Causality vs. interleavings in concurrent game semantics

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Abstract

We investigate relationships between interleaving and causal notions of game semantics for concurrent programming languages, focusing on the existence of canonical compact causal representations of the interleaving game semantics of programs.

We perform our study on an affine variant of Idealized Parallel Algol (IPA), for which we present two games model: an interleaving model (an adaptation of Ghica and Murawski's fully abstract games model for IPA up to may-testing), and a causal model (a variant of Rideau and Winskel's games on event structures). Both models are sound and adequate for affine IPA. Then, we relate the two models. First we give a causality-forgetting operation mapping functorially the causal model to the interleaving one. We show that from an interleaving strategy we can reconstruct a causal strategy, from which it follows that the interleaving model is the observational quotient of the causal one. Then, we investigate several reconstructions of causal strategies from interleaving ones, showing finally that there are programs which are inherently causally ambiguous, with several distinct minimal causal representations.

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1 Introduction

Game semantics present a program as a representation of its behaviour under execution, against any execution environment. This interpretation is computed compositionally, following the methodology of denotational semantics. Game semantics and interactive semantics in general have been developed for a variety of programming language features. They are an established theoretical tool in the foundational study of logic and programming languages, with a growing body of research on applications to various topics, *e.g.* model-checking [1, 10], hardware [4] or software [12] compilation, for higher-order programs. These works exploit the ability of game semantics to provide compositionally a clean and elegant presentation of the operational behaviour of a program, which can then give an invariant for program transformations, or be exploited for analysis.

One subject where game semantics particularly shine is for reasoning about program equivalence. Indeed, game semantics models are often *fully abstract*: they characterise programs up to contextual equivalence, meaning that two programs behave in the same way in all contexts if and only if the corresponding strategies have the same plays. Concurrent languages are no exception: Ghica and Murawski's games model for IPA [5] is fully abstract *wrt.* may-testing. Although, in this language, contextual equivalence is undecidable even for second-order programs, decidability can be recovered for a restricted language [6]. But Ghica and Murawski's model represents concurrent programs with *interleavings*, so whether one works in a decidable fragment or simply uses non automated tools, reasoning on the

fully abstract model requires one to explore *all possible interleavings*. This is the so-called *state explosion problem* familiar in the verification of concurrent systems [7].

Partial order methods provide good tools to alleviate this problem. They provide more compact representations of concurrent programs, avoiding the enumeration of all interleavings. For IPA, recent advances in partial-order based game semantics [11, 3] allow us to restate Ghica and Murawski's model based on partial orders or *event structures*. But can we get back full abstraction this way? Since the interleaving model is fully abstract, the question is: can we give a clean, compact, presentation of the interleaving games model of IPA *via* partial orders? As it is, the interpretation of IPA in *e.g.* [3] is certainly not fully abstract since it retains intensional information (such as the point of non-deterministic branching) invisible up to may-testing. But can we rework it so it yields canonical partial-order representatives for strategies in the interleaving model? In this paper, we show that already in an *affine* setting, the answer is no.

Our contributions are the following. We describe an affine variant of IPA – it is mostly there to provide illustrations and an operational light. For this affine IPA, we give two new categories of games. The first is an affine version of Ghica and Murawski's model. The second draws inspiration from Rideau and Winskel's category of strategies as event structures, without the information on the point of non-deterministic branching, which is irrelevant up to may-testing. Via a collapse of the causal model into the interleaving one, we show that the latter is the observational quotient of the former. We describe several causal reconstructions from an interleaving strategy, aiming for minimality. Finally, we show that interleaving strategies have in general no canonical minimal causal representation.

On the game semantics front, our two models are arena-based, in the spirit of HO games [8]. They both operate on a notion of arenas enriched with conflict, which is required in an affine setting. Our interleaving model is *not* fully abstract for affine IPA. Indeed, we have omitted well-bracketing (as well as bad variables and semaphores) in an effort to make the presentation lighter. These aspects are orthogonal to the problem at hand, and our developments would apply just as well with those. Apart from well-bracketing, our interleaving model is fully compatible with Ghica and Murawski's – strategies in our sense can easily be read as strategies in their sense, as pointers can be uniquely recovered.

2 Affine IPA and its interleaving game semantics

In this section we introduce affine IPA, and the category **GM** of interleaving strategies.

2.1 Affine IPA

▸ **Definition 1.** The **types** of affine IPA are $A, B ::= \mathbb{B} \mid \mathbf{com} \mid A \multimap B \mid \mathbf{ref}_r \mid \mathbf{ref}_w$.

We have types for booleans, commands, and a linear function space. Finally we have two types $\mathbf{ref}_r$ and $\mathbf{ref}_w$ for read-only and write-only variables (this splitting of **ref** is necessary to make the variables non-trivial in an affine setting).

The **terms** of affine IPA are the following:

$$\begin{array}{rcl} M, N & ::= & x \mid M\,N \mid \lambda x.\,M \mid \mathbf{tt} \mid \mathbf{ff} \mid \mathbf{if}\,M\,N_1\,N_2 \mid \bot \\ & & \mid \mathbf{skip} \mid M; N \mid \mathbf{newref}\,v\,\mathbf{in}\,M \mid M := \mathbf{tt} \mid !M \mid M \parallel N \end{array}$$

References are considered initialized to **ff**. As they can only be read once, the only useful value to write is **tt**, hence the restricted assignment command. Typing rules are standard,

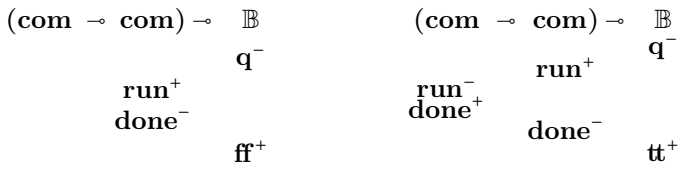


Figure 1 Maximal plays of the alternating game semantics of **strict**

we only mention a few. Firstly, affine function application and boolean elimination.

$$\frac{\Gamma \vdash M : A \multimap B \qquad \Delta \vdash N : A}{\Gamma, \Delta \vdash M\, N : B} \qquad \frac{\Gamma \vdash M : \mathbb{B} \qquad \Delta \vdash N_1 : A \qquad \Delta \vdash N_2 : A}{\Gamma, \Delta \vdash \mathbf{if}\, M\, N_1\, N_2 : A}$$

Crucially the first rule treats the context multiplicatively, making the language affine. Secondly, here are the rules for reference manipulation.

$$\frac{\Gamma, r : \mathbf{ref}_r, r : \mathbf{ref}_w \vdash M : \mathbb{B}}{\Gamma \vdash \mathbf{newref}\, r\, \mathbf{in}\, M : \mathbb{B}} \qquad \frac{\Gamma \vdash M : \mathbf{ref}_r}{\Gamma \vdash !M : \mathbb{B}} \qquad \frac{\Gamma \vdash M : \mathbf{ref}_w}{\Gamma \vdash M := \mathbf{tt} : \mathbf{com}}$$

Splitting between the read and write capabilities of the variable type is necessary for the variables to be used in a non-trivial way. For example, the following term is typable:

$$\mathbf{strict} = \lambda f^{\mathbf{com} \multimap \mathbf{com}}.\, \mathbf{newref}\, r\, \mathbf{in}\, (f\, (r := \mathbf{tt})); !r : (\mathbf{com} \multimap \mathbf{com}) \multimap \mathbb{B}$$

The language is equipped with the same operational semantics as in [5] – we skip the details. The operational semantics yields an *evaluation* relation: for $\vdash M : \mathbb{B}$, we write $M \Downarrow_{\text{may}} b$ to mean that M *may* evaluate to the boolean b, or just $M \Downarrow_{\text{may}}$ to mean that M *may* converge. From the combination of concurrency and state, affine IPA is a non-deterministic language.

