^{\mathrm{th}}$ place;
- $\mathbf{F} = \mathrm{diag}(-f_1, \dots, -f_M)$ where $f_i$ are chosen positive;
- $\mathbf{Id}$ denotes the $M \times M$ identity matrix;
- $\Pi_M : \mathbb{C}^N \to \mathbb{C}^M$ denotes the projection on the $M$-first components (more precisely, if $\mathbf{z} = (z_1, \cdots, z_M, z_{M+1}, \cdots, z_N)^T \in \mathbb{C}^N$, then $\Pi_M \mathbf{z} = (z_1, \cdots, z_M)^T \in \mathbb{C}^M$);
- $\gamma$ is a positive real number (gain coefficient);
- the $^*$ stands for the conjugate transpose.

Let us emphasize that the adaptive state estimate (2.11)-(2.12) is the one proposed by Kreisselmeier (see equation (9) in [9]) in the particular case where the initial data of the observer is zero and where the dynamics of $\theta$ is driven by a quadratic error criterion $L$ (see equation (7) in [9]). However, the available measurement error here is not $\hat{\mathbf{p}}(t) - \mathbf{p}^N(t)$ (since $\mathbf{p}^N(t) = \mathbf{q}^N(t) - \mathbf{q}_-^N(t)$ is unknown) but only $\hat{\mathbf{p}}(t) - \Pi_M \mathbf{q}^N(t)$, and this generates additional difficulties. In particular, this is why the observer $\hat{\mathbf{p}}(t) \in \mathbb{C}^M$ is computed using the output vector $\Pi_M \mathbf{q}^N(t) = (q_1^N(t), \dots, q_M^N(t))^T \in \mathbb{C}^M$, the latter being obtained from the $N > M$ measurements collected in the vector $\mathbf{y}^N(t) \in \mathbb{C}^N$. As will be seen later, this construction, which might seem unnatural, ensures that the remainder term $\mathbf{q}_-^N(t)$ decays fast enough to 0 and hence guarantees the convergence of the observer.

Let $\boldsymbol{\theta}^{\boldsymbol{\infty}} := (\lambda_1 + f_1, \cdots, \lambda_M + f_M)^T = \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right) \mathbf{1} \in \mathbb{C}^M$, where $\mathbf{1}$ denotes the vector of $\mathbb{C}^M$ whose all components are equal to 1 and where $\boldsymbol{\Lambda}^M = \mathrm{diag}(\lambda_1, \dots, \lambda_M)$.

We are now in position to state the two main results of the paper.

**Theorem 2.2.** *Let $R > 0$ and $\varepsilon > 0$ be given. Assume that the initial data $p_0$ satisfies*

$$\|p_0\|_X \leqslant R \tag{2.13}$$

*and*

$$|p_n^0| > \varepsilon, \qquad \forall n = 1, \dots, M. \tag{2.14}$$

*Assume also that $N$ is chosen such that* (2.4) *is satisfied. Finally, let $\hat{\mathbf{p}}(t)$ and $\boldsymbol{\theta}(t)$ be defined by* (2.11) *and* (2.12).

*Then we can choose the matrix $\mathbf{F}$ such that there exist $\kappa > 0$, $\omega > 0$ satisfying*

$$\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\boldsymbol{\infty}}\|_{\mathbb{C}^M} \leqslant \kappa\, e^{-\omega t} \quad \text{and} \quad \|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\|_{\mathbb{C}^M} \leqslant \kappa\, e^{-\omega t} \qquad (t > 0).$$

**Remark 2.3.** *Notice that $\boldsymbol{\theta}^{\boldsymbol{\infty}}$ provides an estimate for the $M$ first eigenvalues of $A$, and hence of the unknown diffusion coefficient $k$. In particular, $\theta_1(t)$ converges exponentially to $\theta_1^\infty = \lambda_1 + f_1 = \lambda_1^0 - k\lambda_1^D + f_1$.*

**Theorem 2.4.** *Let $\hat{\mathbf{p}}(t) = (\hat{p}_1(t), \dots, \hat{p}_M(t))^T \in \mathbb{C}^M$ and $\boldsymbol{\theta}(t) = (\theta_1(t), \dots, \theta_M(t))^T \in \mathbb{C}^M$ be defined by* (2.11) *and* (2.12) *and set*

$$\hat{p}(t) = \sum_{n=1}^{M} \hat{p}_n(t) \varphi_n. \tag{2.15}$$

*Then, under the assumptions of Theorem 2.2, we can choose the matrix*

$$\mathbf{F} = \mathrm{diag}(-f_1, \dots, -f_M), \qquad f_i > 0,\ i = 1, \dots, M,$$

