Expected Length of the Voronoi Path in a High Dimensional Poisson-Delaunay Triangulation

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Abstract

Let X be a d dimensional Poisson point process. We prove that the expected length of the Voronoi path between two points at distance 1 in the Delaunay triangulation associated with X is $\sqrt{\frac{2d}{\pi}} + O(d^{-\frac{1}{2}})$ when $d \to \infty$. In any dimension, we also provide a precise interval containing the actual value; in 3D the expected length is between 1.4977 and 1.50007.

MSC: 68 - Computer science. 60 - Probability theory and stochastic processes.

Keywords: Random distribution; Walking strategies; Routing; Point location.

1 Introduction

Finding paths in a Delaunay triangulation is a classical problem in computational geometry [8]. In the context of random points, several kind of paths have been studied in two dimensions such as straight walks [2, 9], cone walks [3], visibility walks [6], shortest paths [4], or Voronoi paths [1].

In this paper we take interest in the stretch ratio of a particular path in the Delaunay triangulation – the Voronoi path – and study its expected length in dimension d when the point set is a Poisson point process. The Voronoi path links the seeds of the Voronoi regions intersected by a line segment. An illustration for dimension 2 is given in Figure 1. The main result of this paper is the computation of upper and lower bounds on the expected length of the Voronoi path. These bounds show that the asymptotic behavior of the length is $\sqrt{\frac{2d}{\pi}} + O(d^{-\frac{1}{2}})$. Table 1 provides the values of our bounds for small dimensions as well as the approximated actual values obtained from numerical integration.

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Obviously, the length of the Voronoi path gives an upper bound on the length of the shortest path and provides an upper bound on the expected stretch ratio of long walks in the Delaunay triangulation of a random point set.

Previous results provide values only for dimension two, where the expected length of the Voronoi path is $\frac{4}{\pi} \simeq 1.27$ [1, 7].

2 Voronoi Path

We define the Voronoi path $VP_{\chi,s,t}$ as the list of closest neighbors in a set of points $\chi \subset \mathbb{R}^d$ of a point moving linearly from $s = (0, 0, \ldots, 0)$ to $t = (1, 0, \ldots, 0)$. This path is x-monotone, it starts at the closest neighbor of s and reaches the closest neighbor of t. It uses a sequence of edges of DT_χ the Delaunay triangulation of χ.

Notice that one can consider $VP_{\chi\cup\{s,t\},s,t}$ to obtain a path that actually goes from s to t. This path differs from $VP_{\chi,s,t}$ only by few edges around s and around t.

An edge p_1p_2 belongs to $VP_{\chi,s,t}$ iff the unique ball $B_x(p_1, p_2)$ centered on the x-axis and having p_1 and p_2 on its boundary is centered on the segment $[st]$ and does not contain any other point of χ (see Figure 1). Thus, denoting $M(p_1, p_2)$ the center of $B_x(p_1, p_2)$, we can write the length of the Voronoi path as

$$\ell(VP_{\chi,s,t}) = \frac{1}{2} \sum_{(p_1,p_2)\in\chi^2} \mathbb{1}_{[B_x(p_1,p_2)\cap\chi=\emptyset]} \mathbb{1}_{[M(p_1,p_2)\in[st]]} \|p_1p_2\|, \quad (1)$$

where the $\frac{1}{2}$ arises because each edge is counted twice, once for each orientation.

Now we turn our interest to the case where the point set is a Poisson point process. We first remark that the expected length is independent from the density: let X_λ be a Poisson point process of density λ; by a simple scaling argument we get

$$\mathbb{E}\left[\ell(VP_{X_\lambda,s,t})\right] = \lambda^{\frac{1}{d}}\, \mathbb{E}\left[\ell(VP_{X_1,s,\lambda^{-\frac{1}{d}}t})\right].$$

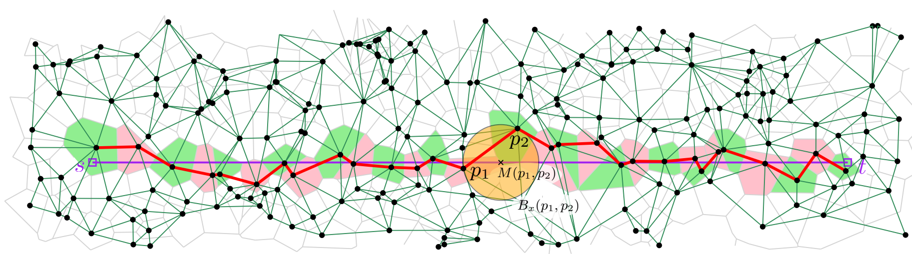

Figure 1: The Voronoi path.

dimension	1	2	3	4	5	6	7	8	$d \to \infty$
lower bound			1.497	1.682	1.835	1.918	2.077	2.224	$\sim \sqrt{\frac{2d}{\pi}}$
exact value (numerical approximation)	1	$\frac{4}{\pi} \simeq 1.27^{\dagger}$	$1.500^{\star}$	$1.698^{\star}$	$1.875^{\star}$	$2.04^{\star}$	$2.2^{\star}$	$2.3^{\star}$	$\sim \sqrt{\frac{2d}{\pi}}$
upper bound			1.5001	1.699	1.881	2.078	2.225	2.364	$\sim \sqrt{\frac{2d}{\pi}}$

† [1] ⋆ obtained from numerical integration

Table 1: Lower and upper bounds for the expected length of the Voronoi path.

On the other hand, if s, t, and u are collinear with t between s and u, the Voronoi paths can be concatenated:

$$VP_{X_\lambda,s,u} = VP_{X_\lambda,s,t} \oplus VP_{X_\lambda,t,u}.$$

Since the process is invariant by translation, we deduce for any $\gamma > 0$:

$$\mathbb{E}\left[\ell(VP_{X_1,s,\gamma t})\right] = \gamma\,\mathbb{E}\left[\ell(VP_{X_1,s,t})\right]$$

and conclude

$$\mathbb{E}\left[\ell(VP_{X_\lambda,s,t})\right] = \mathbb{E}\left[\ell(VP_{X_1,s,t})\right].$$

Thus, we can restrict our attention to Poisson point process of unit intensity. From now on, we will shorten the notation $VP_{X_1,s,t}$ in VP_X.

Using Slivnyak-Mecke formula [10, Theorem 3.3.5], Equation (1) becomes:

$$\mathbb{E}\left[\ell(VP_X)\right] = \frac{1}{2}\int_{(\mathbb{R}^d)^2} \mathbb{P}\left[B_x(p_1,p_2) \cap X = \emptyset\right] \mathbb{1}_{[M(p_1,p_2)\in[st]]} \|p_1p_2\| dp_1 dp_2. \quad (2)$$

The path $VP_{X_\lambda \cup\{s,t\},s,t}$ has about the same length when the density λ of the Poisson point process is high; namely,

$$\mathbb{E}\left[\left|\ell(VP_{X_\lambda,s,t}) - \ell(VP_{X_\lambda\cup\{s,t\},s,t})\right|\right] = O\left(\lambda^{-\frac{1}{d}}\right)$$

can be proven as an easy generalization from dimension two [7].

3 Voronoi Path in Dimension 3

We start by illustrating our method in three dimensions.