2.2 Arenas

In game semantics, one interprets a program as a set of *interactions*, usually called *plays*, with its execution environment. For instance, some maximal plays of the interpretation $[\![\mathbf{strict}]\!]$ of the term $\mathbf{strict} : (\mathbf{com} \multimap \mathbf{com}) \multimap \mathbb{B}$ defined above are displayed in Figure 1. Those diagrams are read from top to bottom, and moves have polarity either Player (+, Program) or Opponent (−, Environment). In the first play of Figure 1 Opponent behaves like a constant, where in Figure 1 he is strict. Although the programs are stateful, plays do not carry state: instead, we only see how the state influences Player's behaviour.

To make this formal, we first extract from the type the computational events on which plays such as the above are formed. These are organized into *arenas*.

▸ **Definition 2.** An **event structure with polarities** is a tuple $(A, \leq_A, \sharp_A, \mathrm{pol}_A)$ where A is a set of **moves** or **events**, $\leq_A$ is a partial order on A such that for any $a \in A$, $[a] = \{a' \in A \mid a' \leq_A a\}$ is finite, $\sharp_A$ is an irreflexive symmetric **conflict** relation such that for all $a \sharp_A a'$, for all $a' \leq_A a'_0$, we also have $a \sharp_A a'_0$. Finally, $\mathrm{pol}_A : A \to \{-, +\}$ is a polarity function.

Apart from the fact that we only have binary conflict, this is the same notion of event structures with polarities as in [11]. A **configuration** of A, written $x \in \mathscr{C}(A)$, is a finite

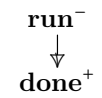
$x \subseteq A$ which is *down-closed* (if $a \in x$ and $a' \leq_A a$, then $a' \in x$ as well) and *consistent* (for all $a_1, a_2 \in x$, $\neg(a_1 \natural_A a_2)$). For $a_1, a_2 \in A$, we say that a_1 **immediately causes** a_2, written $a_1 \rightarrowtriangle a_2$, when $a_1 <_A a_2$ and for all $a_1 \leq a \leq a_2$ we have either $a_1 = a$ or $a = a_2$. We also write $a_1 \sim a_2$ if a_1 and a_2 are in **immediate conflict**, meaning $a_1 \natural_A a_2$ and for all $a_1' \leq_A a_1$, $a_2' \leq_A a_2$ (with *at least* one of them strict), we have $\neg(a_1' \natural_A a_2')$. Finally, we write $\min(A)$ for the set of minimal events of A.

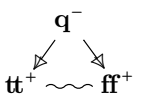
Arenas are certain event structures with polarities:

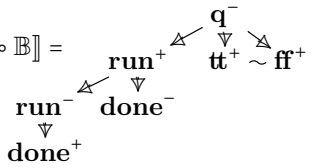
▸ **Definition 3.** An **arena** is an event structure with polarities such that $\leq_A$ is a *forest* (for all $a_1, a_2 \leq_A a$, either $a_1 \leq_A a_2$ or $a_2 \leq_A a_1$), is *alternating* (for all $a_1 \rightarrowtriangle a_2$, $\mathrm{pol}_A(a_1) \neq \mathrm{pol}_A(a_2)$), and *race-free* (if $a_1 \sim a_2$, then $\mathrm{pol}(a_1) = \mathrm{pol}(a_2)$).

Although our formulation is slightly different, our arenas are very close to the standard notion of [8]: the three differences is that we have no Question/Answer distinction, our arenas are not necessarily negative, and we have a conflict relation.

▸ **Example 4.** We display below the arenas for some types of IPA.

$[\![\mathbf{com}]\!] =$ 

$[\![\mathbb{B}]\!] =$ 

$[\![(\mathbf{com} \multimap \mathbf{com}) \multimap \mathbb{B}]\!] =$ 

On $[\![\mathbf{com}]\!]$, Opponent may start running the command ($\mathbf{run}^-$), which may or may not terminate ($\mathbf{done}^+$). On $[\![\mathbb{B}]\!]$, Opponent may interrogate the boolean ($\mathbf{q}^-$), and Player may or may not answer. If he does, it will be with exactly *one* of the incompatible $\mathbf{tt}^+$ and $\mathbf{ff}^+$.

We will see later on how to systematically interpret types of IPA as arenas. For now on though, we give two simple constructions on arenas.

▸ **Definition 5.** Let A be an arena. Its **dual**, written $A^\perp$, has the same data as A but polarity reversed. If A and B are arenas, then their **parallel composition** $A \parallel B$, also written $A \otimes B$ for the **tensor**, has components:

- *Events/moves.* the disjoint union $\{1\} \times A \cup \{2\} \times B$,
- *Causality, conflict.* Inherited from A and B.

In this paper, we will define two categories **GM** and **PO** with arenas as objects.

2.3 Interleaving-based game semantics on arenas

Now, we define a compact closed category of games called **GM**, by reference to Ghica and Murawski's model of IPA [5]. Our category will be much simpler though, as it will be an affine version of theirs, without bracketing conditions. Firstly, we need to define *plays*.

▸ **Definition 6.** Let A be an arena. A **play** s on A, written $s \in \mathcal{P}_A$, is a total order $s = (|s|, \leq_s)$ of moves of A such that $|s| \in \mathscr{C}(A)$, and for any $a, b \in s$, if $a \leq_A b$ then $a \leq_s b$. We write $s \sqsubseteq t$ for the usual prefix ordering on plays.

In [5], strategies are closed under some *saturation conditions*: for instance, if $sa^+b^- \in \sigma$ and b does not actually depend on a in the game, then σ can always *delay* a until after b was played. In other words, we have $sba \in \sigma$ as well. In our affine variant, we will have a slightly different formulation of saturation. First we define an order on plays.

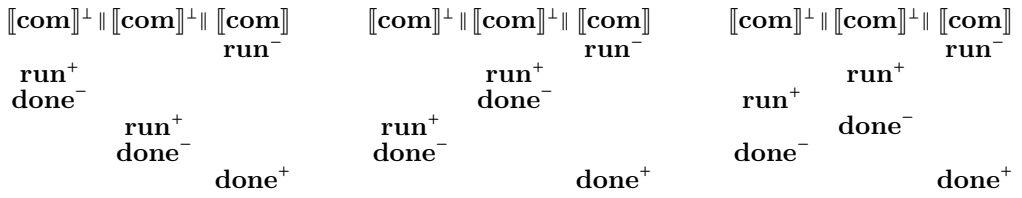


Figure 2 Some plays in $[\![\|]\!]_{\mathbf{GM}}$

▸ **Definition 7.** Let $s, t \in \mathcal{P}_A$ for A an arena. Then we say that $s \preceq t$ iff $|s| \subseteq |t|$, and:

- If $a_1^+ \leq_s a_2^-$, then $a_1 \leq_t a_2$.
- For $a_2^+ \in |s|$, if $a_1{}^- \leq_t a_2^+$, then $a_1 \in |s|$ and $a_1 \leq_s a_2$.