*such that there exist $\kappa > 0$ and $\omega > 0$ satisfying*

$$\|\hat{p}(t) - p(t)\|_X \leqslant \kappa\, e^{-\omega t} \qquad (t > 0).$$

*Moreover, we have*

$$k = \frac{1}{\lambda_1^D}\left(\lambda_1^0 + f_1 - \lim_{t\to+\infty}\theta_1(t)\right).$$

*Proof.* Let us define $P^M$, $X^M$, $X_-^M$, $p^M$, $p_-^M$ as above, but replacing $N$ by $M$. Using (2.6) written with $M$ instead of $N$, we have

$$\|\hat{p}(t) - p(t)\|_X^2 \leqslant 2\,\|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\|_{\mathbb{C}^M}^2 + 2\left\|e^{tA}p_-^M(0)\right\|_X^2.$$

On the other hand,

$$\|e^{tA}p_-^M(0)\|_X \leqslant K\,\|p_-^M(0)\|_X\, e^{-\alpha_M t},$$

where $K > 0$ and $\alpha_M$ may be any positive constant such that $-\alpha_M > \operatorname{Re}\lambda_{M+1}$. The result follows then from Theorem 2.2. □

# 3 Proof of Theorem 2.2

We split up the proof of Theorem 2.2 into three lemmas and two propositions.

**Lemma 3.1.** *Let* $\boldsymbol{\theta}^{\boldsymbol{\infty}} = \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\mathbf{1} \in \mathbb{C}^M$. *Then* $\mathbf{p}(t) \in \mathbb{C}^M$ *can be written as*

$$\mathbf{p}(t) = \mathbf{M}(t)\mathbf{E}\,\boldsymbol{\theta}^{\boldsymbol{\infty}} + e^{t\mathbf{F}}\mathbf{p}(0) - \int_0^t e^{(t-s)\mathbf{F}}\left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{q}_-^N(s)\,\mathrm{d}s \qquad (t > 0). \quad (3.1)$$

*Proof.* We set

$$\tilde{\mathbf{p}}(t) := \mathbf{M}(t)\mathbf{E}\,\boldsymbol{\theta}^{\boldsymbol{\infty}} + e^{t\mathbf{F}}\mathbf{p}(0) - \int_0^t e^{(t-s)\mathbf{F}}\left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{q}_-^N(s)\,\mathrm{d}s,$$

and we want to prove that $\tilde{\mathbf{p}}(t) = \mathbf{p}(t)$ for any $t > 0$. We have

$$\frac{d}{dt}\,\tilde{\mathbf{p}}(t) = \dot{\mathbf{M}}(t)\mathbf{E}\,\boldsymbol{\theta}^{\boldsymbol{\infty}} + \mathbf{F}e^{t\mathbf{F}}\mathbf{p}(0) - \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{q}_-^N(t) - \int_0^t \mathbf{F}e^{(t-s)\mathbf{F}}\left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{q}_-^N(s)\,\mathrm{d}s,$$

which implies, using (2.12a), that

$$\frac{d}{dt}\,\tilde{\mathbf{p}}(t) = \mathbf{F}\tilde{\mathbf{p}}(t) + [q_1^N(t)\mathbf{Id}, \ldots, q_M^N(t)\mathbf{Id}]\mathbf{E}\,\boldsymbol{\theta}^{\boldsymbol{\infty}} - \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{q}_-^N(t).$$

Moreover, we can easily verify that

$$[q_1^N(t)\mathbf{Id}, \ldots, q_M^N(t)\mathbf{Id}]\mathbf{E}\,\boldsymbol{\theta}^{\boldsymbol{\infty}} = \sum_{n=1}^M q_n^N(t)\mathbf{E}_n\,\boldsymbol{\theta}^{\boldsymbol{\infty}} = \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{q}^N(t).$$