Lemma 1. *In dimension 3,*

$$1.4977 \leq \mathbb{E}\left[\ell(VP_X)\right] \leq 1.5001$$

Proof. The integral in Equation (2) is computed by substitution. The points p_1 and p_2 are defined by their sphere $B_x(p_1,p_2)$ and their spherical coordinates on that sphere. Let Φ be the function

$$\begin{array}{rccc} \Phi: & \mathbb{R}\times\mathbb{R}_+\times([0,\pi)\times[0,2\pi))^2 & \longrightarrow & (\mathbb{R}^3)^2 \\ & (x,r,\alpha_1,\beta_1,\alpha_2,\beta_2) & \longmapsto & (p_1,p_2), \end{array}$$

with

$$p_i = \begin{pmatrix} x + r\cos\alpha_i \\ r\sin\alpha_i\cos\beta_i \\ r\sin\alpha_i\sin\beta_i \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + ru_i \qquad \text{with} \quad u_i = \begin{pmatrix} \cos\alpha_i \\ \sin\alpha_i\cos\beta_i \\ \sin\alpha_i\sin\beta_i \end{pmatrix}.$$

Φ is a C^1-diffeomorphism up to a null set. Its Jacobian J_Φ has as determinant:

$$\begin{aligned}\det(J_\Phi) &= \begin{vmatrix} 1 & \cos\alpha_1 & -r\sin\alpha_1 & 0 & 0 & 0 \\ 0 & \sin\alpha_1\cos\beta_1 & r\cos\alpha_1\cos\beta_1 & -r\sin\alpha_1\sin\beta_1 & 0 & 0 \\ 0 & \sin\alpha_1\sin\beta_1 & r\cos\alpha_1\sin\beta_1 & r\sin\alpha_1\cos\beta_1 & 0 & 0 \\ 1 & \cos\alpha_2 & 0 & 0 & -r\sin\alpha_2 & 0 \\ 0 & \sin\alpha_2\cos\beta_2 & 0 & 0 & r\cos\alpha_2\cos\beta_2 & -r\sin\alpha_2\sin\beta_2 \\ 0 & \sin\alpha_2\sin\beta_2 & 0 & 0 & r\cos\alpha_2\sin\beta_2 & r\sin\alpha_2\cos\beta_2 \end{vmatrix} \\ &= r^4\sin\alpha_1\sin\alpha_2(\cos\alpha_1 - \cos\alpha_2).\end{aligned}$$

Then, we substitute the new variables:

$$\begin{aligned}\mathbb{E}\left[\ell(VP_X)\right] &= \tfrac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi} \mathbb{P}\left[B((x,0,0),r)\cap X=\emptyset\right]\mathbb{1}_{[(x,0,0)\in[st]]} \\ &\qquad \cdot r\|u_1u_2\||\det(J_\Phi)|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}r\mathrm{d}x \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-\frac{4}{3}\pi r^3}\mathbb{1}_{[x\in[0,1]]}\cdot r\|u_1u_2\| \\ &\qquad \cdot r^4\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}r\mathrm{d}x.\end{aligned}$$

Separating the different variables, the shape of the integral[1] on r matches Equation (9):

$$\begin{aligned}\mathbb{E}\left[\ell(VP_X)\right] &= \tfrac{1}{2}\left(\int_0^1 dx\right)\left(\int_0^{\infty} e^{-\frac{4}{3}\pi r^3} r^5\mathrm{d}r\right) \\ &\qquad \cdot\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\cdot\|u_1u_2\|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\ &= \tfrac{1}{2}\frac{3}{16\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\cdot\|u_1u_2\|\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2. \qquad (3)\end{aligned}$$

Unfortunately, we cannot compute formally this integral with the exact value of $\|u_1u_2\|$. Using the trivial bound $\|u_1u_2\| \leq 2$ we get

$$\begin{aligned}&\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\sin\alpha_1\sin\alpha_2|\cos\alpha_1-\cos\alpha_2|\cdot 2\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\ &= \left(2\int_0^{\pi}\int_0^{\alpha_2}\sin\alpha_1\sin\alpha_2\left(\cos\alpha_1-\cos\alpha_2\right)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right)\cdot\left(\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\beta_1\mathrm{d}\beta_2\right) \\ &= \frac{8}{3}\cdot 4\pi^2 = \frac{32\pi^2}{3},\end{aligned}$$

where the $\frac{8}{3}$ factor comes from Equation (11). Plugging this bound in Equation (3) gives $\mathbb{E}\left[\ell(VP_X)\right] \leq 2$.

Using a better bound for $\|u_1u_2\|$, we can improve on this result. Expressing $\|u_1u_2\|$ in terms of the spherical coordinates we have

$$\begin{aligned}\|u_1u_2\| &= \sqrt{(u_1-u_2)^2} = \sqrt{u_1^2+u_2^2-2u_1\cdot u_2} = \sqrt{2-2u_1\cdot u_2} \\ &= \sqrt{2}\sqrt{1-\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2\cos(\beta_1-\beta_2)} \\ &= \sqrt{2}\sqrt{1-\delta} \qquad (4) \\ \text{with} \quad & \delta = \left(\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\cos(\beta_1-\beta_2)\right). \qquad (5)\end{aligned}$$

[1]Useful integrals on exponential or trigonometric functions are given in appendix.

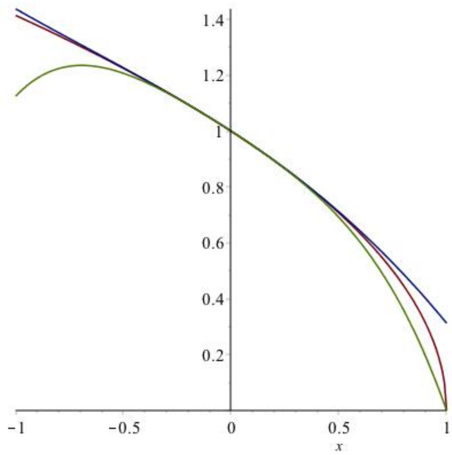

Figure 2: Bounding $\sqrt{1-y}$ from above and below using Taylor expansions.

We can use Taylor expansion to bound $\|u_1 u_2\|$. If $T_k(y)$ is the Taylor expansion up to degree k of $\sqrt{1-y}$, for k odd and $y \in [-1, 1]$ we have

$$T_k(y) - T_k(1)y^{k+1} \leq \sqrt{1-y} \leq T_k(y). \tag{6}$$

Figure 2 illustrates this for $k = 3$:

$$1 - \tfrac{y}{2} - \tfrac{y^2}{8} - \tfrac{y^3}{16} - \tfrac{5y^4}{16} \leq \sqrt{1-y} \leq 1 - \tfrac{y}{2} - \tfrac{y^2}{8} - \tfrac{y^3}{16}.$$

One can compute the following integrals (e.g., using Maple, see [5]):

$$\begin{aligned}
2\int_0^{\pi}\int_0^{\alpha_2}\int_0^{2\pi}\int_0^{2\pi} & \sin\alpha_1 \sin\alpha_2 |\cos\alpha_1 - \cos\alpha_2| T_{41}(\delta) \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\
&= \tfrac{26299475949008588023023238 1}{23281266760547350903521280}\pi^2, \\
2\int_0^{\pi}\int_0^{\alpha_2}\int_0^{2\pi}\int_0^{2\pi} & \sin\alpha_1 \sin\alpha_2 |\cos\alpha_1 - \cos\alpha_2| \left(T_{41}(\delta) - T_{41}(1)\delta^{42}\right) \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\
&= \tfrac{15077901215708368860256545 03}{1332651821465813879305011 20}\pi^2.
\end{aligned}$$

Plugging these values in Equation (3) and using Equations (4) and (6) yields the bounds announced in the lemma statement. $\square$

A numerical evaluation of the integral at Equation (3) gives a value in that interval, pretty close to $\frac{3}{2}$ and we conjecture that $\frac{3}{2}$ is the correct value.