Clearly, $\preceq$ is a partial order on $\mathcal{P}_A$. Intuitively, going upwards in $\preceq$ corresponds to strengthening causal information by pushing Opponent moves behind Player moves, hence implying that those Opponent moves were not true dependencies for the Player moves. The partial order $\preceq$ is generated by elementary permutations, as in the saturation conditions in [5], along with the prefix ordering. We now define:

▸ **Definition 8.** A **GM**-strategy on arena A, written $\sigma : A$, is a set $\sigma \subseteq \mathcal{P}_A$ which is:

- *Saturated:* if $s \in \sigma$ and $t \preceq s$, then $t \in \sigma$ as well,
- *Receptive:* if $s \in \sigma$ and $|s| \subset |s| \cup \{a^-\} \in \mathscr{C}(A)$, then $sa \in \sigma$ as well.

▸ **Example 9.** The **GM**-strategy $[\![\|]\!]_{\mathbf{GM}} : [\![\mathbf{com}]\!]^\perp \parr [\![\mathbf{com}]\!]^\perp \parr [\![\mathbf{com}]\!]$ comprises all plays on $[\![\mathbf{com}]\!]^\perp \parr [\![\mathbf{com}]\!]^\perp \parr [\![\mathbf{com}]\!]$ such that:

- If $\mathbf{run}^+$ appears on either occurrence of $[\![\mathbf{com}]\!]^\perp$, then $\mathbf{run}^-$ must appear before,
- If $\mathbf{done}^+$ appears, then both $\mathbf{done}^-$ must appear before.

Figure 2 displays several plays of $[\![\|]\!]_{\mathbf{GM}}$. In total, $[\![\|]\!]_{\mathbf{GM}}$ has six maximal plays.

As usual in play-based game semantics, operations on **GM**-strategies rely crucially on a notion of *restriction* of plays. Consider A an arena, $s \in \mathcal{P}_A$, and B some sub-component on A (we leave the notion of sub-component intentionally somewhat vague: for instance A is a subcomponent of $A \otimes B$, and $A_1 \parr B_1$ is a sub-component of $A_1 \otimes A_2 \parr B_1 \otimes B_2$). The **restriction** $s \upharpoonright B \in \mathcal{P}_B$ is the subsequence of s of moves in component B, in the same order.

Using that, we can now define the **copycat strategy** on A to be:

$$\mathbb{c}_A = \{s \in \mathcal{P}_{A^\perp \parr A} \mid \forall s' \sqsubseteq s,\ \forall a \in |s' \upharpoonright A^\perp|,\ \mathrm{pol}_{A^\perp}(a) = + \implies a \in |s' \upharpoonright A|,\ \forall a \in |s' \upharpoonright A|, \mathrm{pol}_A(a) = + \implies a \in |s' \upharpoonright A^\perp|\}$$

It is a **GM**-strategy. Using the usual *parallel composition plus hiding* mechanism, we can also define *composition*. Given $\sigma : A^\perp \parr B$ and $\tau : B^\perp \parr C$, first define their **interaction** $\tau \circledast \sigma = \{u \in \mathcal{P}_{A \parr B \parr C} \mid u \upharpoonright A \parr B \in \sigma \ \&\ u \upharpoonright B \parr C \in \tau\}$. The **composition** $\tau \odot \sigma : A^\perp \parr C$ is obtained through *hiding*, by $\tau \odot \sigma = \{u \upharpoonright (A \parr C) \mid u \in \tau \circledast \sigma\}$. Altogether:

▸ Proposition 1. There is a compact closed category **GM** with arenas as objects, and as morphisms from A to B, **GM**-strategies $\sigma : A^\perp \parr B$. We also write $\sigma : A \overset{\mathbf{GM}}{\to} B$.

Proof. The operation $\otimes$ on arenas is extended to **GM**-strategies by setting, for $\sigma_1 : A_1 \overset{\mathbf{GM}}{\to} B_1$ and $\sigma_2 : A_2 \overset{\mathbf{GM}}{\to} B_2$, $\sigma_1 \otimes \sigma_2 = \{s \in \mathcal{P}_{(A_1 \otimes A_2)^\perp \parr B_1 \otimes B_2} \mid s \upharpoonright A_1^\perp \parr B_1 \in \sigma_1 \ \&\ s \upharpoonright A_2^\perp \parr B_2 \in \sigma_2\}$. Rather than detailing explicitly the rest of the structure, we will inherit it from the forthcoming category **PO**. All laws will then follow from Proposition 4. ◂

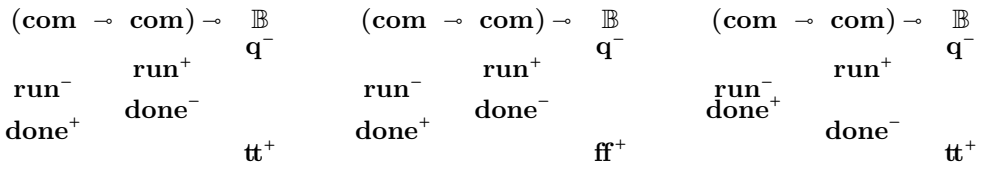


Figure 3 Some maximal plays of the non-alternating game semantics of **strict**

For now we do not show how to interpret affine IPA in **GM** – for that one actually needs a symmetric monoidal closed subcategory of *negative* arenas, which seems difficult to define without appealing to **PO**. However, we illustrate this interpretation by revisiting Figure 1.

▸ **Example 10.** The **GM**-strategy corresponding to **strict** will contain, among others, the maximal plays described in Figure 3.

Although **strict** is a sequential program, the fact that in **GM**, Opponent may *not* be sequential (and, in this case, non well-bracketed either) allows us to observe new behaviours from **strict**. For instance, in the first two plays of Figure 3, Opponent concurrently *answers* and *asks for the argument* on $\mathbf{com} \multimap \mathbf{com}$. This triggers a race between the subterms $r := \mathbf{tt}$ and $!r$ of **strict**. As a consequence, one can observe both **tt** and **ff** as final results of the computation. However, if Opponent was to answer only *after* $r := \mathbf{tt}$ was evaluated (as in the third play of Figure 3), the only possible final result would be **tt**.

There are, in total, ten maximal non-alternating plays in the **GM**-strategy for **strict**.

3 Causal game semantics for affine IPA

We give a *causal* variant of **GM**, where plays are partial orders. This yields a category **PO**, close to the category of *concurrent games* of Rideau and Winskel [11] – the main difference is that strategies in **PO** omit information about the point of non-deterministic branching.

3.1 Po-plays and po-strategies

First, we define the notion of partially ordered play.