Consequently, we have

$$\frac{d}{dt}\,\tilde{\mathbf{p}}(t) = \mathbf{F}\tilde{\mathbf{p}}(t) + \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\left(\mathbf{q}^N(t) - \mathbf{q}_-^N(t)\right),$$

and by (2.10)

$$\frac{d}{dt}\,\tilde{\mathbf{p}}(t) = \mathbf{F}\tilde{\mathbf{p}}(t) + \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right)\Pi_M\mathbf{p}^N(t) = \mathbf{F}\left(\tilde{\mathbf{p}}(t) - \mathbf{p}(t)\right) + \boldsymbol{\Lambda}^M\mathbf{p}(t),$$

since $\Pi_M\mathbf{p}^N = \mathbf{p}$ by definition. Using (2.8), we deduce that

$$\frac{d}{dt}\left(\tilde{\mathbf{p}}(t) - \mathbf{p}(t)\right) = \mathbf{F}\left(\tilde{\mathbf{p}}(t) - \mathbf{p}(t)\right).$$

Since we have on the other hand that $\tilde{\mathbf{p}}(0) = \mathbf{M}(0)\mathbf{E}\,\boldsymbol{\theta}^{\boldsymbol{\infty}} + \mathbf{p}(0) = \mathbf{p}(0)$, we can conclude that $\tilde{\mathbf{p}}(t) = \mathbf{p}(t)$ for any $t > 0$. □

**Remark 3.2.**

1. *Equation* (3.1) *provides an explicit formula to compute* $\mathbf{p}(t)$ *if we knew* $\mathbf{p}(0)$, $\boldsymbol{\theta}^\infty$ *(and thus* $\lambda_n$*) and* $\Pi_M \mathbf{q}_-^N(t)$. *This is not the case in the problem studied here.*
2. *If* $\mathbf{q}_-^N = 0$ *(i.e.* $\mathbf{q}^N = \mathbf{p}^N$*) in* (3.1), *we recover exactly the adaptive observer proposed by Kreisselmeier [9] (see also [8]).*

**Lemma 3.3.** *Assume that* $N$ *is chosen according to condition* (2.4) *and that* $p_0$ *satisfies* (2.13)-(2.14). *For* $f_n$ *large enough* ($n = 1, \dots, M$)*, there exist positive constants* $m_0$, $m_1$, $m_2$ *independent of* $p_0$ *such that, for any* $t > t_0 := m_0/\mathrm{Re}\,\lambda_M$, *the two following inequalities hold:*

$$|(\mathbf{E}^*\mathbf{M}(t)^*\mathbf{M}(t)\mathbf{E})_{nn}|^{1/2} \geqslant m_1(\mathrm{Re}\,(\lambda_M)\,t - m_0), \qquad n = 1, \dots, M, \tag{3.2}$$

$$\|\mathbf{M}(t)\mathbf{E}\| \leqslant m_2 e^{\lambda_1 t}. \tag{3.3}$$

*Proof.* We first compute explicitly the matrix $\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E}$. Writing

$$\mathbf{M}(t) = [\mathbf{M}_1(t), \dots, \mathbf{M}_M(t)],$$

where the matrices $\mathbf{M}_n$ are $M \times M$ matrices, we have $\mathbf{M}(t)\mathbf{E} = \sum_{n=1}^M \mathbf{M}_n(t)\mathbf{E}_n$. Moreover the matrices $\mathbf{M}_n(t)$ satisfy the following differential equations

$$\dot{\mathbf{M}}_n(t) = \mathbf{F}\mathbf{M}_n(t) + q_n^N(t)\mathbf{Id}, \qquad n = 1, \dots, M\,.$$

As $\mathbf{M}(0) = 0$, we have by (2.10)

$$\mathbf{M}_n(t) = \int_0^t q_n^N(s)\, e^{(t-s)\mathbf{F}} \mathrm{d}s = \int_0^t p_n^N(s)\, e^{(t-s)\mathbf{F}} \mathrm{d}s + \int_0^t q_{-,n}^N(s)\, e^{(t-s)\mathbf{F}} \mathrm{d}s.$$

Now, recalling that $p_n^N(t) = p_n^0 e^{\lambda_n t}$ are the components of $\mathbf{p}^N(t)$, we have

$$\begin{aligned}\int_0^t p_n^N(s)\, e^{(t-s)\mathbf{F}} \mathrm{d}s &= \int_0^t p_n^0\, e^{\lambda_n s} e^{(t-s)\mathbf{F}} \mathrm{d}s \\ &= p_n^0 \,\mathrm{diag}\left(\frac{e^{\lambda_n t} - e^{-f_1 t}}{\lambda_n + f_1}, \dots, \frac{e^{\lambda_n t} - e^{-f_M t}}{\lambda_n + f_M}\right).\end{aligned}$$

