# A New Road Towards Universal Logic?

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## Abstract

A generic logic called ‘Gaggle logic’ is introduced. It is based on Gaggle theory and deals with connectives of arbitrary arity that are related to each other by abstract laws of residuation. We list the 96 binary connectives and the 16 unary connectives of Gaggle logic. We provide a sound and complete calculus for Gaggle logic which enjoys strong cut elimination and the display property. We show that Gaggle logic is decidable and satisfies the properties of conservativity and interpolation. We also introduce specific inference rules called ‘protoanalytic’ inference rules. These rules are such that, when added to the calculus of Gaggle logic, we obtain a calculus which still enjoys strong cut elimination and the display property. If the language considered contains conjunction and disjunction, then the interpolation theorem also transfers to these extensions of Gaggle logic. In a second part of the report, we generalize the Kracht’s correspondence results established for the basic tense logic to Gaggle logic. We prove that a logic extending Gaggle logic is axiomatizable by means of so-called ‘protoanalytic’ inference rules if, and only if, the class of frames on which such a logic is based is definable by specific first-order frame conditions, also called ‘protoanalytic’. We provide algorithms that compute the corresponding protoanalytic inference rules from the protoanalytic first-order frame conditions, and vice versa. We illustrate these algorithms on well-known structural inference rules and we show in particular how we can recover classical logic from Gaggle logic by the addition of protoanalytic inference rules that refine the standard classical inference rules.

# Contents

# Part I

# Gaggle logic: the Logic of Gaggle Theory

# 1 Introduction

A wide variety of non–classical logics have been introduced over the past decades, such as relevant logics, linear logics and Lambek calculi, to name just a few. On the one hand, this diversity is an asset since each logic has an interest for a specific purpose, and one can pick and resort to some of them for reasoning about a given applicative issue [42]. In fact, many of these non–classical logics have been developed for solving concrete problems in computer science: for example, dynamic logics [28], Hoare and separation logics [29, 49] for reasoning about computer programs, and description logics [2] for formalizing ontologies of the semantic web. Acknowledging and dealing with this plurality and diversity of logics is in a sense at the origin of the development of a philosophical stance in logic called "logical pluralism" [4]. On the other hand, and from a theoretical point of view, this plurality can be felt as problematic because it threatens the unity and the unifying power of logic. Indeed, all logics already have in common to share the same terminology and notions, such as truth, validity, conservativity or interpolation for example, and this is also an asset. Nevertheless, one could argue that non–classical logics are still disorganized and scattered and somehow miss a common formal ground.

In response to that situation, a number of efforts have been made by some logicians to provide a genuine unity to logic (see [52] for instance). This lead to the rise of a research thread sometimes referred to as "Universal Logic". Many kinds of semantics, such as algebraic, categorial, pretopological and relational semantics have been introduced, sometimes only in order to tackle this issue. Within that line of research, Dunn's Gaggle theory [17, 18, 6] is one of the most well–known frameworks based on the relational semantics that deals with that problem. In a nutshell, Dunn's Gaggle Theory is an attempt to understand the Kripke semantics of non-classical logics in a disciplined, systematic way. From another perspective, as we shall see, one could present Gaggle theory as providing formal methods to generate systematically the relational semantics of any non–classical logic, when such a semantics exists.

We share the ideal and the objective of "Universal Logic", but in our view Gaggle theory is only a first step. Indeed, that theory does not really introduce an actual logic or logical framework that could serve as a foundation for non–classical logics, just as the Lambek calculus is sometimes presented as the foundational logic of the varied substructural logics [48]. However, as we will show, Gaggle theory provides formal methods to define such a generic logic. In fact, it allows us to define a logic that can handle connectives of arbitrary arity. Building on (partial) Gaggle theory, we define a generic and abstract logic called 'Gaggle logic' that generalizes the Lambek calculus and other logics in many directions.

The connectives of Gaggle logic are of arbitrary arity and they are related to each other by abstract laws of residuation. As it turns out, residuations are also at the heart of display calculi and strong connections between Gaggle theory and display calculi [5] have already been identified [47, 25]. We push these connections further and we introduce the notions of 'protodisplayability' and 'protoanalytic' inference rules which are generalizations in our logical framework of the notions of 'displayability' and 'analyticity' of inference rules for display calculi. Our notions are set in such a way that we obtain similar meta–theoretical properties as display calculi such as (strong) cut elimination and displayability, but also, as we shall see, interpolation.

*Organization of Part 1.* In Section 2, we recall the basic results of (partial) Gaggle theory. In Section 3.1, we show how to define logical functions providing semantics to connectives from the results of (partial) Gaggle theory. This leads us in Section 3.2 to introduce 'Gaggle logic'. We then list in Section 3.3 and 3.4 the 96 binary connectives and the 16 unary connectives of Gaggle logic. In Section 4, we introduce a calculus which is sound and complete for Gaggle logic and we prove the completeness of this calculus. In Section 5.1, we recall the definition of the well-studied basic tense logic and we show in Section 5.2 and 5.3 that it can serve as a 'Lingua Franca' for reformulating and expressing the various connectives of Gaggle logic. In Section 10.1, we introduce our notion of 'protoanalytic' inference rule that somehow generalizes the display inference rules of display calculi. Then, thanks to our embedding of Gaggle logic into the basic tense logic, we show in Section 6.2 that Gaggle logic enjoys the (strong) cut elimination and satisfies the display property as well. As a consequence of the cut elimination, we prove in Section 6.3 the decidability of the validity problem of Gaggle logic and in Section 6.4 that Gaggle logic and all the other logics extending Gaggle logic with protoanalytic inference rules satisfy the interpolation property (if they contain the classical conjunction and disjunction).

# 2 The Core of Partial Gaggle Theory

For our purpose, the presentation of (partial) Gaggle theory is slightly different from the usual presentation of this theory. The definitions are the same (altough they are sometimes instantiated) but the results of this theory are differently set. Our results can nevertheless easily be obtained from the original presentation [18].

In this section, we consider given an integer $n \in \mathbb{N}$ and a non-empty set $W$. A *$n$–ary function $f$ on $\mathcal{P}(W)$* is a function $f : \mathcal{P}(W)^n \to \mathcal{P}(W)$ and a *$n$–ary relation $R$ over $W$* is a subset of $W^n$. We write $Rw_1 \dots w_n$ for $(w_1, \dots, w_n) \in R$. For all $m, n \in \mathbb{N}$, the expression $[\![m; n]\!]$ denotes the set $\{m, \dots, n\}$ if $m \leq n$, and the empty set $\emptyset$ otherwise.

**Definition 1** (Polarity monoid)**.** Let $(x, y)$ be an ordered pair. The *polarity monoid associated to* $(x, y)$ is the monoid $(\{x, y\}, \cdot)$ such that $x \cdot y = y \cdot x = y$ and $x \cdot x = y \cdot y = x$. ⊣

In the sequel, $(+, -)$ and $(\rightarrow, \leftarrow)$ are two ordered pairs. We will resort to their associated polarity monoids and for all $\pm, \pm' \in \{+, -\}$ (resp. $\leftrightarrow, \leftrightarrow' \in \{\rightarrow, \leftarrow\}$), we write $\pm\pm'$ for $\pm \cdot \pm'$ (resp. $\overset{\leftrightarrow'}{\leftrightarrow}$ for $\leftrightarrow \cdot \leftrightarrow'$).

**Definition 2** (Trace)**.** A *($n$–ary) trace* is a tuple of $\{+, -\}^{n+1}$. A trace $t$ is often denoted $t = (\pm_1, \dots, \pm_n) \mapsto \pm$ (with $\pm_1, \dots, \pm_n, \pm \in \{+, -\}$) and $\pm$ is called the *output* of the trace, denoted $out(t)$. We say that two traces of the same arity *agree* if they have the same output symbol.

Let $t, t'$ be two $n$–ary traces. We say that $t$ and $t'$ are *contrapositive* (with respect to their $j^{th}$ argument) when, if $t = (\pm_1, \dots, \pm_j, \dots, \pm_n) \mapsto \pm$ then $t' = (\pm_1, \dots, -\pm, \dots, \pm_n) \mapsto -\pm_j$. In that case, we write $t' = t^{-j}$. A *sequence of contrapositive $n$–ary traces* is a tuple of $n$–ary traces $(t_1, \dots, t_k)$ such that for all $i \in [\![1, k-1]\!]$, $t_i$ and $t_{i+1}$ are contrapositive. ⊣

Note that $t$ and $t'$ are contrapositive traces if, and only if, $t'$ and $t$ are contrapositive traces.

**Definition 3** (Relation transformations)**.** Let $R$ be an arbitrary $n + 1$–ary relation over $W$. Then, for all $j \in \{1, \dots, n\}$, we define the $n + 1$–ary relations $R^{-j}$ and $\overleftarrow{R}$ as follows: for all $w_1, \dots, w_n, w \in W$,

$$R^{-j} w_1 \dots w_n w \text{ iff } Rw_1 \dots w \dots w_n w_j$$
$$\overleftarrow{R} w_1 \dots w_n w \text{ iff } (w_1, \dots, w_n, w) \in W^{n+1} \backslash R$$

Moreover, we define $\overrightarrow{R} \triangleq R$ and $R^{\leftrightarrow,\sigma}$ denotes $\overset{\leftrightarrow}{R}{}^{\sigma}$ (where $\leftrightarrow \in \{\rightarrow, \leftarrow\}$). If $t$ and $t'$ are contrapositive $n$–ary traces (w.r.t. their $j^{th}$ argument), we define the $n+1$–ary relation $(t', t)(R)$

over $W$ as follows:

$$(t',t)(R) \triangleq \begin{cases} R^{-j} & \text{if } t \text{ and } t' \text{ agree;} \\ \overset{\leftarrow -j}{R} & \text{otherwise.} \end{cases}$$

$\mathfrak{S}_{n+1}$ denotes the set of permutations of the set $[\![1;n+1]\!]$. Let $\sigma, \tau \in \mathfrak{S}_{n+1}$ be such that $\tau$ is the inverse permutation of $\sigma$. We define the $n+1$–ary relation $R^{\sigma}$ as follows: for all $w_1, \ldots, w_{n+1} \in W$,

$$R^{\sigma} w_1 \ldots w_{n+1} \text{ iff } R w_{\tau(1)} \ldots w_{\tau(n+1)} \qquad \dashv$$

**Definition 4** (Functions associated to a trace and a relation)**.** Let $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ be a $n$–ary trace and let $R$ be a $n+1$–ary relation on $W$. We define the *$n$–ary function $f$ on $\mathcal{P}(W)$ associated to $t$ and $R$* as follows:

- If $n = 0$, $f \triangleq R$;
- If $n > 0$, then for all $W_1, \ldots, W_n \in \mathcal{P}(W)$,

$$f(W_1, \ldots, W_n) \triangleq \{w \in W \mid \mathcal{E}_t(W_1, \ldots, W_n, w, R)\}$$

where the expression $\mathcal{E}_t(W_1, \ldots, W_n, w, R)$ is:

- if $\pm = +$: "for all $w_1, \ldots, w_n \in W$, we have $w_1 \pitchfork W_1$ or … or $w_n \pitchfork W_n$ or $Rw_1 \ldots w_n w$";
- if $\pm = -$: "there are $w_1, \ldots, w_n \in W$ such that $w_1 \pitchfork W_1$ and … and $w_n \pitchfork W_n$ and $Rw_1 \ldots w_n w$";

where, if $j \in [\![1;n]\!]$, $w_j \pitchfork W_j \triangleq \begin{cases} w_j \in W_j & \text{if } \pm_j \pm = +; \\ w_j \notin W_j & \text{if } \pm_j \pm = -. \end{cases}$ $\dashv$

**Definition 5** (Isotonic and antitonic functions)**.** Let $f$ be a $n$–ary function on $\mathcal{P}(W)$. We say that $f$ is *isotonic (resp. antitonic) with respect to the $j^{th}$ argument*, written $tn(f,j) = +$ (resp. $tn(f,j) = -$), when for all $W_1, \ldots, W_{j-1}, W_{j+1}, \ldots, W_n, X, Y \in \mathcal{P}(W)$,

$$\text{if } X \subseteq Y \text{ then } f(W_1, \ldots, W_{j-1}, X, W_{j+1}, \ldots, W_n) \subseteq f(W_1, \ldots, W_{j-1}, Y, W_{j+1}, \ldots, W_n)$$
$$\text{resp. } f(W_1, \ldots, W_{j-1}, Y, W_{j+1}, \ldots, W_n) \subseteq f(W_1, \ldots, W_{j-1}, X, W_{j+1}, \ldots, W_n).$$

$\dashv$

**Theorem 1.** *Let $R$ be a $n+1$–ary relation over $W$. Let $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ and $t' = (\pm'_1, \ldots, \pm'_n) \mapsto \pm'$ be two contrapositive $n$–ary traces w.r.t. their $j^{th}$ argument. Let $f$ (resp. $f'$) be the $n$–ary function on $\mathcal{P}(W)$ associated to $t$ and $R$ (resp. associated to $t'$ and $(t',t)(R)$). Then, if $n > 0$:*

- *for all $j \in [\![1;n]\!]$, $tn(f,j) = \pm_j \pm$ (and thus $tn(f',j) = \pm'_j \pm'$ too);*
- *$f$ and $f'$ satisfy the* abstract law of residuation w.r.t. their $j^{th}$ argument*: for all $W_1, \ldots, W_n, X \in \mathcal{P}(W)$,*

$$S(f, W_1, \ldots, W_j, \ldots, W_n, X) \quad \textit{iff} \quad S(f', W_1, \ldots, X, \ldots, W_n, W_j).$$

*where* $S(f, W_1, \ldots, W_n, X) \triangleq \begin{cases} f(W_1, \ldots, W_n) \subseteq X & \textit{if } \pm = - \\ X \subseteq f(W_1, \ldots, W_n) & \textit{if } \pm = +. \end{cases}$

# 3 From Partial Gaggles to Gaggle logic

In this section, we show how Gaggle theory, and in particular Definition 4, leads to the definition of finite families of logical connectives of arbitrary arities which are related to each other by the abstract laws of residuation of Theorem 1.

## 3.1 Logical Functions

We consider given a fixed $n+1$–ary relation $R$ on $W$.

**Definition 6** (Generated logical function)**.** A *generated $n$–ary logical function* is a $n$–ary function $f$ on $\mathcal{P}(W)$ for which there exist $k \in \mathbb{N}$ and a sequence of contrapositive $n$–ary traces $(t_0, \ldots, t_k)$ such that $f$ is the function associated to $t_k$ and $(t_k, t_{k-1})(\ldots(t_1, t_0))(R)$ or $(t_k, t_{k-1})(\ldots(t_1, t_0))(\overleftarrow{R})$. $\dashv$

**Definition 7** (Logical function)**.** A *$n$–ary logical function* is a $n$–ary function on $\mathcal{P}(W)$ associated to a $n$–ary trace $t$ and a $n+1$–ary relation $R^\sigma$ or $\overleftarrow{R}^\sigma$ over $W$, for some $\sigma \in \mathfrak{S}_{n+1}$. It is denoted $f_R^{\sigma,t}$ or $f_{\overleftarrow{R}}^{\sigma,t}$ respectively. $\dashv$

**Lemma 1.** *Every permutation $\sigma \in \mathfrak{S}_{n+1}$ can be decomposed into a finite number of transpositions of the form $(n+1\ j)$. Moreover, if $\sigma = (n+1\ j_1)\ldots(n+1\ j_k) = (n+1\ i_1)\ldots(n+1\ i_l)$, then for all $n$–ary traces $t$, $\left(\left(t^{-j_k}\right)^{-j_{k-1}}\ldots\right)^{-j_1} = \left(\left(t^{-i_l}\right)^{-i_{l-1}}\ldots\right)^{-i_1}$. This defines the $n$–ary trace $\sigma^-(t)$.*

*Proof.* It is well-known in algebra that every permutation $\sigma \in \mathfrak{S}_{n+1}$ can be decomposed into a finite number of transpositions $\sigma = (i_1\ j_1)\ldots(i_m\ j_m)$. Moreover, every transposition $(i\ j)$ is itself decomposable into a triple of transpositions of the form $(n+1\ i)(n+1\ j)(n+1\ i)$. Hence, every permutation $\sigma \in \mathfrak{S}_{n+1}$ can be decomposed into a sequence of transpositions of the form $(n+1\ j)$. The second part of the lemma follows from the fact that the trace that we obtain is the same whether we decompose $(i\ j)$ into $(n+1\ i)(n+1\ j)(n+1\ i)$ or into $(n+1\ j)(n+1\ i)(n+1\ j)$. $\square$

**Theorem 2.** *Every generated logical function is a logical function, and vice versa. The number of $n$–ary logical functions is $(n+1)! \cdot 2^{n+2}$.*

*Proof.* The right to left direction is due to Lemma 1. It follows from this lemma that for all $\sigma \in \mathfrak{S}_{n+1}$, there is $k \in \mathbb{N}$ and a sequence $(j_1, \ldots, j_k) \in [\![1; n]\!]^k$ such that $R^\sigma = R^{-j_1} \circ \ldots \circ R^{-j_k}$. For the left to right direction, it suffices to observe that for all $k \in \mathbb{N}$ and all sequences $(j_1, \ldots, j_k) \in [\![1; n]\!]^k$, if we consider the permutation $\sigma \triangleq (n+1\ j_1)(n+1\ j_2)\ldots(n+1\ j_k)$, then $R^\sigma = R^{-j_1} \circ \ldots \circ R^{-j_k}$. $\square$

## 3.2 Gaggle logic

Our introduction of ‘Gaggle logic’, like many semantic-based logics, is made in three parts: first, we define its language (Definition 9), then its class of models (Definition 11) and finally its satisfaction relation (Definition 12).

**Definition 8** (Propositional letters, logical and structural connectives)**.** The set of *basic propositional letters* $\mathbb{P}_0$ and *basic logical connectives* $\mathbb{C}_0$ (with $\mathbb{P}_0 \subseteq \mathbb{C}_0$) are:

$$\mathbb{P}_0 \triangleq \mathfrak{S}_1 \times \{+,-\} \times \{\leftarrow, \rightarrow\} \qquad \mathbb{C}_0 \triangleq \bigcup_{n \in \mathbb{N}^*} \{\mathfrak{S}_n \times \{+,-\}^n \times \{\leftarrow, \rightarrow\}\}.$$

We consider a countable number of copies of the basic connective and basic propositional letters:

$$\mathbb{P} \triangleq \bigcup_{\star \in \mathbb{P}_0} \{\star_i \mid i \in \mathbb{N}\} \qquad\qquad \mathbb{C} \triangleq \bigcup_{\star \in \mathbb{C}_0} \{\star_i \mid i \in \mathbb{N}\} .$$

$\mathbb{P}$ is called the set of *propositional letters* and $\mathbb{C}$ is called the set of *logical connectives* ($\mathbb{P} \subseteq \mathbb{C}$).

The set of *structural connectives*, denoted $[\mathbb{C}]$ and the *set of structural connectives associated to* $\mathsf{C} \subseteq \mathbb{C}$, denoted $[\mathsf{C}]$, are defined as follows:

$$[\mathbb{C}] \triangleq \{[\star] \mid \star \in \mathbb{C}\} \qquad\qquad [\mathsf{C}] \triangleq \bigcup_{\star_0 \in \mathsf{C}} \{[\star] \mid \star \in \mathbb{C} \text{ and } \mathsf{a}(\star) = \mathsf{a}(\star_0)\} .$$

Note that the structural connectives $[\mathbb{C}]$ are simply a copy of the logical connectives $\mathbb{C}$.

Propositional connectives are denoted $p, p_1, p_2, \ldots$, logical connectives are denoted $\star, \star_1, \star_2, \ldots$ and structural connectives are denoted $[p], [p_1], [p_2], \ldots$ and $[\star], [\star_1], [\star_2], \ldots$ For all $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ with $t \in \{+, -\}^{n+1}$, the *arity* of both $\star$ and $[\star]$, denoted $\mathsf{a}(\star)$ and $\mathsf{a}([\star])$ respectively, is $n$, the *trace* of both $\star$ and $[\star]$ is $t$ and the *signature* of both $\star$ and $[\star]$ is $(\sigma, t, \leftrightarrow)$. Except for Definition 11, *all the sets of logical connectives* $\mathsf{C} \subseteq \mathbb{C}$ *will be such that* $\mathsf{C} \cap \mathbb{P} \neq \emptyset$. $\dashv$

We introduce copies of the basic connectives because in general we need a countable number of propositional letters or, like in some modal logics, we need multiple modalities of the same type. The 0–ary connectives can be seen as the propositional letters of propositional or modal logic. They have a special status and their traces will not matter since the tonicity conditions do not apply for 0–ary connectives.

**Definition 9** (Protolanguage, structures and consecutions)**.** The *protolanguage* $\mathcal{L}$ is the smallest set that contains the propositional letters and that is closed under the other connectives. That is,

- $\mathbb{P} \subseteq \mathcal{L}$;
- for all $\star \in \mathbb{C}$ of arity $n > 0$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$, we have $\star(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}$.

The *structural protolanguage* $[\mathcal{L}]$ is the smallest set that contains the protolanguage $\mathcal{L}$ as well as $[\mathbb{P}]$ and that is closed under the structural connectives of $[\mathbb{C}]$.

A $\mathcal{L}$*–consecution* (resp. $[\mathcal{L}]$*–consecution*) is an expression of the form $\varphi \vdash \psi$ (resp. $X \vdash Y$), where $\varphi, \psi \in \mathcal{L}$ (resp. $X, Y \in [\mathcal{L}]$). The set of all $\mathcal{L}$–consecutions (resp. $[\mathcal{L}]$–consecutions) is denoted $\mathcal{S}$ (resp. $[\mathcal{S}]$).

If $\mathsf{C} \subseteq \mathbb{C}$ then an element of $\mathcal{L}_{\mathsf{C}}$ (resp. $[\mathcal{L}]_{\mathsf{C}}$, $\mathcal{S}_{\mathsf{C}}$, $[\mathcal{S}]_{\mathsf{C}}$) is an element of $\mathcal{L}$ (resp. $[\mathcal{L}]$, $\mathcal{S}$, $[\mathcal{S}]$) that contains only connectives of $[\mathsf{C}]$. Elements of $\mathcal{L}$ (resp. $[\mathcal{L}]$ and $[\mathcal{S}]$) are called *formulas* (resp. *structures* and *consecutions*); they are denoted $\varphi, \psi, \alpha, \ldots$ (resp. $X, Y, A, B, \ldots$ and $X \vdash Y, A \vdash B, \ldots$). $\dashv$

**Definition 10** (Formula associated to a structure)**.** The *formula of* $\mathcal{L}$ *associated to a structure* $X \in [\mathcal{L}]$, denoted $\tau(X)$, is defined inductively as follows:

$$\begin{aligned} \tau(\varphi) &\triangleq \varphi \\ \tau([\star](X_1, \ldots, X_n)) &\triangleq \star(\tau(X_1), \ldots, \tau(X_n)) \end{aligned}$$

This definition is extended to $[\mathcal{L}]_{\mathsf{C}}$–consecution as follows:

$$\tau\left(X \vdash Y\right) \triangleq \tau(X) \vdash \tau(Y) \qquad \dashv$$

**Definition 11** (C–models and C–frames)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ (possibly with $\mathsf{C} \cap \mathbb{P} = \emptyset$). A *C–model* is a tuple $M = (W, \sqsubseteq, \mathcal{R})$ such that $W$ is a non-empty set, $\sqsubseteq$ is a partial order over $W$ and $\mathcal{R}$ is a set of $n$–ary relations over $W$ such that

1. each connective $\star$ of $\mathsf{C}$ of arity $n$ can be associated to a relation of arity $n+1$, denoted $R_\star$, and vice versa;
2. two connectives $\star_1 = (\sigma_1, t_1, \leftrightarrow_1)_{i_1}$ and $\star_2 = (\sigma_2, t_2, \leftrightarrow_2)_{i_2}$ of $\mathsf{C}$ are such that $R_{\star_1} = R_{\star_2}$ if, and only if, $i_1 = i_2$, $\sigma_1^-(t_1) = \sigma_2^-(t_2)$ and $\leftrightarrow_1 = \leftrightarrow_2$;
3. for all $R \in \mathcal{R}$ and all $\star \in \mathsf{C}$ whose signature is $(\sigma, t, \leftrightarrow)$ and such that $R = R_\star$, if $(\pm_1, \ldots, \pm_n) \mapsto \pm$ is the trace $\sigma^-(t)$ then for all $w_1, \ldots, w_n, w_1', \ldots, w_n', v, v' \in W$,

$$\text{if } \overset{\leftrightarrow}{R} w_1 \ldots w_n v, w_1 \square w_1', \ldots, w_n \square w_n' \text{ and } v \sqsubseteq v' \text{ then } \overset{\leftrightarrow}{R} w_1' \ldots w_n' v' \qquad \text{(Tonicity)}$$

where, for all $i \in [\![1; n]\!]$, $w_i \square w_i' \triangleq \begin{cases} w_i \sqsubseteq w_i' & \text{if } \pm_i \pm = -; \\ w_i \sqsupseteq w_i' & \text{if } \pm_i \pm = +. \end{cases}$

We abusively write $w \in M$ for $w \in W$. A *pointed C–model* $(M, w)$ is a C–model $M$ together with a state $w \in M$. The class of all pointed C–models is denoted $\mathcal{M}_\mathsf{C}$ and the class of all pointed $\mathbb{C}$–models is denoted $\mathcal{M}$.

A *C–frame* is a $\mathsf{C} \backslash \mathbb{P}$–model. The class of all pointed $\mathbb{C}$–frames is denoted $\mathcal{F}$. $\dashv$

In case $\star \in \mathbb{P}$ has signature $(Id, \pm, \leftrightarrow)$ with $\leftrightarrow = \rightarrow$ and if $P$ denotes $R_\star$ then the (Tonicity) condition simply rewrites as follows:

$$\text{if } P(w) \text{ and } w \sqsubseteq w' \text{ then } P(w') \qquad \text{(Heredity)}$$

Note also that, in case $\leftrightarrow = \leftarrow$, the (Tonicity) condition rewrites as follows:

$$\text{if } R w_1' \ldots w_n' v', w_1 \square w_1', \ldots, w_n \square w_n' \text{ and } v \sqsubseteq v' \text{ then } R w_1 \ldots w_n v$$

**Definition 12** (Gaggle logic)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M = (W, \mathcal{R})$ be a C–model. We define the *extension function of* $[\mathcal{L}]_\mathsf{C}$ *in* $M$, denoted $[\![\cdot]\!]^M : [\mathcal{L}]_\mathsf{C} \mapsto \mathcal{P}(W)$, inductively as follows: for all $p \in \mathsf{C} \cap \mathbb{P}$, all $\star \in \mathsf{C}$ of arity $n > 0$, both of signature denoted $(\sigma, t, \leftrightarrow)$, for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_\mathsf{C}$ and all $X_1, \ldots, X_n \in [\mathcal{L}]_\mathsf{C}$,

$$\begin{array}{rcl} [\![p]\!]^M & \triangleq & \overset{\leftrightarrow}{R}_p \\ [\![\star(\varphi_1, \ldots, \varphi_n)]\!]^M & \triangleq & f^{\sigma,t}_{\overset{\leftrightarrow}{R}_\star}([\![\varphi_1]\!]^M, \ldots, [\![\varphi_n]\!]^M) \\ [\![[p]]\!]^M & \triangleq & \overset{\leftrightarrow}{R}_p \\ [\![[\star](X_1, \ldots, X_n)]\!]^M & \triangleq & f^{\sigma,t}_{\overset{\leftrightarrow}{R}_\star}([\![X_1]\!]^M, \ldots, [\![X_n]\!]^M) \end{array}$$

We extend the definition of the extension function $[\![\cdot]\!]^M$ to C–frames as follows: for all $X \in [\mathcal{L}]_\mathsf{C}$ and all C–frames $F$,

$$[\![X]\!]^F \triangleq \bigcap_{\mathcal{P}} \left\{ [\![X]\!]^{(F,\mathcal{P})} \mid (F, \mathcal{P}) \text{ is a C–model} \right\}$$

If $\mathcal{E}_0$ is a class of C–models or C–frames, we define the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_0 \times [\mathcal{S}]_\mathsf{C}$ as follows. First, for all $X \in [\mathcal{L}]_\mathsf{C}$ and all $(M, w) \in \mathcal{E}_0$, we set $(M, w) \Vdash X$ if, and only if, $w \in [\![X]\!]^M$. Second, we extend the satisfaction relation $\Vdash$ to $[\mathcal{L}]_\mathsf{C}$–consecutions as follows:

for all $X, Y \in [\mathcal{L}]_{\mathsf{C}}$ and all $(M, w) \in \mathcal{M}$, we set $(M, w) \Vdash X \vdash Y$ iff $(M, w) \Vdash X$ implies that $(M, w) \Vdash Y$. We also write $M \Vdash X \vdash Y$ when for all $w \in M$, we have that $(M, w) \Vdash X \vdash Y$.