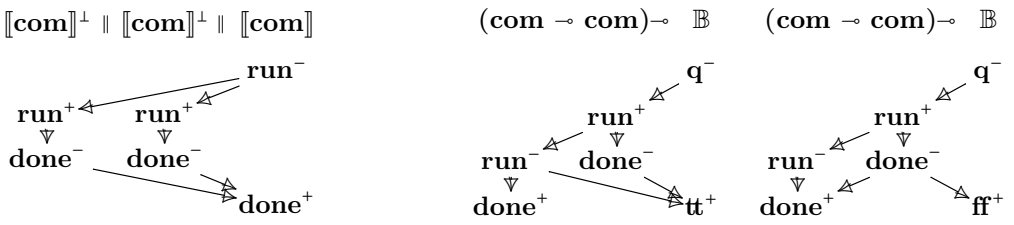
▸ **Definition 11.** A **partially ordered play (po-play)** on arena A is a partial order $\mathbf{q} = (|\mathbf{q}|, \leq_{\mathbf{q}})$ where $|\mathbf{q}| \in \mathscr{C}(A)$, and $\mathbf{q}$ satisfies the following properties:

- *Respects the game:* for $a_1, a_2 \in |\mathbf{q}|$, if $a_1 \leq_A a_2$ then $a_1 \leq_{\mathbf{q}} a_2$,
- *Is courteous:* if $a_1^+ \rightarrowtriangle_{\mathbf{q}} a_2$ then $a_1 \rightarrowtriangle_A a_2$, and if $a_1 \rightarrowtriangle_{\mathbf{q}} a_2^-$, then $a_1 \rightarrowtriangle_A a_2$.

We write $\mathcal{P}_A^{\copyright}$ for the set of po-plays on arena A.

Unlike usual (alternating or non-alternating) plays, po-plays are not chronologically ordered, but carry *causal* information about Player's choices. Hence, a po-play cannot express that an Opponent event happens *after* a given event, unless that dependency is already present in the arena. In fact, a po-play cannot force a dependency between two Player moves either: such a dependency may be broken by an asynchronous execution environment.

Although one po-play may carry information about many interleavings, representing a **GM**-strategy might take *several*. Indeed, a po-play is by itself only able to represent a



(a) A po-play for parallel composition

(b) The two maximal po-plays of $[\![\mathbf{strict}]\!]_{\mathbf{PO}}$

Figure 4 Some po-plays

process which is deterministic *up to the choice of the scheduler* (note that parallel composition is indeed deterministic up to the choice of the scheduler, it is only via its interaction with *e.g.* a shared memory that non-determinism arises). For instance, the **GM**-strategy $\mathbf{coin} : [\![\mathbb{B}]\!] = \{\epsilon, \mathbf{q}^-, \mathbf{q}^-\mathbf{tt}^+, \mathbf{q}^-\mathbf{ff}^+\}$ can only be represented via *two* maximal po-plays: $\mathbf{q}^- \nrightarrow \mathbf{tt}^+$ and $\mathbf{q}^- \nrightarrow \mathbf{ff}^+$. It features actual non-determinism, independent from the scheduler.

To express such non-determinism, Rideau and Winskel [11] formalize strategies as *event structures* rather than partial orders. Our causal notion of strategies builds on their work; but since the present paper is only interested in relating causal with interleaving game semantics (therefore with may-testing), we drop the explicit non-deterministic branching point and consider po-strategies to be certain *sets of partial orders*. For that we first define:

▸ **Definition 12.** Let $\mathbf{q}, \mathbf{q}'$ be two partial orders. We say that $\mathbf{q}$ is **rigidly included** in $\mathbf{q}'$, or that $\mathbf{q}$ is a **prefix** of $\mathbf{q}'$, written $\mathbf{q} \hookrightarrow \mathbf{q}'$, if we have the inclusion $|\mathbf{q}| \subseteq |\mathbf{q}'|$, for any $a_1, a_2 \in |\mathbf{q}|$ we have $a_1 \leq_{\mathbf{q}} a_2$ iff $a_1 \leq_{\mathbf{q}'} a_2$, and $\mathbf{q}$ is down-closed in $\mathbf{q}'$.

We are now in position to define **PO**-strategies.

▸ **Definition 13.** A **PO-strategy** on A, written $\sigma :: A$, is a non-empty prefix-closed $\sigma \subseteq \mathcal{P}^{\copyright}_A$, which is additionally **receptive**: for all $\mathbf{q} \in \sigma$, if $|\mathbf{q}| \in \mathscr{C}(A)$ extends to $|\mathbf{q}| \cup \{a^-\} \in \mathscr{C}(A)$, then there is $\mathbf{q} \hookrightarrow \mathbf{q}' \in \sigma$ such that $|\mathbf{q}'| = |\mathbf{q}| \cup \{a\}$.

It follows by courtesy that $\mathbf{q}'$ is necessarily unique: the immediate dependency of a in $\mathbf{q}'$ is forced by its immediate dependency in A.

Clearly, the set of prefixes of the po-play of Figure 4a gives a **PO**-strategy. For a non-trivial non-deterministic example, we give in Figure 4b the two maximal (up to prefix / rigid inclusion) po-plays of the **PO**-strategy corresponding to **strict**. This gives a quite compact representation of all of the ten maximal plays of the **GM**-strategy for **strict** of Example 10.

3.2 The compact closed category PO

To construct **PO** we start with the causal copycat, which is – configuration-wise – as in [11].

▸ **Definition 14.** Let A be an arena. We define a partial order $\leq_{\mathbb{C}_A}$ on $A^\perp \parr A$:

$$\leq_{\mathbb{C}^{\copyright}_A} \quad = \quad (\{((1,a),(1,a')) \mid a \leq_A a'\} \cup \{((2,a),(2,a')) \mid a \leq_A a'\} \cup \\ \{((1,a),(2,a)) \mid \mathrm{pol}_A(a) = +\} \cup \{((2,a),(1,a)) \mid \mathrm{pol}_A(a) = -\})^+$$

where $(-)^+$ denotes the transitive closure of a relation. Then, $\mathbb{c}^{\copyright}_A :: A^\perp \parr A$ comprises all $x \parr y \in \mathscr{C}(A^\perp \parr A)$ down-closed for $\leq_{\mathbb{C}^{\copyright}_A}$, with the induced partial order.

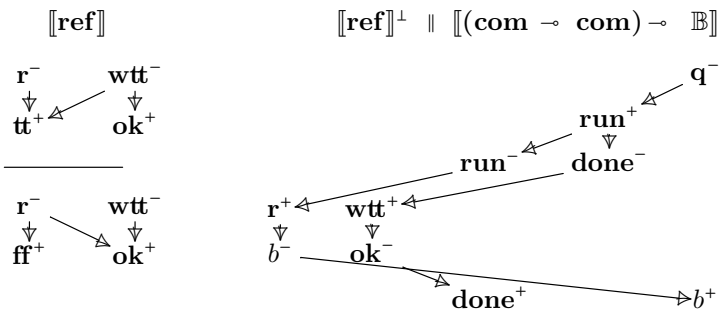


Figure 5 **cell** :: ⟦**ref**⟧ and $\llbracket \lambda f^{\mathbf{com} \multimap \mathbf{com}}.\, f\,(r := \mathbf{tt});\, !r \rrbracket :: \llbracket \mathbf{ref} \rrbracket^{\perp} \parr \llbracket (\mathbf{com} \multimap \mathbf{com}) \multimap \mathbb{B} \rrbracket$.