Therefore

$$\mathbf{M}_n(t) = p_n^0 \,\mathrm{diag}\left(\frac{e^{\lambda_n t} - e^{-f_1 t}}{\lambda_n + f_1}, \dots, \frac{e^{\lambda_n t} - e^{-f_M t}}{\lambda_n + f_M}\right) + \mathbf{M}_n^-(t)$$

where

$$\mathbf{M}_n^-(t) = \mathrm{diag}\left(\int_0^t e^{-(t-s)f_1} q_{-,n}^N(s)\,\mathrm{d}s, \dots, \int_0^t e^{-(t-s)f_M} q_{-,n}^N(s)\,\mathrm{d}s\right).$$

We deduce that

$$\mathbf{M}_n(t)\mathbf{E}_n = \left(p_n^0\, \frac{e^{\lambda_n t} - e^{-f_n t}}{\lambda_n + f_n} + \int_0^t e^{-(t-s)f_n} q_{-,n}^N(s)\,\mathrm{d}s\right)\mathbf{E}_n. \tag{3.4}$$

As $\mathbf{E}_n^* = \mathbf{E}_n$ and $\mathbf{E}_n\mathbf{E}_m = \delta_{nm}\mathbf{E}_n$ ($\delta_{mn}$ denotes the Kronecker symbol), we obtain

$$\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E} = \sum_{n=1}^M \left(p_n^0\, \frac{e^{\lambda_n t} - e^{-f_n t}}{\lambda_n + f_n} + \int_0^t e^{-(t-s)f_n} q_{-,n}^N(s)\,\mathrm{d}s\right)^2 \mathbf{E}_n\,.$$

We shall seek for an upper-bound for the diagonal terms of $\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E}$. Denoting by $\rho_n$ the real part of $\lambda_n$, we have

$$|(\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E})_{nn}|^{1/2} \geq \frac{|p_n^0|}{|\lambda_n + f_n|}(e^{\rho_n t} - e^{-f_n t}) - \int_0^t e^{-(t-s)f_n} |q_{-,n}^N(s)| \, \mathrm{d}s.$$

It follows from the fact that the semigroup $e^{tA}$ generated on $X$ by $A$ is compact for $t \geqslant a^*$ and the definition of $p_-^N$ that there exists $\kappa > 0$ such that

$$\|p_-^N(t)\|_X \leqslant \kappa \|p_-^N(0)\|_X e^{-\alpha t} \qquad (t > 0),$$

where $\alpha$ satisfies (2.5).

From the definition of $\mathbf{y}_-^N$, we infer that there exists a positive constant $L'$ such that

$$\|\mathbf{y}_-^N(t)\|_{\mathbb{C}^N} \leqslant L' \, \|p_-^N(0)\|_X \, e^{-\alpha t} \qquad (t > 0).$$

As $\mathbf{q}_-^N(t) := (\mathbf{C}^N)^{-1}\mathbf{y}_-^N(t)$, we can write an analogous inequality for its components $q_{-,n}^N$: there exists a positive constant $L$ such that

$$\left|q_{-,n}^N(t)\right| \leqslant L \|p_-^N(0)\|_X \, e^{-\alpha t}, \qquad \text{for } n = 1, \ldots, M\,. \tag{3.5}$$

Thus, if the $f_n$'s are chosen greatest than $\alpha$, we can write

$$|(\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E})_{nn}|^{1/2} \geqslant K_n(e^{\rho_n t} - e^{-f_n t}) - L_n(e^{-\alpha t} - e^{-f_n t})$$

where

$$K_n = \frac{|p_n^0|}{|\lambda_n + f_n|}\,, \qquad L_n = \frac{L\|p_-^N(0)\|_X}{f_n - \alpha}\,.$$