4 Voronoi Path in Dimension d

Theorem 2. *In dimension d, the expected length of the Voronoi path is bounded by:*

$$\frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5} d}{\pi^2(2d-2)!}\left(1-\frac{d-1}{4d^2-1}\right)\sqrt{2} \leq \mathbb{E}\left[\ell(VP_X)\right] \leq \frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5} d}{\pi^2(2d-2)!}\left(1+\frac{1}{4d-2}\right)\sqrt{2},$$

and the behavior when $d \to \infty$ is:

$$\sqrt{\frac{2d}{\pi}} - \frac{1}{4\sqrt{2d\pi}} + O\left(d^{-\frac{3}{2}}\right) \leq \mathbb{E}\left[\ell(VP_X)\right] \leq \sqrt{\frac{2d}{\pi}} + \frac{3}{4\sqrt{2d\pi}} + O\left(d^{-\frac{3}{2}}\right).$$

The rest of the section is devoted to the proof of Theorem 2.

As in dimension 3, we compute by substitution the integral in Equation (2) defining the points p_1 and p_2 by their sphere $B_x(p_1,p_2)$ and their spherical coordinates on that sphere. Let Φ be the function

$$\begin{array}{llll}\Phi: & \mathbb{R}\times\mathbb{R}_+\times\left([0,\pi]^{d-1}\times[0,2\pi)\right)^2 & \longrightarrow & (\mathbb{R}^d)^2 \\ & (x,r,\alpha_{1,1},\ldots,\alpha_{1,d-1},\alpha_{2,1},\ldots,\alpha_{2,d-1}) & \longmapsto & (p_1,p_2),\end{array}$$

with

$$p_i = \begin{pmatrix} x\\0\\0\\ \vdots\\0\\ \vdots\\0\\0\end{pmatrix} + ru_i \qquad \text{with} \quad u_i = \begin{pmatrix} \cos\alpha_{i,1}\\ \sin\alpha_{i,1}\cos\alpha_{i,2}\\ \sin\alpha_{i,1}\sin\alpha_{i,2}\cos\alpha_{i,3}\\ \vdots\\ \left(\prod_{l=1}^{j-1}\sin\alpha_{i,l}\right)\cos\alpha_{i,j}\\ \vdots\\ \left(\prod_{l=1}^{d-2}\sin\alpha_{i,l}\right)\cos\alpha_{i,d-1}\\ \left(\prod_{l=1}^{d-2}\sin\alpha_{i,l}\right)\sin\alpha_{i,d-1}\end{pmatrix}.$$

Φ is a C^1-diffeomorphism up to a null set. Its Jacobian J_Φ has as determinant:

$$\begin{aligned}\det(J_\Phi) &= r^{2(d-1)}\mathcal{J}(\alpha_{1:2,1:d-2}) \\ \text{with}\quad \mathcal{J}(\alpha_{1:2,1:d-2}) &= \prod_{i=2}^{d-2}\sin^{d-i-1}(\alpha_{1,i})\sin^{d-i-1}(\alpha_{2,i})\cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})|\cos\alpha_{1,1}-\cos\alpha_{2,1}|.\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\ell(VP_X)\right] &= \tfrac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{(\mathbb{S}_{d-1})^2}\mathbb{P}\left[B((x,0,\ldots,0),r)\cap X=\emptyset\right]\mathbb{1}_{[(x,0,\ldots,0)\in[st]]} \\ &\qquad \cdot r\|u_1u_2\||\det(J_\Phi)|\mathrm{d}\alpha_{1,1}\ldots\mathrm{d}\alpha_{1,d-1}\mathrm{d}\alpha_{2,1}\ldots\mathrm{d}\alpha_{2,d-1}\mathrm{d}r\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^1\mathrm{d}x\times\int_0^\infty e^{-\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}r^d}r^{2d-1}\mathrm{d}r \\ &\quad\times\int_{(\mathbb{S}_{d-1})^2}\|u_1u_2\|\,\mathcal{J}(\alpha_{1:2,1:d-2})\mathrm{d}\alpha_{1:2,1:d-1}.\end{aligned}$$

The volume of the unit ball in d dimensions is known to be $\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$. Using Equation (9), we get:

$$\mathbb{E}\left[\ell(VP_X)\right] = \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d} \times \int_{(\mathbb{S}_{d-1})^2} \|u_1 u_2\| \, \mathcal{J}(\alpha_{1:2,1:d-2}) \mathrm{d}\alpha_{1:2,1:d-1} \quad (7)$$

We use the Taylor expansion of Equation (6) with $k = 1$ to bound $\|u_1 u_2\|$:

$$\sqrt{2}\left(1 - \frac{\delta}{2} - \frac{\delta^2}{2}\right) \leq \|u_1 u_2\| \leq \sqrt{2}\left(1 - \frac{\delta}{2}\right) \quad (8)$$

with $\delta = u_1 \cdot u_2$. We write δ as a sum of three terms $\delta = \delta_A + \delta_B + \delta_C$:

$$\delta_A = \left(\prod_{j=1}^{d-2} \sin\alpha_{1,j} \sin\alpha_{2,j}\right) \cdot \cos(\alpha_{1,d-1} - \alpha_{2,d-1}),$$

$$\delta_B = \sum_{i=2}^{d-2} \left(\prod_{j=1}^{i-1} \sin\alpha_{1,j} \sin\alpha_{2,j}\right) \cos\alpha_{1,i} \cos\alpha_{2,i},$$

$$\delta_C = \cos\alpha_{1,1} \cos\alpha_{2,1}.$$

Replacing $\|u_1 u_2\|$ in Equation (7) by its bounds yields the computation of the following integral for $k \in \{0, 1, 2\}$:

$$\mathcal{I}_k = \int_{(\mathbb{S}_{d-1})^2} \delta^k \mathcal{J}(\alpha_{1:2,1:d-2}) \mathrm{d}\alpha_{1:2,1:d-1}.$$

4.1 Computation of $\mathcal{I}_0$

$$\begin{aligned}
\mathcal{I}_0 &= \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\alpha_{1,d-1}\mathrm{d}\alpha_{2,d-1} \times \\
&\quad \prod_{i=2}^{d-2}\int_0^{\pi} \sin^{d-i-1}(\alpha_{1,i})\mathrm{d}\alpha_{1,i} \times \prod_{i=2}^{d-2}\int_0^{\pi} \sin^{d-i-1}(\alpha_{2,i})\mathrm{d}\alpha_{2,i} \times \\
&\quad \int_0^{\pi}\int_0^{\pi} \sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})|\cos\alpha_{1,1} - \cos\alpha_{2,1}|\mathrm{d}\alpha_{1,1}\mathrm{d}\alpha_{2,1} \\
&= 4\pi^2 \times \left(\prod_{i=2}^{d-2} \frac{\Gamma\left(\frac{d-i}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)}\right)^2 \pi^{d-3} \\
&\quad \times \int_0^{\pi}\int_0^{\pi} \sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})|\cos\alpha_{1,1} - \cos\alpha_{2,1}|\mathrm{d}\alpha_{1,1}\mathrm{d}\alpha_{2,1}
\end{aligned}$$

using Equation (10). The product can be reduced by a telescoping argument and the integral is computed using Equation (11). We get:

$$\begin{aligned}
\mathcal{I}_0 &= 4\pi^2 \times \frac{1}{\Gamma\left(\frac{d-1}{2}\right)^2} \times \pi^{d-3} \times \frac{2^{2d}((d-2)!)^2}{(2d-2))!} \\
&= \frac{2^{2d+2}\pi^{d-1}((d-2)!)^2}{\Gamma\left(\frac{d-1}{2}\right)^2 (2d-2))!}.
\end{aligned}$$