We will see in Proposition 4 that this is indeed a causal version of $\mathbb{c}_A : A^{\perp} \parr A$. Now, we define composition of **PO**-strategies. We first define composition of po-plays (via interaction plus hiding, essentially as in [11]), before lifting it component-wise to **PO**-strategies.

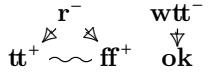
▸ **Definition 15.** Two dual po-plays $\mathbf{q} \in \mathcal{P}^{\copyright}_{A}$, $\mathbf{q}' \in \mathcal{P}^{\copyright}_{A^{\perp}}$ such that $|\mathbf{q}| = |\mathbf{q}'|$ are **causally compatible** if $(\leq_{\mathbf{q}} \cup \leq_{\mathbf{q}'})^*$ is a partial order, *i.e.* is acyclic. Then we write $\mathbf{q} \wedge \mathbf{q}' = (|\mathbf{q}|, \leq_{\mathbf{q} \wedge \mathbf{q}'})$ for the resulting partial order.

If $\mathbf{q}$ and $\mathbf{q}'$ are causally compatible po-plays on dual games as above, the events of $\mathbf{q} \wedge \mathbf{q}'$ have no well-defined polarity, so it is not a po-play. If $\mathbf{q} \in \mathcal{P}^{\copyright}_{A^{\perp} \parr B}$ and $\mathbf{q}' \in \mathcal{P}^{\copyright}_{B^{\perp} \parr C}$ are not dual but composable, we say that they are **causally compatible** if $|\mathbf{q}| = x_A \parr x_B$, $|\mathbf{q}'| = x_B \parr x_C$, plus $(\mathbf{q} \parr x_C)$ and $(x_A \parr \mathbf{q}')$ are causally compatible (where x_A, x_C inherit the order from A, C – in particular, x_A is regarded as a member of $\mathcal{P}^{\copyright}_{A}$, and x_C as a member of $\mathcal{P}^{\copyright}_{C^{\perp}}$), we define their open interaction $\mathbf{q}' \circledast \mathbf{q} = (\mathbf{q} \parr x_C) \wedge (x_A \parr \mathbf{q}')$.

In that case we define $\mathbf{q}' \odot \mathbf{q} \in \mathcal{P}^{\copyright}_{A^{\perp} \parr C}$ as the **projection** $\mathbf{q}' \circledast \mathbf{q} \downarrow A^{\perp} \parr C$, with events those of $\mathbf{q}' \circledast \mathbf{q}$ that are in A or C, and partial order as in $\leq_{\mathbf{q}' \circledast \mathbf{q}}$. This being a po-play is a variation on the stability by composition of courtesy in [11] (there called *innocence*).

▸ **Definition 16.** Let $\sigma :: A^{\perp} \parr B$ and $\tau :: B^{\perp} \parr C$ be **PO**-strategies. Their **composition** is $\tau \odot \sigma = \{\mathbf{q}' \odot \mathbf{q} \mid \mathbf{q}' \in \tau \;\&\; \mathbf{q} \in \sigma \text{ causally compatible}\}$. Then, $\tau \odot \sigma :: A^{\perp} \parr C$ is a **PO**-strategy.

The construction is a simplification of [11]: po-plays are certain concurrent strategies, and their composition is close to the composition of concurrent strategies with the simplification that events of po-plays are those of the games rather than only *labeled* by the game.

▸ **Example 17.** Consider $\llbracket \mathbf{ref}_r \rrbracket \otimes \llbracket \mathbf{ref}_w \rrbracket =$ 

, for the type of references.

By abuse of notation, we write ⟦**ref**⟧ for $\llbracket \mathbf{ref}_w \rrbracket \otimes \llbracket \mathbf{ref}_r \rrbracket$. The **PO**-strategy interpreting **strict** is the composition of the **PO**-strategy with maximal po-play at the right hand side of Figure 5 (interpreting $r : \mathbf{ref}_w, r : \mathbf{ref}_r \vdash \lambda f^{\mathbf{com} \multimap \mathbf{com}}.\, f\,(r := \mathbf{tt});\, !r$ following Section 3.3), and **cell** :: ⟦**ref**⟧ for the memory cell (with maximal po-plays at the left hand side of Figure 5). Performing composition as above produces the two maximal po-plays of Figure 4b.

▸ Proposition 2. There is a compact closed category **PO** with *arenas* as objects, and **PO**-strategies $\sigma :: A^{\perp} \parr B$ as morphisms from A to B, also written $\sigma : A \overset{\mathbf{PO}}{\to} B$.

Proof. The tensor $\mathbf{q}_1 \otimes \mathbf{q}_2$ of $\mathbf{q}_1 \in \mathcal{P}^{\copyright}_{A_1^\perp \parallel B_1}$ and $\mathbf{q}_2 \in \mathcal{P}^{\copyright}_{A_2^\perp \parallel B_2}$ is the obvious inherited partial order on $(A_1 \parallel A_2)^\perp \parallel (B_1 \parallel B_2)$. The tensor $\sigma_1 \otimes \sigma_2$ of **PO**-strategies $\sigma_1 :: A_1^\perp \parallel B_1$ and $\sigma_2 : A_2^\perp \parallel B_2$ is defined component-wise. Structural morphisms are copycat **PO**-strategies.

PO simplifies (omitting explicit non-deterministic branching information) the bicategory of concurrent games [11], whose compact closed structure is established with details in [2]. ◂

3.3 Interpretation of affine IPA

For completeness, we succinctly describe how one can define the interpretation of affine IPA in **PO**. In fact, affine IPA will not be interpreted directly in **PO**, which does not support weakening of variables as the empty arena 1, unit for the tensor, is not terminal (since **PO**-strategies can have minimal positive events, there are in general several **PO**-strategies on $A^\perp \parallel 1$ as soon as A has at least one minimal negative event). We have to restrict to a proper subcategory of **PO**, defined as follows.

▸ **Definition 18.** An event structure with polarities A is **negative** if $\mathrm{pol}(\min(A)) \subseteq \{-\}$.

The category $\mathbf{PO}^-$ is the subcategory of **PO** with objects **negative arenas**, and morphisms the **negative PO-strategies** whose po-plays are all negative.

The empty arena 1 is terminal in $\mathbf{PO}^-$: if A is negative then $A^\perp \parallel 1$ has no negative minimal event. Therefore a negative $\sigma :: A^\perp \parallel 1$ must be empty, as a potential minimal event would be in particular minimal in $A^\perp \parallel 1$. However, restricting to $\mathbf{PO}^-$ has a price: we lose the closure $A^\perp \parallel B$, which is in general not negative and hence not an object of $\mathbf{PO}^-$. Thus we build a negative version, where the minimal events of A depend on those of B.

▸ **Definition 19.** Let A, B be two negative arenas. The arena $A \multimap B$ has:

- *Events/polarity:* $(\|_{b \in \min(B)} A^\perp) \parallel B$.
- *Causality:* $(\|_{b \in \min(B)} A^\perp) \parallel B$, enriched with $((2, b), (1, (b, a)))$ for $a \in A$ and $b \in \min(B)$.
- *Conflict:* $(\|_{b \in \min(B)} A^\perp) \parallel B$, plus those inherited by $(1, (b_1, a)) \sim (1, (b_2, a))$ for $b_1 \neq b_2$.