The triple $([\mathcal{S}]_{\mathsf{C}}, \mathcal{E}_0, \Vdash)$ is a logic called the *Gaggle logic associated to $\mathcal{E}_0$ and $\mathsf{C}$* and $([\mathcal{S}], \mathcal{M}, \Vdash)$ is simply called *Gaggle logic*. ⊣

The relation $\sqsubseteq$ was originally introduced by Dunn [18] in order to deal with neutral elements of the algebraic approach. Intuitively, we want to set $w \sqsubseteq v$ when all the formulas which are true at $w$ are also true at $v$. Our definitions of $\mathsf{C}$–models are such that this property, often called 'Heredity', indeed holds in Gaggle logic.

**Proposition 1** (Heredity)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M$ be a $\mathsf{C}$–model. Then, for all $w, v \in M$ and all $X \in [\mathcal{L}]_{\mathsf{C}}$, if $w \in [\![X]\!]^M$ and $w \sqsubseteq v$ then $v \in [\![X]\!]^M$.*

Finally, note that Boolean negation is not among the 16 unary connectives introduced in Figure 2. However, the Boolean negation of any formula can be defined systematically on a case by case basis, as follows.

**Definition 13** (Boolean negation)**.** Let $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ be a logical connective whose trace is $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$. The *Boolean negation* of $\star$ is the logical connective $\overline{\star} \triangleq (\sigma, \overline{t}, \overleftrightarrow{\leftrightarrow})_i \in \mathbb{C}$, where $\overline{t} = (\pm_1, \ldots, \pm_n) \mapsto -\pm$. If $X \in [\mathcal{L}]$, the *Boolean negation* of $X$, denoted $\overline{X}$, is defined by:

$$\overline{X} \triangleq \begin{cases} \overline{p} & \text{if } X = p; \\ [\overline{p}] & \text{if } X = [p]; \\ \overline{\star}(\varphi_1, \ldots, \varphi_n) & \text{if } X = \star(\varphi_1, \ldots, \varphi_n); \\ [\overline{\star}]\,(X_1, \ldots, X_n) & \text{if } X = [\star]\,(X_1, \ldots, X_n). \end{cases}$$ ⊣

**Proposition 2.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M$ be a $\mathsf{C}$–model. Then, for all $w \in W$ and all $X \in [\mathcal{L}]_{\mathsf{C}}$,*

$$w \in [\![\overline{X}]\!]^M \textit{ iff } w \notin [\![X]\!]^M$$

### 3.3 Binary Connectives of Gaggle logic

The truth conditions of the 2–ary basic logical connectives of Gaggle logic are given in Figures 3, 4 and 5. In fact, it is only half of all the binary basic logical connectives of Gaggle logic. We obtain the other half by replacing $R$ by $\overleftarrow{R}$ and $\overleftarrow{R}$ by $R$ everywhere. All in all, this yields $96 = (2+1)! \cdot 2^{(2+2)}$ binary basic logical connectives (as predicted by Theorem 2).

Many of these binary connectives have already been introduced in the literature [26, 48, 1]. For example, $(\sigma_1, t_1, \rightarrow)$, $(\sigma_5, t_3, \leftarrow)$ and $(\sigma_3, t_2, \leftarrow)$ correspond to the fusion $\circ$, implication $\backslash$ and co-implication $/$ connectives of the Lambek calculus [36] and the heads of the 'stroke' and 'dagger' family, $(\sigma_1, t_7, \rightarrow)$ and $(\sigma_1, t_8, \rightarrow)$, correspond to Sheffer's stroke and dagger [26] (the substructural analogues of the classical NAND and NOR). Other examples of well-known modal and substructural connectives are provided in Figure 13.

### 3.4 Unary Connectives of Gaggle logic

The truth conditions of the 1–ary basic logical connectives of Gaggle logic are given in Figure 7. In fact, it is only half of all the unary basic logical connectives of Gaggle logic. We obtain the other half by replacing $R$ by $\overleftarrow{R}$ and $\overleftarrow{R}$ by $R$ everywhere. All in all, this yields $16 = (1 + 1)! \cdot 2^{(1+2)}$ unary basic logical connectives (as predicted by Theorem 2).

$$
\begin{tikzcd}
 & (-,+) \mapsto + \arrow[r, dash, "\text{-1}"] & (-,+) \mapsto + \arrow[dr, dash, "\text{-2}"] & \\
(-,-) \mapsto - \arrow[ur, dash, "\text{-2}"] \arrow[dr, dash, "\text{-1}"'] & & & (-,-) \mapsto - \\
 & (+,-) \mapsto + \arrow[r, dash, "\text{-2}"'] & (+,-) \mapsto + \arrow[ur, dash, "\text{-1}"'] &
\end{tikzcd}
$$

Figure 1: The founded family of 2–ary traces whose head is $(-,-) \mapsto -$ and their contrapositive relationships

| Permutations of $\mathfrak{S}_3$ | Families of 2–ary traces |
|---|---|
| $\sigma_1 = (1,2,3) = Id$ | $t_1 = (-,-) \mapsto - = t_2^{-1} = t_3^{-2}$ |
| $\sigma_2 = (3,2,1) = (1\ 3) \circ \sigma_1$ | $t_2 = (+,-) \mapsto + = t_1^{-1} = t_2^{-2}$ |
| $\sigma_3 = (3,1,2) = (2\ 3) \circ \sigma_2$ | $t_3 = (-,+) \mapsto + = t_1^{-2} = t_3^{-1}$ |
| $\sigma_4 = (2,1,3) = (1\ 3) \circ \sigma_3$ | $t_4 = (+,+) \mapsto + = t_5^{-1} = t_6^{-2}$ |
| $\sigma_5 = (2,3,1) = (2\ 3) \circ \sigma_4$ | $t_5 = (-,+) \mapsto - = t_4^{-1} = t_5^{-2}$ |
| $\sigma_6 = (1,3,2) = (1\ 3) \circ \sigma_5$ | $t_6 = (+,-) \mapsto - = t_4^{-2} = t_6^{-1}$ |
| | $t_7 = (+,+) \mapsto - = t_7^{-1} = t_7^{-2}$ |
| | $t_8 = (-,-) \mapsto + = t_8^{-1} = t_8^{-2}$ |

Figure 2: Permutations of $\mathfrak{S}_3$ and families of 2–ary traces

| Connective | Truth condition |
|---|---|
| The ‘conjunction’ founded family | |
| $\varphi\ (\sigma_1, t_1, \rightarrow)\ \psi$ | $\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruvw)$ |
| $\varphi\ (\sigma_2, t_2, \leftarrow)\ \psi$ | $\forall uv\left(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee \overleftarrow{R}wvu\right)$ |
| $\varphi\ (\sigma_3, t_2, \leftarrow)\ \psi$ | $\forall uv\left(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee \overleftarrow{R}vwu\right)$ |
| $\varphi\ (\sigma_4, t_1, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_1, t_1, \rightarrow)\ \varphi)$ | $\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rvuw)$ |
| $\varphi\ (\sigma_5, t_3, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_2, t_2, \leftarrow)\ \varphi)$ | $\forall uv\left(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee \overleftarrow{R}wuv\right)$ |
| $\varphi\ (\sigma_6, t_3, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_3, t_2, \leftarrow)\ \varphi)$ | $\forall uv\left(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee \overleftarrow{R}uwv\right)$ |
| The ‘not but’ founded family | |
| $\varphi\ (\sigma_1, t_6, \rightarrow)\ \psi$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruvw)$ |
| $\varphi\ (\sigma_2, t_6, \rightarrow)\ \psi$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwvu)$ |
| $\varphi\ (\sigma_3, t_4, \leftarrow)\ \psi$ | $\forall uv\left(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee \overleftarrow{R}vwu\right)$ |
| $\varphi\ (\sigma_4, t_5, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_1, t_6, \rightarrow)\ \varphi)$ | $\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvuw)$ |
| $\varphi\ (\sigma_5, t_5, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_2, t_6, \rightarrow)\ \varphi)$ | $\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwuv)$ |
| $\varphi\ (\sigma_6, t_4, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_3, t_4, \leftarrow)\ \varphi)$ | $\forall uv\left(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee \overleftarrow{R}uwv\right)$ |
| The ‘but not’ founded family | |
| $\varphi\ (\sigma_1, t_5, 1)\ \psi$ | $\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Ruvw)$ |
| $\varphi\ (\sigma_2, t_4, \leftarrow)\ \psi$ | $\forall uv\left(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee \overleftarrow{R}wvu\right)$ |
| $\varphi\ (\sigma_3, t_6, \rightarrow)\ \psi$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rvwu)$ |
| $\varphi\ (\sigma_4, t_6, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_1, t_5, 1)\ \varphi)$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rvuw)$ |
| $\varphi\ (\sigma_5, t_4, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_2, t_4, \leftarrow)\ \varphi)$ | $\forall uv\left(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee \overleftarrow{R}wuv\right)$ |
| $\varphi\ (\sigma_6, t_5, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_3, t_6, \rightarrow)\ \varphi)$ | $\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Ruwv)$ |

Figure 3: The 2–ary basic logical connectives

| Connective | Truth condition |
|---|---|
| **The ‘disjunction’ founded family** | |
| $\varphi\ (\sigma_1, t_4, \rightarrow)\ \psi$ | $\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee Ruvw)$ |
| $\varphi\ (\sigma_2, t_5, \leftarrow)\ \psi$ | $\exists uv \left(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge \overleftarrow{R}wvu\right)$ |
| $\varphi\ (\sigma_3, t_5, \leftarrow)\ \psi$ | $\exists uv \left(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge \overleftarrow{R}vwu\right)$ |
| $\varphi\ (\sigma_4, t_4, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_1, t_4, \rightarrow)\ \varphi)$ | $\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee Rvuw)$ |
| $\varphi\ (\sigma_5, t_6, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_2, t_5, \leftarrow)\ \varphi)$ | $\exists uv \left(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge \overleftarrow{R}wuv\right)$ |
| $\varphi\ (\sigma_6, t_6, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_3, t_5, \leftarrow)\ \varphi)$ | $\exists uv \left(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge \overleftarrow{R}uwv\right)$ |
| **The ‘implication’ founded family** | |
| $\varphi\ (\sigma_1, t_3, \rightarrow)\ \psi$ | $\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee Ruvw)$ |
| $\varphi\ (\sigma_2, t_3, \rightarrow)\ \psi$ | $\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee Rwvu)$ |
| $\varphi\ (\sigma_3, t_1, \leftarrow)\ \psi$ | $\exists uv \left(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge \overleftarrow{R}vwu\right)$ |
| $\varphi\ (\sigma_4, t_2, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_1, t_3, \rightarrow)\ \varphi)$ | $\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee Rvuw)$ |
| $\varphi\ (\sigma_5, t_2, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_2, t_3, \rightarrow)\ \varphi)$ | $\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee Rwuv)$ |
| $\varphi\ (\sigma_6, t_1, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_3, t_1, \leftarrow)\ \varphi)$ | $\exists uv \left(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge \overleftarrow{R}uwv\right)$ |
| **The ‘co-implication’ founded family** | |
| $\varphi\ (\sigma_1, t_2, \rightarrow)\ \psi$ | $\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee Ruvw)$ |
| $\varphi\ (\sigma_2, t_1, \leftarrow)\ \psi$ | $\exists uv \left(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge \overleftarrow{R}wvu\right)$ |
| $\varphi\ (\sigma_3, t_3, \rightarrow)\ \psi$ | $\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee Rvwu)$ |
| $\varphi\ (\sigma_3, t_3, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_1, t_2, \rightarrow)\ \varphi)$ | $\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee Rvuw)$ |
| $\varphi\ (\sigma_5, t_1, \leftarrow)\ \psi$<br>$(\psi\ (\sigma_2, t_1, \leftarrow)\ \varphi)$ | $\exists uv \left(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge \overleftarrow{R}wuv\right)$ |
| $\varphi\ (\sigma_6, t_2, \rightarrow)\ \psi$<br>$(\psi\ (\sigma_3, t_3, \rightarrow)\ \varphi)$ | $\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee Ruwv)$ |

Figure 4: The 2–ary basic logical connectives

| Connective | Truth condition |
|---|---|
| The 'stroke' founded family | |
| $\varphi\ (\sigma_1, t_7, \to)\ \psi$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Ruvw)$ |
| $\varphi\ (\sigma_2, t_7, \to)\ \psi$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwvu)$ |
| $\varphi\ (\sigma_3, t_7, \to)\ \psi$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvwu)$ |
| $\varphi\ (\sigma_4, t_7, \to)\ \psi$<br>$(\psi\ (\sigma_1, t_7, \to)\ \varphi)$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvuw)$ |
| $\varphi\ (\sigma_5, t_7, \to)\ \psi$<br>$(\psi\ (\sigma_2, t_7, \to)\ \varphi)$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwuv)$ |
| $\varphi\ (\sigma_6, t_7, \to)\ \psi$<br>$(\psi\ (\sigma_3, t_7, \to)\ \varphi)$ | $\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Ruwv)$ |
| The 'dagger' founded family | |
| $\varphi\ (\sigma_1, t_8, \to)\ \psi$ | $\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruvw)$ |
| $\varphi\ (\sigma_2, t_8, \to)\ \psi$ | $\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwvu)$ |
| $\varphi\ (\sigma_3, t_8, \to)\ \psi$ | $\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvwu)$ |
| $\varphi\ (\sigma_4, t_8, \to)\ \psi$<br>$(\psi\ (\sigma_1, t_8, \to)\ \varphi)$ | $\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvuw)$ |
| $\varphi\ (\sigma_5, t_8, \to)\ \psi$<br>$(\psi\ (\sigma_2, t_8, \to)\ \varphi)$ | $\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwuv)$ |
| $\varphi\ (\sigma_6, t_8, \to)\ \psi$<br>$(\psi\ (\sigma_3, t_8, \to)\ \varphi)$ | $\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruwv)$ |

Figure 5: The 2–ary basic logical connectives

| Permutations of $\mathfrak{S}_2$ | Families of 1–ary traces |
|---|---|
| $\sigma_1 = (1, 2) = Id$<br>$\sigma_2 = (2, 1) = (1\ 2) \circ \sigma_1$ | $t'_1 = (-) \mapsto - = t_2^{'-1}$<br>$t'_2 = (+) \mapsto + = t_1^{'-1}$ |
| | $t'_3 = (-) \mapsto + = t_3^{'-1}$ |
| | $t'_4 = (+) \mapsto - = t_4^{'-1}$ |

Figure 6: Permutations of $\mathfrak{S}_2$ and families of 1–ary traces

| Connective | Truth condition |
|---|---|
| The ‘sometimes positive’ founded family | |
| $\left(\sigma_1, t'_1, \rightarrow\right)\ \varphi$ | $\exists v\,(v \in [\![\varphi]\!] \wedge Rvw)$ |
| $\left(\sigma_2, t'_2, \leftarrow\right)\ \varphi$ | $\forall v\left(v \in [\![\varphi]\!] \vee \overleftarrow{R}wv\right)$ |
| The ‘always positive’ founded family | |
| $\left(\sigma_1, t'_2, \rightarrow\right)\ \varphi$ | $\forall v\,(v \in [\![\varphi]\!] \vee Rvw)$ |
| $\left(\sigma_2, t'_1, \leftarrow\right)\ \varphi$ | $\exists v\left(v \in [\![\varphi]\!] \wedge \overleftarrow{R}wv\right)$ |
| The ‘always negative’ founded family | |
| $\left(\sigma_1, t'_3, \rightarrow\right)\ \varphi$ | $\forall v\,(v \notin [\![\varphi]\!] \vee Rvw)$ |
| $\left(\sigma_2, t'_3, \rightarrow\right)\ \varphi$ | $\forall v\,(v \notin [\![\varphi]\!] \vee Rwv)$ |
| The ‘sometimes negative’ founded family | |
| $\left(\sigma_1, t'_4, \rightarrow\right)\ \varphi$ | $\exists v\,(v \notin [\![\varphi]\!] \wedge Rvw)$ |
| $\left(\sigma_2, t'_4, \rightarrow\right)\ \varphi$ | $\exists v\,(v \notin [\![\varphi]\!] \wedge Rwv)$ |

Figure 7: The 1–ary basic logical connectives

# 4 Calculus for Gaggle logic

After some general definitions in Section 4.1 which will be used in the rest of the report, we introduce in Section 4.2 a calculus for Gaggle logic whose form generalizes the form of display calculi. Unlike display calculi, we do not resort to an explicit logical or structural connective for Boolean negation in our calculus. Hence, our notions of protoantecedant part and protoconsequent part generalize the notions of antecedant part and consequent part of display calculi.

## 4.1 Preliminary Definitions

These definitions are very general and apply to any kind of logical formalism.

**Definition 14** (Logic)**.** A *logic* is a triple $\mathsf{L} \triangleq (\mathcal{L}, E, \models)$ where

- $\mathcal{L}$ is a *logical language* defined as a set of well-formed expressions built from a set of *connectives* $\mathsf{C}$ and a set of *propositional letters* $\mathbb{P}$;
- $E$ is a *class of pointed models or frames*;
- $\models$ is a *satisfaction relation* which relates in a compositional manner elements of $\mathcal{L}$ to models of $E$ by means of so-called *truth conditions*. $\dashv$

Our definition of a calculus and of an inference rule is taken from [41].

**Definition 15** (Conservativity)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ and $\mathsf{L}' = (\mathcal{L}', E', \models')$ be two logics such that $\mathcal{L} \subseteq \mathcal{L}'$. We say that $\mathsf{L}'$ is a *conservative extension* of $\mathsf{L}$ when $\left\{\varphi \in \mathcal{L} \mid \models_{\mathsf{L}}\varphi\right\} = \mathcal{L} \cap \left\{\varphi' \in \mathcal{L}' \mid \models'_{\mathsf{L}'}\varphi'\right\}$. $\dashv$

**Definition 16** (Calculus and sequent calculus )**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. A *calculus* $\mathsf{P}$ for $\mathcal{L}$ is a set of elements of $\mathcal{L}$ called *axioms* and a set of *inference rules*. Most often, one can effectively decide whether a given element of $\mathcal{L}$ is an axiom. To be more precise, an *inference rule* $R$ for $\mathcal{L}$ is a relation among elements of $\mathcal{L}$ such that there is a unique $l \in \mathbb{N}^*$ such that, for all $\varphi, \varphi_1, \ldots, \varphi_l \in \mathcal{L}$, one can effectively decide whether $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$. The elements $\varphi_1, \ldots, \varphi_l$ are called the *premises* and $\varphi$ is called the *conclusion* and we say that $\varphi$ is a *direct consequence of* $\varphi_1, \ldots, \varphi_l$ by virtue of $R$. Let $\Gamma \subseteq \mathcal{L}$ and let $\varphi \in \mathcal{L}$. We say that $\varphi$ is *provable* (from $\Gamma$) in $\mathsf{P}$ or a *theorem* of $\mathsf{P}$, denoted $\vdash_{\mathsf{P}} \varphi$ (resp. $\Gamma \vdash_{\mathsf{P}} \varphi$), when there is a *proof* of $\varphi$ (from $\Gamma$) in $\mathsf{P}$, that is, a finite sequence of formulas ending in $\varphi$ such that each of these formulas is:

1. either an instance of an axiom of $\mathsf{P}$ (or a formula of $\Gamma$);

2. or the direct consequence of preceding formulas by virtue of an inference rule $R$.

If $\mathcal{S}$ is a set of $\mathcal{L}$–consecutions, this set $\mathcal{S}$ can be viewed as a logical language. Then, we call *sequent calculus for* $\mathcal{S}$ a calculus for $\mathcal{S}$.

Axioms and inference rules are often represented by means of *axiom schemas* and *inference rule schemas*, that is, expressions of the following form, depending on whether we deal with formulas of $\mathcal{L}$ or $\mathcal{L}$–consecutions:

Axiom schemas:

$\alpha$ $\qquad\qquad$ $A \vdash B$

Inference rule schemas:

$$\frac{\alpha_1 \quad \ldots \quad \alpha_n}{\alpha} \qquad\qquad \frac{A_1 \vdash B_1 \quad \ldots \quad A_n \vdash B_n}{A \vdash B}$$

where $\alpha_1, \ldots, \alpha_n, \alpha$ are built up from *variables* often denoted $\varphi, \psi, \ldots$ and the connectives of $\mathsf{F}$ and, likewise, $A_1, \ldots, A_n, B_1, \ldots, B_n, A, B$ are built up from *variables* often denoted $X, Y, \ldots$ and the connectives of $\mathsf{F}$. In this representation, inference rules and axioms schemas are closed by *uniform substitution*: each variable can be replaced uniformly by *any* well-formed expression of $\mathcal{L}$. $\dashv$

**Definition 17** (Truth, validity, logical consequence)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $M \in E$, $\varphi \in \mathcal{L}$, $R$ be an inference rule for $\mathcal{L}$ and $S, S'$ be either inference rules for $\mathcal{L}$ or formulas of $\mathcal{L}$. If $\Gamma$ is a set of formulas or inference rules, we write $M \models \Gamma$ when for all $\varphi \in \Gamma$, we have $M \models \varphi$. Then, we say that

- $\varphi$ is *true* (*satisfied*) at $M$ or $M$ is a *model* of $\varphi$ when $M \models \varphi$;

- $\varphi$ is *valid*, denoted $\models_{\mathsf{L}} \varphi$, when for all models $M \in E$, we have $M \models \varphi$;

- $R$ is *true* (*satisfied*) at $M$ or $M$ is a *model* of $R$, denoted $M \models R$, when for all $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$, if $M \models \varphi_i$ for all $i \in \{1, \ldots, l\}$, then $M \models \varphi$.

- $S$ is *equivalent* to $S'$ in $E$, denoted $S \equiv_E S'$, when for all $M \in E$, $M \models S$ if, and only if, $M \models S'$. $\dashv$

**Definition 18** (Soundness and completeness)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $\mathsf{P}$ be a calculus for $\mathcal{L}$. Then,

- $\mathsf{P}$ is *sound* for the logic $\mathsf{L}$ when for all $\varphi \in \mathcal{L}$, if $\vdash_{\mathsf{P}} \varphi$, then $\models_{\mathsf{L}} \varphi$.

- $\mathsf{P}$ is *(strongly) complete* for the logic $\mathsf{L}$ when for all $\varphi \in \mathcal{L}$ (and all $\Gamma \subseteq \mathcal{L}$), if $\models_{\mathsf{L}} \varphi$, then $\vdash_{\mathsf{P}} \varphi$ (resp. if $\Gamma \models_{\mathsf{L}} \varphi$, then $\Gamma \vdash_{\mathsf{P}} \varphi$). $\dashv$

### 4.2 A Calculus for Gaggle logic

We introduce a calculus for Gaggle logic. The main inference rules of this calculus (that is, $R^{-j}$ and $R^K_\star$) are direct translations in Gaggle logic of the abstract laws of residuations and the tonicity relations of Theorem 1 respectively.

**Definition 19.** Let $\mathsf{C} \subseteq \mathbb{C}$. We denote by $\mathsf{Galog}(\mathsf{C})$ the sub-calculus of Figure 8 consisting only in the axioms and inference rules that contain the logical connectives of $\mathsf{C}$ and the structural connectives of $[\mathsf{C}]$. When $\mathsf{C} = \mathbb{C}$, then $\mathsf{Galog}(\mathsf{C})$ is also denoted $\mathsf{Galog}$. $\dashv$

**Proposition 3.**

1. *The following rule is admissible in* $\mathsf{Galog}$*:*

$$\frac{X \vdash Y}{\overline{\overline{Y} \vdash \overline{X}}}\ \overline{R}$$

2. *For all $\varphi \in \mathcal{L}$, $\varphi \vdash \varphi$ is provable in* $\mathsf{Galog}$.

*Proof.* We prove each item separately.

1. We prove the first item by induction on the length $n$ of the proof of $X \vdash Y$.

   $n = 0$. In that case, $X \vdash Y$ is of the form $[p] \vdash p$ or $p \vdash [p]$ and we can indeed prove that $[\overline{p}] \vdash \overline{p}$ or $\overline{p} \vdash [\overline{p}]$ (since these are axioms).

   $n + 1$. For the induction step, we consider the last rule applied in the proof of $X \vdash Y$ and we apply the Induction Hypothesis to the premise(s) of that rule. Then, as it turns out, the conclusion that we obtain by the application of the same rule to the (reversed) premise(s) is $\overline{Y} \vdash \overline{X}$.

2. The proof of the second item is by Induction on the structure of $\varphi$. The proof is standard.

$\square$

**Theorem 3.** *The calculus* $\mathsf{Galog}$ *is sound and complete for Gaggle logic.*

*Proof.* See Section 4.3. $\square$

### 4.3 Soundness and Completeness Proof

In this section, we provide the soundness and completeness proof of Theorem 3.

**Definition 20** (Good trace)**.** A *good trace* is a trace $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ such that:

- if $\pm = -$ then for all $j \in [\![1; n]\!]$, $\pm_j = -$;
- if $\pm = +$ then there is $j \in [\![1; n]\!]$ such that $\pm_j = +$. $\dashv$

**Definition 21** (Canonical model)**.** The *canonical model of Gaggle logic* is the tuple $M^c = (W^c, \sqsubseteq^c, \mathcal{R}^c)$ such that:

- $W^c \triangleq \{w_\varphi \mid \varphi \in \mathcal{L}\} /_\sim$ where $\sim$ is the equivalence relation over $\{w_\varphi \mid \varphi \in \mathcal{L}\}$ defined by $w_\varphi \sim w_\psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are both provable in $\mathsf{Galog}$;
- $\widetilde{w}_\varphi \sqsubseteq^c \widetilde{w}_\psi$ holds if, and only if, $\psi \vdash \varphi$ is provable in $\mathsf{Galog}$;

$$\frac{X \vdash \varphi \quad \varphi \vdash Y}{X \vdash Y}\ cut$$

*Axioms and introduction rules:*

$$\frac{}{S\left(\left[p\right], p\right)}\ R_p^K \qquad \frac{S\left(\left[p\right], X\right)}{S\left(p, X\right)}\ R_p^A$$

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left(\left[\star\right], X_1, \ldots, X_n, \star\left(\varphi_1, \ldots, \varphi_n\right)\right)}\ R_\star^K \qquad \frac{S\left(\left[\star\right], \varphi_1, \ldots, \varphi_n, X\right)}{S\left(\star, \varphi_1, \ldots, \varphi_n, X\right)}\ R_\star^A$$

*Structural rules:*

$$\frac{S\left(\left[\star\right], X_1, \ldots, X_n, X\right)}{S\left(\left[\star^j\right], X_1, \ldots, \overline{X_j}, \ldots, X_n, X\right)}\ R^j \qquad \frac{S\left(\left[\star\right], X_1, \ldots, X_j, \ldots, X_n, Y\right)}{S\left(\left[\star^{-j}\right], X_1, \ldots, Y, \ldots, X_n, X_j\right)}\ R^{-j}$$

(the inferences $R^j$ and $R^{-j}$ are drawn with double lines)

---

where for all $p = (Id, \pm, \leftrightarrow)_i \in \mathbb{P}$, all $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$, with $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$:

1) for all $j \in [\![1; n]\!]$, we have $(U_j, V_j) \triangleq \begin{cases} (X_j, \varphi_j) & \text{if } \pm_j \pm = + \\ (\varphi_j, X_j) & \text{if } \pm_j \pm = - \end{cases}$

2) $S(\circledast, X_1, \ldots, X_n, X) \triangleq \begin{cases} \circledast(X_1, \ldots, X_n) \vdash X & \text{if } \pm = - \\ X \vdash \circledast\left(X_1, \ldots, X_n\right) & \text{if } \pm = + \end{cases}$

with $\circledast$ which is either $\star$, $[\star]$, $p$ or $[p]$. So, if $\circledast$ is $p$ or $[p]$, then

$S(\circledast, X) \triangleq \begin{cases} \circledast \vdash X & \text{if } \pm = - \\ X \vdash \circledast & \text{if } \pm = +. \end{cases}$

3) $\star^j \triangleq (\sigma, t^j, \leftrightarrow)_i$ with $t^j \triangleq (\pm_1, \ldots, -\pm_j, \ldots, \pm_n) \mapsto \pm$

4) $\star^{-j} \triangleq (\sigma, t^{-j}, \leftrightarrow^{-j})_i$ with $t^{-j} \triangleq (\pm_1, \ldots, -\pm, \ldots, \pm_n) \mapsto -\pm_j$ *and*

$\leftrightarrow^{-j} \triangleq \begin{cases} \leftrightarrow & \text{if } -\pm_j = \pm \\ \overleftarrow{\leftrightarrow} & \text{otherwise.} \end{cases}$

5) $\overline{\star} \triangleq (\sigma, \overline{t}, \overleftarrow{\leftrightarrow})_i$ with $\overline{t} \triangleq (\pm_1, \ldots, \pm_n) \mapsto -\pm$

6) $\overline{X} \triangleq \begin{cases} \overline{p} & \text{if } X = p \\ \overline{\star}(\varphi_1, \ldots, \varphi_n) & \text{if } X = \star(\varphi_1, \ldots, \varphi_n) \\ [\overline{p}] & \text{if } X = [p] \\ [\overline{\star}]\left(X_1, \ldots, X_n\right) & \text{if } X = [\star]\left(X_1, \ldots, X_n\right) \end{cases}$

Figure 8: Calculus Galog

- $\mathcal{R}^c \triangleq \{R_\star \mid \star \in \mathbb{C}\}$ such that for all $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ such that $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ is a good trace, for all $\widetilde{w}_{\varphi_1}, \ldots, \widetilde{w}_{\varphi_n}, \widetilde{w}_\varphi \in W^c$,

$$R_\star^{\sigma,\leftrightarrow} \widetilde{w}_{\varphi_1} \ldots \widetilde{w}_{\varphi_n} \widetilde{w}_\varphi \text{ iff } \varphi \vdash \star(\varphi_1, \ldots, \varphi_n) \tag{1}$$

$\dashv$

In particular, note that for all $\widetilde{w}_\varphi \in W^c$ and all $p = (Id, \pm, \leftrightarrow)_i \in \mathbb{P}$, Expression 1 rewrites as follows:

$$\overset{\leftrightarrow}{R}_p \widetilde{w}_\varphi \text{ iff } \varphi \vdash p.$$

Moreover, the canonical model of Gaggle logic is well-defined. In particular, the choice of representants of the equivalence classes $\widetilde{w}_\varphi, \widetilde{w}_{\varphi_1}, \ldots \widetilde{w}_{\varphi_n}$ is irrelevant w.r.t. the truth of Expression (1). That is, for all $\varphi, \psi \in \widetilde{w}_\varphi$ and all $\varphi_1, \psi_1 \in \widetilde{w}_{\varphi_1}, \ldots, \varphi_n, \psi_n \in \widetilde{w}_{\varphi_n}$, we have that $\varphi \vdash \star(\varphi_1, \ldots, \varphi_n)$ is provable in Gaggle logic iff $\psi \vdash \star(\psi_1, \ldots, \psi_n)$ is provable in Gaggle logic (partly because of the cut rule).