4.2 Computation of $\mathcal{I}_1$

Using the above decomposition of δ into $\delta_A + \delta_B + \delta_C$, we can write $\mathcal{I}_1 = \mathcal{I}_1^A + \mathcal{I}_1^B + \mathcal{I}_1^C$.

$$\mathcal{I}_1^A = \int_{(\mathbb{S}_{d-1})^2} \overbrace{\left(\prod_{j=1}^{d-2} \sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cdot\cos(\alpha_{1,d-1}-\alpha_{2,d-1})}^{\delta_A} \mathcal{J}(\alpha_{1:2,1:d-2})\mathrm{d}\alpha_{1:2,1:d-1},$$

$$\mathcal{I}_1^B = \sum_{i=2}^{d-2}\int_{(\mathbb{S}_{d-1})^2} \overbrace{\left(\prod_{j=1}^{i-1} \sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cos\alpha_{1,i}\cos\alpha_{2,i}}^{\delta_B} \mathcal{J}(\alpha_{1:2,1:d-2})\mathrm{d}\alpha_{1:2,1:d-1},$$

$$\mathcal{I}_1^C = \int_{(\mathbb{S}_{d-1})^2} \overbrace{\cos\alpha_{1,1}\cos\alpha_{2,1}}^{\delta_C} \mathcal{J}(\alpha_{1:2,1:d-2})\mathrm{d}\alpha_{1:2,1:d-1}.$$

We have $\mathcal{I}_1^A = 0$ since $\mathcal{J}$ does not depends on $\alpha_{1,d-1}$ and $\alpha_{2,d-1}$ thus integrating over $\alpha_{1,d-1}$ and $\alpha_{2,d-1}$ create a null factor according to Equation (12). We also have $\mathcal{I}_1^B = 0$ since variables $\alpha_{1,i}$ appear only within sinus inside $\mathcal{J}$, thus integrating over $\alpha_{1,i}$ creates a null factor in each term of the sum using Equation (13).

To compute $\mathcal{I}_1^C$, we can integrate all variables different from $\alpha_{1,1}\alpha_{2,1}$ in the same way as we have done for computing $\mathcal{I}_0$ and get

$$\begin{aligned}\mathcal{I}_1 = \mathcal{I}_1^C =& \int_{(\mathbb{S}_{d-1})^2} \cos\alpha_{1,1}\cos\alpha_{2,1}\left(\prod_{j=2}^{d-2}\sin^{d-j-1}(\alpha_{1,j})\sin^{d-j-1}(\alpha_{2,j})\right) \\ &\cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1:2,1:d-1} \\ =& \frac{4\pi^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)^2} \\ &\times\int_0^\pi\int_0^\pi \sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cos(\alpha_{1,1})\cos(\alpha_{2,1})|\cos\alpha_{1,1}-\cos\alpha_{2,1}|\mathrm{d}\alpha_{1,1}\mathrm{d}\alpha_{2,1}.\end{aligned}$$

Then using Equation (14) one can finally compute $\mathcal{I}_1$. Observing that the result of Equation (14) is the same as Equation (11) up to a factor $-(2d-1)$ we get

$$\mathcal{I}_1 = -\frac{1}{2d-1}\mathcal{I}_0.$$

4.3 The Upper Bound

Using the values of $\mathcal{I}_0$ and $\mathcal{I}_1$ in Equation (7), we get

$$
\begin{aligned}
\mathbb{E}\left[\ell(VP_X)\right] &\leq \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\sqrt{2}\left(\mathcal{I}_0 - \tfrac{1}{2}\mathcal{I}_1\right) \\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\mathcal{I}_0\left(1+\frac{1}{4d-2}\right)\sqrt{2} \\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\frac{4\pi^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)^2}\frac{2^{2d}((d-2)!)^2}{(2d-2))!}\left(1+\frac{1}{4d-2}\right)\sqrt{2} \\
&= \frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5}d}{\pi^2(2d-2)!}\left(1+\frac{1}{4d-2}\right)\sqrt{2} \\
&= \sqrt{\frac{2d}{\pi}} + \frac{3}{4\sqrt{2d\pi}} + O\left(d^{-\frac{3}{2}}\right), \qquad \text{when } d \to \infty.
\end{aligned}
$$

4.4 Computation of $\mathcal{I}_2$

We can easily get an upper bound of integral $\mathcal{I}_2$ by noticing that $\delta^2 \leq 1$ and thus $\mathcal{I}_2 \leq \mathcal{I}_0$, but this yields an unsatisfactory lower bound for the length of the Voronoi path. So we compute $\mathcal{I}_2$ exactly.

The integral $\mathcal{I}_2$ can be split in 6 terms according to the development of δ^2:

$$\delta^2 = \delta_A^2 + \delta_B^2 + \delta_C^2 + 2\delta_A\delta_B + 2\delta_A\delta_C + 2\delta_B\delta_C.$$

As for the computation of $\mathcal{I}_1$, because of Equations 12 and 13, the three terms corresponding to $\delta_A\delta_B$, $\delta_A\delta_C$, and $\delta_B\delta_C$ yields null integrals. Thus $\mathcal{I}_2 = \mathcal{I}_2^A + \mathcal{I}_2^B + \mathcal{I}_2^C$, where

$$
\begin{aligned}
\mathcal{I}_2^A \quad &= \int_{(\mathbb{S}_{d-1})^2} \overbrace{\left(\left(\prod_{j=1}^{d-2}\sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cdot\cos(\alpha_{1,d-1}-\alpha_{2,d-1})\right)^2}^{\delta_A^2} \\
&\quad \overbrace{\left(\prod_{j=2}^{d-2}\sin^{d-j-1}(\alpha_{1,j})\sin^{d-j-1}(\alpha_{2,j})\right)\cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|}^{\mathcal{J}(\alpha_{1:2,1:d-2})}\,\mathrm{d}\alpha_{1:2,1:d-1} \\
&= \int_{(\mathbb{S}_{d-1})^2}\left(\prod_{i=2}^{d-2}\sin^{d-i+1}(\alpha_{1,i})\sin^{d-i+1}(\alpha_{2,i})\right)\cos^2(\alpha_{1,d-1}-\alpha_{2,d-1}) \\
&\qquad\cdot\left(\sin^d(\alpha_{1,1})\sin^d(\alpha_{2,1})|\cos(\alpha_{1,1})-\cos(\alpha_{2,1})|\right)\mathrm{d}\alpha_{1:2,1:d-1} \\
&= \left(\prod_{i=2}^{d-2}\int_0^{\pi}\sin^{d-i+1}(\alpha_{1,i})\mathrm{d}\alpha_{1,i}\right)^2 \\
&\qquad\cdot\int_0^{2\pi}\int_0^{2\pi}\cos^2(\alpha_{1,d-1}-\alpha_{2,d-1})\mathrm{d}\alpha_{2,d-1}\mathrm{d}\alpha_{1,d-1} \\
&\qquad\cdot\int_0^{\pi}\int_0^{\pi}\sin^d(\alpha_{1,1})\sin^d(\alpha_{2,1})|\cos(\alpha_{1,1})-\cos(\alpha_{2,1})|\mathrm{d}\alpha_{2,1}\mathrm{d}\alpha_{1,1}.
\end{aligned}
$$