Now, if $L_n \leqslant K_n$, we have

$$\begin{aligned} K_n(e^{\rho_n t} - e^{-f_n t}) - L_n(e^{-\alpha t} - e^{-f_n t}) &\geqslant K_n(e^{\rho_n t} - e^{-\alpha t}) + (L_n - K_n)e^{-f_n t} \\ &\geqslant K_n(e^{\rho_n t} - 1) + L_n - K_n \\ &\geqslant K_n\left(\rho_n t - \frac{K_n - L_n}{K_n}\right), \end{aligned}$$

and if $L_n > K_n$, we have

$$\begin{aligned} K_n(e^{\rho_n t} - e^{-f_n t}) - L_n(e^{-\alpha t} - e^{-f_n t}) &\geqslant K_n e^{\rho_n t} - L_n e^{-\alpha t} \\ &\geqslant K_n(\rho_n t + 1) - L_n \\ &= K_n\Big(\rho_n t - \frac{L_n - K_n}{K_n}\Big). \end{aligned}$$

Hence, in both cases we have

$$|(\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E})_{nn}|^{1/2} \geqslant K_n(\rho_n t - K_n'), \qquad n = 1, \ldots, M$$

with $K_n' = |L_n - K_n| \, K_n^{-1}$.

Due to assumptions (2.13)-(2.14), there exist constants $\ell, \ell' > 0$ independent of $p_0$ such that for all $n = 1, \ldots, M$:

$$\ell'\varepsilon \leqslant K_n \leqslant \ell R, \qquad L_n \leqslant \ell R, \qquad K_n' \leqslant \ell \frac{R}{\varepsilon}.$$

Hence, $K_n(\rho_n t - K_n') \geqslant K_n(\rho_M t - m_0)$ where $m_0 := \ell R/\varepsilon$. Letting $m_1 := \ell'\varepsilon$ and $t_0 = m_0/\rho_M$, we have

$$|(\mathbf{E}^*\mathbf{M}^*(t)\mathbf{M}(t)\mathbf{E})_{nn}|^{1/2} \geqslant m_1(\rho_M t - m_0), \qquad \forall t > t_0.$$

We now prove the second point of the lemma. From (3.4) and (3.5), we have

$$|(\mathbf{M}(t)\mathbf{E})_{nn}| \leqslant K_n\,|e^{\lambda_n t} - e^{-f_n t}| + L_n(e^{-\alpha t} - e^{-f_n t}) \leqslant (2K_n + L_n)e^{\rho_n t},$$

where, for the last inequality, we have used $-f_n \leqslant -\alpha \leqslant \rho_n$, $n = 1,\dots,M$. Hence, since $\mathbf{M}(t)\mathbf{E}$ is a diagonal matrix,

$$\|\mathbf{E}^*\mathbf{M}^*(t)\| = \|\mathbf{M}(t)\mathbf{E}\| = \max_{n=1,\dots,M} |(\mathbf{M}(t)\mathbf{E})_{nn}| \leqslant m_2\, e^{\lambda_1 t}$$

where $m_2 = 3\ell R$. Notice that $\lambda_1$ is real and is equal to the largest value taken by the numbers $\rho_n$ $(n = 1,\dots,M)$. □

**Proposition 3.4.** *Assume that $N$ is chosen according to condition* (2.4) *and that $p_0$ satisfies* (2.13)-(2.14)*. For $f_n$ large enough ($n = 1,\dots,M$), the function $V$ defined by*

$$V(t) = \|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty}\|^2_{\mathbb{C}^M} \qquad (t > 0)$$

*satisfy for all $t > t_0 = m_0/\mathrm{Re}\,\lambda_M$:*

$$\dot V(t) \leqslant -2\gamma\, m_1^2(\mathrm{Re}\,(\lambda_M)\,t - m_0)^2 V(t) + 2\gamma\, m_6 e^{(\lambda_1-\alpha)t} V(t)^{1/2}, \tag{3.6}$$

*where $m_6$ is a positive constant independent of $p_0$ and where $\alpha$ satisfies* (2.5)*.*