**Lemma 2.** *The canonical model is a $\mathbb{C}$–model.*

*Proof sketch.* We need to show that the Tonicity conditions are fulfilled and that $\sqsubseteq^c$ is a partial order. The Tonicity conditions hold because of the rule $R_\star^K$. As for the partial order, $\sqsubseteq^c$ is clearly a preorder by the cut rule and because of item 2 of Proposition 3. The antisymmetry of $\sqsubseteq^c$ is proved thanks to the fact that our states are equivalence classes stemming from the provability equivalence relationship. $\square$

**Lemma 3.** *For all $\psi \in \mathcal{L}$ and all $\widetilde{w}_\varphi \in W^c$, if $M^c, \widetilde{w}_\varphi \Vdash \psi$ then $\varphi \vdash \psi$ is provable in Galog.*

*Proof.* We prove it by induction on $\psi$. Assume that $M^c, \widetilde{w}_\varphi \Vdash \psi$. That is, $\widetilde{w}_\varphi \in f^{\sigma,t}_{R_\star^{\leftrightarrow}}([\![\varphi_1]\!]^{M^c}, \ldots, [\![\varphi_n]\!]^{M^c})$. The base case $\psi = p \in \mathbb{P}$ holds by definition. For the induction step, assume that $\psi = \star(\varphi_1, \ldots, \varphi_n)$ with $\star = (\sigma, t, \leftrightarrow)_i$.

- Assume that $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ *is* a good trace.

- If $\pm = -$: then $t = (-, \ldots, -) \mapsto -$ because $t$ is a good trace. So, there are $\widetilde{w}_{\chi_1}, \ldots, \widetilde{w}_{\chi_n} \in W^c$ such that $\widetilde{w}_{\chi_1} \pitchfork [\![\varphi_1]\!]^{M^c}$ and $\ldots$ and $\widetilde{w}_{\chi_n} \pitchfork [\![\varphi_n]\!]^{M^c}$ and $R_\star^{\sigma,\leftrightarrow} \widetilde{w}_{\chi_1} \ldots \widetilde{w}_{\chi_n} \widetilde{w}_\varphi$, that is, there are $\widetilde{w}_{\chi_1}, \ldots, \widetilde{w}_{\chi_n} \in W^c$ such that $\widetilde{w}_{\chi_1} \in [\![\varphi_1]\!]^{M^c}$ and $\ldots$ and $\widetilde{w}_{\chi_n} \in [\![\varphi_n]\!]^{M^c}$ and $\varphi \vdash \star(\chi_1, \ldots, \chi_n)$, because $t = (-, \ldots, -) \mapsto -$. Therefore, by Induction Hypothesis, for all $j \in [\![1; n]\!]$, we have that $\chi_j \vdash \varphi_j$. Moreover, $\varphi \vdash \star(\chi_1, \ldots, \chi_n)$ (1). Thus, because for all $j \in [\![1; n]\!]$ it holds that $tn(f^{\sigma,t}_{R_\Gamma^{\leftrightarrow}}, j) = \pm_j \pm = +$, we have that $\star(\chi_1, \ldots, \chi_n) \vdash \star(\varphi_1, \ldots, \varphi_n)$ (2) by Rules $R_\star^K$ and $R_\star^A$. Hence, by the cut Rule applied to (1) and (2) above, we finally have that $\varphi \vdash \star(\varphi_1, \ldots, \varphi_n)$.

- If $\pm = +$: for all $\widetilde{w}_{\chi_1}, \ldots, \widetilde{w}_{\chi_n} \in W^c$, either $\widetilde{w}_{\chi_1} \pitchfork [\![\varphi_1]\!]^{M^c}$ or $\ldots$ or $\widetilde{w}_{\chi_n} \pitchfork [\![\varphi_n]\!]^{M^c}$ or $R_\star^{\sigma,\leftrightarrow} \widetilde{w}_{\chi_1} \ldots \widetilde{w}_{\chi_n} \widetilde{w}_\varphi$, that is, for all $\widetilde{w}_{\chi_1}, \ldots, \widetilde{w}_{\chi_n} \in W^c$, either $\widetilde{w}_{\chi_{i_1}} \notin [\![\varphi_{i_1}]\!]^{M^c}$ or $\ldots$ or $\widetilde{w}_{\chi_{i_k}} \notin [\![\varphi_{i_k}]\!]^{M^c}$ or $\widetilde{w}_{\chi_{i_{k+1}}} \in [\![\varphi_{i_{k+1}}]\!]^{M^c}$ or $\ldots$ or $\widetilde{w}_{\chi_{i_n}} \in [\![\varphi_{i_n}]\!]^{M^c}$ or $\varphi \vdash \star(\chi_1, \ldots, \chi_n)$. For all $l > k$, take $\chi_{i_l} \triangleq \star^{-i_l}(\varphi_1, \ldots, \varphi_{i_l - 1}, \varphi, \varphi_{i_l + 1}, \ldots, \varphi_n)$ with $\star^{-i_l} \triangleq (\sigma, t^{-i_l}, \leftrightarrow)_i$. For all $l \leq k$, take $\chi_{i_l} = \varphi_{i_l}$; so $\widetilde{w}_{\chi_{i_l}} \in [\![\varphi_{i_k}]\!]^{M^c}$ and therefore, by Induction Hypothesis, $\chi_{i_l} \vdash \varphi_{i_l}$. First, assume that $\varphi \vdash \star(\chi_1, \ldots, \chi_n)$ holds. That is, $\varphi \vdash \star(\psi_1, \ldots, \psi_n)$ holds, where

$$\psi_i \triangleq \begin{cases} \varphi_i & \text{if } i \in \{i_1, \ldots, i_k\} \\ \chi_i & \text{otherwise.} \end{cases} \tag{2}$$

Now, for all $i \notin \{i_1, \dots, i_k\}$, $\chi_i \models \varphi_i$ and $tn(\star, i) = +$. So, finally, $\varphi \models \star(\varphi_1, \dots, \varphi_n)$, that is, $\varphi \models \psi$. Second, assume that $\varphi \models \star(\chi_1, \dots, \chi_n)$ does not hold. Then, since ($\widetilde{w}_{\chi_{i_1}} \notin [\![\varphi_{i_1}]\!]^{M^c}$ or $\dots$ or $\widetilde{w}_{\chi_{i_k}} \notin [\![\varphi_{i_k}]\!]^{M^c}$) does *not* hold, this entails that either $\widetilde{w}_{\chi_{i_{k+1}}} \in [\![\varphi_{i_{k+1}}]\!]^{M^c}$ holds or $\dots$ or $\widetilde{w}_{\chi_{i_n}} \in [\![\varphi_{i_n}]\!]^{M^c}$ holds. That is, for some $l > k$, $\star^{-i_l}(\varphi_1, \dots, \varphi_{i_l-1}, \varphi, \varphi_{i_l+1}, \dots, \varphi_n) \models \varphi_{i_l}$ holds. Hence, by Rule $R^{-j}$, we have that $\varphi \models \star(\varphi_1, \dots, \varphi_n)$, that is, $\varphi \models \psi$.

• Assume that $t = (\pm_1, \dots, \pm_n) \mapsto \pm$ is *not* a good trace.

- If $\pm = -$: assume that $\{\pm_{i_1}, \dots, \pm_{i_k}\} = \{-\}$ and $\{\pm_{i_{k+1}}, \dots, \pm_{i_n}\} = \{+\}$ with $\{i_1, \dots, i_n\} = \{1, \dots, n\}$. Then, $M^c, \widetilde{w}_\varphi \Vdash \star(\varphi_1, \dots, \varphi_n)$ holds

iff $M^c, \widetilde{w}_\varphi \Vdash \star'(\varphi_1, \dots, \overline{\varphi_{i_{k+1}}}, \dots, \overline{\varphi_{i_n}}, \dots, \varphi_n)$ where $\star' = (\sigma, t_0, \leftrightarrow)_i$ with $t_0 = (-, \dots, -) \mapsto -$, by soundness of Rule $R^j$;

iff $\varphi \models \star'(\varphi_1, \dots, \overline{\varphi_{i_{k+1}}}, \dots, \overline{\varphi_{i_n}}, \dots, \varphi_n)$ is provable in Galog, by the first part of the proof because $t_0$ is a good trace;

iff $\varphi \models \star(\varphi_1, \dots, \varphi_n)$ is provable in Galog, by Rule $R^j$.

- If $\pm = +$: then $t = (-, \dots, -) \mapsto +$. Let us consider any $j \in [\![1; n]\!]$. Then, $M^c, \widetilde{w}_\varphi \Vdash \star(\varphi_1, \dots, \varphi_n)$ holds

iff $M^c, \widetilde{w}_\varphi \Vdash \star^j(\varphi_1, \dots, \overline{\varphi_j}, \dots, \varphi_n)$ where $\star^j = (\sigma, t^j, \leftrightarrow)_i$ with $t^j = (-, \dots, -, +, -, \dots, -) \mapsto +$, by soundness of Rule $R^j$;

iff $\varphi \models \star^j(\varphi_1, \dots, \overline{\varphi_j}, \dots, \varphi_n)$ is provable in Galog, by the first part of the proof because $t^j$ is a good trace;

iff $\varphi \models \star(\varphi_1, \dots, \varphi_n)$ is provable in Galog, by Rule $R^j$.

- If $\pm = +$: then $t = (-, \dots, -) \mapsto +$. The proof is then completely similar to the previous case, we use again one of the rules $R^j$. □

**Lemma 4.** *Let $\varphi \in \mathcal{L}$. If it is impossible to prove $\varphi \models \overline{\varphi}$ in Galog, then $M^c, \widetilde{w}_\varphi \Vdash \varphi$.*

*Proof.* We prove it by induction on $\varphi$. The base case $\varphi = p \in \mathbb{P}$ holds by definition. Assume that $\varphi = \star(\varphi_1, \dots, \varphi_n)$ with $\star = (\sigma, t, \leftrightarrow)_i$.

• Assume that $t = (\pm_1, \dots, \pm_n) \mapsto \pm$ is a good trace.

- If $\pm = -$: Because $t$ is a good trace, $t = (-, \dots, -) \mapsto -$. So, $M^c, \widetilde{w}_\varphi \Vdash \varphi$ holds if, and only if, there are $\widetilde{w}_1, \dots, \widetilde{w}_n \in W^c$ such that $R_\star^{\sigma,\leftrightarrow} \widetilde{w}_1 \dots \widetilde{w}_n \widetilde{w}_\varphi$ and $M^c, \widetilde{w}_1 \Vdash \varphi_1$ and $\dots$ and $M^c, \widetilde{w}_n \Vdash \varphi_n$. It suffices to take $(\widetilde{w}_1, \dots, \widetilde{w}_n) = (\widetilde{w}_{\varphi_1}, \dots, \widetilde{w}_{\varphi_n})$. Indeed, we have that $R_\star^{\sigma,\leftrightarrow} \widetilde{w}_{\varphi_1} \dots \widetilde{w}_\varphi$ because $\varphi \models \star(\varphi_1, \dots, \varphi_n)$ is provable by Proposition 3. Moreover, by Induction Hypothesis, we have that $M^c, \widetilde{w}_{\varphi_1} \Vdash \varphi_1$ and $\dots$ and $M^c, \widetilde{w}_{\varphi_n} \Vdash \varphi_n$.

- If $\pm = +$: Because $t$ is a good trace, it is of the form $(\pm_1, \dots, \pm_n) \mapsto +$ where $\{\pm_{i_1}, \dots, \pm_{i_k}\} = \{-\}$ and $\{\pm_{i_{k+1}}, \dots, \pm_{i_n}\} = \{+\}$. So, $M^c, \widetilde{w}_\varphi \Vdash \varphi$ holds if, and only if, for all $\widetilde{w}_1, \dots, \widetilde{w}_n \in W^c$ such that $R_\star^{\sigma,\leftrightarrow} \widetilde{w}_1 \dots \widetilde{w}_n \widetilde{w}_\varphi$, if $M^c, \widetilde{w}_{i_1} \Vdash \varphi_{i_1}$ and $\dots$ and $M^c, \widetilde{w}_{i_k} \Vdash \varphi_{i_k}$ then $M^c, \widetilde{w}_{i_{k+1}} \Vdash \varphi_{i_{k+1}}$ or $\dots$ or $M^c, \widetilde{w}_{i_n} \models \varphi_{i_n}$. So, let $\widetilde{w}_{\chi_1}, \dots, \widetilde{w}_{\chi_n} \in W^c$ be such that $R_\star^{\sigma,\leftrightarrow} \widetilde{w}_{\chi_1} \dots \widetilde{w}_{\chi_n} \widetilde{w}_\varphi$ and assume towards a contradiction that $M^c, \widetilde{w}_{\chi_{i_1}} \Vdash \varphi_{i_1}$ and $\dots$ and $M^c, \widetilde{w}_{\chi_{i_k}} \Vdash \varphi_{i_k}$ but not $M^c, \widetilde{w}_{\chi_{i_{k+1}}} \Vdash \varphi_{i_{k+1}}$ and $\dots$ and not $M^c, \widetilde{w}_{w_{i_n}} \Vdash \varphi_{i_n}$. Then, $M^c, \widetilde{w}_\varphi \Vdash \star'(\psi_1, \dots \psi_n)$ where $\star' = (\sigma, t', \leftrightarrow')$ is such that $t' = (-, \dots, -) \mapsto -$, $\leftrightarrow' = \overleftrightarrow{\leftrightarrow}$ and for all $i \in [\![1; n]\!]$,

$$\psi_i \triangleq \begin{cases} \varphi_i & \text{if } i \in \{i_1, \dots, i_k\} \\ \overline{\varphi_i} & \text{if } i \in \{i_{k+1}, \dots, i_n\} \end{cases}$$

So, from Lemma 3, we obtain that $\varphi \vdash \star' (\psi_1, \ldots, \psi_n)$. Therefore, from the admissibility of rule $\overline{R}$, we have that $\overline{\star'}(\psi_1, \ldots, \psi_n) \vdash \overline{\varphi}$. So, by successive application of Rules $\star^{i_{k+1}}, \ldots, \star^{i_n}$, we have that $\star(\varphi_1, \ldots, \varphi_n) \vdash \overline{\varphi}$ is provable in Galog, that is, $\varphi \vdash \overline{\varphi}$ is provable in Galog. This is impossible by assumption. Hence, we have proved that $M^c, \widetilde{w}_\varphi \Vdash \varphi$.

• Assume that $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ is *not* a good trace. In that case, the proof is the same as in Lemma 3, we use the rule $R^j$. □

**Lemma 5.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $X \vdash Y$ be a $\mathcal{L}_{\mathsf{C}}$–consecution. If $M^c \Vdash X \vdash Y$ then for all $\mathsf{C}$–models $M$, we have that $M \Vdash X \vdash Y$.*

*Proof.* We reason by contraposition. Assume that there is a C–models $M$ for which it is not the case that $M \Vdash X \vdash Y$. Then, by soundness, this entails that $X \vdash Y$ is not provable, and therefore, by the truth lemma, that it is not the case that $M^c \Vdash X \vdash Y$. □

*Proof of Theorem 3.* We first prove the soundness. The soundness of the axiom $R_p^K$ is clear, as well as the soundness of rules $R_p^A$, $R_\star^A$ and $R^j$. As for rules $R_\star^K$ and $R^{-j}$, their soundness follows respectively from the two items of Theorem 1.

As for completeness, we need to show that if a statement $X \vdash Y$ is true in all C–models, then $X \vdash Y$ is provable in Galog. Because one can easily prove that $\tau(X) \vdash \tau(Y)$ is provable in Galog if, and only if, $X \vdash Y$ is provable in Galog, this amounts to prove that if a statement $\varphi \vdash \psi$ is true in all C–models, then $\varphi \vdash \psi$ is derivable in Galog. But if $\varphi \vdash \psi$ is true in all C–models, then it is true in the canonical model as well, so we have $M^c \Vdash \varphi \vdash \psi$. In other words, for any world $\widetilde{w}_\chi \in M^c$, we have that if $M^c, \widetilde{w}_\chi \Vdash \varphi$ then $M^c, \widetilde{w}_\chi \Vdash \psi$ (1). If this holds for any world $\widetilde{w}_\chi$ then in particular for world $\widetilde{w}_\varphi$.

1. Assume that $M^c, \widetilde{w}_\varphi \Vdash \varphi$. Then, $M^c, \widetilde{w}_\varphi \Vdash \psi$, by (1). So, by Lemma 3,we have that $\varphi \vdash \psi$ is provable in Galog.

2. Assume that it is not the case that $M^c, \widetilde{w}_\varphi \Vdash \varphi$. Then, by Lemma 4, we have that $\varphi \vdash \overline{\varphi}$ is provable in Galog (2). Therefore, it is impossible that $\varphi$ is satisfied in $M^c$, because otherwise, by soundness, we would also have both $M^c, \widetilde{w} \Vdash \varphi$ and $M^c, \widetilde{w} \Vdash \overline{\varphi}$, which is impossible. Hence, $M^c \Vdash \overline{\varphi}$. So, in particular, we have that $M^c, \widetilde{w}_{\overline{\psi}} \Vdash \overline{\varphi}$ holds. Thus, by Lemma 3, we obtain that $\overline{\psi} \vdash \overline{\varphi}$ is provable in Galog. So, by the admissibility of Rule $\overline{R}$ (see Proposition 3), we can conclude that $\varphi \vdash \psi$ is provable in Galog.

Hence, in all cases, we obtain that $\varphi \vdash \psi$ is provable in Galog. □

# 5 Intermezzo: Tense Logic as a Lingua Franca

In this section, we recall the definition of the basic tense logic (Section 5.1) and we show that a specific fragment of a multi-modal version of this logic, that we call 'tense Gaggle logic', is as expressive as Gaggle logic (Section 5.2) with respect to specific consecutions, called 'protoanalytic' consecutions. The idea to resort to tense logic as a lingua franca for display calculi is not new. It has already been used to prove completeness results for subintuitionistic logics w.r.t. the Kripke semantics [58]. However, our exact method and the reasons for which we follow it are slightly different from those of Wansing [58].

## 5.1 Basic Tense Logic

**Notation 1.** $\mathbb{P}$ is a set of propositional letters and

$$\mathbb{T} \triangleq \{\top, \bot, \neg, \wedge, \vee, \to, \Diamond, \Box, \Diamond^-, \Box^-\} \qquad\qquad [\mathbb{T}] \triangleq \{*, \bullet, , \} \qquad \dashv$$

**Definition 22** (Tense language)**.** The *tense language* $\mathcal{L}(\mathbb{P}, \mathbb{T})$ is the smallest set that contains the propositional letters $\mathbb{P}$ and that is closed under the tense connectives $\mathbb{T}$. The *structural tense language* $\mathcal{L}(\mathbb{P}, [\mathbb{T}])$ is the smallest set that contains the tense language and that is closed under the structural connectives of $[\mathbb{T}]$.

A $\mathcal{L}(\mathbb{P}, [\mathbb{T}])$*–consecution* is an expression of the form $X \vdash Y$, where $X, Y \in \mathcal{L}(\mathbb{P}, [\mathbb{T}])$. The class of all $\mathcal{L}(\mathbb{P}, [\mathbb{T}])$–consecutions is denoted $\mathcal{S}(\mathbb{P}, [\mathbb{T}])$. $\dashv$

**Definition 23** (Kripke frame)**.** A *Kripke frame* is a pair $F = (W, R)$ where $W$ is a non-empty set and $R$ is a binary relation on $W$. A *Kripke model* is a tuple $M = (W, R, \mathcal{P})$ where $(W, R)$ is a Kripke frame and $\mathcal{P}$ is a set of unary relations $P$, one for each $p \in \mathbb{P}$. We write $w \in F$ and $w \in M$ for $w \in W$. A *pointed Kripke frame (resp. model)* $(F, w)$ (resp. $(M, w)$) is a Kripke frame $F$ (resp. model $M$) with a state $w \in F$ (resp. $w \in M$). The class of pointed Kripke frames (models) is denoted $\mathcal{F}$ (resp. $\mathcal{K}$). $\dashv$

**Definition 24** (Tense logic)**.** We define the *satisfaction relation* $\models \subseteq \mathcal{K} \times \mathcal{L}(\mathbb{P}, \mathbb{T})$ inductively as follows. Let $(M, w) \in \mathcal{K}$ be a pointed Kripke model and let $\varphi, \psi \in \mathcal{L}(\mathbb{P}, \mathbb{T})$.

$$\begin{array}{lll}
M, w \models \top & & \text{always} \\
M, w \models \bot & & \text{never} \\
M, w \models p & \text{iff} & w \in P \\
M, w \models \neg\varphi & \text{iff} & \text{it is not the case that } M, w \models \varphi \\
M, w \models \varphi \wedge \psi & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \varphi \vee \psi & \text{iff} & M, w \models \varphi \text{ or } M, w \models \psi \\
M, w \models \varphi \to \psi & \text{iff} & \text{if } M, w \models \varphi \text{ then } M, w \models \psi \\
M, w \models \Box\varphi & \text{iff} & \text{for all } v \in M \text{ such that } Rwv, M, v \models \varphi \\
M, w \models \Diamond\varphi & \text{iff} & \text{there is } v \in M \text{ such that } Rwv \text{ and } M, v \models \varphi \\
M, w \models \Box^-\varphi & \text{iff} & \text{for all } v \in M \text{ such that } Rvw, M, v \models \varphi \\
M, w \models \Diamond^-\varphi & \text{iff} & \text{there is } v \in M \text{ such that } Rvw \text{ and } M, v \models \varphi
\end{array}$$

We extend the scope of the evaluation relation $\models$ simultaneously in two different ways in order to also relate worlds to structures. The *antecedent evaluation relation* $\overset{A}{\models} \subseteq \mathcal{K} \times \mathcal{L}(\mathbb{P}, [\mathbb{T}])$ and the *consequent evaluation relation* $\overset{K}{\models} \subseteq \mathcal{K} \times \mathcal{L}(\mathbb{P}, [\mathbb{T}])$ are defined inductively as follows. The truth conditions for the formulas are defined as above.

$$\begin{array}{lll}
M, w \overset{A}{\models} *X & \text{iff} & \text{it is not the case that } M, w \overset{K}{\models} X \\
M, w \overset{K}{\models} *X & \text{iff} & \text{it is not the case that } M, w \overset{A}{\models} X \\
M, w \overset{A}{\models} \bullet X & \text{iff} & \text{there is } v \in M \text{ such that } Rvw \text{ and } M, v \overset{A}{\models} X \\
M, w \overset{K}{\models} \bullet X & \text{iff} & \text{for all } v \in M \text{ such that } Rwv, M, v \overset{K}{\models} X
\end{array}$$

We extend the scope of the relation $\Vdash$ to also relate points to $\mathcal{L}(\mathbb{P}, [\mathbb{T}])$–consecutions:

$$M, w \Vdash X \vdash Y \quad \text{iff} \quad \text{if } M, w \overset{A}{\Vdash} X, \text{ then } M, w \overset{K}{\Vdash} Y.$$

Moreover, we define $[\![\varphi]\!]_M \triangleq \{w \in M \mid M, w \models \varphi\}$. If $\mathcal{M}_0$ is a class of pointed Kripke models, the triple $(\mathcal{L}(\mathbb{P}, \mathbb{T}), \mathcal{M}_0, \models)$ is a logic called the *tense logic associated to* $\mathcal{M}_0$. $\dashv$

**Definition 25** (Translations $t_1$ and $t_2$). We define the translations $t_1 : \mathcal{L}(\mathbb{P}, [\mathbb{T}]) \to \mathcal{L}(\mathbb{P}, \mathbb{T})$ and $t_2 : \mathcal{L}(\mathbb{P}, [\mathbb{T}]) \to \mathcal{L}(\mathbb{P}, \mathbb{T})$ inductively as follows:

$$\begin{array}{llll} t_1(\varphi) & \triangleq \varphi & t_2(\varphi) & \triangleq \varphi \\ t_1(\bullet X) & \triangleq \Diamond^- t_1(X) & t_2(\bullet X) & \triangleq \Box t_2(X) \\ t_1(*X) & \triangleq \neg t_2(X) & t_2(*X) & \triangleq \neg t_1(X) \\ t_1(X,Y) & \triangleq t_1(X) \wedge t_1(Y) & t_2(X,Y) & \triangleq t_2(X) \vee t_2(Y) \end{array}$$

$\dashv$

## 5.2 Tense Gaggle logic

Tense Gaggle logic is a multi-modal extension of the basic tense logic. Only the syntax of its language is different, the semantics of tense Gaggle logic being the same as the basic tense logic.