The different factors are computed using Equations (10), and (15), and

(11). We get

$$\begin{aligned}
\mathcal{I}_2^A &= \left(\prod_{i=2}^{d-2} \frac{\Gamma\left(\frac{d-i}{2}+1\right)}{\Gamma\left(\frac{d-i+1}{2}+1\right)}\right)^2 \pi^{d-3} \cdot 2\pi^2 \cdot \frac{2^{2d+4} d!^2}{(2d+2)!} \\
&= \frac{2\pi^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)^2} \cdot \frac{2^{2d+4}(d!)^2}{(2d+2)!} = \frac{2^{2d+5}\pi^{d-1}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2 (2d+2)!},
\end{aligned}$$

is obtained by a telescoping argument. The term corresponding to δ_B^2 is

$$\begin{aligned}
\mathcal{I}_2^B &= \int_{(\mathbb{S}_{d-1})^2} \overbrace{\left(\sum_{i=2}^{d-2}\left(\prod_{j=1}^{i-1} \sin\alpha_{1,j}\sin\alpha_{2,j}\right)\cos\alpha_{1,i}\cos\alpha_{2,i}\right)^2}^{\delta_B^2} \\
&\quad \overbrace{\left(\prod_{j=2}^{d-2} \sin^{d-j-1}(\alpha_{1,j})\sin^{d-j-1}(\alpha_{2,j})\right) \cdot \sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1}) \cdot |\cos\alpha_{1,1} - \cos\alpha_{2,1}|}^{\mathcal{J}(\alpha_{1:2,1:d-2})} \mathrm{d}\alpha_{1:2,1:d-1} \\
&= \int_{(\mathbb{S}_{d-1})^2} \left(\prod_{i=2}^{d-2} \sin^{d-i-1}(\alpha_{1,i})\sin^{d-i-1}(\alpha_{2,i}) \times \sum_{i=2}^{d-2}\left(\prod_{j=2}^{i-1}\sin^2(\alpha_{1,j})\sin^2(\alpha_{2,j})\right)\cos^2(\alpha_{1,i})\cos^2(\alpha_{2,i})\right) \\
&\quad \times \left(\sin^d(\alpha_{1,1})\sin^d(\alpha_{2,1})|\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})|\right) \mathrm{d}\alpha_{1:2,1:d-1}.
\end{aligned}$$

Since all terms where $\cos\alpha_{1,i}$ is not squared have null integral by Equation (13), $\mathcal{I}_2^B$ reduces to:

$$\begin{aligned}
\mathcal{I}_2^B &= \left(\sum_{i=2}^{d-2}\left(\prod_{j=2}^{i-1}\int_0^\pi \sin^{d-j+1}(\alpha_{1,j})\mathrm{d}\alpha_{1,j}\right)^2\right. \\
&\qquad \left.\left(\int_0^\pi \sin^{d-i-1}(\alpha_{1,i})\cos^2(\alpha_{1,i})\mathrm{d}\alpha_{1,i}\right)^2 \left(\prod_{j=i+1}^{d-2}\int_0^\pi \sin^{d-j-1}(\alpha_{1,j})\mathrm{d}\alpha_{1,j}\right)^2\right) \\
&\quad \times \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\alpha_{2,d-1}\mathrm{d}\alpha_{1,d-1} \times \int_0^\pi\int_0^\pi \sin^d(\alpha_{1,1})\sin^d(\alpha_{2,1})|\cos(\alpha_{1,1}) - \cos(\alpha_{2,1})|\mathrm{d}\alpha_{2,1}\mathrm{d}\alpha_{1,1}.
\end{aligned}$$

The different factors are computed using Equations (10), (16), and (14) and we get:

$$\begin{aligned}
\mathcal{I}_2^B &= \left(\sum_{i=2}^{d-2}\left(\prod_{j=2}^{i-1}\frac{\Gamma\left(\frac{d-j+2}{2}\right)}{\Gamma\left(\frac{d-j+3}{2}\right)}\right)^2\left(\frac{1}{d-i+1}\cdot\frac{\Gamma\left(\frac{d-i}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)}\right)^2\right.\\
&\qquad\left.\left(\prod_{j=i+1}^{d-2}\frac{\Gamma\left(\frac{d-j}{2}\right)}{\Gamma\left(\frac{d-j+1}{2}\right)}\right)^2\right)\pi^{d-3}\times 4\pi^2\times\frac{2^{2d+4}d!^2}{(2d+2)!}\\
&= \left(\frac{1}{\Gamma\left(\frac{d+1}{2}\right)^2}\sum_{i=3}^{d-1}\left(\frac{\Gamma\left(1+\frac{i}{2}\right)}{i\Gamma\left(\frac{i}{2}\right)}\right)^2\right)\times 4\pi^{d-1}\frac{2^{2d+4}d!^2}{(2d+2)!}\\
&= \left(\frac{1}{\Gamma\left(\frac{d+1}{2}\right)^2}\sum_{i=3}^{d-1}\left(\frac{1}{2}\right)^2\right)\times 4\pi^{d-1}\frac{2^{2d+4}d!^2}{(2d+2)!}\\
&= \left(\frac{(d-3)/4}{\Gamma\left(\frac{d+1}{2}\right)^2}\right)\times 4\pi^{d-1}\frac{2^{2d+4}d!^2}{(2d+2)!} = \frac{\pi^{d-1}(d-3)2^{2d+4}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2(2d+2)!}.
\end{aligned}$$

Finally, the term corresponding to δ_C^2 is

$$\begin{aligned}
\mathcal{I}_2^C &= \int_{(\mathbb{S}_{d-1})^2}\overbrace{(\cos\alpha_{1,1}\cos\alpha_{2,1})^2}^{\delta_C^2}\\
&\quad\overbrace{\left(\prod_{i=2}^{d-2}\sin^{d-i-1}(\alpha_{1,j})\sin^{d-i-1}(\alpha_{2,i})\right)\cdot\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cdot|\cos\alpha_{1,1}-\cos\alpha_{2,1}|}^{\mathcal{J}(\alpha_{1:2,1:d-2})}\,\mathrm{d}\alpha_{1:2,1:d-1}\\
&= \left(\prod_{i=2}^{d-2}\int_0^\pi\sin^{d-i-1}\alpha_{1,i}\mathrm{d}\alpha_{1,i}\right)^2\times\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\alpha_{2,d-1}\mathrm{d}\alpha_{1,d-1}\\
&\quad\times\int_0^\pi\int_0^\pi\sin^{d-2}(\alpha_{1,1})\sin^{d-2}(\alpha_{2,1})\cos^2(\alpha_{1,1})\cos^2(\alpha_{2,1})|\cos(\alpha_{1,1})-\cos(\alpha_{2,1})|\mathrm{d}\alpha_{2,1}\mathrm{d}\alpha_{1,1},
\end{aligned}$$

which gives, using Equations (10), and (17):

$$\begin{aligned}
\mathcal{I}_2^C &= \left(\prod_{i=2}^{d-2}\frac{\Gamma\left(\frac{d-i}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)}\right)^2\pi^{d-3}\times 4\pi^2\times\frac{2^{2d+2}(7d-1)d!^2}{(d-1)^2d(2d+2)!}\\
&= \frac{4\pi^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)^2}\cdot\frac{2^{2d+4}(d!)^2}{(2d+2)!}\cdot\frac{7d-1}{16d} = \frac{2^{2d+2}\pi^{d-1}(d!)^2(7d-1)}{d\,\Gamma\left(\frac{d+1}{2}\right)^2(2d+2)!}.
\end{aligned}$$