*Proof.* Let us compute $\dot V$, the derivative of $V$ along the trajectories of system (2.12). Using the expression of $\mathbf{q}^N$ given by (2.10), we have

$$\begin{aligned}
\dot V(t) &= 2(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty})^*\dot{\boldsymbol{\theta}}(t)\\
&= -2\gamma(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty})^*\mathbf{E}^*\mathbf{M}^*(t)(\hat{\mathbf{p}}(t) - \Pi_M\mathbf{q}^N(t))\\
&= -2\gamma(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty})^*\mathbf{E}^*\mathbf{M}^*(t)(\hat{\mathbf{p}}(t) - \mathbf{p}(t) - \Pi_M\mathbf{q}^N_-(t))\\
&= -2\gamma(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty})^*\mathbf{E}^*\mathbf{M}^*(t)\,\Big(\mathbf{M}(t)\mathbf{E}(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty})\\
&\qquad -e^{tF}\mathbf{p}(0) + \int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M\mathbf{q}^N_-(s)\mathrm{d}s - \Pi_M\mathbf{q}^N_-(t)\Big),
\end{aligned} \tag{3.7}$$

using, for the last formula, (2.11) and (3.1). Substituting (3.2) in (3.7), we obtain

$$\begin{aligned}
\dot V(t) \leq{}& -2\gamma m_1^2(\mathrm{Re}\,(\lambda_M)\,t - m_0)^2 V(t)\\
&+ 2\gamma\|\mathbf{E}^*\mathbf{M}^*(t)\|\,\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty}\|\Big\{\|e^{t\mathbf{F}}\|\,\|\mathbf{p}(0)\|\\
&+ \Big\|\int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M\mathbf{q}^N_-(s)\mathrm{d}s\Big\| + \|\Pi_M\mathbf{q}^N_-(t)\|\Big\}
\end{aligned} \tag{3.8}$$

for every $t > t_0$ (the norms are taken in $\mathbb{C}^M$). We need an upperbound for the second term in the hand-right side of this inequality. From the definition of $\mathbf{q}^N_-$ and from inequalities (3.5), there exists a positive constant $m_3$ independent of $p_0$ such that

$$\|\Pi_M\mathbf{q}^N_-(t)\|_{\mathbb{C}^M} \leqslant m_3\, e^{-\alpha t} \quad (t > 0). \tag{3.9}$$

We deduce from this inequality and from $f_n > \alpha$ that we have

$$\Big\|\int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M\mathbf{q}^N_-(s)\mathrm{d}s\Big\|_{\mathbb{C}^M} \leqslant m_3\,\frac{\lambda_1 + f_\infty}{f_0 - \alpha}(e^{-\alpha t} - e^{-f_\infty t}) \leqslant m_4\, e^{-\alpha t} \tag{3.10}$$

where $f_0 = \min_{n=1,\dots,M} f_n$, $f_\infty = \max_{n=1,\dots,M} f_n$ and $m_4 = m_3(\lambda_1 + f_\infty)(f_0 - \alpha)^{-1}$. We deduce from inequalities (3.9), (3.10) and from $f_n > \alpha$ that

$$\|e^{t\mathbf{F}}\|_{\mathbb{C}^M}\,\|\mathbf{p}(0)\|_{\mathbb{C}^M} + \Big\|\int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M\mathbf{q}^N_-(s)\mathrm{d}s\Big\|_{\mathbb{C}^M} + \|\Pi_M\mathbf{q}^N_-(t)\|_{\mathbb{C}^M} \leqslant m_5 e^{-\alpha t}, \tag{3.11}$$

where $m_5 = R + m_4 + m_3$ . Substituting inequalities (3.3) and (3.11) in (3.8), we obtain

$$\dot{V}(t) \le -2\gamma m_1^2(\rho_M t - m_0)^2 V(t) + 2\gamma m_6 e^{(\lambda_1-\alpha)t} V(t)^{1/2}$$

with $m_6 = m_2 m_5$ .

$\square$

**Lemma 3.5.** *Assume that $N$ is chosen according to condition* (2.4) *and that $p_0$ satisfies* (2.13)-(2.14)*. The function*

$$W(t) := V(t)^{1/2} = \|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty\|_{\mathbb{C}^M}$$

*is continuous and right differentiable on $\mathbb{R}_+$. Moreover, denoting by $\dot{W}_r$ the right derivative of $W$, $\dot{W}_r$ satisfies the following inequality for every $t > t_0 = m_0/\mathrm{Re}\,\lambda_M$:*

$$\dot{W}_r(t) \le -\gamma m_1^2(\mathrm{Re}\,(\lambda_M)\,t - m_0)^2 W(t) + \gamma\, m_6\, e^{(\lambda_1-\alpha)t}, \tag{3.12}$$

*where $\alpha$ satisfies* (2.5)

*Proof.* Notice first that as $t \mapsto \langle Cp(t), C\varphi\rangle_Y$ is a $C^1$ function (for every $\varphi \in X$), $\mathbf{y}^N$ and $\mathbf{q}^N$ are also $C^1$ function (by definition), and so $\boldsymbol{\theta}$ is a $C^2$ function due to (2.12). Thus $W$ is continuous.