If A, B are conflict-free and B has a unique minimal event, then $A \multimap B$ coincides with the usual arrow arena construction in Hyland-Ong games [8]. In general if B has a unique minimal event, then $A \multimap B$ does not introduce new conflicts or copies of A, and only differs from $A^\perp \parallel B$ by the fact that events of $A^\perp$ now depend on the minimal event of B – see Example 4 for such an arrow arena. However, if B has several minimal events, then multiple copies of A are created; fortunately we can use conflict to maintain linearity.

The arena $A \multimap B$ does not yet give a closure with respect to the tensor. The issue is that there are more **PO**-strategies in $A \multimap B$ than in $A^\perp \parallel B$. Indeed, consider a **PO**-strategy $\sigma :: \mathbb{B}^\perp \parallel (\mathbb{B} \otimes \mathbb{B})$, that plays $\mathbf{q}^+$ in the left hand side occurrence of $\mathbb{B}$ whenever Opponent plays $\mathbf{q}^-$ in *both* right hand side occurrences of $\mathbb{B}$. Then on $\mathbb{B} \multimap (\mathbb{B} \otimes \mathbb{B})$ there are *two* ways to replicate this, as they are two copies of the left hand side $\mathbb{B}$ in the arena. To get back a closed structure, we need to restrict the category further.

▸ **Definition 20.** A negative **PO**-strategy $\sigma :: A$ is **well-threaded** iff, for any $\mathbf{q} \in \sigma$, $\mathbf{q}$ has at most one minimal event. Copycat is well-threaded and well-threaded **PO**-strategies are stable under composition – they form a subcategory $\mathbf{PO}^-_{\mathrm{wt}}$ of $\mathbf{PO}^-$.

Up to renaming of events, negative well-threaded strategies on $(A \parallel B)^\perp \parallel C$ exactly coincide with those on $A^\perp \parallel B \multimap C$. Leveraging the compact closed structure of **PO**, it follows that $\mathbf{PO}^-_{\mathrm{wt}}$ is symmetric monoidal closed (where the monoidal unit 1 is terminal). As such, it supports the interpretation of the affine λ-calculus: any term $x_1 : A_1, \ldots, x_n : A_n \vdash$

$M : B$ is interpreted as a **PO**-strategy $[\![M]\!] : [\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!] \overset{\mathbf{PO}^-_{\mathrm{wt}}}{\to} [\![B]\!]$. Along with the **PO**-strategy with unique po-play that of Figure 4a for parallel composition, the interpretation of the **newref** construct as sketched in Example 17, and the obvious **PO**-strategies for the other affine IPA combinators, we get an interpretation $[\![-]\!]$ of affine IPA into $\mathbf{PO}^-_{\mathrm{wt}}$, which is a subcategory of **PO**. Standard techniques entail:

▸ Proposition 3. The interpretation $[\![-]\!]$ is sound and adequate for affine IPA, *i.e.* for $\vdash M :$ **com**, we have $M \Downarrow_{\mathrm{may}}$ iff $[\![M]\!]$ contains a positive event.

4 From PO to GM and back

We finally enter the final section of this paper, and relate the two semantics.

4.1 Forgetting causality

We start with the easy part: that **PO** can be embedded into **GM**. As partial orders are more informative than plays, it is easy to move from the former to the latter.

▸ **Definition 21.** Let $\mathbf{q} \in \mathcal{P}^{\copyright}_A$. A **play** in $\mathbf{q}$ is $s \in \mathcal{P}_A$ such that $|s| \subseteq |\mathbf{q}|$, and such that for all a_2 in $|s|$, if $a_1 \leq_{\mathbf{q}} a_2$, then $a_1 \in |s|$ and $a_1 \leq_s a_2$. We write $\mathrm{Plays}(\mathbf{q})$ for the set of plays in $\mathbf{q}$.

From courtesy of $\mathbf{q}$ it follows that $\mathrm{Plays}(\mathbf{q})$ satisfies the saturation condition of Definition 8. For $\sigma :: A$ a **PO**-strategy, we have $\mathrm{Plays}(\sigma) = \bigcup\{\mathrm{Plays}(\mathbf{q}) \mid \mathbf{q} \in \sigma\}$ a **GM**-strategy, as receptivity follows from receptivity of σ. In fact, we have:

▸ Proposition 4. There is an identity-on-object functor $\mathrm{Plays} : \mathbf{PO} \to \mathbf{GM}$.

This is a direct verification. As in Section 2.2 we have by anticipation defined the compact closed structure of **GM** to be the image of that of **PO** through Plays, this functor preserves the compact closed structure by construction. Combined with the interpretation $[\![-]\!]$ of affine IPA in **PO**, this gives a sound and adequate interpretation $\mathrm{Plays} \circ [\![-]\!]$ of affine IPA in **GM**. Providing a direct sound interpretation to **GM** without **PO** would be awkward, as it is unclear how to define well-threaded **GM**-strategies with no access to causality.

As emphasized in the introduction, the interpretation $\mathrm{Plays} \circ [\![-]\!]$ is *not* fully abstract for affine IPA. However, let us emphasize again that we are not interested in full abstraction for affine IPA; rather this serves as a simpler setting in which to study the relationship between the fully abstract model for IPA [5] and its causal variant in *e.g.* [3].

4.2 Recovering causality

We now investigate how one can recover a **PO**-strategy from a **GM**-strategy.

4.2.1 A naive causal reconstruction

As a first step, we simply reverse the construction of Definition 21.

▸ **Definition 22.** A **causal resolution** $\sigma : A$ is any $\mathbf{q} \in \mathcal{P}^{\copyright}_A$ such that $\mathrm{Plays}(\mathbf{q}) \subseteq \sigma$.

Because some **GM**-strategies (such as $\mathbf{coin} : \mathbb{B}$) are inherently non-deterministic, it is hopeless to try to describe them with a unique maximal causal resolution. A first rough causal reconstruction for a **GM**-strategy consists simply in taking *all* causal resolutions.

▸ Proposition 5. Let $\sigma : A$ be a **GM**-strategy. Then, $\mathrm{Caus}(\sigma) = \{\mathbf{q} \in \mathcal{P}^{\copyright}_A \mid \mathrm{Plays}(\mathbf{q}) \subseteq \sigma\}$ is a **PO**-strategy such that $\mathrm{Plays}(\mathrm{Caus}(\sigma)) = \sigma$. Moreover, this yields a lax functor $\mathrm{Caus} : \mathbf{GM} \to \mathbf{PO}$, *i.e.* we have $\mathbb{c}^{\copyright}_A \subseteq \mathrm{Caus}(\mathbb{c}_A)$ and $\mathrm{Caus}(\tau) \odot \mathrm{Caus}(\sigma) \subseteq \mathrm{Caus}(\tau \odot \sigma)$ for all $\sigma : A^{\perp} \parr B$ and $\tau : B^{\perp} \parr C$ (but neither of the other inclusions hold).

Proof. Each causal resolution is courteous by definition; receptivity and closure under prefix are immediate. Each play $s \in \sigma$ appears in a causal resolution $\mathbf{q}_s$, whose plays are exactly those $t \preceq s$ obtained by saturation from s. Finally, lax functoriality is straightforward.