Summing the three terms together and simplifying gives:

$$
\begin{aligned}
\mathcal{I}_2 &= \frac{2^{2d+5}\pi^{d-1}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2(2d+2)!} + \frac{\pi^{d-1}(d-3)2^{2d+4}(d!)^2}{\Gamma\left(\frac{d+1}{2}\right)^2(2d+2)!} + \frac{2^{2d+2}\pi^{d-1}(d!)^2(7d-1)}{d\,\Gamma\left(\frac{d+1}{2}\right)^2(2d+2)!} \\
&= \frac{2^{2d+2}\pi^{d-1}((d-2)!)^2}{\Gamma\left(\frac{d-1}{2}\right)^2(2d-2))!} \cdot \frac{d^2(d-1)^2}{\left(\frac{d-1}{2}\right)^2(2d+2)(2d+1)2d(2d-1)}\left(8+4(d-3)+\frac{7d-1}{d}\right) \\
&= \frac{2^{2d+2}\pi^{d-1}((d-2)!)^2}{\Gamma\left(\frac{d-1}{2}\right)^2(2d-2))!} \cdot \frac{4d^2}{(2d+2)(2d+1)2d(2d-1)} \cdot \frac{(d+1)(4d-1)}{d} \\
&= \mathcal{I}_0 \cdot \frac{4d-1}{4d^2-1}.
\end{aligned}
$$

4.5 The Lower Bound

Using the values of $\mathcal{I}_0$, $\mathcal{I}_1$, and $\mathcal{I}_2$ in Equation (7), we get

$$
\begin{aligned}
\mathbb{E}\left[\ell(VP_X)\right] &\geq \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\sqrt{2}\left(\mathcal{I}_0 - \tfrac{1}{2}\mathcal{I}_1 - \tfrac{1}{2}\mathcal{I}_2\right) \\
&= \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{2d\pi^d}\mathcal{I}_0\left(1+\frac{1}{4d-2}-\frac{4d-1}{4d^2-1}\right)\sqrt{2} \\
&= \frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5}d}{\pi^2(2d-2)!}\left(1-\frac{d-1}{4d^2-1}\right)\sqrt{2} \\
&= \sqrt{\frac{2d}{\pi}} - \frac{1}{4\sqrt{2d\pi}} + O\left(d^{-\frac{3}{2}}\right) \qquad \text{when } d \to \infty.
\end{aligned}
$$

5 Small Dimensions

For d small and similarly to what we have done in dimension 3, we can use symbolic computation to get formal better bounds using higher order Taylor expansions. We also used numerical integration to get an approximation of the actual values. Results are in Table 2, Maple code is available [5].

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d	k	lower bound	$\simeq$	exact value	upper bound	$\simeq$
3	41	$\frac{788984278470257640690697143}{745000536337515228912680960}\sqrt{2}$	1.49770	1.500	$\frac{452337036471251065807696350 9}{426448582869060441377603584 0}\sqrt{2}$	1.50007
4	7	$\frac{102494570}{8729721}\frac{\sqrt{2}}{\pi^2}$	1.6823	1.698	$\frac{121774997}{10270260}\frac{\sqrt{2}}{\pi^2}$	1.6990
5	3	$\frac{135}{104}\sqrt{2}$	1.8357	1.875	$\frac{21305}{16016}\sqrt{2}$	1.8812
6	1	$\frac{3014656}{225225}\frac{\sqrt{2}}{\pi^2}$	1.9179	2.04	$\frac{753664}{51975}\frac{\sqrt{2}}{\pi^2}$	2.0778
7	1	$\frac{210}{143}\sqrt{2}$	2.0768	2.2	$\frac{225}{143}\sqrt{2}$	2.2252
8	1	$\frac{2080374784}{134008875}\frac{\sqrt{2}}{\pi^2}$	2.2244	2.3	$\frac{130023424}{7882875}\frac{\sqrt{2}}{\pi^2}$	2.3635

Table 2: Lower and upper bounds for the expected length of the Voronoi path using Taylor expansion of order k and exact values from numerical integration.

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Appendix: Useful Integrals

In the paper we used several integrals whose expressions are given here. For completeness, the proofs are given in this appendix, although they are essentially boring technical computations. For d small enough, Maple or WolframAlpha can compute most of such integrals. However, for symbolic d, such software may handle only a couple of them. Proofs are below.

$$\int_0^\infty e^{-cr^d} r^{2d-1} \mathrm{d}r = \frac{1}{d \cdot c^2}. \tag{9}$$

$$\int_0^\pi \sin^d \alpha \, \mathrm{d}\alpha = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1+\frac{d}{2}\right)} \cdot \sqrt{\pi}. \tag{10}$$

$$\int_0^\pi \int_0^\pi \sin^d \alpha_1 \sin^d \alpha_2 |\cos \alpha_1 - \cos \alpha_2| \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 = \frac{2^{2d+4} d!^2}{(2d+2)!}. \tag{11}$$

$$\int_0^{2\pi} \int_0^{2\pi} \cos(\alpha_1 - \alpha_2) \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 = 0. \tag{12}$$

$$\int_0^\pi \sin^d \alpha \cos \alpha \mathrm{d}\alpha = 0. \tag{13}$$

$$\int_0^\pi \int_0^\pi \sin^d \alpha_1 \sin^d \alpha_2 |\cos \alpha_1 - \cos \alpha_2| \cos \alpha_1 \cos \alpha_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 = -\frac{2^{2d+4} d!^2}{(2d+3)!}. \tag{14}$$

$$\int_0^{2\pi} \int_0^{2\pi} \cos^2(\alpha_1 - \alpha_2) \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 = 2\pi^2. \tag{15}$$

$$\int_0^\pi \sin^d(\alpha) \cos^2(\alpha) \mathrm{d}\alpha = \left(\frac{1}{d+2}\right) \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1+\frac{d}{2}\right)} \cdot \sqrt{\pi}. \tag{16}$$

$$\begin{aligned} &\int_0^\pi \int_0^\pi \sin^d \alpha_1 \sin^d \alpha_2 \cos^2 \alpha_1 \cos^2 \alpha_2 |\cos \alpha_1 - \cos \alpha_2| \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ &\quad = \frac{2^{2d+6}(7d+13)(d+2)!^2}{(d+1)^2(d+2)(2d+6)!}. \end{aligned} \tag{17}$$

$$\int_a^b \sin^d \alpha \, \mathrm{d}\alpha = \begin{cases} -\displaystyle\sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \frac{1}{i} \left(\frac{\Gamma(\frac{d+1}{2}) \Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2}) \Gamma(\frac{1+i}{2})} \right) \sin^{i-1}(\alpha) \cos(\alpha) \Big|_a^b, & \text{if } d \text{ is odd}, \\ -\displaystyle\sum_{\substack{2 \le i \le d \\ i \text{ even}}} \frac{1}{i} \left(\frac{\Gamma(\frac{d+1}{2}) \Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2}) \Gamma(\frac{1+i}{2})} \right) \sin^{i-1}(\alpha) \cos(\alpha) \Big|_a^b + \frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})} \frac{\alpha}{\sqrt{\pi}} \Big|_a^b, & \text{if } d \text{ is even}. \end{cases} \tag{18}$$

$$\int_0^\pi \alpha \sin^d \alpha \cos \alpha \, \mathrm{d}\alpha = -\left(\frac{1}{d+1}\right) \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(1+\frac{d+1}{2}\right)} \cdot \sqrt{\pi}. \tag{19}$$