Moreover if $t_1$ is such that $V(t_1) \ne 0$, then $W$ is differentiable at $t = t_1$ and so is right differentiable. In this case we have

$$\dot{W}_r(t_1) = \frac{\dot{V}(t_1)}{2V(t_1)^{1/2}}\,.$$

By (3.6), (3.12) holds.

If $V(t_1) = 0$, from formula (3.7), we obtain that $\dot{V}(t_1) = 0$. So we can write

$$V(t) = \frac{\beta}{2}(t-t_1)^2 + o((t-t_1)^2)$$

with $\beta = \ddot{V}(t_1) \geqslant 0$ because $V(t) \geqslant 0$ near $t_1$. From this equality, and taking $h > 0$, we have

$$\frac{W(t_1+h) - W(t_1)}{h} = \frac{\sqrt{\beta h^2/2 + o(h^2)}}{h} = \sqrt{\beta/2 + o(1)},$$

which proves that $W$ is right differentiable at $t = t_1$ and that $\dot{W}_r(t_1) = \sqrt{\beta/2}$.
Notice that if $\beta > 0$, then $V(t) > 0$ for $t$ in some interval $(t_1, t_1+\tau)$ (here $\tau > 0$), so in this case we have

$$\frac{\dot{V}(t_1+h)}{2V(t_1+h)^{1/2}} = \frac{\beta h + o(h)}{2\sqrt{\beta h^2/2 + o(h^2)}} = \frac{\beta + o(1)}{2\sqrt{\beta/2 + o(1)}}$$

which proves that

$$\lim_{\substack{t\to t_1\\ t>t_1}} \frac{\dot{V}(t)}{2V(t)^{1/2}} = \dot{W}_r(t_1)\,.$$

Thus, in the case where $V(t_1) = 0$ and $\beta > 0$, inequality (3.12) can be obtained obtained by dividing inequality (3.6) by $V(t)^{1/2}$ and by taking the limit as $t$ tends to $t_1^+$.
Finally, if $V(t_1) = 0$ and $\beta = 0$, we have $\dot{W}_r(t_1) = W(t_1) = 0$ and the inequality of the lemma is obvious. $\square$

**Proposition 3.6.** *Assume that $N$ is chosen according to condition* (2.4) *and that $p_0$ satisfies* (2.13)-(2.14)*. For $f_n$ large enough ($n = 1, \dots, M$), there exist $m_7 > 0$ and $t_1 \geqslant t_0 = m_0/\mathrm{Re}\,\lambda_M$ independent of $p_0$ such that, for any $t > t_1$, we have*

$$\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty\|_{\mathbb{C}^M} \leqslant m_7\, e^{(\lambda_1-\alpha)(t+t_0)/2}, \tag{3.13}$$

*where $\alpha$ satisfies* (2.5)*.*

*Proof.* Let us set $\rho_M = \mathrm{Re}\,(\lambda_M)$ and consider the function $\mathcal{W}$ defined as

$$\mathcal{W}(t) = \chi(t)W(t),$$

where $W$ is defined in Lemma 3.5 and (recall that $m_0 = \rho_M t_0$),

$$\chi(t) = \exp\Big(\frac{\gamma m_1^2}{3\rho_M}(\rho_M t - m_0)^3\Big) = \exp\Big(\frac{\gamma m_1^2 \rho_M^2}{3}(t-t_0)^3\Big).$$

Then, $\mathcal{W}$ is right differentiable on $\mathbb{R}_+$ and for every $t > t_0$, we get from (3.12):

$$\dot{\mathcal{W}}_r(t) = \Big\{\dot{W}_r(t) + \gamma m_1^2(\rho_M t - m_0)^2 W(t)\Big\}\,\chi(t) \leqslant \gamma m_6 e^{(\lambda_1-\alpha)t}\chi(t).$$