To see why $\mathrm{Caus}(-)$ is only lax functorial, take $A = \{a^-\}$, $C = \{c^-\}$ and $B = 1$. Take the **PO**-strategy $\sigma :: A^{\perp} \parr B$ to have as only non-empty po-play the singleton a^+, while $\tau :: B^{\perp} \parr C$ has only non-empty po-play the singleton c^-. Then the **GM**-strategy $\mathrm{Plays}(\tau) \odot \mathrm{Plays}(\sigma)$ admits $c^- \twoheadrightarrow a^+$ as a causal resolution, which is therefore a po-play of $\mathrm{Caus}(\mathrm{Plays}(\tau) \odot \mathrm{Plays}(\sigma))$. On the other hand, $\mathrm{Caus}(\mathrm{Plays}(\tau)) \odot \mathrm{Caus}(\mathrm{Plays}(\sigma)) = \tau \odot \sigma$ has only one maximal po-play, with causally independent c^- and a^+. ◂

In particular, each **GM**-strategy is *definable* as a **PO**-strategy. Along with Proposition 4, and the fact that (just as in [5]) two distinct **GM**-strategies can always be distinguished by a **GM**-strategy, this entails that **GM** is the **observational quotient** of **PO**, in the sense that for $\sigma_1, \sigma_2 :: A$, $\mathrm{Plays}(\sigma_1) = \mathrm{Plays}(\sigma_2)$ iff for all $\alpha :: A^{\perp} \parr \mathbf{com}$, $\alpha \odot \sigma_1 = \alpha \odot \sigma_2$.

There are in general many **PO**-strategies corresponding to one **GM**-strategy, as **GM**-strategies only remember the *observable* behaviour. Some **PO**-strategies are more succinct than others for a fixed **GM**-strategy; and the causal reconstruction $\mathrm{Caus}(-)$ is not very economical as it constructs the *biggest* such causal representation. For instance, the **PO**-strategy $\mathrm{Caus}(\llbracket \parr \rrbracket_{\mathbf{GM}})$ not only comprises the po-play of Figure 4a, but also the linear po-play of *sequential* command composition.

4.2.2 Extremal causal resolutions

As we have seen, the construction $\mathrm{Caus}(-)$ presented above does not yield a satisfactory causal representation of a **GM**-strategy because it is not *minimal*. Seeking a minimal canonical causal representation of a **GM**-strategy, we now investigate when certain causal resolutions are subsumed by others, and hence can be removed without changing $\mathrm{Plays}(\sigma)$.

For $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{P}^{\copyright}_A$ with $|\mathbf{q}_1| = |\mathbf{q}_2|$, considering $\mathbf{q}_1$ subsumed by $\mathbf{q}_2$ when $\mathrm{Plays}(\mathbf{q}_1) \subseteq \mathrm{Plays}(\mathbf{q}_2)$ is a bit too naive. Indeed, consider $\mathbf{cell} :: \llbracket \mathbf{ref} \rrbracket$ of Figure 5. We have:

$$\mathrm{Plays}\left(\begin{array}{ccc} \mathbf{r}^- & & \mathbf{wtt}^- \\ & \searrow & \downarrow \\ & & \mathbf{ok}^+ \end{array}\right) \subseteq \mathrm{Plays}\left(\begin{array}{ccc} \mathbf{r}^- & & \mathbf{wtt}^- \\ & & \downarrow \\ & & \mathbf{ok}^+ \end{array}\right)$$

However, moving from the former to the latter does not preserve the future: namely, whereas any play in the left hand side can only be extended with $\mathbf{ff}^+$, there are plays in the right hand side that can be extended with $\mathbf{tt}^+$ as well. So, the left hand side has to be kept.

To address this relaxation of causality while taking account of the future, for $|\mathbf{q}_1| = |\mathbf{q}_2|$ with $\mathrm{Plays}(\mathbf{q}_1) \subseteq \mathrm{Plays}(\mathbf{q}_2)$, we will say that $\mathbf{q}_2$ *relaxes* $\mathbf{q}_1$ if the inclusion of plays is automatically transferred to all possible rigid extensions of $\mathbf{q}_1$. More formally:

▸ **Definition 23.** We define a partial order $\leq$ called **relaxation** coinductively, by $\mathbf{q}_1 \leq \mathbf{q}_2$ iff $|\mathbf{q}_1| = |\mathbf{q}_2|$, $\mathrm{Plays}(\mathbf{q}_1) \subseteq \mathrm{Plays}(\mathbf{q}_2)$, and for all $\mathbf{q}_1 \hookrightarrow \mathbf{q}_1'$, there exists $\mathbf{q}_2 \hookrightarrow \mathbf{q}_2'$ such that $\mathbf{q}_1' \leq \mathbf{q}_2'$.

For $\sigma : A$ a **GM**-strategy and $\mathbf{q} \in \mathrm{Caus}(\sigma)$, we say that $\mathbf{q}$ is **extremal** in σ iff $\mathbf{q}$ is $\leq$-maximal. Let $\mathrm{Extr}(\sigma)$ be denote the set of extremal po-plays in σ.

▸ Proposition 6. For any $\sigma : A$, we have $\mathrm{Extr}(\sigma) :: A$ such that $\mathrm{Plays}(\mathrm{Extr}(\sigma)) = \sigma$.

The operation $\mathrm{Extr}(-)$ performs well on many examples: for instance, it recovers the proper **PO**-strategies for all the examples of **GM**-strategies in this paper until now. It also properly reverses $\mathrm{Plays}(-)$ for *deterministic* **PO**-strategies, with only one maximal po-play. In that case, it matches the previously known correspondence between Rideau and Winskel's *deterministic concurrent strategies* [13] and Melliès and Mimram's category of *receptive ingenuous strategies* [9].

In the general case however, $\mathrm{Extr}(-)$ is not even lax functorial. But more importantly, it turns out that $\mathrm{Extr}(\sigma)$ is still not necessarily a minimal causal representation of σ. We present an example outside of the interpretation of affine IPA as it is more succinct, but it is easy to find similar examples within the interpretation.

▸ **Example 24.** Let A be a non-negative arena, with two concurrent events $\ominus$ and $\oplus$. Consider the **GM**-strategy $\sigma : A_1 \parallel A_2$ with plays (annotations are for disambiguation):

$$\sigma = \mathrm{Plays}\left(\begin{array}{cc}\ominus_1 & \ominus_2\\ \downarrow & \downarrow\\ \oplus_1 & \oplus_2\end{array}\right) \cup \mathrm{Plays}\left(\begin{array}{cc}\ominus_1 & \ominus_2\\ & \searrow\\ \oplus_1 & \oplus_2\end{array}\right) \cup \mathrm{Plays}\left(\begin{array}{cc}\ominus_1 & \ominus_2\\ \swarrow & \\ \oplus_1 & \oplus_2\end{array}\right)$$

All three po-plays are extremal in σ. However, despite being extremal, the first po-play is **redundant**: it can be removed, yielding the same **GM**-strategy. Indeed, call the three po-plays above $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$; and take $s \in \mathrm{Plays}(\mathbf{q}_1)$. If $s \notin \mathrm{Plays}(\mathbf{q}_2)$, then $\oplus_2 \leq_s \ominus_1$ as this is the only constraint in $\mathbf{q}_2$. Likewise, $s \notin \mathrm{Plays}(\mathbf{q}_3)$ means that $\oplus_1 \leq_s \ominus_2$. But these constraints, put together with those of $\mathbf{q}_1$, yield a contradiction. Therefore $s \in \mathrm{Plays}(\mathbf{q}_2) \cup \mathrm{Plays}(\mathbf{q}_3)$. The two extremal po-plays $\mathbf{q}_2, \mathbf{q}_3$ yield a smaller representation of σ.