Proofs

Proof of Equation (9). Classic. □

Proof of Equation (18). Integration by parts gives

$$\int_a^b \sin^d \alpha \, \mathrm{d}\alpha = -\cos(\alpha) \sin^{d-1}(\alpha) \Big|_a^b + \int_a^b \cos^2 \alpha (d-1) \sin^{d-2}(\alpha) \, \mathrm{d}\alpha.$$

Using the identity $\cos^2(\alpha) = 1 - \sin^2(\alpha)$ and simplifying gives

$$\int_a^b \sin^d \alpha \, \mathrm{d}\alpha = -\left(\frac{1}{d}\right) \cos(\alpha) \sin^{d-1}(\alpha)\Big|_a^b + \left(\frac{d-1}{d}\right) \int_a^b \sin^{d-2}(\alpha) \, \mathrm{d}\alpha.$$

Now, using $\int_a^b \sin^0 \alpha \, \mathrm{d}\alpha = \alpha\Big|_a^b$ and the identity

$$\prod_{k=1}^{k} \frac{2i-1}{2i} = \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(k+1)\sqrt{\pi}},$$

by induction on d, we get that, for d even,

$$\int_a^b \sin^d \alpha \, \mathrm{d}\alpha = -\sum_{\substack{2 \le i \le d \\ i \text{ even}}} \frac{1}{i} \left(\frac{\Gamma(\frac{d+1}{2}) / \left(\Gamma(1+\frac{d}{2})\sqrt{\pi}\right)}{\Gamma(\frac{i+1}{2}) / \left(\Gamma(1+\frac{i}{2})\sqrt{\pi}\right)} \right) \sin^{i-1}(\alpha) \cos(\alpha)\Big|_a^b + \frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})} \frac{\alpha}{\sqrt{\pi}}\Big|_a^b.$$

Analogously, for d odd, we use $\int_a^b \sin^1 \alpha \, \mathrm{d}\alpha = -\cos(\alpha)\Big|_a^b$ and the identity

$$\prod_{k=1}^{k} \frac{2i}{2i+1} = \frac{\Gamma(d+1)\sqrt{\pi}}{\Gamma(d+\frac{3}{2})},$$

to get by induction

$$\int_a^b \sin^d \alpha \, \mathrm{d}\alpha = -\sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \frac{1}{i} \left(\frac{\left(\Gamma(\frac{d+1}{2})\sqrt{\pi}\right) / \Gamma(1+\frac{d}{2})}{\left(\Gamma(\frac{i+1}{2})\sqrt{\pi}\right) / \Gamma(1+\frac{i}{2})} \right) \sin^{i-1}(\alpha) \cos(\alpha)\Big|_a^b.$$

Grouping the two expressions above completes the proof. □

Proof of Equation (10). It is a direct corollary of Equation 18. □

Proof of Equation (19). Integration by parts combined with Equation (10) gives

$$\begin{aligned}
\int_0^\pi \alpha \sin^d \alpha \cos \alpha \, \mathrm{d}\alpha &= \frac{1}{d+1} \alpha \sin^{d+1}(\alpha)\Big|_0^\pi - \frac{1}{d+1} \int_0^\pi \sin^{d+1} \alpha \, \mathrm{d}\alpha \\
&= -\frac{1}{d+1} \int_0^\pi \sin^{d+1} \alpha \, \mathrm{d}\alpha \\
&= -\left(\frac{1}{d+1}\right) \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(1+\frac{d+1}{2}\right)} \cdot \sqrt{\pi}.
\end{aligned}$$

□

Proof of Equation (11). First, we use the symmetry with respect to the line $\alpha_1 = \alpha_2$ to split the integral in two equal parts without absolute values.

$$\begin{aligned}
\text{Integral} &= 2 \iint_{\substack{\alpha_1, \alpha_2 \in [0,\pi]^2 \\ \alpha_1 < \alpha_2}} \sin^d \alpha_1 \sin^d \alpha_2 (\cos \alpha_1 - \cos \alpha_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\
&= 2 \left(\int_0^\pi \sin^d \alpha_1 \cos \alpha_1 \int_{\alpha_1}^\pi \sin^d \alpha_2 \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 - \int_0^\pi \sin^d \alpha_2 \cos \alpha_2 \int_0^{\alpha_2} \sin^d \alpha_1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \right) \\
&= 2 \int_0^\pi \sin^d x \cos x \left(\int_x^\pi \sin^d y \mathrm{d}y - \int_0^x \sin^d y \mathrm{d}y \right).
\end{aligned}$$

Let $f_d(x) = \left(\int_x^\pi \sin^d y \mathrm{d}y - \int_0^x \sin^d y \mathrm{d}y\right)$. Then, by Equation 18, we get

$$f_d(x) = \begin{cases} 2 \sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \frac{1}{i} \left(\frac{\Gamma(\frac{d+1}{2})\Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2})\Gamma(\frac{1+i}{2})} \right) \sin^{i-1} x \cos x, & \text{if } d \text{ is odd,} \\ 2 \sum_{\substack{2 \le i \le d \\ i \text{ even}}} \frac{1}{i} \left(\frac{\Gamma(\frac{d+1}{2})\Gamma(1+\frac{i}{2})}{\Gamma(1+\frac{d}{2})\Gamma(\frac{1+i}{2})} \right) \sin^{i-1} x \cos x + \frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})} \sqrt{\pi} \left(1 - \frac{2x}{\pi}\right), & \text{if } d \text{ is even.} \end{cases}$$

First, suppose d is odd. Then, replacing the equation above in the simplified integral gives

$$2 \int_0^\pi \sin^d x \cos x \left(2 \sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)} \frac{1}{i} \sin^{i-1} x \cos x \right) \mathrm{d}x,$$

which simplifies into

$$4 \sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)} \frac{1}{i} \int_0^\pi \sin^{d+i-1} x \cos^2 x \mathrm{d}x.$$

Using Equation 16 gives

$$4 \sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \frac{1}{i} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)} \left(\frac{1}{d+i+1} \right) \frac{\Gamma\left(\frac{d+i}{2}\right)}{\Gamma\left(\frac{d+i+1}{2}\right)} \sqrt{\pi}.$$

Taking $d = 2k-1$, $i = 2j-1$, replacing in the equation above and summing, gives

$$4 \frac{(4k+1)\sqrt{\pi}\Gamma(k)\Gamma(2k)}{\Gamma(2k+\frac{3}{2})\Gamma(1+k)}.$$

Finally, replacing k by $\frac{d+1}{2}$ and simplifying gives

$$2\sqrt{\pi} \frac{\Gamma(d+1)}{\left(\frac{d+1}{2}\right)\Gamma(d+1+\frac{1}{2})}.$$

Now, suppose d is even. Then, again, replacing f_d in the simplified integral gives

$$2 \int_0^\pi \sin^d x \cos x \left(\underbrace{2 \sum_{\substack{2 \le i \le d \\ i \text{ even}}} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)} \frac{1}{i} \sin^{i-1} x \cos x}_{A} + \underbrace{\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})} \sqrt{\pi} \left(1 - \frac{2x}{\pi}\right)}_{B} \right) \mathrm{d}x,$$

We handle the A term first, which is pretty similar to the d odd case. Again, for the A term, expanding the sum and using Equation 16 gives

$$4 \sum_{\substack{2 \le i \le d \\ i \text{ even}}} \frac{1}{i} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)} \left(\frac{1}{d+i+1} \right) \frac{\Gamma\left(\frac{d+i}{2}\right)}{\Gamma\left(\frac{d+i+1}{2}\right)} \sqrt{\pi}.$$