Applying the mean value inequality, we deduce that

$$\mathcal{W}(t) - \mathcal{W}(t_0) \leqslant \gamma m_6 \int_{t_0}^{t} e^{(\lambda_1-\alpha)s}\chi(s)\,\mathrm{d}s$$

and hence, since $W(t_0) = \mathcal{W}(t_0)$

$$W(t) \leqslant \frac{W(t_0)}{\chi(t)} + \gamma m_6 \frac{1}{\chi(t)}\int_{t_0}^{t} e^{(\lambda_1-\alpha)s}\chi(s)\,\mathrm{d}s. \tag{3.14}$$

The first term in the right-hand side of this inequality tends to zero as $t$ tends to infinity. We will see that the same is true for the second term. To this end we divide the integral appearing in the second term into two parts. As $\lambda_1 < \alpha$ we first have

$$\begin{aligned}
\int_{t_0}^{(t+t_0)/2} e^{(\lambda_1-\alpha)s}\chi(s)\,\mathrm{d}s &\leqslant \int_{t_0}^{(t+t_0)/2} \chi(s)\,\mathrm{d}s \\
&= \int_0^{\frac{\rho_M^3}{8}(t-t_0)^3} \frac{\exp\Big(\frac{\gamma m_1^2}{3\rho_M}\,\sigma\Big)}{3\rho_M \sigma^{2/3}}\,\mathrm{d}\sigma \\
&\leqslant \exp\Big(\frac{\gamma m_1^2\rho_M^2}{24}(t-t_0)^3\Big)\int_0^{\frac{\rho_M^3}{8}(t-t_0)^3} \frac{\mathrm{d}\sigma}{3\rho_M\sigma^{2/3}} \\
&= \exp\Big(\frac{\gamma m_1^2\rho_M^2}{24}(t-t_0)^3\Big)\,\frac{t-t_0}{2}\,.
\end{aligned} \tag{3.15}$$

On the other hand

$$\int_{(t+t_0)/2}^{t} e^{(\lambda_1-\alpha)s}\chi(s)\,\mathrm{d}s \leqslant \chi(t)\int_{(t+t_0)/2}^{t} e^{(\lambda_1-\alpha)s}\mathrm{d}s \leqslant \frac{\chi(t)}{\alpha-\lambda_1}\,e^{(\lambda_1-\alpha)(t+t_0)/2}. \tag{3.16}$$

Substituting inequalities (3.15) and (3.16) into (3.14), we obtain

$$W(t) \leqslant \frac{W(t_0)}{\chi(t)} + \gamma m_6\Big\{\exp\Big(-\frac{7\gamma m_1^2\rho_M^2}{24}(t-t_0)^3\Big)\frac{t-t_0}{2} + \frac{1}{\alpha-\lambda_1}\,e^{(\lambda_1-\alpha)(t+t_0)/2}\Big\}.$$

Therefore there exist $t_1 \geqslant t_0$ and a constant $m_7 > 0$ such that if $t > t_1$ we have

$$\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty\|_{\mathbb{C}^M} = W(t) \leqslant m_7\, e^{(\lambda_1-\alpha)(t+t_0)/2}.$$

□

We are now in position to prove Theorem 2.2.

*Proof of Theorem 2.2.* Proposition 3.6 obviously implies that $\boldsymbol{\theta}(t)$ tends exponentially to $\boldsymbol{\theta}^{\infty}$ as $t \to \infty$ since $\lambda_1 < \alpha$.

Concerning the convergence of $\hat{\mathbf{p}} - \mathbf{p}$, from (2.11) and (3.1), we obtain (taking the norms in $\mathbb{C}^M$)

$$\|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\| \leqslant \|\mathbf{M}(t)\mathbf{E}\| \, \|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^{\infty}\| + \|e^{t\mathbf{F}}\| \, \|\mathbf{p}(0)\| \\ + \left\| \int_0^t e^{(t-s)\mathbf{F}} \left(\boldsymbol{\Lambda}^M - \mathbf{F}\right) \Pi_M \mathbf{q}_-^N(s) \, \mathrm{d}s \right\|.$$

Gathering (3.3), (3.13) and (3.11), we have, for every $t > t_1$,

$$\|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\|_{\mathbb{C}^M} \leqslant m_2 m_7 \, e^{(\lambda_1 - \alpha)t_0/2} e^{(3\lambda_1 - \alpha)t/2} + m_5 e^{-\alpha t},$$

which implies that $\hat{\mathbf{p}}(t) - \mathbf{p}(t)$ tends exponentially to zero due to (2.5), and this ends the proof. □

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