**Definition 26** (Associated tense connectives). Let $\mathsf{C} \subseteq \mathbb{C}$. The set of *tense connectives associated to* $\mathsf{C}$, denoted $\mathsf{C}_\blacklozenge$, is defined by:

$$\mathsf{C}_\blacklozenge \triangleq (\mathsf{C} \cap \mathbb{P}) \cup \mathbb{T} \cup \bigcup_{\star \in \mathsf{C} \setminus \mathbb{P}} \left\{ \blacklozenge^-_{\star,i}, \blacksquare^-_{\star,i}, \blacklozenge_{\star,i}, \blacksquare_{\star,i} \mid \mathsf{a}(\star) = n, i \in [\![1;n]\!] \right\}$$

where for all $i \in [\![1;n]\!]$, $\blacklozenge_{\star,i} \triangleq (\sigma'_2, t'_1, \to)_i$, $\blacklozenge^-_{\star,i} \triangleq (\sigma'_1, t'_1, \to)_i$, $\blacksquare_{\star,i} \triangleq (\sigma'_1, t'_2, \leftarrow)_i$ and $\blacksquare^-_{\star,i} \triangleq (\sigma'_2, t'_2, \leftarrow)_i$. The set of *structural tense connectives associated to* $\mathsf{C}$ is defined by:

$$[\mathsf{C}_\blacklozenge] \triangleq \{*, \bullet, ,\} \cup \bigcup_{\star \in \mathsf{C} \setminus \mathbb{P}} \{\bullet_{\star,i} \mid \mathsf{a}(\star) = n, i \in [\![1;n]\!]\}$$

where $\bullet_{\star,i} \triangleq \left[\blacklozenge^-_{\star,i}\right]$. $\dashv$

**Definition 27** (Tense protolanguage). The *tense protolanguage* $\mathcal{L}_{\mathbb{C}_\blacklozenge}$ is the smallest set such that:

- for all $p = (Id, t, \leftrightarrow)_i \in \mathbb{P}$, we have that
  - if $\leftrightarrow = \to$ then $\blacklozenge^- p \in \mathcal{L}_{\mathbb{C}_\blacklozenge}$;
  - if $\leftrightarrow = \leftarrow$ then $\blacksquare^- p \in \mathcal{L}_{\mathbb{C}_\blacklozenge}$.
- for all $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ such that $\mathsf{a}(\star) = n > 0$ and $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$, for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathbb{C}_\blacklozenge}$,
  - if $\pm = -$ then the following belongs to $\mathcal{L}_{\mathbb{C}_\blacklozenge}$:
$$\blacklozenge^-_{\star,n}\big(\varphi_n \wedge \blacklozenge^-_{\star,n-1}\big(\varphi_{n-1} \wedge \ldots \wedge \blacklozenge^-_{\star,1}\varphi_1\big)\big) \tag{3}$$
  - if $\pm = +$ then the following belongs to $\mathcal{L}_{\mathbb{C}_\blacklozenge}$:
$$\blacksquare^-_{\star,n}\big(\varphi_n \vee \blacksquare^-_{\star,n-1}\big(\varphi_{n-1} \vee \ldots \vee \blacksquare^-_{\star,1}\varphi_1\big)\big) \tag{4}$$

  The *structural tense protolanguage* $[\mathcal{L}]_{\mathbb{C}_\blacklozenge}$ is the smallest set that contains the tense protolanguage and such that for all $\star \in \mathsf{C}$ and all $X_1, \ldots, X_n \in [\mathcal{L}]_{\mathbb{C}_\blacklozenge}$, the following belong to $[\mathcal{L}]_{\mathbb{C}_\blacklozenge}$:

$$\bullet_{\star,n}\,(X_n,\ \bullet_{\star,n-1}\,(X_{n-1},\ \ldots,\ \bullet_{\star,1}\,X_1)) \tag{5}$$
$$*\bullet_{\star,n} * (X_n,\ *\bullet_{\star,n-1} * (X_{n-1},\ \ldots,\ *\bullet_{\star,1} * X_1)) \tag{6}$$

Let $\mathsf{C} \subseteq \mathbb{C}$. The *(structural) tense protolanguage associated to $\mathsf{C}$*, denoted $\mathcal{L}_{\mathsf{C}_\blacklozenge}$ (resp. $[\mathcal{L}]_{\mathsf{C}_\blacklozenge}$), is the tense protolanguage $\mathcal{L}_{\mathbb{C}_\blacklozenge}$ (resp. $[\mathcal{L}]_{\mathbb{C}_\blacklozenge}$) restricted to the connectives of $\mathsf{C}_\blacklozenge$ (resp. $[\mathsf{C}_\blacklozenge]$). A $[\mathcal{L}]_{\mathsf{C}_\blacklozenge}$*–consecution* is an expression of the form $X \vdash Y$ where $X, Y \in [\mathcal{L}]_{\mathsf{C}_\blacklozenge}$. The class of all $[\mathcal{L}]_{\mathsf{C}_\blacklozenge}$–consecutions is denoted $\mathcal{S}_{\mathsf{C}_\blacklozenge}$. ⊣

**Definition 28** (Tense $\mathsf{C}$–model)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. A *tense $\mathsf{C}$–model* is a tuple $M_\blacklozenge = (W, \sqsubseteq, \mathcal{R}, \mathcal{P})$ where $W$ is a non-empty set, $\sqsubseteq$ is a partial order over $W$, $\mathcal{R}$ is a set of sequences $(R^1_\star, \ldots, R^n_\star)$ of binary relations over $W$ and $\mathcal{P}$ is a set of unary relations such that

1. each connective $\star \in \mathsf{C}$ such that $\mathsf{a}(\star) = n$ can be associated to a sequence of binary relations of $\mathcal{R}$ denoted $(R^1_\star, \ldots, R^n_\star)$ (if $n > 0$) or to a unary relation of $\mathcal{P}$ denoted $P_\star$ (if $n = 0$), and vice versa;
2. the set of sequences of binary relations $(R^1_\star, \ldots, R^n_\star)$ (associated to each connective $\star$ of $\mathsf{C}$) and the set of connectives $\star$ of $\mathsf{C}$ are in bijection;
3. for all $(R^1, \ldots, R^n) \in \mathcal{R}$,
   (a) for all $i \in [\![1; n-1]\!]$ and all $w, v \in W$, if $R^i wv$ then there is $u \in W$ such that $R^{i+1} vu$;
   (b) for all $i \in [\![2; n]\!]$ and all $w, v \in W$, if $R^i vw$ then there is $u \in W$ such that $R^{i-1} uv$;
4. for all $\star \in \mathsf{C}$ such that $(R^1, \ldots, R^n) = (R^1_\star, \ldots, R^n_\star)$ and whose signature is $(\sigma, t, \leftrightarrow)$, if $(\pm_1, \ldots, \pm_n) \mapsto \pm$ is the trace $\sigma^-(t)$ then for all $i \in [\![1; n]\!]$ and all $w_i, w_{i+1}, w'_i, w'_{i+1} \in W$, $R$ denoting $R^i$, we have:

$$\text{if } R^{\leftrightarrow} w_i w_{i+1}, w_i \square w'_i \text{ and } w_{i+1} \square w'_{i+1} \text{ then } R^{\leftrightarrow} w'_i w'_{i+1} \qquad (\text{Tonicity}_\blacklozenge)$$

where, if $j \in \{i, i+1\}$, $w_j \square w'_j \triangleq \begin{cases} w_j \sqsubseteq w'_j & \text{if } \pm_{\tau(j)}\pm = -; \\ w_j \sqsupseteq w'_j & \text{if } \pm_{\tau(j)}\pm = +; \\ w_j \sqsubseteq w'_j & \text{if } \tau(j) = n+1. \end{cases}$

We abusively write $w \in M_\blacklozenge$ for $w \in W$. A *pointed tense $\mathsf{C}$–model* $(M_\blacklozenge, w)$ is a tense $\mathsf{C}$–model $M_\blacklozenge$ with a state $w \in M_\blacklozenge$. The class of pointed tense $\mathsf{C}$–models is denoted $\mathcal{M}_{\mathsf{C}_\blacklozenge}$. ⊣

**Definition 29** (Tense Gaggle logic)**.** Let $\mathsf{C} \subseteq \mathbb{C}$, let $M_\blacklozenge$ be a tense $\mathsf{C}$–model and let $\mathcal{M}_\blacklozenge$ be a class of tense $\mathsf{C}$–models. We define the *extension function of $\mathcal{L}_{\mathsf{C}_\blacklozenge}$ in $M_\blacklozenge$* and we also define the *satisfaction relation* $\models \subseteq \mathcal{M}_\blacklozenge \times \mathcal{L}_{\mathsf{C}_\blacklozenge}$ as follows. For each sequence of binary relations $(R^1_\star, \ldots, R^n_\star)$ associated to a $\star \in \mathsf{C}$, we associate for all $i \in [\![1; n]\!]$ the modalities $\blacklozenge^-_{\star,i}$, $\blacksquare^-_{\star,i}$, $\blacklozenge_{\star,i}$, $\blacksquare_{\star,i}$ to the relation $R^i_\star$ and we define the truth conditions for these modalities like in Definition 24. The truth conditions for the modalities $\Diamond$, $\Box$, $\Diamond^-$, $\Box^-$ are defined as follows:

| | | |
|---|---|---|
| $M, w \models \Box\varphi$ | iff | for all $v \in M$ such that $w \sqsubseteq v$, $M, v \models \varphi$ |
| $M, w \models \Diamond\varphi$ | iff | there is $v \in M$ such that $w \sqsubseteq v$ and $M, v \models \varphi$ |
| $M, w \models \Box^-\varphi$ | iff | for all $v \in M$ such that $v \sqsubseteq w$, $M, v \models \varphi$ |
| $M, w \models \Diamond^-\varphi$ | iff | there is $v \in M$ such that $v \sqsubseteq w$ and $M, v \models \varphi$ |

Likewise, we extend the scope of the evaluation relation $\models$ simultaneously in two different ways in order to also relate worlds to structures of of $[\mathcal{L}]_{\mathsf{C}_\blacklozenge}$. The *antecedent evaluation relation*

$\overset{A}{\models} \subseteq \mathcal{M}_{\blacklozenge} \times [\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$ and the *consequent evaluation relation* $\overset{K}{\models} \subseteq \mathcal{M}_{\blacklozenge} \times [\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$ are defined inductively as follows. The truth conditions for the formulas are defined as above.

$$
\begin{array}{lll}
M, w \overset{A}{\models} * X & \text{iff} & \text{it is not the case that } M, w \overset{K}{\models} X \\
M, w \overset{K}{\models} * X & \text{iff} & \text{it is not the case that } M, w \overset{A}{\models} X \\
M, w \overset{A}{\models} \bullet X & \text{iff} & \text{there is } v \in M \text{ such that } v \sqsubseteq w \text{ and } M, v \overset{A}{\models} X \\
M, w \overset{K}{\models} \bullet X & \text{iff} & \text{for all } v \in M \text{ such that } w \sqsubseteq v, M, v \overset{K}{\models} X \\
M, w \overset{A}{\models} \bullet_{\star,i} X & \text{iff} & \text{there is } v \in M \text{ such that } R^i_{\star} vw \text{ and } M, w \overset{A}{\models} X \\
M, w \overset{K}{\models} \bullet_{\star,i} X & \text{iff} & \text{for all } v \in M \text{ such that } R^i_{\star} wv, M, v \overset{K}{\models} X
\end{array}
$$

We extend the scope of the relation $\models$ to also relate points to $[\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$–consecutions:

$$M, w \models X \vdash Y \quad \text{iff} \quad \text{if } M, w \overset{A}{\models} X, \text{ then } M, w \overset{K}{\models} Y.$$

We define $[\![\varphi]\!]_M \triangleq \{w \in M \mid M, w \models \varphi\}$. The triple $(\mathcal{S}_{\mathsf{C}_{\blacklozenge}}, \mathcal{M}_{\blacklozenge}, \models)$ is called the *tense Gaggle logic associated to* $\mathcal{M}_{\blacklozenge}$. When $\mathsf{C}_{\blacklozenge} = \mathbb{C}_{\blacklozenge}$, it is simply called the *tense Gaggle logic*. $\dashv$

Hence, tense Gaggle logic is simply a *multi–modal* tense logic (as defined in Definition 24).

## 5.3 From Gaggle logic to Tense Gaggle logic, and Back

In this section, we show that Gaggle logic and tense Gaggle logic are in fact equally expressive. To prove it, we exhibit the corresponding translations from one language to the other.

**Definition 30** (Translation $\tau_1$). We define inductively the translation $\tau_1 : [\mathcal{L}]_{\mathbb{C}} \to [\mathcal{L}]_{\mathbb{C}_{\blacklozenge}}$ as follows: for all $p = (\sigma, t, \leftrightarrow)_i \in \mathbb{P}$ and all $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ such that $\mathsf{a}(\star) > 0$ and $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$,

$$\tau_1(p) = \tau_1([p]) \triangleq \begin{cases} \blacklozenge^- p & \text{if } \leftrightarrow = \rightarrow \\ \blacksquare^- p & \text{if } \leftrightarrow = \leftarrow \end{cases}$$

$$\tau_1\big(\star(\varphi_1, \ldots, \varphi_n)\big) \triangleq \begin{cases} \blacklozenge^-_{\star,n}\big(\tau_1(\varphi_n) \wedge \blacklozenge^-_{\star,n-1}\big(\tau_1(\varphi_{n-1}) \wedge \ldots \wedge \blacklozenge^-_{\star,1}\tau_1(\varphi_1)\big)\big) \\ \text{if } \pm = - \\ \blacksquare^-_{\star,n}\big(\tau_1(\varphi_n) \vee \blacksquare^-_{\star,n-1}\big(\tau_1(\varphi_{n-1}) \vee \ldots \vee \blacksquare^-_{\star,1}\tau_1(\varphi_1)\big)\big) \\ \text{if } \pm = + \end{cases}$$

$$\tau_1\big([\star](X_1, \ldots, X_n)\big) \triangleq \begin{cases} \bullet_{\star,n}\left(\tau_1(X_n),\ \bullet_{\star,n-1}\left(\tau_1(X_{n-1}),\ \ldots,\ \bullet_{\star,1}\ \tau_1(X_1)\right)\right) \\ \text{if } \pm = - \\ * \bullet_{\star,n} * \left(\tau_1(X_n),\ * \bullet_{\star,n-1} * \left(\tau_1(X_{n-1}),\ \ldots,\ * \bullet_{\star,1} * \tau_1(X_1)\right)\right) \\ \text{if } \pm = +. \end{cases}$$

We extend this translation $\tau_1$ to $[\mathcal{L}]_{\mathbb{C}}$–consecutions:

$$\tau_1(X \vdash Y) \triangleq \tau_1(X) \vdash \tau_1(Y)$$

and then to inference rules for $[\mathcal{L}]_{\mathbb{C}}$–consecutions:

$$\tau_1(R) \triangleq \{(\tau_1(C_1), \ldots, \tau_1(C_l), \tau_1(C)) \mid (C_1, \ldots, C_l, C) \in R\}. \qquad \dashv$$

**Definition 31** (Translation $\tau_1^-$)**.** We define inductively the translation $\tau_1^- : [\mathcal{L}]_{\mathbb{C}_\blacklozenge} \rightarrow [\mathcal{L}]_{\mathbb{C}}$ as follows: for all $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ such that $\mathsf{a}(\star) > 0$,

$$\tau_1^-(\blacklozenge^- p) \triangleq p$$
$$\tau_1^-(\blacksquare^- p) \triangleq p$$
$$\tau_1^-(\text{Exp. (3)}) = \tau_1^-(\text{Exp. (4)}) \triangleq \star(\tau_1^-(\varphi_1), \ldots, \tau_1^-(\varphi_n))$$
$$\tau_1^-(\text{Exp. (5)}) = \tau_1^-(\text{Exp. (6)}) \triangleq [\star](\tau_1^-(X_1), \ldots, \tau_1^-(X_n))$$

We extend this translation $\tau_1^-$ to $[\mathcal{L}]_{\mathbb{C}_\blacklozenge}$–consecutions:

$$\tau_1^-(X \models Y) \triangleq \tau_1^-(X) \models \tau_1^-(Y)$$

and then to inference rules for $[\mathcal{L}]_{\mathbb{C}_\blacklozenge}$–consecutions:

$$\tau_1^-(R) \triangleq \left\{(\tau_1^-(C_1), \ldots, \tau_1^-(C_l), \tau_1^-(C)) \mid (C_1, \ldots, C_l, C) \in R\right\}. \qquad \dashv$$

**Proposition 4.** *The translations $\tau_1$ and $\tau_1^-$ are inverse bijections. In particular, if $R$ is an inference rule for $[\mathcal{S}]_{\mathbb{C}}$ and $R'$ is an inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$, then $\tau_1\left(\tau_1^-\left(R'\right)\right) = R'$ and $\tau_1^-(\tau_1(R)) = R$.*

*Proof.* It suffices to examine the base and inductive steps defining $\tau_1$ and $\tau_1^-$ to notice that these mappings are inverse of each other. □

# 6 Protoanalytic Inference Rule and Cut Elimination

In this section, we introduce a generalization in Gaggle logic of the so–called 'analytic' or 'display' rules of display calculi [5], that we call 'protoanalytic inference rules'. These rules turn out to have similar properties, but our protoanalytic inference rules preserve the interpolation property of Gaggle logic as well.

## 6.1 Protoanalytic Inference Rule

'Protoantecedent part' and 'protoconsequent part' are a generalization of the notions of 'antecedent part' and 'consequent part' of display calculi.

**Definition 32** (Protoantecedent and protoconsequent part)**.** Let $X, Y, Z \in [\mathcal{L}]$ be structures. If $Z$ is a substructure of $X$, then $tn(X, Z)$ is defined inductively as follows:

- if $X = Z$ then $tn(X, Z) \triangleq +$;
- if $X = [\star](X_1, \ldots, X_n)$ and $Z$ appears in $X_j$ then $tn(X, Z) \triangleq tn(\star, j) tn(X_j, Z)$.

If $X \models Y$ is a $[\mathcal{L}]$–consecution, then $X$ is called the *antecedent* and $Y$ is called the *consequent* of $X \models Y$. If $Z$ is a substructure of $X$ or $Y$, $Z$ is called a *protoantecedent part* (resp. *protoconsequent part*) of $X \models Y$ when $tn(X, Z) = +$ or $tn(Y, Z) = -$ (resp. $tn(X, Z) = -$ or $tn(Y, Z) = +$). $\dashv$

As we shall see later, protoantecedent (resp. protoconsequent) parts of a $[\mathcal{L}]$–consecution $X \models Y$ can always be 'displayed' (appear) as the sole antecedent (resp. consequent) of a provably equivalent consecution in $\mathsf{Galog}$.

**Definition 33** (Protoanalytic structures, consecutions and inference rules)**.**

1. A *protoanalytic structure* is a structure $A \in [\mathcal{L}]$ containing only structural connectives and such that either $A \in [\mathbb{P}]$ or for all substructures $[\star]\,(X_1, \ldots, X_n)$ of $A$ with $\star$ of trace $(\pm_1, \ldots, \pm_n) \mapsto \pm$, for all $j \in [\![1;n]\!]$, if $X_j = [\star_j]\,(Y_1, \ldots, Y_{n_j})$ with $\star_j \in [\mathbb{C}] \setminus [\mathbb{P}]$ of trace $(\pm^j_1, \ldots, \pm^j_{n_j}) \mapsto \pm^j$ then $\pm_j = \pm^j$.

2. A *protoanalytic consecution* is a consecution $A \vdash B \in [\mathcal{S}]$ where $A$ and $B$ are protoanalytic structures such that:

   (a) for all substructures $[p] = [(Id, \pm, \leftrightarrow)_i] \in [\mathbb{P}]$ of $A \vdash B$, we have that $\leftrightarrow = \rightarrow$ if, and only if, $[p]$ is a protoantecedent part of $A \vdash B$;

   (b) if $A = [\star_1]\,(X_1, \ldots, X_{n_1})$ and $B = [\star_2]\,(Y_1, \ldots, Y_{n_2})$ then $out(\star_1) = -$ and $out(\star_2) = +$.

3. A *protoanalytic inference rule* is an inference rule $R$ for $[\mathcal{S}]$ of the following form:

$$\frac{A_1 \vdash B_1 \quad \ldots \quad A_n \vdash B_n}{A \vdash B}\ R$$

   where $A_1 \vdash B_1, \ldots, A_n \vdash B_n$ and $A \vdash B$ are protoanalytic consecutions such that:

   (a) for all $[p] \in [\mathbb{P}]$ appearing in some premise $A_i \vdash B_i$, either $[p]$ or $[\overline{p}]$ appears in the conclusion $A \vdash B$;

   (b) for all $[p] \in [\mathbb{P}]$ appearing both in the conclusion $A \vdash B$ and in some premise $A_i \vdash B_i$, $[p]$ is protoantecedent part of $A_i \vdash B_i$ if, and only if, $[p]$ is protoantecedent part of $A \vdash B$;

   (c) there is a common $[p] \in [\mathbb{P}]$ appearing in all consecutions of the inference rule;

   (d) there is no $[p] \in [\mathbb{P}]$ appearing twice in $A \vdash B$. $\dashv$

## 6.2 Cut Elimination and Display Property

In this section, we show that the cut rule can always be eliminated from any proof of Galog and from any extension of Galog with protoanalytic rules.

**Lemma 6.**

- *$R$ is a protoanalytic inference rule if, and only if, $\tau_1(R)$ is equivalent to a special inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ containing only structural connectives;*
- *$R$ is a special inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ containing only structural connectives if, and only if, $\tau_1^-(R)$ is a protoanalytic inference rule.*

*Proof.* It suffices to prove the first item, the second item following from Proposition 4.

Let $R$ be a protoanalytic inference rule. Because of the rule $R^{-j}$ and Condition 3)(b) of Definition 33, $R$ is equivalent to a protoanalytic inference rule of one of the following form:

$$\frac{A_1 \vdash B \quad \ldots \quad A_n \vdash B}{A \vdash B}\ R' \qquad\qquad \frac{A \vdash B_1 \quad \ldots \quad A \vdash B_n}{A \vdash B}\ R'' \tag{7}$$

where $A$ or $B$ is the common $[p]$ appearing in all consecutions of $R$. Then, because of Corollary 3, $R''$ is also equivalent to a rule of the same form as $R'$. Moreover, by Proposition 4, $\tau_1(R)$ is equivalent to a special inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ iff $\tau_1(R')$ is equivalent to a special inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$. So, w.l.o.g., we only consider the protoanalytic inference rules of the form of $R'$.

To show that $\tau_1(R')$ is equivalent to a special inference rule, it suffices to show that $\tau_2(\tau_1(R'))$ is a primitive formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$, because of [35]. Now, because of Condition 2)(b) of Definition 33, the output of the trace of the outermost connective of $A_1, \ldots, A_n$ and $A$ is $-$. So, by Definition of $\tau_1$, the translations $\tau_1(A_1), \ldots, \tau_1(A_n)$ and $\tau_1(A)$ are of the form of Expression 5:

$$\bullet_{\star,n}\left(\tau_1\left(X_n\right),\ \bullet_{\star,n-1}\left(\tau_1\left(X_{n-1}\right),\ \ldots,\ \bullet_{\star,1}\,\tau_1\left(X_1\right)\right)\right) \tag{8}$$

Moreover, again by Definition of $\tau_1$ and Condition 1) of Definition 33, $\tau_1(X_1), \ldots, \tau_1(X_n)$ are again of the form (8). Finally, together with Condition 2) (a), we obtain that $\tau_1(A_1), \ldots, \tau_1(A_n)$ and $\tau_1(A)$ are expressions belonging to the language defined by the following grammar in BNF:

$$X \ ::= \ \blacklozenge^{-}p \ \mid \ \bullet_{\star,n}\left(X,\ \bullet_{\star,n-1}\left(X,\ \ldots,\ \bullet_{\star,1}\,X\right)\right) \tag{9}$$

where $p = (Id, \pm, \leftrightarrow)_i \in \mathbb{P}$ with $\leftrightarrow = \rightarrow$ and $\star \in \mathbb{C}$. Thus, $\tau_2(\tau_1(R')) = t_1(\tau_1(A)) \rightarrow t_1(\tau_1(A_1)) \vee \ldots \vee t_1(\tau_1(A_n))$ is a formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$ built up only with propositionnal letters $p = (Id, \pm, \rightarrow) \in \mathbb{P}$ and the 'sometimes positive' connectives $\left\{\blacklozenge^{-}, \blacklozenge^{-}_{\star,n}, \blacklozenge^{-}_{\star,n-1}, \ldots\right\}$ of $\mathbb{C}_\blacklozenge$ such that there is no $p \in \mathbb{P}$ appearing twice in the antecedent of $\tau_2(\tau_1(R'))$. Thus, $\tau_2(\tau_1(R'))$ is a primitive formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$. Hence, $\tau_1(R')$, and thus $\tau_1(R)$, are special inference rules for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$.

To prove the converse, that is, if $\tau_1(R)$ is a special inference rule then $R$ is a protoanalytic inference rule, it suffices to prove that if $R$ is a special inference rule of $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$, then $\tau_1^{-}(R)$ is a protoanalytic inference rule, because of Proposition 4. Because $\tau_2^{-}(\tau_2(R))$ is equivalent to $R$ by Proposition 8, we can assume without loss of generality that $R$ is of the form of $R'$ with $B = [(Id, \pm, \leftarrow)_i] \in [\mathbb{P}]$. If Condition 1) of Definition 33 is not fulfilled, then there is a substructure $X = [\star]\,(X_1, \ldots, X_n)$ of $A_1, \ldots, A_n$ or $A$ such that $\star$ is of trace $(\pm_1, \ldots, \pm_n) \mapsto \pm$ and $X_j = [\star_j]\,(Y_1, \ldots, Y_{n_j})$ is such that $\star_j$ is of trace $(\pm^j_1, \ldots, \pm^j_{n_j}) \mapsto \pm^j$ with $\pm_j = \pm^j$. Let us consider such a substructure $X$. Then, the translation of $X$ by $t_1 \circ \tau_1$ and $t_2 \circ \tau_1$ will be such that we will have an alternation of modalities of the form $\blacklozenge\blacksquare$ or $\blacksquare\blacklozenge$. Such an alternation is not allowed in a primitive formula. Hence, $\tau_2 \circ \tau_1(R)$ cannot be a primitive formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$. So, $\tau_1(R)$ cannot be a special rule of $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ by Theorem 12. Therefore, if $\tau_1(R)$ is a special rule of $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$, then Condition 1) of Definition 33 is fulfilled. The proof for Condition 2) of Definition 33 follows the same pattern as Condition 1). In that case, if Condition 2) was not fulfilled, we would obtain a subformula of the form $\blacksquare\overline{p}$ or more generally of the form $\blacksquare\varphi$ which cannot occur in a primitive formula. The proof for Condition 3) follows from the Conditions $(C1)$–$(C8)$ fulfilled by all special inference rules. Condition 3) (a) follows from $(C1)$. Condition 3) (b) follows from $(C4)$. Condition 3) (c) follows from the fact that in the special rule of the form above, the structure $B$ is the common consequent of all consecutions. Condition 3)(d) follows from $(C3)$. □

**Lemma 7.** *Let $\tau_1$(Galog) denote the calculus for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ obtained from Galog by translating all its axioms and inference rules for $[\mathcal{S}]_{\mathbb{C}}$ with the mapping $\tau_1$ (of Definition 30). Then, $\tau_1$(Galog) is a proper display calculus for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ which is sound and complete for the tense Gaggle logic $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}_{\mathbb{C}_\blacklozenge}, \Vdash\right)$. Moreover, each proof in $\tau_1$(Galog) can be mapped to a proof in Galog, and vice versa.*

*Proof.* It suffices to observe that all the structural rules of $\mathsf{Galog}$ are protoanalytic and that the translation of a protoanalytic inference rule by $\tau_1$ is an inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$ that satisfies the conditions (C1)–(C8) of Belnap [5]: the notions of protoantecedent and protoconsequent parts are just replaced by the notions of antecedent and consequent parts. Hence, $\tau_1(\mathsf{Galog})$ is a proper display calculus. Moreover, because $\tau_1$ is a bijection by Lemma 6, each proof in $\tau_1(\mathsf{Galog})$ can be mapped to a proof in $\mathsf{Galog}(\mathbb{C})$, and vice versa. □

**Theorem 4** (Display property)**.** *The calculus $\mathsf{Galog}$ satisfies the* display property*: for each $[\mathcal{L}]$–consecution $X \models Y$ and each protoantecedent part (resp. protoconsequent part) $Z$ of $X \models Y$, if $X \models Y$ is provable in $\mathsf{Galog}$ then there exists a structure $W \in [\mathcal{L}]$ such that $Z \models W$ (resp. $W \models Z$) is provable in $\mathsf{Galog}$.*

*Proof.* It follows from Lemma 7 and the fact that every proper display calculus satisfies the display property. □

**Theorem 5** (Cut elimination)**.** *$\mathsf{Galog}$ enjoys strong cut elimination.*

*Proof.* It follows from Lemma 7 and the fact that every proper display calculus enjoys strong cut elimination. □

### 6.3 Consequences of Cut Elimination

As usual in proof theory and ever since Gentzen [23], the fact that the cut rule can be eliminated from any proof is of practical and theoretical importance and we easily obtain a number of significant results about our logics: decidability, conservativity and interpolation.