Taking $d = 2k$, $i = 2j$, replacing in the equation above and summing, gives

$$4 \left(\frac{\sqrt{\pi}\Gamma(2k+1)}{(2k+1)\Gamma(2k+\frac{3}{2})} - \frac{2}{(2k+1)^2} \right).$$

Finally, replacing k by $d/2$ and simplifying gives

$$2\sqrt{\pi}\frac{\Gamma(d+1)}{\left(\frac{d+1}{2}\right)\Gamma(d+1+\frac{1}{2})} - 2\frac{1}{\left(\frac{d+1}{2}\right)^2}.$$

Now, we handle the B term, which is

$$2\int_0^\pi \sin^d x \cos x \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(1+d/2)}\right)\sqrt{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x.$$

Expanding it, gives

$$2\int_0^\pi \sin^d x \cos x \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(1+d/2)}\right)\sqrt{\pi}\mathrm{d}x - 2\cdot\frac{2}{\pi}\cdot\int_0^\pi x\sin^d x \cos x \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(1+d/2)}\right)\sqrt{\pi}\mathrm{d}x.$$

The left term is null because of Equation 13. And plugging Equation 19 in the right term above gives

$$2\cdot\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+d/2)}\sqrt{\pi}\cdot\left(\frac{1}{d+1}\right)\cdot\frac{\Gamma(1+d/2)}{\Gamma(1+\frac{d+1}{2})}\cdot\sqrt{\pi}\cdot\frac{2}{\pi} = 2\frac{1}{\left(\frac{d+1}{2}\right)^2}.$$

Summing A term with B term gives

$$2\sqrt{\pi}\frac{\Gamma(d+1)}{\left(\frac{d+1}{2}\right)\Gamma(d+1+\frac{1}{2})}.$$

Therefore, the expression above holds for both odd and even d. Moreover, it can be simplified to the following nicer expression

$$\frac{2^{2d+4}(d!)^2}{(2d+2)!}.$$

□

Proof of Equation (12). Symmetry on the circle gives 0. □

Proof of Equation (13). Symmetry with respect to $\pi/2$ gives 0. □

Proof of Equation (14). As in the proof of Equation 11, we simplify this integral to get rid of the absolute value and isolate each of its variable.

$$\begin{aligned}
\text{Integral} &= 2\iint_{\substack{\alpha_1,\alpha_2\in[0,\pi]^2\\ \alpha_1<\alpha_2}} \sin^d\alpha_1 \sin^d\alpha_2(\cos\alpha_1-\cos\alpha_2)\cos\alpha_1\cos\alpha_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\
&= 2\left(-\int_0^\pi \sin^d\alpha_1\cos\alpha_1\int_{\alpha_1}^\pi \sin^d\alpha_2\cos^2\alpha_2\mathrm{d}\alpha_2\mathrm{d}\alpha_1 + \int_0^\pi \sin^d\alpha_2\cos\alpha_2\int_0^{\alpha_2}\sin^d\alpha_1\cos^2\alpha_1\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right)\\
&= 2\int_0^\pi \sin^d x\cos x\left(-\int_x^\pi \sin^d y\cos^2 y\mathrm{d}y + \int_0^x \sin^d y\cos^2 y\mathrm{d}y\right).\\
&= -2\int_0^\pi \sin^d x\cos x f_d(x)\mathrm{d}x + 2\int_0^\pi \sin^d x\cos x f_{d+2}(x)\mathrm{d}x,
\end{aligned}$$

where f_d is defined in the proof of Equation 11. The first integral is exactly the opposite as the one in Equation 11, the second integral is quite similar, for

d odd it gives:

$$2\int_0^\pi \sin^d x\cos x\left(2\sum_{\substack{1\le i\le d+2\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+3}{2}\right)\Gamma(1+i/2)}{\Gamma(2+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\sin^{i-1}x\cos x\right)\mathrm{d}x$$

$$= 4\sum_{\substack{1\le i\le d+2\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+3}{2}\right)\Gamma(1+i/2)}{\Gamma(2+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\int_0^\pi \sin^{d+i-1}x\cos^2 x$$

$$= 4\sum_{\substack{1\le i\le d+2\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+3}{2}\right)\Gamma(1+i/2)}{\Gamma(2+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\left(\frac{1}{d+i+1}\right)\frac{\Gamma\left(\frac{d+i}{2}\right)}{\Gamma\left(\frac{d+i+1}{2}\right)}\sqrt{\pi}=\frac{\Gamma(d-1)}{\Gamma\left(d+\frac{1}{2}\right)}.$$

Substracting the two integrals yields the claimed result. The case with d even can be solved similarly. □

Proof of Equation (15). Simple integration. □

Proof of Equation (16). Use $\cos^2 x = 1-\sin^2 x$ and Equation 10, then simplify. □

Proof of Equation (17). As in the proof of Equations 11 and 14, we simplify this integral to get rid of the absolute value and isolate each of its variable.

$$\begin{aligned}
\text{Integral} &= 2\iint_{\substack{\alpha_1,\alpha_2\in[0,\pi]^2\\ \alpha_1<\alpha_2}} \sin^d\alpha_1\sin^d\alpha_2(\cos\alpha_1-\cos\alpha_2)\cos^2\alpha_1\cos^2\alpha_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\
&= 2\left(\int_0^\pi \sin^d\alpha_1\cos^3\alpha_1\int_{\alpha_1}^\pi \sin^d\alpha_2\cos^2\alpha_2\mathrm{d}\alpha_2\mathrm{d}\alpha_1-\int_0^\pi\sin^d\alpha_2\cos^3\alpha_2\int_0^{\alpha_2}\sin^d\alpha_1\cos^2\alpha_1\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right)\\
&= 2\int_0^\pi \sin^d x\cos x(1-\sin^2 x)\left(\int_x^\pi \sin^d y(1-\sin^2 y)\mathrm{d}y-\int_0^x\sin^d y(1-\sin^2 y)\mathrm{d}y\right).\\
&= 2\int_0^\pi \sin^d x\cos x f_d(x)\mathrm{d}x-2\int_0^\pi \sin^d x\cos x f_{d+2}(x)\mathrm{d}x\\
&\qquad -2\int_0^\pi \sin^{d+2}x\cos x f_d(x)\mathrm{d}x+2\int_0^\pi\sin^{d+2}x\cos x f_{d+2}(x)\mathrm{d}x,
\end{aligned}$$

where f_d is defined in the proof of Equation 11. The first and fourth integrals are the same as the ones in Equation 11 (with d replaced by $d+2$ in the fourth integral), the second integral is the same as the one in Equation 14, the third integral is quite similar, for d odd the third integral gives:

$$2\int_0^\pi \sin^{d+2}x\cos x\left(2\sum_{\substack{1\le i\le d\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\sin^{i-1}x\cos x\right)\mathrm{d}x$$

$$= 4\sum_{\substack{1\le i\le d\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\int_0^\pi\sin^{d+i+1}x\cos^2 x$$

$$= 4\sum_{\substack{1\le i\le d\\ i\text{ odd}}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(1+i/2)}{\Gamma(1+d/2)\Gamma\left(\frac{i+1}{2}\right)}\frac{1}{i}\left(\frac{1}{d+i+3}\right)\frac{\Gamma\left(\frac{d+i+2}{2}\right)}{\Gamma\left(\frac{d+i+3}{2}\right)}\sqrt{\pi}$$

$$= \frac{4\,\Gamma(d+3)}{(d+3)\Gamma\left(d+\frac{7}{2}\right)}.$$

Adding the four integrals yields the claimed result. The case with d even can be solved similarly. □