In the example above, $\{\mathbf{q}_2, \mathbf{q}_3\}$ is the *unique* minimal causal representation for σ. But can we always reach such a canonical representation by removing redundant extremals?

4.2.3 Causally ambiguous GM-strategies

Until this point, and including Example 24, all the examples of **GM**-strategies considered in this paper have a unique minimal causal representation, *i.e.* a unique set of extremal po-plays with minimal cardinality. They are all *causally unambiguous*:

▸ **Definition 25.** For A a finite arena, a **GM**-strategy $\sigma : A$ is **causally ambiguous** if there are (at least) two distinct sets of extremal po-plays of minimal cardinality $X = \{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ and $Y = \{\mathbf{q}'_1, \ldots, \mathbf{q}'_n\}$, such that $\sigma = \bigcup_{1 \leq i \leq n} \mathrm{Plays}(\mathbf{q}_i) = \bigcup_{1 \leq i \leq n} \mathrm{Plays}(\mathbf{q}'_i)$.

To conclude this paper, we show the following result.

▸ **Theorem 26.** *There is a term of affine IPA:*

$$\vdash M : ((\mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com}) \multimap \mathbf{com}) \multimap \mathbf{com}$$

such that $[\![M]\!]_{\mathbf{GM}}$ *is causally ambiguous.*

Proof. We first exhibit a causally ambiguous **GM**-strategy outside of the interpretation of affine IPA, and then sketch how the same phenomenon can be replicated via a term.

Figure 6 displays five po-plays $\mathbf{q}_1, \ldots, \mathbf{q}_5$, generating a **GM**-strategy $\sigma = \cup_{1 \leq i \leq 5} \mathrm{Plays}(\mathbf{q}_i)$ – the game A is the same as in Example 24. A rather tedious but direct verification ensures that they are all extremal: for that, it suffices to check that for each of these po-plays,

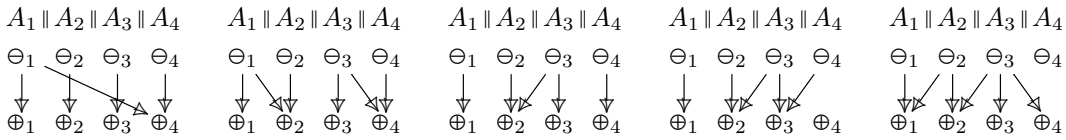


Figure 6 Extremal generators $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ and $\mathbf{q}_5$ of a causally ambiguous **GM**-strategy.

dropping any of the causal links unlocks a play not yet in σ. For instance, dropping the diagonal immediate causal link in $\mathbf{q}_1$ unlocks the play $\ominus_4 \oplus_4 \ominus_2 \oplus_2 \notin \sigma$.

Then, we note that $\mathbf{q}_2$ is redundant. Indeed, $\mathrm{Plays}(\mathbf{q}_2) \subseteq \mathrm{Plays}(\mathbf{q}_1) \cup \mathrm{Plays}(\mathbf{q}_3)$: as in Example 24, we cannot have at the same time $\oplus_4 \leq_s \ominus_1$ and $\oplus_2 \leq_s \ominus_3$ in $s \in \mathrm{Plays}(\mathbf{q}_2)$. Perhaps less obviously, $\mathbf{q}_3$ is redundant as well: we have $\mathrm{Plays}(\mathbf{q}_3) \subseteq \mathrm{Plays}(\mathbf{q}_2) \cup \mathrm{Plays}(\mathbf{q}_4) \cup \mathrm{Plays}(\mathbf{q}_5)$. Indeed, take $s \in \mathrm{Plays}(\mathbf{q}_3)$. If $s \notin \mathrm{Plays}(\mathbf{q}_4)$, then $\oplus_3 \leq_s \ominus_4$. If $s \notin \mathrm{Plays}(\mathbf{q}_5)$, then either $\oplus_1 \leq_s \ominus_2$ or $\oplus_4 \leq_s \ominus_3$, but the latter is incompatible as the constraints we already have on $\ominus_3, \oplus_3, \ominus_4, \oplus_4$ yield a cycle. Thus $\oplus_1 \leq_s \ominus_2$. But then if $s \notin \mathrm{Plays}(\mathbf{q}_2)$, then $\oplus_2 \leq_s \ominus_1$ or $\oplus_4 \leq_s \ominus_3$, but both possibilities yield a cycle; absurd.

None of $\mathbf{q}_1, \mathbf{q}_4, \mathbf{q}_5$ are redundant: only $\mathbf{q}_2$ and $\mathbf{q}_3$. Removing *both* $\mathbf{q}_2$ and $\mathbf{q}_3$ leads to the loss of the play $\ominus_3 \oplus_3 \ominus_4 \oplus_4 \ominus_1 \oplus_1$. There are two distinct minimal sets of extremals $\{\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5\}$ and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_4, \mathbf{q}_5\}$, both generating σ – so σ is causally ambiguous.

We replicate this in affine IPA. First, we replace each A with $\mathbf{com}$. However, $\mathbf{q}_4$ and $\mathbf{q}_5$ do not have the causal link $\ominus_4 \rightarrowtriangle \oplus_4$; so we need *five* occurrences of $\mathbf{com}$, organised as $\mathbf{com}_1 \parallel \mathbf{com}_2 \parallel \mathbf{com}_3 \parallel \mathbf{com}_4 \parallel \mathbf{com}'_4$, where $\mathbf{run}'_4$, $\mathbf{done}_4$ play the role of $\ominus_4, \oplus_4$ and $\oplus'_4$ is ignored. This yields $\sigma' : \mathbf{com}_1 \parallel \mathbf{com}_2 \parallel \mathbf{com}_3 \parallel \mathbf{com}_4 \parallel \mathbf{com}'_4$ causally ambiguous. This is not a type of affine IPA (and σ' is not well-threaded), so instead we lift σ' to:

$$\sigma'' : [\![((\mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com} \multimap \mathbf{com}) \multimap \mathbf{com}) \multimap \mathbf{com}]\!]$$

Using variables, one can implement in affine IPA each of the po-plays corresponding in this type to the $\mathbf{q}_i$s above. It is also easy to define a non-deterministic choice operation in affine IPA, using which these are put together to define M such that $[\![M]\!]_{\mathbf{GM}} = \sigma''$. ◂

5 Conclusions

The phenomenon presented here is fairly robust, and causally ambiguous strategies would most likely emerge as well in other concurrent programming languages. Since interleaving games models are inherently related with observational equivalence as they exactly capture the observable behaviour of programs, it seems that unfortunately we cannot use the causal model presented here or those of *e.g.* [11, 3] to give canonical compact representations of concurrent programs up to contextual equivalence.

Causal structures are however still very relevant for other purposes (*e.g.* model-checking, error diagnostics, weak memory models, ...), and constructing them compositionally from programs remains an interesting challenge.

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