**Theorem 6.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{Galog}(\mathsf{C})$ is sound and complete for the logic $([\mathcal{S}]_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

*Proof.* It is standard. See for example [45] for details. □

**Theorem 7** (Decidability)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $X, Y \in [\mathcal{L}]_{\mathsf{C}}$. The problem of determining whether $X$ or $X \models Y$ are valid in the logics $([\mathcal{L}]_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ and $([\mathcal{S}]_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ (respectively) is decidable.*

*Proof.* It suffices to observe that the set of consecutions that can lead to a *cut-free* proof of $X \models Y$ in $\mathsf{Galog}(\mathsf{C})$ is finite. The problem of finding a proof of $X \models Y$ thus boils down to a graph reachability problem in a finite graph whose edges are labeled by the rules, which is decidable. We then obtain the result by completeness of $\mathsf{Galog}(\mathsf{C})$ for $([\mathcal{L}]_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ and $([\mathcal{S}]_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$. □

Note that the dual problem, the satisfiability problem for structures $X$, is also decidable, because of Proposition 2.

**Theorem 8** (Conservativity)**.** *If $\mathsf{C} \subseteq \mathsf{C}' \subseteq \mathbb{C}$ then the logic $\left([\mathcal{S}]_{\mathsf{C}'}, \mathcal{M}_{\mathsf{C}'}, \Vdash\right)$ is a conservative extension of the logic $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash\right)$.*

*Proof.* It is standard. See for example [45] for details. □

Interpolation is intimately connected with consistency, compactness and definability [19]. In computer science, it plays an important role in systems that need to be decomposed or refined into simpler components. Interpolation has been used in problems such as invariant generation [40], type inference [31], model checking [38, 39] and the decomposition of complex ontologies

[33]. Thus, whether a given logic satisfies interpolation is of practical importance in computer science as well as of theoretical importance in logic.

It turns out that there are various versions of the notion of interpolation (see the discussion in [56] for example). Here, we deal with the original notion of interpolation which was originally introduced for first-order logic by Craig [16]. Our proof method is different from the method to prove interpolation of display calculi of Brotherston and Goré [9].

**Theorem 9** (Interpolation)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{Galog}(\mathsf{C})$ satisfies the* interpolation property*: for all $X, Y \in [\mathcal{L}]_{\mathsf{C}}$, if $X \models Y$ is provable in $\mathsf{Galog}(\mathsf{C})$, then there exists a formula $\varphi \in \mathcal{L}_{\mathsf{C}}$ called the* interpolant *such that*

1. *$X \models \varphi$ and $\varphi \models Y$ are both provable in $\mathsf{Galog}(\mathsf{C})$;*
2. *$V(\varphi) \subseteq V(X) \cap V(Y)$;*

*where for all $Z \in [\mathcal{L}]_{\mathsf{C}}$ $V(Z)$ is the set of propositional letters and their Boolean negation (possibly appearing in propositional structures) of $Z$. Moreover, the interpolant can be effectively constructed from the cut–free proof of $X \models Y$ in $\mathsf{Galog}(\mathsf{C})$.*

*Proof.* By induction on the length $n$ of the proof of $X \models Y$.

$n = 0$. In that case, $X \models Y$ is of the form $[p] \models p$ or $p \models [p]$. In both cases, the propositional letter $p$ is the interpolant since $p \models p$ is provable in $\mathsf{Galog}(\mathsf{C})$.

$n+1$. For the induction step, we consider the last rule applied in the proof of $X \models Y$ and we apply the Induction Hypothesis to the premises of that rule. Then, by suitable combinations, we show that an interpolant of the conclusion of that rule can also be defined from the interpolant of the premises.

Let $\star = (\sigma, t, \leftrightarrow)_i \in \mathbb{C}$ with $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$.

- Rule $R^K_\star$: $\dfrac{U_1 \models V_1 \quad \ldots \quad U_n \models V_n}{S([\star], X_1, \ldots, X_n, \star(\varphi_1, \ldots, \varphi_n))}$

  where for all $j \in [\![1; n]\!]$, $(U_j, V_j) \triangleq \begin{cases} (X_j, \varphi_j) & \text{if } \pm_j \pm = + \\ (\varphi_j, X_j) & \text{if } \pm_j \pm = - \end{cases}$.

  By Induction Hypothesis, there are formulas $\chi_1, \ldots, \chi_n \in \mathcal{L}_{\mathsf{C}}$ such that for all $j \in [\![1; n]\!]$, $U_j \models \chi_j$ and $\chi_j \models V_j$ with $V(\chi_j) \subseteq V(U_j) \cap V(V_j)$. Now, let us define $\chi \triangleq \star(\chi_1, \ldots, \chi_n)$. We are going to show that $\chi$ is an interpolant of $X \models Y$.

  Without loss of generality, assume that $\pm = -$. Then, $S([\star], X_1, \ldots, X_n, \star(\varphi_1, \ldots, \varphi_n))$ is $[\star](X_1, \ldots, X_n) \models \star(\varphi_1, \ldots, \varphi_n)$. Let us define for all $j \in [\![1; n]\!]$ the following consecutions:

  $$U'_j \models V'_j \triangleq \begin{cases} X_j \models \chi_j & \text{if } \pm_j \pm = + \\ \chi_j \models X_j & \text{if } \pm_j \pm = - \end{cases}$$

  Then, applying the rule $R^K_\star$ on the premises $U'_1 \models V'_1, \ldots, U'_n \models V'_n$, we obtain that the following consecution is provable in $\mathsf{Galog}(\mathsf{C})$:

  $$[\star](X_1, \ldots, X_n) \models \star(\chi_1, \ldots, \chi_n). \tag{10}$$

  Likewise, if we define for all $j \in [\![1; n]\!]$ the following consecutions:

  $$U''_j \models V''_j \triangleq \begin{cases} \chi_j \models \varphi_j & \text{if } \pm_j \pm = + \\ \varphi_j \models \chi_j & \text{if } \pm_j \pm = - \end{cases}$$

then, applying the rule $R_\star^K$ on the premises $U_1'' \vdash V_1'', \ldots, U_n'' \vdash V_n''$ we obtain that $[\star](\chi_1, \ldots, \chi_n) \vdash \star(\varphi_1, \ldots, \varphi_n)$ and therefore, by rule $R_\star^A$, the following consecution is provable in $\mathsf{Galog}(\mathsf{C})$:

$$\star(\chi_1, \ldots, \chi_n) \vdash \star(\varphi_1, \ldots, \varphi_n). \tag{11}$$

Moreover,

$$\begin{aligned} V(\star(\chi_1, \ldots, \chi_n)) &= V(\chi_1) \cup \ldots \cup V(\chi_n) \\ &\subseteq (V(X_1) \cap V(\varphi_1)) \cup \ldots \cup (V(X_n) \cap V(\varphi_n)) \\ &\subseteq (V(X_1) \cup \ldots \cup V(X_n)) \cap (V(\varphi_1) \cup \ldots \cup V(\varphi_n)) \\ &= V([\star](X_1, \ldots, X_n)) \cap V(\star(\varphi_1, \ldots, \varphi_n)) \end{aligned} \tag{12}$$

From Expressions (10), (11) and (12), we obtain the expected result.

- Rule $R_\star^A$: $\dfrac{[\star](\varphi_1, \ldots, \varphi_n) \vdash X}{\star(\varphi_1, \ldots, \varphi_n) \vdash X}$. By Induction Hypothesis, there is $\chi \in \mathcal{L}(\mathsf{C})$ such that $[\star](\varphi_1, \ldots, \varphi_n) \vdash \chi$, $\chi \vdash X$ and $V(\chi) \subseteq V([\star](\varphi_1, \ldots, \varphi_n)) \cap V(X)$. So, again by application of $R_\star^A$, we have that $\star(\varphi_1, \ldots, \varphi_n) \vdash \chi$ and $\chi \vdash X$ with $V(\chi) \subseteq V(\star(\varphi_1, \ldots, \varphi_n)) \cap V(X)$.

- Rule $R^j$: $\dfrac{S([\star], X_1, \ldots, X_n, X)}{\overline{S\left(\left[\star^j\right], X_1, \ldots, \overline{X_j}, \ldots, X_n, X\right)}}$. It follows the same reasoning as rule $R_\star^A$, because $V(X) = V(\overline{X})$.

- Rule $R^{-j}$: $\dfrac{S([\star], X_1, \ldots, X_j, \ldots, X_n, Y)}{\overline{S\left(\left[\star^{-j}\right], X_1, \ldots, Y, \ldots, X_n, X_j\right)}}$. First, assume that $\pm = -$ and $\pm_j = +$. Then, rule $R^{-j}$ rewrites as follows: $\dfrac{[\star](X_1, \ldots, X_j, \ldots, X_n) \vdash Y}{\overline{\left[\star^{-j}\right](X_1, \ldots, Y, \ldots, X_n) \vdash X_j}}$. By Induction Hypothesis, there is $\chi \in \mathcal{L}(\mathsf{C})$ such that $[\star](X_1, \ldots, X_j, \ldots, X_n) \vdash \chi$ and $\chi \vdash Y$ with $V(\chi) \subseteq V(Y) \cup V([\star](X_1, \ldots, X_n))$. Then, by Rule $R_\star^A$, we have that

$$\star^j(\psi_1, \ldots, \chi, \ldots, \psi_n) \vdash X_j \tag{13}$$

where for all $i \in [\![1;n]\!] \backslash \{j\}$, $\psi_i \triangleq \tau(X_i)$ ($\tau$ is introduced in Definition 10). Moreover, we have that $tn(\star^{-j}, j) = \pm_j \pm = +- = -$. So, because we have that $\chi \vdash Y$, it holds that $\left[\star^{-j}\right](X_1, \ldots, Y, \ldots, X_n) \vdash \left[\star^{-j}\right](X_1, \ldots, \chi, \ldots, X_n)$, and therefore by rule $R_\star^K$ we have that

$$\left[\star^{-j}\right](X_1, \ldots, Y, \ldots, X_n) \vdash \star^{-j}(\psi_1, \ldots, \chi, \ldots, \psi_n). \tag{14}$$

Besides,

$$\begin{aligned} V(\star^{-j}(\psi_1, \ldots, \chi, \ldots, \psi_n) &= V(\chi) \cup V(\psi_1) \cup \ldots \cup V(\psi_n) \\ &\subseteq V(Y) \cup V([\star](X_1, \ldots, X_n)) \end{aligned} \tag{15}$$

From Expressions (13), (14) and (15), we obtain the expected result. The proof is similar for all the other cases: $\pm = -$ and $\pm_j = -$, $\pm = +$ and $\pm_j = +$, $\pm = +$ and $\pm_j = -$. $\square$

### 6.4 Protodisplay Logics

**Definition 34** (Protodisplay logic)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. A logic for $[\mathcal{S}]_{\mathsf{C}}$ is *protodisplay* when there exists a calculus extending $\mathsf{Galog}(\mathsf{C})$ with protoanalytic inference rules which is sound and complete for that logic. $\dashv$

**Theorem 10.** *Every protodisplay logic enjoys strong cut elimination and satisfies the display property. Moreover, if the protodisplay logic contains the conjunction $\wedge$ and disjunction $\vee$ connectives of classical logic in its language, then it also satisfies the interpolation property.*

*Proof.* It is similar to the proofs given for Gaggle logic of the same properties. To prove the strong cut elimination and the display property, we only need to show that if $R$ is a protoanalytic inference rule then $\tau_1(R)$ is a special inference rule for $[\mathcal{S}]_{\mathbb{C}_{\blacklozenge}}$, and this is exactly what Lemma 6 tells us.

To prove the interpolation property, we need to show that the induction step in the proof of Theorem 9 holds also for protoanalytic inference rules. So, let $R$ be a protoanalytic inference rule. Because of the rule $R^{-j}$ and Condition 3)(b) of Definition 33, $R$ is equivalent to a protoanalytic inference rule of one of the following form:

$$\frac{A_1 \models B \quad \ldots \quad A_n \models B}{A \models B}\,R' \qquad\qquad \frac{A \models B_1 \quad \ldots \quad A \models B_n}{A \models B}\,R''$$

where $A$ or $B$ is the common $[p]$ appearing in all consecutions of $R$.

First, let us consider a rule of the form $R'$. Then, by Induction hypothesis, there are $\varphi_1, \ldots \varphi_n \in \mathcal{L}$ such that $A_1 \models \varphi_1$ and $\varphi_1 \models B$, …, $A_n \models \varphi_n$ and $\varphi_n \models B$ where, for all $j \in [\![1; n]\!]$, we have that $V(\varphi_j) \subseteq V(A_j) \cap V(B)$. If we define $\varphi \triangleq \varphi_1 \vee \ldots \vee \varphi_n$, then the following holds: $A_1 \models \varphi, \ldots, A_n \models \varphi$. Hence, from rule $R'$, we have that $A \models \varphi$. Moreover, we also have that $\varphi \models B$. So, we only need to prove that $V(\varphi) \subseteq V(A) \cap V(B)$. First, $V(\varphi) = V(\varphi_1) \cup \ldots \cup V(\varphi_n) \subseteq V(B)$ because $V(\varphi_j) \subseteq V(B)$ for all $j \in [\![1; n]\!]$ by assumption. Second, $V(\varphi) = V(\varphi_1) \cup \ldots \cup V(\varphi_n) \subseteq V(A_1) \cup \ldots \cup V(A_n)$ and $V(A_1) \cup \ldots \cup V(A_n) \subseteq V(A)$ because of Conditions 3) (a) of Definition 33. So, $V(\varphi) \subseteq V(A) \cap V(B)$. Hence, $\varphi$ is the interpolant of $A \models B$.

The proof for the case of a rule of the form $R''$ is similar, except that the interpolant to consider is $\varphi \triangleq \varphi_1 \wedge \ldots \wedge \varphi_n$. $\square$

# 7 Conclusion

### 7.1 Related Work

Generalized Lambek Calculus is a generalization of the Lambek calculus to connectives of arbitrary arities also based on the abstract laws of residuation [32, 11]. It is in fact a protodisplay logic, but it is less expressive and general than our Gaggle logic.

A number of results about interpolation for modal, intuitionistic, substructural and display logics already exist [21, 22, 45]. Failure of interpolation for a given logic implies that this logic is not protodisplay. Hence, ticket entailement and many relevance logics are not protodisplay [53] as well as some fragments of linear logics [50]. On the other hand, many displayable logics satisfy the interpolation property [9], because the notion of protodisplayability is a generalization of the notion of displayability.

## 7.2 Concluding Remarks

We proved the decidability of Gaggle logic thanks to proof–theoretical methods. However, even if Gaggle logic is decidable, this might not be the case in general for every protodisplay logic. *A priori*, this depends on the shape and nature of the protoanalytic inference rule(s) added to Gaggle logic (for example, if rules are *expansive* or not [48]).

This work and our results have applications in computer science. Indeed, if we show that a logic is protodisplay then it satisfies the interpolation property, which is of practical importance in computer science (as argued in Section 6.3). For example, bunched logics $\{BI, BBI, dMBI, CBI\}$ [44, 10, 8, 30] are closely related to separation logic [49] and are used for proving properties of programs and memory models. These logics have been provided with a ternary relational semantics [37] and a display calculus [8]. For that reason, they may be all protodisplay and hence satisfy the interpolation property. Note that this goes against the conjecture of Brotherston and Goré [9] according to which interpolation should fail for bunched logics such as BI.

Protodisplay logics are only a subset of all non–classical logics (see the previous section). However, we should not forget that protodisplay logics are only some of the logics that extend Gaggle logic. Other kinds of extensions of Gaggle logic are obviously possible. Given the generality and abstract nature of Gaggle logic, other non–classical logics could still be axiomatized as extensions of Gaggle logic by means of less restrictive inference rules than our protoanalytic inference rules.

# Part II

# A Characterization of Protodisplay Logics

# 8 Introduction

There are three possible equivalent approaches for formally introducing and defining a logic equipped with a calculus. We are going to set them out. In the first two approaches, we start by defining a *logical language* which consists of a set of well-formed formulas. Then, there are two different alternatives to follow: a *semantically-driven* alternative and a *syntactically-driven* alternative.

1. In the *semantically-driven* alternative, we start by providing a semantics to the well-formed formulas by means of a class of frames or models and a *satisfaction relation*. This semantics gives meaning to well-formed formulas and defines at the same time a set of *validities*: the well-formed formulas which are satisfied or true in every model. To capture the set of validities, we define a *calculus*, which is a (finite) set of *axiom schemas* and *inference rules* from which we can derive specific well-formed formulas called *theorems*.

2. In the *syntactically-driven* alternative, we start by providing a calculus that defines a set of theorems. This set of theorems is another means to characterize and define the logic. Then, we define a semantics for this logic which defines in turn a set of validities.

The equality of the set of validities and the set of theorems is captured by the notions of *soundness* and *completeness* (one notion for each inclusion). In that case, we say that the calculus *axiomatizes* the logical language for the semantics that we have defined.

3. In the third approach, we proceed the other way around. We first define a semantics consisting of a specific class of models and then a logical language with a satisfaction relation to 'talk about' these models. This defines in turn a set of validities that we can axiomatize with a calculus, as in the first approach.

In all cases, the three approaches lead to the same outcome: a logical language equipped with a syntax and semantics, together with a calculus axiomatizing the set of validities of the logic (*i.e.*, a calculus such that its set of theorems is the set of validities of the logic).

Correspondence theory investigates to what extent specific properties of the semantics can be reformulated in terms of the validity of specific formulas or inference rules. More precisely, it addresses the following kinds of questions. When does the truth of a given formula in a frame corresponds to a first-order property in that frame ? When does the validity of a formula on a class of frames correspond to the fact that this class of frames satisfies a specific first-order property ? Which class of frames defined by a first-order property can be defined by a formula ? The answers to these questions are already well-known for the basic tense logic with respect to the Kripke semantics: the Sahlqvist–van Benthem and Kracht theorems answer the first two questions and the Goldblatt–Thomason theorem answers the third question (see [7, 54] for more details on correspondence theory for modal and tense logics).

In this second part, we develop a correspondence theory for Gaggle logic using the correspondence results already known for basic tense logic. Gaggle logic introduces connectives of arbitrary arity related to each other by abstract laws of residuation. We prove in that article that a logic extending Gaggle logic is axiomatizable with so-called 'protoanalytic' inference rules if, and only if, the class of frames on which such a logic is based is definable by specific first-order frame conditions, called 'protoanalytic' as well. We also provide algorithms that compute the protoanalytic inference rules corresponding to some given protoanalytic first-order frame conditions and, vice versa, we provide algorithms that compute the protoanalytic first-order frame conditions corresponding to some given protoanalytic inference rules.

*Organization of Part II.* In Section 2, we recall the basic results of (partial) Gaggle theory. In Section 3.1, we show how to define logical functions providing semantics to connectives from the results of (partial) Gaggle theory. This leads us in Section 3.2 to introduce 'Gaggle logic'. In Section 4, we introduce a calculus which is sound and complete for Gaggle logic. In Section 5, we recall the main results of correspondence theory for the basic tense logic. This leads us in Section 10 to develop our correspondence theory for Gaggle logic thanks to the results for the basic tense logic and the embedding of Gaggle logic into basic tense logic. Our main result, Theorem 14, is stated in Section 10.2. It characterizes completely in semantical terms the protodisplayble logics. Then, in Section 11, we provide examples of translations from protoanalytic inference rules to first-order frame conditions (Section 11.1), and vice versa (Section 11.2). Using these correspondence results, we show in Section 12 how one can recover the logics captured by Gaggle theory as extensions of the fragment of Gaggle logic with binary and unary connectives by adding suitable protoanalytic inference rules. Finally, in Section 13, we discuss related works and then conclude.

# 9 Correspondence Theory for Basic Tense Logic

Correspondence theory investigates to what extent specific properties of accessibility relations can be reformulated in terms of the validity of specific formulas or inference rules. It addresses the following kinds of questions: when does the truth of a given (modal or tense) formula in a frame corresponds to a first-order property in this frame ? (Sahlqvist correspondence theorem); and when does the validity of a (modal or tense) formula on a class of frames corresponds to the fact that this class of frames satisfies a specific first-order property (and vice versa) ? (Sahlqvist and Kracht theorems) (see [7, 54] for more details on correspondence theory for modal and tense logic).

**Definition 35** (Binary FOL and SOL frame languages)**.** The *(binary) first-order frame language*, denoted $\mathcal{L}_{\mathbb{T}}^{\text{FOL}}$, is the first-order language that has the identity symbol $=$ and the binary relation $R$. The *(binary) second-order frame language*, denoted $\mathcal{L}_{\mathbb{T}}^{\text{SOL}}$, is the second-order language obtained by augmenting $\mathcal{L}_{\mathbb{T}}^{\text{FOL}}$ with a collection of monadic predicate variables $P_1, P_2, \dots$ associated to each $p_1, p_2, \dots \in \mathbb{P}$. The satisfaction relations $\models_{\text{FOL}}$ and $\models_{\text{SOL}}$ for the languages $\mathcal{L}_{\mathbb{T}}^{\text{FOL}}$, $\mathcal{L}_{\mathbb{T}}^{\text{SOL}}$ respectively on the class of all pointed Kripke frames $\mathcal{F}$ are defined as usual (see [41] for example). $\dashv$

**Notation 2.** Like in [35], we introduce the following abbreviations:

$$(\forall y \rhd x)\alpha(y) \triangleq \forall y(Rxy \to \alpha(y))$$
$$(\forall y \lhd x)\alpha(y) \triangleq \forall y(Ryx \to \alpha(y))$$
$$(\exists y \rhd x)\alpha(y) \triangleq \exists y(Rxy \wedge \alpha(y))$$
$$(\exists y \lhd x)\alpha(y) \triangleq \exists y(Ryx \wedge \alpha(y))$$

We call the constructs $(\forall y \rhd x), (\forall y \lhd x)$ and $(\exists y \rhd x), (\exists y \lhd x), (\exists yz \rhd x)$ *restricted universal (resp. existential) quantifiers.* $\dashv$

**Definition 36** (Standard translation for tense formulas)**.** For all (free) variables $x$, we define the *standard translation* $ST_x : \mathcal{L}(\mathbb{P}, \mathbb{T}) \to \mathcal{L}_{\mathbb{T}}^{\text{SOL}}$ inductively as follows:

$$\begin{array}{lcl}
ST_x(\top) & \triangleq & (x = x) \\
ST_x(\bot) & \triangleq & \neg(x = x) \\
ST_x(p) & \triangleq & P(x) \\
ST_x(\neg\varphi) & \triangleq & \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) & \triangleq & ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x(\varphi \vee \psi) & \triangleq & ST_x(\varphi) \vee ST_x(\psi) \\
ST_x(\varphi \to \psi) & \triangleq & ST_x(\varphi) \to ST_x(\psi) \\
ST_x(\Diamond\varphi) & \triangleq & (\exists y \rhd x) ST_y(\varphi) \\
ST_x(\Box\varphi) & \triangleq & (\forall y \rhd x) ST_y(\varphi) \\
ST_x(\Diamond^-\varphi) & \triangleq & (\exists y \lhd x) ST_y(\varphi) \\
ST_x(\Box^-\varphi) & \triangleq & (\forall y \lhd x) ST_y(\varphi)
\end{array}$$

where $P$ is a monadic predicate variable. ⊣

**Definition 37** (Definability and local frame correspondence)**.** Let $(\mathcal{L}(\mathbb{P}, \mathsf{F}), \mathcal{F}_0, \models)$ be a tense logic associated to a class $\mathcal{F}_0$ of pointed Kripke frames. Let $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{F})$, let $\Sigma$ be a set of inference rules for $\mathcal{L}(\mathbb{P}, \mathsf{F})$ and let $\Theta(x)$ be a (set of) formula(s) of $\mathcal{L}_{\mathbb{T}}^{\text{FOL}}$ such that $x$ is supposed to be the only free variable of $\Theta(x)$.

We say that $\Theta(x)$ *defines* a class of pointed Kripke frames $\mathcal{F}_0$ or that $\mathcal{F}_0$ is *defined by* $\Theta(x)$ when the following holds:

$$\mathcal{F}_0 = \left\{(F, w) \mid (F, w) \in \mathcal{F} \text{ and } F \models_{\text{FOL}} \Theta[w]\right\}$$

We say that $\varphi$ (resp. $\Sigma$) and $\Theta(x)$ are *local frame correspondents* when for all pointed Kripke frames $(F, w) \in \mathcal{F}$,

$$\begin{array}{lcl}
F, w \models \varphi & \text{iff} & F \models_{\text{FOL}} \Theta[w] \\
F, w \models \Sigma & \text{iff} & F \models_{\text{FOL}} \Theta[w]
\end{array}$$

where $F \models_{\text{FOL}} \Theta[w]$ means that $\Theta(x)$ is interpreted in $F$ with respect to an assignment that assigns $w$ to the free variable $x$. We say that $\varphi$ (resp. $\Sigma$) and a sentence $\Theta$ of $\mathcal{L}_{\mathbb{T}}^{\text{FOL}}$ are *global frame correspondents* when for all Kripke frames $F$,

$$\begin{array}{lcl}
F \models \varphi & \text{iff} & F \models_{\text{FOL}} \Theta \\
F \models \Sigma & \text{iff} & F \models_{\text{FOL}} \Theta
\end{array}$$

⊣

Then, one can easily show that:

**Fact 1.** *If* $(F, w)$ *is a pointed Kripke frame and* $\varphi \in \mathcal{L}(\mathbb{P}, \mathbb{T})$*, then*

$$F, w \models \varphi \quad \textit{iff} \quad F \models_{\text{FOL}} ST_x(\varphi)[w]$$

*where* $F \models_{\text{FOL}} ST_x(\varphi)[w]$ *means that* $ST_x(\varphi)$ *is interpreted in* $F$ *with respect to an assignment that assigns* $w$ *to the free variable* $x$*.*

One would want instead to have an equivalent *first-order* formula of $\mathcal{L}_{\mathbb{T}}^{\text{FOL}}$ and not a *second-order* formula of $\mathcal{L}_{\mathbb{T}}^{\text{SOL}}$, if it exists. This cannot be the case in general and we therefore introduce a fragment of the formulas of $\mathcal{L}(\mathbb{P}, \mathbb{T})$ called the *Sahlqvist fragment* which consists of formulas for which we can always find a first-order equivalent. Larger fragments exist, such as the class of *inductive formulas* [24], but they will not be needed here.

**Definition 38** (Primitive formulas of $\mathcal{L}(\mathbb{P},\mathbb{T})$)**.** A *primitive modal formula of* $\mathcal{L}(\mathbb{P},\mathbb{T})$ is a formula of the form $\varphi \to \psi$, where $\varphi, \psi \in \mathcal{L}(\mathbb{P},\mathsf{F})$ with $\mathsf{F} \triangleq \{\top, \wedge, \vee, \Diamond, \Diamond^-\}$ and such that $\varphi$ contains each propositional variable of $\mathbb{P}$ at most once. $\dashv$

**Definition 39** (Primitive formulas of $\mathcal{L}_{\mathbb{T}}^{\mathrm{FOL}}$)**.** We say that an occurrence of a variable $y$ in a formula $\alpha$ is *inherently universal* if either $y$ is free, or else $y$ is bound by a restricted universal quantifier which is not in the scope of a (restricted) existential quantifier.

A *Kracht formula of* $\mathcal{L}_{\mathbb{T}}^{\mathrm{FOL}}$ is built up from atomic formulas of the form $x = y$ or $Rxy$ with the help of $\wedge, \vee$ and the restricted quantifiers in such a way that in a subformula $x = y$ or $Rxy$ at least one of $x$ or $y$ is hereditary universal.

A *primitive first-order formula of* $\mathcal{L}_{\mathbb{T}}^{\mathrm{FOL}}$ is a Kracht formula of $\mathcal{L}_{\mathbb{T}}^{\mathrm{FOL}}$ in which no universal quantifier is in the scope of an existential quantifier. $\dashv$

**Definition 40** (Special inference rule)**.** A special inference rule for $\mathcal{L}(\mathbb{P},[\mathbb{T}])$–consecutions is an inference rule that satisfies the conditions (C1)–(C8) of Belnap [5] such that one structure variable is shared as the common consequent or antecedent:

$$\frac{A_1 \vdash B \quad \ldots \quad A_n \vdash B}{A \vdash B}\, R_1 \qquad\qquad \frac{A \vdash B_1 \quad \ldots \quad A \vdash B_n}{A \vdash B}\, R_2$$

$\dashv$

**Definition 41** (Translations $\tau_2$ and $\tau_2^-$)**.** Let $R_1$ and $R_2$ be special inference rules of the form represented in Definition 40. The modal primitive formula of $\mathcal{L}(\mathbb{P},\mathbb{T})$ associated to $R_1$ and $R_2$ are defined as follows. First, all structure variables are uniformly replaced by (fresh) propositional letters, yielding the structures $A^*, A_1^*, \ldots, A_n^*$ and $B^*, B_1^*, \ldots, B_n^*$. Second, we define:

$$\tau_2(R_1) \triangleq t_1(A^*) \to (t_1(A_1^*) \vee \ldots \vee t_1(A_n^*))$$
$$\tau_2(R_2) \triangleq (t_2(B_1^*) \wedge \ldots \wedge t_2(B_n^*)) \to t_2(B^*).$$

The converse algorithm $\tau_2^{0,-}$ can be found in [35, p. 106-107]. It transforms any primitive formula of $\mathcal{L}(\mathbb{P},\mathbb{T})$ into a special inference rule for $\mathcal{L}(\mathbb{P},[\mathbb{T}])$–consecutions which has the same deductive power as the formula.

Kracht's algorithm $\tau_2^{0,-}$ applies to any multi-modal tense logic and in particular to tense Gaggle logic. We modify this algorithm $\tau_2^{0,-}$ by adding a last instruction: every propositional letter $p$ is replaced by $\blacklozenge^- p$ if it is in antecedent part of its consecution (obtained by $\tau_2^{0,-}$) and by $\blacksquare^- \overline{p}$ if it is in consequent part, where $p$ (resp $\overline{p}$) is a (fresh) propositional letter of signature $(Id, \pm, \to)$ (resp. $(Id, \pm, \leftarrow)$). This yields a new algorithm that we denote $\tau_2^-$. $\dashv$

**Definition 42** (Translation $\tau_3$ and $\tau_3^-$)**.** The *Sahlqvist algorithm* for modal primitive formulas, denoted $\tau_3$, is defined in [51]. The *Kracht algorithm* for first-order primitive formulas, denoted $\tau_3^-$, is defined in [34]. $\dashv$

**Theorem 11** (Correspondence [7, 35])**.** *Let $\Theta(x)$ be a first-order primitive formula of $\mathcal{L}_{\mathbb{T}}^{\mathrm{FOL}}$ with one free variable $x$, let $\varphi$ be a modal primitive formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$ and let $R$ be a special inference rule of $\mathcal{S}_{C_\blacklozenge}$. Then, the following pairs are all local frame correspondents: $\Theta(x)$ and $\tau_3^-(\Theta(x))$; $\Theta(x)$ and $\tau_2^-\left(\tau_3^-(\Theta(x))\right)$; $\tau_3(\varphi)$ and $\varphi$; $\tau_3(\tau_2(R))$ and $R$.*

**Theorem 12** (Canonicity [35])**.** *Let $\mathcal{F}_0$ be a class of pointed C–frames. There exists a finite set of special inference rules $\Sigma_\blacklozenge$ in $\mathcal{S}_{C_\blacklozenge}$ such that $\mathbf{DLM}_{\mathrm{t}} + \Sigma_\blacklozenge$ is sound and complete for the logic $(\mathcal{S}(\mathbb{P},[\mathbb{T}]), \mathcal{F}_0, \models)$ if, and only if, there exists a finite set $\Theta(x)$ of primitive formulas of $\mathcal{L}_{\mathbb{T}}^{\mathrm{FOL}}$ with one free variable $x$ that defines $\mathcal{F}_0$.*[1] *Moreover, $\Sigma_\blacklozenge$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma_\blacklozenge$, as follows: $\Theta(x) \triangleq \tau_3(\tau_2(\Sigma_\blacklozenge))$ and $\Sigma_\blacklozenge \triangleq \tau_2^-\left(\tau_3^-(\Theta(x))\right)$.*

[1] The display calculus $\mathbf{DLM}_{\mathrm{t}}$ is recalled in Appendix 15.

# 10 Correspondence Theory for Gaggle logic

This section contains the main results of the report, namely a characterization of protodisplay logics in terms of a description of the class of frames on which they are based. We also show how to compute the protoanalytic inference rule(s) that correspond(s) to a specific protoanalytic first-order frame condition and, vice versa, we show how to compute the protoanalytic first-order frame condition that corresponds to a specific protoanalytic inference rule.

## 10.1 More Translations

**Definition 43** (FOL and SOL $\mathsf{C}$–frame logics)**.** Let $\mathsf{C} \subseteq \mathbb{C}$.

The *FOL (resp. SOL) $\mathsf{C}$–frame language*, denoted $\mathcal{L}^{\mathrm{FOL}}(\mathsf{C})$ (resp. $\mathcal{L}^{\mathrm{SOL}}(\mathsf{C})$), is the first–order (resp. second–order) language that has an identity predicate $=$, a binary predicate $\sqsubseteq$ and a set of $n$–ary predicates $\mathcal{R}$ satisfying the conditions of Definition 11 with $n > 1$ for $\mathcal{L}^{\mathrm{FOL}}(\mathsf{C})$.

Similarly, the *FOL (resp. SOL) tense $\mathsf{C}_\blacklozenge$–frame languages*, denoted $\mathcal{L}^{\mathrm{FOL}}(\mathsf{C}_\blacklozenge)$ (resp. $\mathcal{L}^{\mathrm{SOL}}(\mathsf{C}_\blacklozenge)$), is the first–order (resp. second–order) language that has an identity predicate $=$, a binary predicate $\sqsubseteq$ and a set of $n$–ary predicates $\mathcal{R}$ satisfying the conditions of Definition 28 with $n > 1$ for $\mathcal{L}^{\mathrm{FOL}}(\mathsf{C}_\blacklozenge)$.

$\mathcal{L}^{\mathrm{SOL}}(\mathsf{C})$ and $\mathcal{L}^{\mathrm{SOL}}(\mathsf{C}_\blacklozenge)$ allow moreover quantification over unary predicates. The satisfaction relations for the languages $\mathcal{L}^{\mathrm{FOL}}(\mathsf{C})$ and $\mathcal{L}^{\mathrm{SOL}}(\mathsf{C})$ (resp. $\mathcal{L}^{\mathrm{FOL}}(\mathsf{C}_\blacklozenge)$ and $\mathcal{L}^{\mathrm{SOL}}(\mathsf{C}_\blacklozenge)$) over the class of all pointed $\mathsf{C}$–frames (resp. $\mathsf{C}_\blacklozenge$–frames) are defined as usual (see [41] for instance). $\dashv$

**Notation 3.** We generalize the notations of Notation 2 as follows. If $\mathsf{C} \subseteq \mathbb{C}$ then for all $\mathsf{C}$–frames $F$ and all $\star = (\sigma, t, \leftrightarrow)_i \in \mathsf{C}$ such that $\mathsf{a}(\star) = n$:

$$(\forall y_1 \ldots y_n \rhd^\star x)\alpha(y_1, \ldots, y_n) \triangleq \forall y_1 \ldots y_n(R_\star^{\leftrightarrow,\sigma} y_1 \ldots y_n x \to \alpha(y_1, \ldots, y_n))$$
$$(\exists y_1 \ldots y_n \rhd^\star x)\alpha(y_1, \ldots, y_n) \triangleq \exists y_1 \ldots y_n(R_\star^{\leftrightarrow,\sigma} y_1 \ldots y_n x \wedge \alpha(y_1, \ldots, y_n))$$

where $R_\star$ is the $n+1$–ary relation associated to $\star$ in $F$. We call the constructs $(\forall y_1 \ldots y_n \rhd^\star x)$ and $(\exists y_1 \ldots y_n \rhd^\star x)$ *restricted universal (resp. existential) quantifiers*. This notation also applies to $\mathsf{C}_\blacklozenge$–frames or Kripke frames and for example, for the latter, $(\exists y \rhd^{\diamond^-} x)\alpha(y)$ denotes the expression $(\exists y \lhd x)\alpha(y)$ of Notation 2. $\dashv$

**Definition 44** (Protoanalytic formulas of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$ and $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$)**.** Let $\mathsf{C} \subseteq \mathbb{C}$.

A *Kracht formula of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$* is built up from atomic formulas of the form $x = y$, $x \sqsubseteq y$ and $Ry_1 \ldots y_n x$ with the help of $\wedge, \vee$ and the restricted quantifiers $(\forall y_1 \ldots y_n \rhd^\star x)$ and $(\exists y_1 \ldots y_n \rhd^\star x)$ in such a way that in a subformula $x = y$, $x \sqsubseteq y$ or $Ryx$ at least one of $x$ and $y$ is hereditary universal and in a subformula $R_\star^{\leftrightarrow,\sigma} y_1 \ldots y_n x$ every subsequence $(w, v, u)$ of $(y_1, \ldots, y_n, x)$ is such that either $v$ is hereditary universal or $w$ and $u$ are both hereditary universal (inclusive or). A *protoanalytic formula of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$* is a Kracht formula of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$ in which no universal quantifier is in the scope of an existential quantifier. The set of protoanalytic formulas of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$ is denoted $\mathcal{L}^{\mathrm{FOL},G}_{\mathsf{C}}$.

A *Kracht formula of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$* is built up from atomic formulas of the form $x = y$, $x \sqsubseteq y$ and $Ryx$ such that in a subformula $x \sqsubseteq y$, $Rxy$ or $x = y$, at least one of $x$ and $y$ is hereditary universal. A *protoanalytic formula $\varphi$ of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$* is a Kracht formula $\varphi$ of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$ such that the restricted quantifiers appearing in $\varphi$ follow the patterns defined by Expressions (16) and (17) below. The set of protoanalytic formulas of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$ is denoted $\mathcal{L}^{\mathrm{FOL},G}_{\mathsf{C}_\blacklozenge}$. $\dashv$

Every subformula $R_{\star}^{\leftrightarrow,\sigma} y_1 \dots y_n x$ of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$ translates into a sequence of atomic formulas $R_{\star}^{1} y_1 y_2$, $R_{\star}^{2} y_2 y_3$,…, $R_{\star}^{n-1} y_{n-1} y_n$, $R_{\star}^{n} y_n x$, and in each pair, at least one variable must be hereditary universal. This explains the special condition on subtriples that we need to introduce for Kracht formulas of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$ (and protoanalytic formulas of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$).

**Definition 45** (Translations $\tau_4$ and $\tau_4^-$)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. We define the translation $\tau_4 : \mathcal{L}_{\mathsf{C}_{\blacklozenge}}^{\mathrm{FOL},G} \to \mathcal{L}_{\mathsf{C}}^{\mathrm{FOL},G}$ and its converse $\tau_4^- : \mathcal{L}_{\mathsf{C}}^{\mathrm{FOL},G} \to \mathcal{L}_{\mathsf{C}_{\blacklozenge}}^{\mathrm{FOL},G}$ as follows (from the left to right direction for $\tau_4$ and from the right to left direction for $\tau_4^-$): for all $\star \in \mathsf{C}$ such that $\mathsf{a}(\star) = n$,

$$(\exists y_1 \dots y_n \rhd^{\star} x) :\triangleq: (\exists y_n \rhd^{\blacklozenge_{\star,n}^-} x)(\exists y_{n-1} \rhd^{\blacklozenge_{\star,n-1}^-} y_n) \dots (\exists y_1 \rhd^{\blacklozenge_{\star,1}^-} y_2) \tag{16}$$

$$(\forall y_1 \dots y_n \rhd^{\star} x) :\triangleq: (\forall y_n \rhd^{\blacklozenge_{\star,n}^-} x)(\forall y_{n-1} \rhd^{\blacklozenge_{\star,n-1}^-} y_n) \dots (\forall y_1 \rhd^{\blacklozenge_{\star,1}^-} y_2) \tag{17}$$

where $\blacklozenge_{\star,n}^-$,…,$\blacklozenge_{\star,1}^-$ are (some of) the tense connectives associated to $\star$. ⊣

These two translations relate atomic formulas and the restricted quantifers of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$ to sequences of restricted quantifiers of $\mathcal{L}_{\mathsf{C}_{\blacklozenge}}^{\mathrm{FOL}}$, and vice versa.

**Definition 46** (Translations $\tau_0$ and $\tau_0^-$)**.** We define the translation $\tau_0$ from inference rule *schemas* to inference rules as follows. Let $R$ be an inference rule *schema*:

$$\frac{X_1 \mid\!\!- Y_1 \quad \dots \quad X_n \mid\!\!- Y_n}{X \mid\!\!- Y}\ R$$

The inference rule $\tau_0(R)$ is obtained by replacing uniformly each structural variable $X_0$ occurring in a consecution $U \mid\!\!- V$ of $R$ (that is, either $X_1 \mid\!\!- Y_1, \dots, X_n \mid\!\!- Y_n$ or $X \mid\!\!- Y$) by:

- $[p]$ if $X_0$ is protoantecedent part in $U \mid\!\!- V$;
- $[\overline{p}]$ if $X_0$ is protoconsequent part in $U \mid\!\!- V$;

where $[p] = [(Id, \pm, \to)_i] \in [\mathbb{P}]$ and $[\overline{p}] = [(Id, \pm, \leftarrow)_i] \in [\mathbb{P}]$ are 'fresh' propositional structures.

The inverse translation $\tau_0^-$ from inference rules to inference rules *schemas* is defined by replacing uniformly in an inference rule every occurrence of a structure variable $[p]$ or $[\overline{p}]$ by a structure variable $X$ or a formula variable $\varphi$ (depending on the context).

Then, we say that $R$ is a *protoanalytic inference rule schema* if, and only if, $\tau_0(R)$ is a protoanalytic inference rule. ⊣

## 10.2 Characterization of Protodisplay Logics

**Theorem 13** (Correspondence)**.** *Let $\Sigma$ be a finite set of special inference rules in $[\mathcal{S}]_{\mathbb{C}}$. Then, $\Sigma$ and $\Theta(x) \triangleq \tau_3(\tau_2(\tau_1(\Sigma)))$ are local frame correspondents.*

**Theorem 14** (Characterization)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. A logic $([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \mid\!\mid\!\!-)$ based on a class $\mathcal{F}_0$ of pointed $\mathsf{C}$–frames is protodisplay by a set of protoanalytic inference rules $\Sigma$ if, and only if, $\mathcal{F}_0$ is defined by some finite set $\Theta(x)$ of protoanalytic formulas of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$ with one free variable. Moreover, $\Sigma$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma$, as follows: $\Theta(x) \triangleq \tau_4(\tau_3(\tau_2(\tau_1(\tau_0(\Sigma)))))$ and $\Sigma \triangleq \tau_0^-\left(\tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)\right)$ (see Figure 9).*

Theorem 14 shows that a logic extending Gaggle logic is protodisplay if, and only if, the class of pointed C–frames on which it is defined can be defined by some finite set of protoanalytic first-order formulas. In that case, we have at our disposal algorithms to compute the protoanalytic first-order formulas defining the class of pointed C–frames that correspond to the protoanalytic inference rules of the protodisplay calculus and, vice versa, we also have at our disposal algorithms to compute the protoanalytic inference rules of the display calculus that correspond to the protoanalytic first-order formulas defining the class of pointed C–frames. These different algorithms are listed in Figure 9.

*Remark* 1. Our results also hold if we consider 'plain' C–frames instead of *pointed* C–frames and if validity is defined with respect to classes of (plain) C–frames instead of classes of *pointed* C–frames. This is because Sahlqvist's and Kracht's results hold both with respect to the so-called global *and* local correspondence.

*Remark* 2. Importantly, our algorithms apply not only to unary and binary connectives but in fact to connectives of arbitrary arities. In particular, they apply to nullary connectives and the truth constants $\top$ and $\bot$ can be recovered by adding the usual structural rules for the empty structure $\mathbf{I}$ of the literature (see Rules ($\mathbf{I}^A_{;_\mathrm{i}}$), ($\mathbf{I}^K_{;_\mathrm{i}}$), ($\mathsf{Q}^A_{\sim_\mathrm{j}}$), ($\mathsf{Q}^K_{-_\mathrm{j}}$) and Section 5). In our framework, $\mathbf{I}$ is simply viewed as a structural nullary connective $[p]$, and the addition of some (protoanalytic) structural rules constrains the propositional letter $p$ to be valid (or unsatisfiable).

# 11 Examples of Correspondence Translations

In this section, we provide examples of correspondence translations that use the algorithms $\tau_4, \tau_3, \tau_2, \tau_1$ and $\tau_4^-, \tau_3^-, \tau_2^-, \tau_1^-$ defined in the previous sections. The algorithm $\tau_3^-$ that we use is different from Kracht's algorithm and has been defined specifically for the kind of protoanalytic (in fact primitive) formulas that we consider.

In Figures 10 and 11, we provide a number of global frame correspondents of protoanalytic inference rules. In these figures, we use the following abbreviations and notations: for all $i, j \in \{1, 2, 3\}$,

1. $;_\mathrm{i}$ and $;_\mathrm{j}$ range over $\{[\otimes_3], [\otimes_2], [\otimes_1]\}$ and $,_\mathrm{i}$ and $,_\mathrm{j}$ range over $\{[\oplus_3], [\oplus_2], [\oplus_1]\}$;
2. $\prec_\mathrm{i}$ ranges over $\{[\prec_1], [\prec_2], [\prec_3]\}$ and $\succ_\mathrm{i}$ ranges over $\{[\succ_1], [\succ_2], [\succ_3]\}$;
3. $<_\mathrm{i}$ ranges over $\{[\subset_3], [\subset_2], [\subset_1]\}$ and $>_\mathrm{i}$ ranges over $\{[\supset_3], [\supset_2], [\supset_1]\}$;
4. $\sim_\mathrm{i}$ and $\sim_\mathrm{j}$ range over $\{[\sim_1], [\sim_2]\}$ and $-_\mathrm{i}$and $-_\mathrm{j}$ range over $\{[-_1], [-_2]\}$.

Structural connectives are generally denoted $[\star_i] = [(\sigma_i, t_i, \leftrightarrow)]$ (or $[\star_j] = [(\sigma_j, t_j, \leftrightarrow)]$). We also use the following notations:

$$R(xy)zw \triangleq \exists u(Rxyu \wedge Ruzw) \qquad Rx(yz)w \triangleq \exists u(Ryzu \wedge Rxuw)$$
$$R \triangleq R^{\leftrightarrow,\sigma_i}_{\star_i} \qquad S \triangleq S^{\leftrightarrow,\sigma_\mathrm{j}}_{\star_\mathrm{j}}$$

## 11.1 From Inference Rules to First-order Frame Conditions

In the rules that we consider to run our algorithms, we use the structural connective $;_3$, that is $[\otimes_3]$, because it corresponds to the the usual structural connective of substructural logic, often denoted ";" [48]. This obviously does not preclude ourselves to apply our algorithms to other structural connectives. For better readability, we also use the notations of Figure 14.

We execute the algorithms $\tau_4, \tau_3, \tau_2, \tau_1, \tau_0$ on the classical inference rules ($\mathsf{K}^A_{;_\mathrm{i}}$) and ($\mathsf{WI}^A_{;_\mathrm{i}}$).

$$
\begin{tikzcd}
\text{Protoanalytic inference rules schemas for } [\mathcal{S}]_{\mathbb{C}} \text{ [Def. 46]} \arrow[d, bend left, "\tau_0 \quad \text{[Def. 46]}"] \\
\text{Protoanalytic inference rules for } [\mathcal{S}]_{\mathbb{C}} \text{ [Def. 33]} \arrow[u, bend left, "\text{[Def. 46]} \quad \tau_0^-"] \arrow[d, bend left, "\tau_1 \quad \text{[Def. 30]}"] \\
\text{Special inference rules for } [\mathcal{S}]_{\mathbb{C}_\blacklozenge} \text{ [Def. 40]} \arrow[u, bend left, "\text{[Def. 31]} \quad \tau_1^-"] \arrow[d, bend left, "\tau_2 \quad \text{[Def. 41]}"] \\
\text{Primitive formulas of } \mathcal{L}_{\mathbb{C}_\blacklozenge} \text{ [Def. 38]} \arrow[u, bend left, "\text{[Def. 41]} \quad \tau_2^-"] \arrow[d, bend left, "\tau_3 \quad \text{[51]}"] \\
\text{Protoanalytic formulas of } \mathcal{L}^{\mathrm{FOL}}_{\mathbb{C}_\blacklozenge} \text{ [Def. 44]} \arrow[u, bend left, "\text{[34]} \quad \tau_3^-"] \arrow[d, bend left, "\tau_4 \quad \text{[Def. 45]}"] \\
\text{Protoanalytic formulas of } \mathcal{L}^{\mathrm{FOL}}_{\mathbb{C}} \text{ [Def. 44]} \arrow[u, bend left, "\text{[Def. 45]} \quad \tau_4^-"]
\end{tikzcd}
$$

Figure 9: Translations from protoanalytic inference rules to first-order frame conditions, and vice versa

| Protoanalytic Rule | Global Frame Correspondent | |
|---|---|---|
| $\dfrac{(X ;_{\mathrm{i}} Y) ;_{\mathrm{i}} Z \vdash U}{X ;_{\mathrm{i}} (Y ;_{\mathrm{i}} Z) \vdash U}$ | $Rx(yz)w \to R(xy)zw$ | $\mathsf{B}^{c,A}_{;_{\mathrm{i}}}$ |
| $\dfrac{U \vdash (X ,_{\mathrm{i}} Y) ,_{\mathrm{i}} Z}{U \vdash X, (Y ,_{\mathrm{i}} Z)}$ | $Rx(yz)w \to R(xy)zw$ | $\mathsf{B}^{c,K}_{,_{\mathrm{i}}}$ |
| $\dfrac{Y ;_{\mathrm{i}} X \vdash Z}{X ;_{\mathrm{i}} Y \vdash Z}$ | $Rxyz \to Ryxz$ | $\mathsf{CI}^{A}_{;_{\mathrm{i}}}$ |
| $\dfrac{Z \vdash Y ,_{\mathrm{i}} X}{Z \vdash X ,_{\mathrm{i}} Y}$ | $Rxyz \to Ryxz$ | $\mathsf{CI}^{K}_{,_{\mathrm{i}}}$ |
| $\dfrac{X ;_{\mathrm{i}} X \vdash Y}{X \vdash Y}$ | $Rxxx$ | $\mathsf{WI}^{A}_{;_{\mathrm{i}}}$ |
| $\dfrac{Y \vdash X ,_{\mathrm{i}} X}{Y \vdash X}$ | $Rxxx$ | $\mathsf{WI}^{K}_{,_{\mathrm{i}}}$ |
| $\dfrac{X \vdash Z}{X ;_{\mathrm{i}} Y \vdash Z}$ | $Rxyz \to x \sqsubseteq z$ | $\mathsf{K}^{A}_{;_{\mathrm{i}}}$ |
| $\dfrac{Z \vdash X}{Z \vdash X ,_{\mathrm{i}} Y}$ | $Rxyz \to x \sqsubseteq z$ | $\mathsf{K}^{K}_{,_{\mathrm{i}}}$ |
| $\dfrac{X \prec_{\mathrm{i}} Y \vdash Z}{X ;_{\mathrm{i}} \sim_{\mathrm{j}} Y \vdash Z}$ | $Ryzx \wedge Swz \wedge w' \sqsubseteq w$ $\to \exists tu \big(Rtux \wedge w' \sqsubseteq u \wedge y \sqsubseteq t\big)$ | $\mathsf{N}^{A}_{1}$ |
| $\dfrac{X \vdash Y <_{\mathrm{i}} Z}{X \vdash Y ,_{\mathrm{i}} -_{\mathrm{j}} Z}$ | $Ryzx \wedge Swz \wedge w' \sqsubseteq w$ $\to \exists tu \big(Rtux \wedge w' \sqsubseteq u \wedge y \sqsubseteq t\big)$ | $\mathsf{N}^{K}_{1}$ |
| $\dfrac{X \succ_{\mathrm{i}} Y \vdash Z}{\sim_{\mathrm{j}} X ;_{\mathrm{i}} Y \vdash Z}$ | $Rzyx \wedge Swz \wedge w' \sqsubseteq w$ $\to \exists tu \big(Rutx \wedge w' \sqsubseteq u \wedge y \sqsubseteq t\big)$ | $\mathsf{N}^{A}_{2}$ |
| $\dfrac{X \vdash Y >_{\mathrm{i}} Z}{X \vdash -_{\mathrm{j}} Y ,_{\mathrm{i}} Z}$ | $Rzyx \wedge Swz \wedge w' \sqsubseteq w$ $\to \exists tu \big(Rutx \wedge w' \sqsubseteq u \wedge y \sqsubseteq t\big)$ | $\mathsf{N}^{K}_{2}$ |
| $\dfrac{X ;_{\mathrm{j}} Y \vdash Z}{X ;_{\mathrm{i}} Y \vdash Z}$ | $R^{\leftrightarrow,\sigma_i}_{\star_i} xyz \to R^{\leftrightarrow,\sigma_j}_{\star_j} xyz$ | $\mathsf{TI}^{A}_{i;\,j}$ |
| $\dfrac{Z \vdash X ,_{\mathrm{j}} Y}{Z \vdash X ,_{\mathrm{i}} Y}$ | $R^{\leftrightarrow,\sigma_j}_{\star_j} xyz \to R^{\leftrightarrow,\sigma_i}_{\star_i} xyz$ | $\mathsf{TI}^{K}_{i,\,j}$ |
| $\dfrac{\sim_{\mathrm{j}} X \vdash Y}{\sim_{\mathrm{i}} X \vdash Y}$ | $Syx \to Sxy$ | $\mathsf{BI}^{A}_{i,j}$ |
| $\dfrac{X \vdash -_{\mathrm{j}} Y}{X \vdash -_{\mathrm{i}} Y}$ | $Syx \to Sxy$ | $\mathsf{BI}^{K}_{i,j}$ |

Figure 10: Global frame correspondents of ‘classical’ protoanalytic inference rules

| Protoanalytic Rule | Global Frame Correspondent | |
|---|---|---|
| $\dfrac{\sim_{\mathrm{j}}\,[f] \vdash X}{[t] \vdash X}$ | $T(x) \to \exists y(Syx \wedge \neg F(y))$ | $\mathsf{Q}^{A}_{\sim_{\mathrm{j}}}$ |
| $\dfrac{X \vdash\ -_{\mathrm{j}}\,[t]}{X \vdash\ [f]}$ | $\neg F(x) \to \exists y(Syx \wedge T(y))$ | $\mathsf{Q}^{K}_{-_{\mathrm{j}}}$ |
| $\dfrac{[t]\ ;_{\mathrm{i}}\ X \vdash Y}{X \vdash Y}$ | $\exists y\,(Ryxx \wedge T(y))$ | $\mathbf{I}^{A}_{;_{\mathrm{i}}}$ |
| $\dfrac{Y \vdash\ [f]\ \ ,_{\mathrm{i}}\ X}{Y \vdash X}$ | $\exists y\,(Ryxx \wedge \neg F(y))$ | $\mathbf{I}^{K}_{;_{\mathrm{i}}}$ |

Figure 11: Global frame correspondents of ‘classical’ protoanalytic inference rules

| Protoanalytic Rule | Global Frame Correspondent | |
|---|---|---|
| $\dfrac{(X\,;Z);Y \vdash U}{(X\,;Y);Z \vdash U}$ | $R(xz)yw \to R(xy)zw$ | ($\mathsf{C}_{;}$ ) |
| $\dfrac{X;(Y\,;Z) \vdash U}{(X\,;Y);Z \vdash U}$ | $R(xy)xw \to Rx(yz)w$ | ($\mathsf{B}_{;}$ ) |
| $\dfrac{X \vdash Y}{X;X \vdash Y}$ | $Rxxy \to x \sqsubseteq y$ | ($\mathsf{M}_{;}$ ) |
| $\dfrac{\bullet X \vdash Y}{X \vdash Y}$ | $Sxx$ | ($\mathsf{T}_{\bullet}$) |
| $\dfrac{\bullet X \vdash Y}{\bullet \bullet X \vdash Y}$ | $Sxy \wedge Syz \to Sxz$ | ($\mathsf{4}_{\bullet}$) |
| $\dfrac{\bullet X \vdash Y}{\bullet X;\ \bullet X \vdash Y}$ | $Sxy \to R(xS)(xS)y$ | ($\mathsf{mWI}_{\bullet,;}$ ) |
| $\dfrac{\bullet(X\,;Y) \vdash Z}{\bullet X;\ \bullet Y \vdash Z}$ | $Rzw(Sx) \to R(zS)(wS)x$ | ($\mathsf{mMP}_{\bullet,;}$ ) |

Figure 12: Global frame correspondents of other protoanalytic inference rules

| Connective | In Gaggle logic | Connective | In Gaggle logic | Permutations and traces |
|---|---|---|---|---|
| $t$ | $(Id, -, \rightarrow)$ | $\otimes_3$ | $(\sigma_1, t_1, \rightarrow)$ | $\sigma_1 = (1, 2, 3)$ |
| $t'$ | $(Id, +, \rightarrow)$ | $\otimes_2$ | $(\sigma_3, t_1, \rightarrow)$ | $\sigma_3 = (3, 1, 2)$ |
| $f$ | $(Id, +, \leftarrow)$ | $\otimes_1$ | $(\sigma_5, t_1, \rightarrow)$ | $\sigma_5 = (2, 3, 1)$ |
| $f'$ | $(Id, -, \leftarrow)$ | $\oplus_3$ | $(\sigma_1, t_4, \leftarrow)$ | $\sigma'_1 = (1, 2)$ |
| $\sim_1$ | $(\sigma'_2, t'_4, \rightarrow)$ | $\oplus_2$ | $(\sigma_3, t_4, \leftarrow)$ | $\sigma'_2 = (2, 1)$ |
| $\sim_2$ | $(\sigma'_1, t'_4, \rightarrow)$ | $\oplus_1$ | $(\sigma_5, t_4, \leftarrow)$ | $t_1 = (-, -) \mapsto -$ |
| $-_1$ | $(\sigma'_2, t'_3, \rightarrow)$ | $\succ_3$ | $(\sigma_1, t_6, \rightarrow)$ | $t_2 = (+, -) \mapsto +$ |
| $-_2$ | $(\sigma'_1, t'_3, \rightarrow)$ | $\succ_2$ | $(\sigma_3, t_6, \rightarrow)$ | $t_3 = (-, +) \mapsto +$ |
| $\diamond$ | $(\sigma'_2, t'_1, \rightarrow)$ | $\succ_1$ | $(\sigma_5, t_6, \rightarrow)$ | $t_4 = (+, +) \mapsto +$ |
| $\diamond^-$ | $(\sigma'_1, t'_1, \rightarrow)$ | $\prec_3$ | $(\sigma_1, t_5, \rightarrow)$ | $t_5 = (-, +) \mapsto -$ |
| $\square$ | $(\sigma'_1, t'_2, \leftarrow)$ | $\prec_2$ | $(\sigma_3, t_5, \rightarrow)$ | $t_6 = (+, -) \mapsto -$ |
| $\square^-$ | $(\sigma'_2, t'_2, \leftarrow)$ | $\prec_1$ | $(\sigma_5, t_5, \rightarrow)$ | $t'_1 = (-) \mapsto -$ |
| | | $\supset_3$ | $(\sigma_1, t_3, \leftarrow)$ | $t'_2 = (+) \mapsto +$ |
| | | $\supset_2$ | $(\sigma_3, t_3, \leftarrow)$ | $t'_3 = (-) \mapsto +$ |
| | | $\supset_1$ | $(\sigma_5, t_3, \leftarrow)$ | $t'_4 = (+) \mapsto -$ |
| | | $\subset_3$ | $(\sigma_1, t_2, \leftarrow)$ | |
| | | $\subset_2$ | $(\sigma_3, t_2, \leftarrow)$ | |
| | | $\subset_1$ | $(\sigma_5, t_2, \leftarrow)$ | |

Figure 13: Some modal and substructural connectives of Gaggle logic

| Notation | Denotation | Notation | Denotation |
|---|---|---|---|
| $\blacklozenge^-_2$ | $\blacklozenge^-_{\otimes_3,2}$ | $\bullet_2$ | $\bullet_{\otimes_3,2}$ |
| $\blacklozenge^-_1$ | $\blacklozenge^-_{\otimes_3,1}$ | $\bullet_1$ | $\bullet_{\otimes_3,1}$ |
| $R$ | $R_{\otimes_3}$ | $S$ | $R_{\diamond^-}$ |
| $R_2$ | $R^2_{\otimes_3}$ | $R_1$ | $R^1_{\otimes_3}$ |
| $(\forall yz \rhd^3 x)$ | $(\forall yz \rhd^{\otimes_3} x)$ | $(\exists yz \rhd^3 x)$ | $(\exists yz \rhd^{\otimes_3} x)$ |
| $(\forall y \lhd x)$ | $(\forall y \rhd^{\diamond^-} x)$ | $(\exists y \lhd x)$ | $(\exists y \rhd^{\diamond^-} x)$ |
| $(\forall y \lhd_0 x)$ | $(\forall y \rhd^{\blacklozenge^-} x)$ | $(\exists y \lhd_0 x)$ | $(\exists y \rhd^{\blacklozenge^-} x)$ |
| $(\forall y \lhd_1 x)$ | $(\forall y \rhd^{\blacklozenge^-_{\otimes_3,1}} x)$ | $(\exists y \lhd_1 x)$ | $(\exists y \rhd^{\blacklozenge^-_{\otimes_3,1}} x)$ |
| $(\forall y \lhd_2 x)$ | $(\forall y \rhd^{\blacklozenge^-_{\otimes_3,2}} x)$ | $(\exists y \lhd_2 x)$ | $(\exists y \rhd^{\blacklozenge^-_{\otimes_3,2}} x)$ |

Figure 14: Notations used in the algorithms

**Inference Rule $\mathsf{K}^{A}_{;\mathrm{i}}$**

Algorithm $\tau_0$:

1. $\dfrac{X \vdash Z}{X \;;_3\; Y \vdash Z}$

2. $\dfrac{[p] \vdash [\overline{r}]}{[p] \;;_3\; [q] \vdash [\overline{r}]}$

Algorithm $\tau_1$:

1. $\dfrac{\blacklozenge^{-} p \vdash \blacksquare^{-} \overline{r}}{\bullet_2(\blacklozenge^{-} q,\ \bullet_1 \blacklozenge^{-} p) \vdash \blacksquare^{-} \overline{r}}$

Algorithm $\tau_2$:

1. $t_1(\bullet_2(\blacklozenge^{-} q,\ \bullet_1 \blacklozenge^{-} p)) \rightarrow t_1(\blacklozenge^{-} p)$

2. $\blacklozenge_2^{-}(\blacklozenge^{-} q \wedge (\blacklozenge_1^{-} \blacklozenge^{-} p)) \rightarrow \blacklozenge^{-} p$

Algorithm $\tau_3$:

1. $\forall PQ \left(\left(\exists y \left(R_2 yx \wedge \exists y' \left(y' \sqsubseteq y \wedge Q(y')\right) \wedge \exists z \left(R_1 zy \wedge \exists z' \left(z' \sqsubseteq z \wedge P(z')\right)\right)\right)\right) \rightarrow \exists x' \left(x' \sqsubseteq x \wedge P(x')\right)\right)$

2. $\forall PQ \left(\left(\exists yzy'z' \left(R_2 yx \wedge R_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge Q(y') \wedge P(z')\right)\right) \rightarrow \exists x' \left(x' \sqsubseteq x \wedge P(x')\right)\right)$

3. We take $\sigma(P) \triangleq \lambda u.z' = u$ and $\sigma(Q) \triangleq \lambda u.y' = u$

4. $\left(\exists yzy'z' \left(R_2 yx \wedge R_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge y' = y' \wedge z' = z'\right)\right) \rightarrow \left(\exists x' \left(x' \sqsubseteq x \wedge z' = x'\right)\right)$

5. $\left(\exists yzy'z' \left(R_2 yx \wedge R_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right)\right) \rightarrow z' \sqsubseteq x$

6. $\forall yzy'z' \left(\left(R_2 yx \wedge R_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right) \rightarrow z' \sqsubseteq x\right)$

7. $(\forall y \lhd_2 x)(\forall z \lhd_1 y)(\forall y' \lhd_0 y)(\forall z' \lhd_0 z) z' \sqsubseteq x$

Algorithm $\tau_4$:

1. $(\forall yz \rhd^3 x)(\forall y' \lhd_0 y)(\forall z' \lhd_0 z) z' \sqsubseteq x$

Finally, translated into plain $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$, we obtain:

$$\forall yzy'z' \left(Rzyx \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \rightarrow z' \sqsubseteq x\right) \tag{18}$$

Because of the (Tonicity) conditions of Definition 11, Expression (18) is equivalent to Expression (19) (for the direction (18) to (19), it suffices to take $y' = y$ and $z' = z$ and the direction (19) to (18) holds because we have $Rz'y'x$ by (Tonicity))

$$\forall yz \, (Rzyx \rightarrow z \sqsubseteq x) \tag{19}$$

Condition (19) is indeed the condition given in [48, Table 11.1, p. 250].

**Inference Rule $\mathsf{WI}^{A}_{;\mathbf{i}}$**

Algorithm $\tau_0$:

1. $\dfrac{X \;;_3\; X \vdash Y}{X \vdash Y}$

2. $\dfrac{[p] \;;_3\; [p] \vdash [\overline{r}]}{[p] \vdash [\overline{r}]}$.

Algorithm $\tau_1$:

1. $\dfrac{\bullet_2(\blacklozenge^- p,\ \bullet_1 \blacklozenge^- p) \vdash \blacksquare^- \overline{r}}{\blacklozenge^- p \vdash \blacksquare^- \overline{r}}$

Algorithm $\tau_2$:

1. $t_1(\blacklozenge^- p) \to t_1(\bullet_2(\blacklozenge^- p,\ \bullet_1 \blacklozenge^- p))$

2. $\blacklozenge^- p \to \blacklozenge^-_2(\blacklozenge^- p \wedge \blacklozenge^-_1 \blacklozenge^- p)$

Algorithm $\tau_3$:

1. $\forall P\left(\exists x'\left(x' \sqsubseteq x \wedge P(x')\right) \to \exists y\left(R_2yx \wedge \left(\exists y'\left(y' \sqsubseteq y \wedge P(y')\right) \wedge \exists z\left(R_1zy \wedge \exists z'\left(z' \sqsubseteq z \wedge P(z')\right)\right)\right)\right)\right)$

2. $\forall P\left(\exists x'\left(x' \sqsubseteq x \wedge P(x')\right) \to \exists yzy'z'\left(R_2yx \wedge R_1zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge P(y') \wedge P(z')\right)\right)$

3. $\forall P\left(\forall x'\left(x' \sqsubseteq x \wedge P(x') \to \exists yzy'z'\left(R_2yx \wedge R_1zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge P(y') \wedge P(z')\right)\right)\right)$

4. We take $\sigma(P) \triangleq \lambda u.x' = u$

5. $\forall x'\left(x' \sqsubseteq x \to \exists yzy'z'\left(R_2yx \wedge R_1zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge x' = y' \wedge x' = z'\right)\right)$

6. $(\forall x' \lhd_0 x)(\exists y \lhd_2 x)(\exists z \lhd_1 y)(x' \sqsubseteq y \wedge x' \sqsubseteq z)$

Algorithm $\tau_4$:

1. $(\forall x' \lhd_0 x)(\exists yz \rhd^3 x)(x' \sqsubseteq y \wedge x' \sqsubseteq z)$.

Finally, translated into plain $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$, we obtain:

$$\forall x'\left(x' \sqsubseteq x \to \exists yz\left(Rzyx \wedge x' \sqsubseteq y \wedge x' \sqsubseteq z\right)\right) \tag{20}$$

Because of the (Tonicity) conditions of Definition 11, Expression (20) is equivalent to Expression (21) (for the direction (20) to (21), take $x' = x$ and we obtain $Rzyx \wedge x \sqsubseteq z \wedge x \sqsubseteq y$, so $Rxxx$; for the direction (21) to (20), take $y = x'$ and $z = x'$):

$$\forall x'(x' \sqsubseteq x \to Rx'x'x) \tag{21}$$

Condition (21) is slightly different from the condition given in [48, Table 11.1, p. 250], which is the following:

$$Rxxx \tag{22}$$

This difference can be explained by the fact that our first-order correspondents are *local* first-order correspondent, which means that they have to be evaluated on *pointed* $\mathsf{C}$–frames, whereas the first-order correspondent in [48] are *global*, which means that they have to be evaluated on *all* the points of the $\mathsf{C}$–frames. As noted in Remark 1, all our results also hold if we consider *global* correspondence and plain $\mathsf{C}$–frames instead of *pointed* $\mathsf{C}$–frames. In that case, we do have that Expression (21) is equivalent to Expression (22), because $\forall x\left(\forall x'(x' \sqsubseteq x \to Rx'x'x)\right)$ is equivalent to $\forall x Rxxx$.

## 11.2 From First-order Frame Conditions to Inference Rules

We execute the algorithms $\tau_4^-, \tau_3^-, \tau_2^-, \tau_1^-, \tau_0^-$ on the corresponding first-order conditions of the inference rules mMP and $\mathsf{B}^c$. These first-order conditions are taken from [48, Table 11.1, p. 250] and it will turn out that our algorithms yield the same inference rules as the ones given in [48, Table 11.1, p. 250]. The algorithm $\tau_3^-$ that we use is different from Kracht's algorithm and has been defined specifically for the kind of protoanalytic (in fact primitive) formulas that we consider.

#### First-order Frame Correspondent of Rule $\mathsf{B}^{c,A}_{;\mathrm{i}}$

Algorithm $\tau_4^-$:

1. $\forall yzwu\,((Ryzu \wedge Rxuw) \rightarrow \exists t\,(Rxyt \wedge Rtzw))$
2. $(\forall wu \rhd^3 x)(\forall yz \rhd^3 u)(\exists tz' \rhd^3 x)(Rwyt \wedge z' = z)$
3. $(\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z)(\exists z' \lhd_2 x)(\exists t \lhd_1 z')\left(R_1wy \wedge R_2yt \wedge z' = z\right)$

Algorithm $\tau_3^-$:

1. $\forall PQR\left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z)\left((P(w) \wedge Q(y) \wedge R(z)) \rightarrow (\exists z' \lhd_2 x)(\exists t \lhd_1 z')\right.\right.$ $\left.\left.(\exists y' \lhd_2 t)(\exists w' \lhd_1 y')\left(P(w') \wedge Q(y') \wedge R(z')\right)\right)\right)$
2. $\forall PQR\left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z)\left((P(w) \wedge Q(y) \wedge R(z)) \rightarrow (\exists z' \lhd_2 x)\left(R(z') \wedge\right.\right.\right.$ $\left.\left.\left.(\exists t \lhd_1 z')(\exists y' \lhd_2 t)\left(Q(y') \wedge (\exists w' \lhd_1 y') \wedge P(w')\right)\right)\right)\right)$
3. $\forall PQR\left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z)\left((P(w) \wedge Q(y) \wedge R(z)) \rightarrow (\exists z' \lhd_2 x)\left(R(z') \wedge\right.\right.\right.$ $\left.\left.\left.(\exists t \lhd_1 z')(\exists y' \lhd_2 t)\left(Q(y') \wedge (\exists w' \lhd_1 y') \wedge P(w')\right)\right)\right)\right)$
4. $\forall PQR\,((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z)\,((P(w) \wedge Q(y) \wedge R(z)) \rightarrow$ $ST_x\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^-\left(\blacklozenge_2^-\left(q \wedge \blacklozenge_1^- p\right)\right)\right)\right)))$
5. $\forall PQR\,(((\exists u \lhd_2 x)\,((\exists z \lhd_2 u)\,(R(z) \wedge (\exists y \lhd_1 z)Q(y)) \wedge (\exists w \lhd_1 u)P(w))) \rightarrow$ $ST_x\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^-\left(\blacklozenge_2^-\left(q \wedge \blacklozenge_1^- p\right)\right)\right)\right))$
6. $\forall PQR\left(ST_x\left(\blacklozenge_2^-\left(\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^- q\right)\right) \wedge \blacklozenge_1^- p\right)\right) \rightarrow ST_x\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^-\left(\blacklozenge_2^-\left(q \wedge \blacklozenge_1^- p\right)\right)\right)\right)\right)$
7. $\forall PQR\left(ST_x\left(\blacklozenge_2^-\left(\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^- q\right)\right) \wedge \blacklozenge_1^- p\right) \rightarrow \blacklozenge_2^-\left(r \wedge \blacklozenge_1^-\left(\blacklozenge_2^-\left(q \wedge \blacklozenge_1^- p\right)\right)\right)\right)\right)$
8. $\blacklozenge_2^-\left(\left(\blacklozenge_2^-\left(\blacklozenge^- r \wedge \blacklozenge_1^- \blacklozenge^- q\right)\right) \wedge \blacklozenge_1^- \blacklozenge^- p\right) \rightarrow \blacklozenge_2^-\left(\blacklozenge^- r \wedge \blacklozenge_1^-\left(\blacklozenge_2^-\left(\blacklozenge^- q \wedge \blacklozenge_1^- \blacklozenge^- p\right)\right)\right)$

Algorithm $\tau_2^-$:

1. $\dfrac{t_1^-\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^- \blacklozenge_2^-\left(q \wedge \blacklozenge_1^- p\right)\right)\right) \vdash t_2^-(s)}{t_1^-\left(\blacklozenge_2^-\left(\blacklozenge_2^-\left(r \wedge \blacklozenge_1^- q\right) \wedge \blacklozenge_1^- p\right)\right) \vdash t_2^-(s)}$
2. $\dfrac{\bullet_2\,(r,\ \bullet_1 \bullet_2\,(q,\ \bullet_1\, p)) \vdash s}{\bullet_2\,(\bullet_2\,(r,\ \bullet_1\, q)\,,\ \bullet_1\, p) \vdash s}$
3. $\dfrac{\bullet_2\left(\blacklozenge^- r,\ \bullet_1 \bullet_2\left(\blacklozenge^- q,\ \bullet_1 \blacklozenge^- p\right)\right) \vdash \blacksquare^- \overline{s}}{\bullet_2\left(\bullet_2\left(\blacklozenge^- r,\ \bullet_1 \blacklozenge^- q\right),\ \bullet_1 \blacklozenge^- p\right) \vdash \blacksquare^- \overline{s}}$

Algorithm $\tau_1^-$:

1 $\dfrac{\tau_1^-\left(\bullet_2\left(\blacklozenge^- r,\ \bullet_1 \bullet_2\left(\blacklozenge^- q,\ \bullet_1 \blacklozenge^- p\right)\right)\right) \vdash \tau_1\left(\blacksquare^- \overline{s}\right)}{\tau_1^-\left(\bullet_2\left(\bullet_2\left(\blacklozenge^- r,\ \bullet_1 \blacklozenge^- q\right),\ \bullet_1 \blacklozenge^- p\right)\right) \vdash \tau_1^-\left(\blacksquare^- \overline{s}\right)}$

2 $\dfrac{\left(\tau_1^-([p])\ ;_3\ \tau_1^-([q])\right)\ ;_3\ \tau_1^-([r]) \vdash \tau_1^-([\overline{s}])}{\tau_1^-([p])\ ;_3\ \left(\tau_1^-([q])\ ;_3\ \tau_1^-([r])\right) \vdash \tau_1^-([\overline{s}])}$

3 $\dfrac{([p]\ ;_3\ [q])\ ;_3\ [r] \vdash [\overline{s}]}{[p]\ ;_3\ ([q]\ ;_3\ [r]) \vdash [\overline{s}]}$

Algorithm $\tau_0^-$:

1 $\dfrac{X\ ,_{\mathrm{i}}\ (Y\ ,_{\mathrm{i}}\ Z) \vdash U}{(X\ ,_{\mathrm{i}}\ Y)\ ,_{\mathrm{i}}\ Z \vdash U}$ $\mathsf{B}^c$.

**First-order Frame Correspondent of Rule $\mathsf{mMP}_{\bullet,;}$**

Algorithm $\tau_4^-$:

1. $\forall x'y'z'(Sx'x \wedge Rz'y'x' \rightarrow \exists zy(Rzyx \wedge Sz'z \wedge Sy'y)$

2. $(\forall x' \lhd x)(\forall z'y' \rhd^3 x')(\exists zy \rhd^3 x)(Sz'z \wedge Sy'y)$

3. $(\forall x' \lhd x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')(\exists y \lhd_2 x)(\exists z \lhd_1 y)(Sz'z \wedge Sy'y)$

Algorithm $\tau_3^-$:

1. $\forall PQ\left((\forall x' \lhd x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $(\exists y \lhd_2 x)(\exists z \lhd_1 y)(\exists t \lhd y)(\exists u \lhd z)(P(t) \wedge Q(u))))$

2. $\forall PQ\left((\forall x' \lhd x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $(\exists y \lhd_2 x)(\exists z \lhd_1 y)\left((\exists u \lhd z)P(t) \wedge (\exists t \lhd y)Q(u)\right)))$

3. $\forall PQ\left((\forall x' \lhd x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $(\exists y \lhd_2 x)(\exists z \lhd_1 y)\left((\exists u \lhd z)P(t) \wedge (\exists t \lhd y)Q(u)\right)))$

4. $\forall PQ\left((\forall x' \lhd x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow ST_x\left(\blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- q)\right)\right)\right)$

5. $\forall PQ\left((\exists x' \lhd x)(\exists y' \lhd_2 x')(\exists z' \lhd_1 y')(P(y') \wedge Q(z')) \rightarrow ST_x\left(\blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- q)\right)\right)$

6. $\forall PQ\left((\exists x' \lhd x)(\exists y' \lhd_2 x')\left(P(y') \wedge (\exists z' \lhd_1 y')Q(z')\right) \rightarrow ST_x\left(\blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- q)\right)\right)$

7. $\forall PQ\left(ST_x\left(\Diamond^- \blacklozenge_2^-\left(p \wedge \blacklozenge_1^- q\right)\right) \rightarrow ST_x\left(\blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- q)\right)\right)$

8. $\forall PQ\left(ST_x\left(\Diamond^- \blacklozenge_2^-\left(p \wedge \blacklozenge_1^- q\right) \rightarrow \blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- q)\right)\right)$

9. $\Diamond^- \blacklozenge_2^-\left(p \wedge \blacklozenge_1^- q\right) \rightarrow \blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- q)$.

Algorithm $\tau_2^-$:

1. $\dfrac{\blacklozenge_2^-(\Diamond^- p \wedge \blacklozenge_1^- \Diamond^- p) \vdash r}{\Diamond^- \blacklozenge_2^-\left(p \wedge \blacklozenge_1^- q\right) \vdash r}$

2. $\dfrac{\bullet_2\left(\bullet q,\ \bullet_1 \bullet p\right) \vdash r}{\bullet \bullet_2\left(q,\ \bullet_1 p\right) \vdash r}$

3. $\dfrac{\bullet_2\left(\bullet\blacklozenge^- q,\ \bullet_1 \bullet\blacklozenge^- p\right) \vdash \blacksquare^- \overline{r}}{\bullet\,\bullet_2\left(\blacklozenge^- q,\ \bullet_1 \blacklozenge^- p\right) \vdash \blacksquare^- \overline{r}}$

Algorithm $\tau_1^-$:

1. $\dfrac{\tau_1^-\left(\bullet_2\left(\bullet\,[q]\,,\ \bullet_1 \bullet [p]\right) \vdash [\overline{r}]\right)}{\tau_1^-\left(\bullet\,\bullet_2\left([q]\,,\ \bullet_1 [p]\right) \vdash [\overline{r}]\right)}$

2. $\dfrac{\bullet\,[p]\ ;_3\ \bullet\,[q] \vdash [\overline{r}]}{\bullet\left([p]\ ;_3\ [q]\right) \vdash [\overline{r}]}$

Algorithm $\tau_0^-$:

1. $\dfrac{\bullet(X\ ,_i\ Y) \vdash Z}{\bullet X\ ,_i\ \bullet Y \vdash Z}.$

## 12 From Gaggle Logic to Classical Logic

In this section, we show how we can recover the classical conjunction $\wedge$ and disjunction $\vee$ as well as the truth constants $\top$ and $\bot$ and the Boolean negation $\neg$ by adding specific protoanalytic inference rules. These rules are in fact refinements of the classical structural rules of classical logic.

**Lemma 8** ([1])**.** *Let $\star$ be a basic connective of the form $(\sigma, t, \rightarrow) \in \{\otimes_3, \otimes_2, \otimes_1\}$ and let $\mathsf{C} \subseteq \mathbb{C}$ be such that $\star \in \mathsf{C}$. Then, any $\mathsf{C}$–frame $F$ satisfying the first-order conditions $\mathsf{B}^{c,A}_{;i}$, $\mathsf{CI}^{A}_{;i}$, $\mathsf{WI}^{A}_{;i}$ and $\mathsf{K}^{A}_{;i}$ is such that for all $w, v, u \in F$,*

$$Ruvw \text{ iff } u \sqsubseteq w \text{ and } v \sqsubseteq w. \tag{23}$$

**Proposition 5.** *Let $\mathsf{C} \subseteq \mathbb{C}$.*

- *If $\mathsf{C}$ contains $\otimes_3$ (or $\oplus_3$) then the calculus $\mathsf{CL}_\wedge$ (resp. $\mathsf{CL}_\vee$) is sound and complete for the logic $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \Vdash\right)$, where $\mathcal{F}_0$ is the class of $\mathsf{C}$–frames such that the ternary relation $R_{\otimes_3}$ (resp. $R_{\oplus_3}$) is the identity relation. Thus, $\otimes_3$ (resp. $\oplus_3$) is the classical conjunction $\wedge$ (resp. disjunction $\vee$).*

- *If $\mathsf{C}$ contains also $t \triangleq (Id, -, \rightarrow)$ (or $f \triangleq (Id, +, \leftarrow)$), then the calculus $\mathsf{CL}_{\wedge,t}$ (resp. $\mathsf{CL}_{\vee,f}$) is sound and complete for the logic $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}'_0, \Vdash\right)$, where $\mathcal{F}'_0$ is $\mathcal{F}_0$ such that $T$ (resp. $F$) is the universal unary relation (resp. empty unary relation). Thus, $t$ is the classical truth constant $\top$ and $f$ is $\bot$.*

- *If $\mathsf{C}$ contains also $\sim_2 \triangleq (\sigma'_1, t'_4, \rightarrow)$ (or $-_2 \triangleq (\sigma'_1, t'_3, \leftarrow)$) then the calculi $\mathsf{CL}_{\wedge,t,\neg}$ and $\mathsf{CL}_{\wedge,t,\neg'}$ (resp. $\mathsf{CL}_{\vee,f,\neg}$ and $\mathsf{CL}_{\vee,f,\neg'}$) are sound and complete for the logic $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}''_0, \Vdash\right)$ where $\mathcal{F}''_0$ is $\mathcal{F}'_0$ such that the binary relations $R_{\sim_2}$ (resp. $R_{-_2}$) associated to $\sim_2$ (resp. $-_2$) is the identity relation. Thus, the connectives $\sim_2$ and $-_2$ are Boolean negation $\neg$ and $\sim_1, \sim_2, -_1, -_2$ all coincide (with Boolean negation).*

$$\mathit{Galog}(\mathit{C}) + \left\{ \mathit{K}^{A}_{;i},\ \mathit{WI}^{A}_{;i},\ \mathit{CI}^{A}_{;i}, \mathit{B}^{c,A}_{;i}, \mathit{TI}^{A}_{i;\,j} \mid i,j \in [\![1;3]\!] \right\} \qquad \mathsf{CL}_{\wedge}$$

$$\mathit{Galog}(\mathit{C}) + \left\{ \mathit{K}^{K}_{,i},\ \mathit{WI}^{K}_{,i},\ \mathit{CI}^{K}_{,i}, \mathit{B}^{c,K}_{,i}, \mathit{TI}^{K}_{i,\,j} \mid i,j \in [\![1;3]\!] \right\} \qquad \mathsf{CL}_{\vee}$$

$$\mathit{CL}_{\wedge} + \{\boldsymbol{I}^{A}_{;i} \mid i \in [\![1;3]\!]\} \qquad \mathsf{CL}_{\wedge,t}$$

$$\mathit{CL}_{\vee} + \{\boldsymbol{I}^{K}_{;i} \mid i \in [\![1;3]\!]\} \qquad \mathsf{CL}_{\vee,f}$$

$$\mathit{CL}_{\wedge,t} + \left\{ \mathit{Q}^{A}_{\sim_j} \right\} + \left\{ \mathit{N}^{A}_{1}, \mathit{BI}^{A}_{i,j} \mid i,j \in [\![1;2]\!] \right\} \qquad \mathsf{CL}_{\wedge,t,\neg}$$

$$\mathit{CL}_{\wedge,t} + \left\{ \mathit{Q}^{A}_{\sim_j} \right\} + \left\{ \mathit{N}^{A}_{2},\ \mathit{BI}^{A}_{i,j} \mid i,j \in [\![1;2]\!] \right\} \qquad \mathsf{CL}_{\wedge,t,\neg'}$$

$$\mathit{CL}_{\vee,f} + \left\{ \mathit{Q}^{K}_{-_j} \right\} + \left\{ \mathit{N}^{K}_{1},\ \mathit{BI}^{K}_{i,j} \mid i,j \in [\![1;2]\!] \right\} \qquad \mathsf{CL}_{\vee,f,\neg}$$

$$\mathit{CL}_{\vee,f} + \left\{ \mathit{Q}^{K}_{-_j} \right\} + \left\{ \mathit{N}^{K}_{2},\ \mathit{BI}^{K}_{i,j} \mid i,j \in [\![1;2]\!] \right\} \qquad \mathsf{CL}_{\vee,f,\neg'}$$

*Proof.* We only prove the proposition for the calculus $\mathsf{Galog}(\mathsf{C}) + \left\{ \mathsf{K}^{A}_{;i}, \mathsf{WI}^{A}_{;i}, \mathsf{CI}^{A}_{;i}, \mathsf{B}^{c,A}_{;i}, \mathsf{TI}^{A}_{i;\,j} \mid i,j \in [\![1;3]\!] \right\} + \left\{ \mathbf{I}^{A}_{;i}, \mathsf{Q}^{A}_{\sim_j} \right\} + \left\{ \mathsf{N}^{A}_{1}, \mathsf{BI}^{A}_{i,j} \right\}$, the proof for the other calculi being similar. Because of Proposition 5, the ternary relation $R_{\wedge}$ is the identity relation. In that case, the condition ($\mathsf{N}^{A}_{1}$) simplifies into: for all $x, y$,

$$R_{\neg}yx \rightarrow y \sqsubseteq x \tag{24}$$

Together with the condition ($\mathsf{BI}^{A}_{i,j}$) ($R_{\neg}yx \rightarrow R_{\neg}xy$) and because $\sqsubseteq$ is antisymmetric, this implies that for all $x, y$, if $R_{\neg}yx$ then $x = y$. Moreover, by conditions $\left\{ \mathbf{I}^{A}_{;i}, \mathsf{Q}^{A}_{\sim_j} \right\}$, we obtain that the relation $R_{\neg}$ is *serial*, that is, for all $x$, there is $y$ such that $R_{\neg}xy$. Therefore, we finally obtain that for all $x, y$, we have that $Rxy$ if, and only if, $x = y$: $R_{\neg}$ is the identity relation. $\square$

In other words, all the calculi of Proposition 5 are sound and complete w.r.t. propositional logic. In fact, if we forget the indices $i, j$, each inference rule of these calculi corresponds to a rule of the display calculus for propositional logic, and vice versa. The effects of the validity of inference rules on the class of $\mathbb{C}$–frames are asymmetric depending on which side of the turnstile $\vdash$ we consider. In fact, we can perfectly introduce only the conjunction or only the disjunction in our language. This allows us to consider as possible extensions of Gaggle logic the logics of *semi–lattice* Gaggles [6].

# 13 Conclusion

## 13.1 Related Work

A characterization result similar to our Theorem 14 has been proved by Ciabattoni & Al [12], but between *formulas* and inference rules of sequent and hypersequent calculi, not between first-order frame conditions and inference rules. This result was based on the Full Lambek Calculus with exchange $\mathsf{FLe}$ as base logic (see *e.g.* [45] for details about this logic). It was later extended and generalized by Ciabattoni and Ramanayake [13] in order to deal with other logics defined by display calculi in a very abstract form (hypersequent calculi can indeed be embedded into display calculi [46, 14]). Ciabattoni and Ramanayake [13] defined properties on formulas from abstractly defined display calculi so that we can obtain a result similar to Kracht's [35, Theorem 16] (and our Theorem 14) for an arbitrary given base display calculus which should nevertheless be *amenable*. Even if Gaggle logic is not *stricto sensu* a display calculus, the notions of display calculi and those introduced in this report are so close that we can transfer the results of [13]. Doing so, if we assume that a set of connectives $\mathsf{C}$ contains the classical $\wedge, \vee, \top$ and $\bot$ (introduced in Section 12), we obtain that every logic based on $\mathsf{C}$ which is protodisplayed by a calculus $\mathsf{Galog}(\mathsf{C}) + \Sigma$ is such that this calculus is '*amenable*' and '*well–behaved*' and *corresponds* to the Hilbert calculus $\tau_1(\mathsf{Galog}(\mathsf{C})) + \tau_1(\Sigma)$ (these notions being defined for display calculi in [13]). Hence, protodisplay logics sit at the top of the hierarchy of 'display' logics elicited by Ciabattoni and Ramanayake [13]. This is not so surprising since the same result holds for display calculi of basic tense logic (as proved in [13]) and our protodisplayble logics are intimately connected with them. However, this result holds only for formulas and not for first–order frame conditions like in our Theorem 14. Morever, it is valid only for (distributive) logics containing the connectives $\wedge, \vee, \top$ and $\bot$ and it does not provide some other common meta–theoretical properties such as interpolation. So, the results that we would obtain by applying the techniques and methods of [13] are strictly subsumed by our Theorem 14.

A characterization result similar to our Theorem 14 has also been proved by Negri [43], but only for normal modal logics and between labelled sequent calculi [20] and universal axioms or so–called *geometric* implications.

Palmigiano & Al [27] have recently applied the tools of unified correspondence [15] to adress the identification of the syntactic shape of axioms which can be translated into analytic structural rules of a display calculus, and the definition of an effective procedure for transforming axioms into such rules. However, they do not provide corresponding frame semantics conditions as we do. Moreover, even if their methodology is generic, it has to be adapted on a case by case basis to each (type of) logic and they do not have a similar characterization theorem nor some common meta-theoretical properties as we do.

Bimbó & Dunn [6] have developed a uniform framework for the relational semantics also based on Gaggle theory. In particular, they provide relational semantics to a wide variety of non–classical logics. Given our characterization result, for each logic, it suffices to examine the definition of its semantics in order to determine whether or not it is protodisplay and, if it is, we can compute a calculus which is sound and complete for that logic. As it turns out, many of the logics presented in [6] are protodisplay, except maybe the logics associated to non–distributive Gaggles (see also Section 7.1). Indeed, the semantics of these logics are based on specific frame conditions which ressemble second–order conditions rather than first–order conditions.

### 13.2 Concluding Remarks

Our approach and results pave the way to many novel research directions. We end our article by listing just two of them. First, obviously, most of the non–classical logics can be revisited and reexamined to determine whether or not they are protodisplay, which would entail in that case a number of interesting meta–theoretical properties (such as conservativity, interpolation, displayability). In that respect, one could consider the uniform framework developed by Bimbó & Dunn [6], also based on Gaggle theory, because it provides a relational semantics for a wide variety of non–classical logics. Second, our unifying framework can also ease the determination of the relative expressiveness of various logics because of the various bridges between syntax and semantics that we have identified.

Finally, even if protodisplay logics are only a part of all non–classical logics, given the generality and abstract nature of Gaggle logic, there is still the hope that other non–classical logics, if not all, could be seen as extensions of Gaggle logic by means of less restrictive constraints than our "protoanalytic" inference rules or, equivalently, our "protoanalytic" first-order frame conditions. This leads us to the following question: could Gaggle logic be considered as a foundation for the "Universal Logic" paradigm ?

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# Part III

# Appendix

$$\begin{tikzcd}
 & (M,w) \arrow[dl, "\text{Tonicity preserving map [Def. 48]}"'] \arrow[dr, "\text{Truth preserving map [Def. 47]}"] & & & \\
(M^0_\blacklozenge, w^0_\blacklozenge) & \otimes \text{ [Def. 49]} & (\vec{M}_\blacklozenge, \vec{w}_\blacklozenge) & = & (M_\blacklozenge, w_\blacklozenge)
\end{tikzcd}$$

Figure 15: Construction of the pointed $\mathsf{C}_\blacklozenge$–model $(M_\blacklozenge, w_\blacklozenge)$ associated to a pointed $\mathsf{C}$–model $(M, w)$

# 14 Proof of Theorem 14

## 14.1 Relative Expressiveness of Gaggle logic and Tense Gaggle logic

To prove that Gaggle logic and tense Gaggle logic are equally expressive with respect to proto-analytic consecutions, we resort to the techniques of *unraveling* of modal logic [7, Section 4.5] and to specific properties of the *product update* of dynamic epistemic logic [3]. The crux of the proof of Proposition 6 consists in showing that for any consecution $X \models Y$ of $[\mathcal{S}]_{\mathsf{C}}$ and for any pointed $\mathsf{C}$–model $(M, w)$ which does not satisfy $X \models Y$, we can always associate a pointed tense $\mathsf{C}_\blacklozenge$–model $(M_\blacklozenge, w_\blacklozenge)$ which does not satisfy $\tau_1(X \models Y)$ either. The construction of this pointed tense $\mathsf{C}_\blacklozenge$–model is summarized in Figure 15.

**Definition 47** (Unraveled $\mathsf{C}$–model and tense unraveled $\mathsf{C}$–model)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $(M, w) = (W, \sqsubseteq, \mathcal{R}, w)$ be a pointed $\mathsf{C}$–model. The *unraveled pointed model associated to* $(M, w)$ is the pointed Kripke model $\vec{M} = (\vec{W}, \vec{\sqsubseteq}, \vec{\mathcal{R}}, \vec{w})$ such that:

1. $\vec{W}$ is a set of sequences of worlds of $W$ of the form $(v_1, \ldots, v_m)$, generally denoted $\vec{v}$. In particular, we define $\vec{w} = (w)$ and we set $\vec{w} \in \vec{W}$;
2. $\vec{\mathcal{R}}$ is a copy of the set $\mathcal{R}$ of $M$ such that for all $\star = (\sigma, t, \leftrightarrow)_i \in \mathsf{C}$, if $R_\star^{\leftrightarrow,\sigma} w_1 \ldots w_n v$ and there is $(\vec{v}, v) \in \vec{W}$ then $(\vec{v}, v, w_n), (\vec{v}, v, w_n, w_{n-1}), \ldots, (\vec{v}, v, w_n, w_{n-1}, \ldots, w_1) \in \vec{W}$ and $\vec{R}_\star^{\leftrightarrow,\sigma}(\vec{v}, v, w_n, \ldots, w_2, w_1)(\vec{v}, v, w_n, \ldots, w_2) \ldots (\vec{v}, v, w_n)(\vec{v}, v)$; in particular, if $\star$ is a propositionnal letter (that is $\mathsf{a}(\star) = 0$), $R_\star^{\leftrightarrow,\sigma}(v)$ holds and $(\vec{v}, v) \in \vec{W}$, then $\vec{R}_\star(\vec{v}, v)$ holds too;
3. $\vec{W}$ and $\vec{\mathcal{R}}$ are the smallest sets satisfying the conditions 1. and 2. above;
4. $\vec{\sqsubseteq} \triangleq \vec{W} \times \vec{W}$.

The *tense unraveled pointed model associated to* $(M, w)$ is the pointed Kripke model $(\vec{M}_\blacklozenge, \vec{w}_\blacklozenge) = (\vec{W}_\blacklozenge, \vec{\sqsubseteq}_\blacklozenge, \vec{\mathcal{R}}_\blacklozenge, \vec{w}_\blacklozenge)$ such that

1. $\vec{W}_\blacklozenge = \vec{W}$, $\vec{\sqsubseteq}_\blacklozenge = \vec{\sqsubseteq}$ and $\vec{w}_\blacklozenge = \vec{w}$;

2. $\vec{\mathcal{R}}_{\blacklozenge}$ is a set of sequences of binary relations $(\overrightarrow{R}^1_{\star}, \ldots, \overrightarrow{R}^n_{\star})$ associated to each $\star \in \mathsf{C}$ such that:

$$R_{\star}^{\overrightarrow{\leftrightarrow},\sigma}(\vec{v}, v, w_n, w_{n-1}, \ldots, w_2, w_1)(\vec{v}, v, w_n, \ldots, w_2) \ldots (\vec{v}, v, w_n, w_{n-1})(\vec{v}, v, w_n)(\vec{v}, v)$$

iff $\overrightarrow{R}^n_{\star}(\vec{v}, v, w_n)(\vec{v}, v)$ and $\overrightarrow{R}^{n-1}_{\star}(\vec{v}, v, w_n, w_{n-1})(\vec{v}, v, w_n)$ and … and

$\overrightarrow{R}^1_{\star}(\vec{v}, v, w_n, \ldots, w_2, w_1)(\vec{v}, v, w_n, \ldots, w_2)$

In particular, if $\star$ is a propositionnal letter (that is $\mathsf{a}(\star) = 0$) then $R_{\star}^{\leftrightarrow,\sigma}(w)$ holds if, and only if, $\overrightarrow{R}_{\star}(w)$ holds. ⊣

We define the satisfaction relation, abusively denoted $\models$, between the set of tense unraveled pointed models and the structures of $[\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$ like in Definition 29.

**Lemma 9.** *Let $\mathsf{C} \subseteq \mathbb{C}$, let $(M, w)$ be a pointed $\mathsf{C}$–model and let $(\vec{M}, \vec{w})$ (and $(\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge})$) be its associated unraveled (resp. tense unraveled) pointed model. Then, for all $X \in [\mathcal{L}]_{\mathsf{C}}$ and all $X_{\blacklozenge} \in [\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$, we have that*

$$(M, w) \Vdash X \text{ iff } (\vec{M}, \vec{w}) \Vdash X \text{ iff } (\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge}) \models \tau_1(X) \tag{25}$$

$$(\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge}) \models X_{\blacklozenge} \text{ iff } (\vec{M}, \vec{w}) \Vdash \tau_1^{-}(X_{\blacklozenge}) \text{ iff } (M, w) \Vdash \tau_1^{-}(X_{\blacklozenge}) \tag{26}$$

*Proof sketch.* The proof of Expression (25) follows from (24) since $\tau_1$ and $\tau_1^{-}$ are inverse bijections (by Proposition 4). The proof of Expression 24 is in two parts. The proof that $(M, w) \Vdash X$ iff $(\vec{M}, \vec{w}) \models X$ is a property of the unravelling operation and follows from [7, Section 4.5]. The proof that $(\vec{M}, \vec{w}) \models X$ iff $(\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge}) \models \tau_1(X)$ is by induction on $X$. This second proof makes essential use of the fact that $(\vec{M}, \vec{w})$ is a tree (possibly infinite). □

However, the tense unraveling may not preserve the (Tonicity$_{\blacklozenge}$) conditions. These conditions are satisfied by the initial pointed $\mathsf{C}$–model $(M, w)$. Hence, we are going to combine the tense unraveled model $(\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge})$ with a tense $\mathsf{C}_{\blacklozenge}$–model $(M^0_{\blacklozenge}, w^0_{\blacklozenge})$ (that preserves the tonicity conditions of $(M, w)$) in order to obtain another tense $\mathsf{C}_{\blacklozenge}$–model (hence also preserving the tonicity conditions) that preserves the protoanalytic consecutions $X \vdash Y$ that were true in $(M, w)$. This combination will be possible thanks to a specific kind of *product update* very similar to the product update of dynamic epistemic logic [3].

**Definition 48** (Tonicity–preserving associated $\mathsf{C}_{\blacklozenge}$–model)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $(M, w)$ be a pointed $\mathsf{C}$–model. The *tonicity–preserving pointed Kripke model associated to* $M = (W, \sqsubseteq, \mathcal{R}, w)$ is the pointed Kripke model $M^0_{\blacklozenge} = (W^0_{\blacklozenge}, \sqsubseteq^0_{\blacklozenge}, \mathcal{R}^0_{\blacklozenge}, w^0_{\blacklozenge})$ such that:

1. $W^0_{\blacklozenge} \triangleq W$ and $w^0_{\blacklozenge} \triangleq w$;
2. $\sqsubseteq^0_{\blacklozenge} \triangleq \sqsubseteq$;
3. $\mathcal{R}^0_{\blacklozenge}$ is a set of sequences of binary relations $(R^{0,1}_{\star,\blacklozenge}, \ldots, R^{0,n}_{\star,\blacklozenge})$ associated to each $\star \in \mathsf{C}$ such that $R_{\star}^{\leftrightarrow,\sigma} w_1 \ldots w_{n+1}$ if, and only if, for all $i \in [\![1; n]\!]$, $R^{0,i}_{\star,\blacklozenge} w_i w_{i+1}$. ⊣

We define the satisfaction relation, abusively denoted $\models$, between the set of tonicity–preserving pointed Kripke models and the set of structures of $[\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$ like in Definition 29.

**Lemma 10.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M$ be a $\mathsf{C}$–model. Then, $M^0_{\blacklozenge}$ is a $\mathsf{C}_{\blacklozenge}$–model and satisfies in particular the conditions of* (Tonicity$_{\blacklozenge}$).

*Proof.* The (Tonicity$_{\blacklozenge}$) conditions are in fact defined from the (Tonicity) conditions so as to satisfy this property. $\square$

**Definition 49** (Associated $\mathsf{C}_{\blacklozenge}$–model)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $(M, w) = (W, \sqsubseteq, \mathcal{R}, w)$ be a pointed $\mathsf{C}$–model. The *pointed model associated to* $(M, w)$ is the tuple $(M_{\blacklozenge}, w_{\blacklozenge}) = (M^0_{\blacklozenge}, w^0_{\blacklozenge}) \otimes (\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge}) \triangleq (W_{\blacklozenge}, \sqsubseteq_{\blacklozenge}, \mathcal{R}_{\blacklozenge}, w_{\blacklozenge})$ such that:

- $W_{\blacklozenge} \triangleq \left\{ \left(x^0_{\blacklozenge}, \vec{x}_{\blacklozenge}\right) \mid x^0_{\blacklozenge} \in W^0_{\blacklozenge}, \vec{x}_{\blacklozenge} \in \vec{W}_{\blacklozenge} \text{ and for all } p \in \mathbb{P}, x^0_{\blacklozenge} \in V(p) \text{ iff } \vec{x}_{\blacklozenge} \in \vec{V}_{\blacklozenge}(p) \right\}$;

- $\mathcal{R}_{\blacklozenge}$ is a set of sequences of binary relations $(R^1_{\star,\blacklozenge}, \ldots, R^n_{\star,\blacklozenge})$ associated to each connective $\star \in \mathsf{C}$ of arity $n$ such that for all $i \in [\![1; n]\!]$,

$$R^i_{\star,\blacklozenge}(x^0_{\blacklozenge}, \vec{x}_{\blacklozenge})(y^0_{\blacklozenge}, \vec{y}_{\blacklozenge}) \text{ iff } R^{0,i}_{\star,\blacklozenge} x^0_{\blacklozenge} y^0_{\blacklozenge} \text{ and } \vec{R}^i_{\star} \vec{x}_{\blacklozenge} \vec{y}_{\blacklozenge};$$

- $(x^0_{\blacklozenge}, \vec{x}_{\blacklozenge}) \sqsubseteq_{\blacklozenge} (y^0_{\blacklozenge}, \vec{y}_{\blacklozenge})$ iff $x^0_{\blacklozenge} \sqsubseteq^0_{\blacklozenge} y^0_{\blacklozenge}$ and $\vec{x}_{\blacklozenge} \vec{\sqsubseteq}_{\blacklozenge} \vec{y}_{\blacklozenge}$;

- $w_{\blacklozenge} \triangleq (w^0_{\blacklozenge}, \vec{w}_{\blacklozenge})$;

where $(\vec{M}_{\blacklozenge}, \vec{w}_{\blacklozenge}) = (\vec{W}_{\blacklozenge}, \vec{\sqsubseteq}_{\blacklozenge}, \vec{\mathcal{R}}_{\blacklozenge}, \vec{w}_{\blacklozenge})$ is the tense unraveled pointed model associated to $(M, w)$ and $(M^0_{\blacklozenge}, w^0_{\blacklozenge}) = (W^0_{\blacklozenge}, \sqsubseteq^0_{\blacklozenge}, \mathcal{R}^0_{\blacklozenge}, w^0_{\blacklozenge})$ is the tonicity–preserving pointed $\mathsf{C}_{\blacklozenge}$–model associated to $(M, w)$.

Then, we define the satisfaction relation, abusively denoted $\models$, between the set of pointed models associated to $\mathsf{C}$–models and the set of structures of $[\mathcal{L}]_{\mathsf{C}_{\blacklozenge}}$ like in Definition 29. $\dashv$

**Lemma 11.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $(M, w) = (W, \sqsubseteq, \mathcal{R}, w)$ be a pointed $\mathsf{C}$–model. Then, the pointed model $(M_{\blacklozenge}, w_{\blacklozenge})$ associated to $(M, w)$ is a pointed tense $\mathsf{C}_{\blacklozenge}$–model. Moreover, for all protoanalytic consecutions $X \vdash Y$, we have that*

$$(M, w) \Vdash X \vdash Y \textit{ iff } (M_{\blacklozenge}, w_{\blacklozenge}) \models \tau_1(X \vdash Y) \tag{27}$$

*Proof.* To prove that $(M_{\blacklozenge}, w_{\blacklozenge})$ is a $\mathsf{C}_{\blacklozenge}$-model, we must prove that it satisfies the (Tonicity$_{\blacklozenge}$) conditions. First, one should observe that these Tonicity$_{\blacklozenge}$ conditions are in fact universal Horn sentences. Indeed, they rewrite as follows: for all $i \in [\![1; n]\!]$,

- if $\leftrightarrow = \rightarrow$: $(\forall w_{i+1} \rhd_0 w_i)(\forall w'_i \bowtie w_i)(\forall w'_{i+1} \bowtie w_{i+1}) R w'_i w'_{i+1}$;

- if $\leftrightarrow = \leftarrow$: $(\forall w'_{i+1} \rhd_0 w'_i)(\forall w_i \bowtie w'_i)(\forall w_{i+1} \bowtie w'_{i+1}) R w_i w_{i+1}$

where, for all $j \in \{i, i+1\}$, (we recall that $\sqsubseteq\!\!\!\!\sqsupseteq$ is defined in Definition 28)

$$w'_j \bowtie w_j \triangleq \begin{cases} w'_j \rhd_0 w_j & \text{if } \sqsubseteq\!\!\!\!\sqsupseteq = \sqsubseteq \\ w'_j \lhd_0 w_j & \text{if } \sqsubseteq\!\!\!\!\sqsupseteq = \sqsupseteq \end{cases} \text{ and } w_j \bowtie w'_j \triangleq \begin{cases} w_j \lhd_0 w'_j & \text{if } \sqsubseteq\!\!\!\!\sqsupseteq = \sqsubseteq \\ w_j \rhd_0 w'_j & \text{if } \sqsubseteq\!\!\!\!\sqsupseteq = \sqsupseteq. \end{cases}$$

Then, we resort to results stemming from dynamic epistemic logic, which is a logic introducing a specific product update between models [3]. This product update has specific properties. In particular, "the only first-order frame conditions that are guaranteed to be preserved [by the product update] are those definable as universal Horn sentences" [55, p. 134]. As it turns out, the product of models defined in Definition 49 is symmetric with respect to its arguments and it can be reformulated as the product update of dynamic epistemic logic (in both directions). Hence, $(M_\blacklozenge, w_\blacklozenge)$ satisfies the (Tonicity$_\blacklozenge$) conditions and it is therefore a $\mathsf{C}_\blacklozenge$-model.

To prove Expression 26, it suffices to prove the following Expression (27), because of Expression (24):

$$(\vec{M}_\blacklozenge, \vec{w}_\blacklozenge) \models \tau_1(X \vdash Y) \text{ iff } (M_\blacklozenge, w_\blacklozenge) \models \tau_1(X \vdash Y) \tag{28}$$

where $(\vec{M}_\blacklozenge, \vec{w}_\blacklozenge)$ is the tense unraveled model associated to $(M, w)$. As explained in the proof of Lemma 6, $\tau_1(X), *\tau_1(Y)$ belongs to the language recognized by the grammar of Expression 9. This entails that $t_1\,(\tau_1(X),\ *\,\tau_1(Y))$ is a *positive existential formula* [7]. Now, we know from results of dynamic epistemic logic [57] that $(\vec{M}_\blacklozenge, \vec{w}_\blacklozenge)$ is a simulation of $(M_\blacklozenge, w_\blacklozenge)$ and we also know from [7, Theorem 2.78] that the only formula which are preserved by a simulation are the positive existential formula. Since we just proved that $t_1\,(\tau_1(X),\ *\,\tau_1(Y))$ is a positive existential formula, we have that

$$(\vec{M}_\blacklozenge, \vec{w}_\blacklozenge) \models t_1\,(\tau_1(X),\ *\,\tau_1(Y)) \text{ iff } (M_\blacklozenge, w_\blacklozenge) \models t_1\,(\tau_1(X),\ *\,\tau_1(Y)) \tag{29}$$

Then, Expression 27 follows easily from Expression (28), by definition of $t_1$: we take the Boolean negation of both terms in (28). □

Now, we prove the other direction: how to construct a $\mathsf{C}$–model associated to a $\mathsf{C}_\blacklozenge$–model that satisfies the same structures.

**Definition 50** ($\mathsf{C}$–model associated to a $\mathsf{C}_\blacklozenge$–model)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M_\blacklozenge = (W_\blacklozenge, \sqsubseteq_\blacklozenge, \mathcal{R}_\blacklozenge)$ be a $\mathsf{C}_\blacklozenge$–model. The *model associated to* $M_\blacklozenge$ is the tuple $M = (W, \sqsubseteq, \mathcal{R})$ such that:

- $W = W_\blacklozenge$ and $\sqsubseteq = \sqsubseteq_\blacklozenge$;
- for all $\star \in \mathsf{C}$ of arity $n$, we associate bijectively to each $R_\star \in \mathcal{R}$ a sequence of binary relations $(R^1_{\star,\blacklozenge}, \ldots, R^n_{\star,\blacklozenge})$ such that $R_\star w_1 \ldots w_n w_{n+1}$ if, and only if, for all $i \in [\![1; n]\!]$, $R^i_{\star,\blacklozenge} w_i w_{i+1}$. ⊣

**Lemma 12.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $(M_\blacklozenge, w_\blacklozenge)$ be a pointed $\mathsf{C}_\blacklozenge$–model. Then, the pointed model $(M, w)$ associated to $(M_\blacklozenge, w_\blacklozenge)$ is a $\mathsf{C}$–model. Moreover, for all structures $X_\blacklozenge$ of $[\mathcal{L}]_{\mathsf{C}_\blacklozenge}$,*

$$(M_\blacklozenge, w_\blacklozenge) \models X_\blacklozenge \textit{ iff } (M, w) \Vdash \tau_1^-(X_\blacklozenge) \tag{30}$$

*Thus, for all protoanalytic consecutions $X_\blacklozenge \vdash Y_\blacklozenge$ of $[\mathcal{S}]_{\mathsf{C}_\blacklozenge}$, we have that $(M_\blacklozenge, w_\blacklozenge) \models X_\blacklozenge \vdash Y_\blacklozenge$ if, and only if, $(M, w) \Vdash \tau_1^-(X_\blacklozenge \vdash Y_\blacklozenge)$*

*Proof.* The proof that $(M, w)$ is a $\mathsf{C}$–model is without particular difficulty, we only need to check that the (Tonicity) conditions are fulfilled. As it turns out, the (Tonicity$_\blacklozenge$) conditions have been defined for that purpose. The proof of Expression 29 is also without particular difficulty and it is by induction on $X_\blacklozenge$. □

**Proposition 6.** *Let $\mathcal{F}_0$ be a class of pointed $\mathsf{C}$–frames and let $X \vdash Y$ be a protoanalytic consecution of $[\mathcal{S}]_{\mathsf{C}}$. Then, $X \vdash Y$ is valid in $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \Vdash\right)$ if, and only if, $\tau_1(X \vdash Y)$ is valid in $\left([\mathcal{S}]_{\mathsf{C}_\blacklozenge}, \mathcal{F}_\blacklozenge, \models\right)$, where $\mathcal{F}_\blacklozenge$ is the class of pointed $\mathsf{C}_\blacklozenge$–frames associated to $\mathcal{F}_0$.*

*Proof.* It follows easily from Lemmas 11 and 12. □

## 14.2 Existence of a Display Calculus for Tense Gaggle logic

**Lemma 13.** *There exists a proper display calculus, denoted $\mathsf{D}_\blacklozenge$, which extends $\textbf{DLM}_t$ (defined in [35]) and which is sound and complete for $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}^-_{\mathbb{C}_\blacklozenge}, \models\right)$, where $\mathcal{M}^-_{\mathbb{C}_\blacklozenge}$ is the class of tense $\mathbb{C}_\blacklozenge$–models whose relation $\sqsubseteq_\blacklozenge$ is reflexive and transitive, but* not necessarily *antisymmetric.*

*Proof.* It suffices to observe that the conditions 3.(a) and 3.(b) and (Tonicity$_\blacklozenge$) of Definition 28 are primitive first-order formulas of $\mathcal{L}^{\text{FOL}}(\mathbb{C})$. Indeed:

- The Tonicity$_\blacklozenge$ conditions have been rewritten as primitive first-order formulas of $\mathcal{L}^{\text{FOL}}(\mathbb{C})$ in the proof of Lemma 11.
- Condition 3.(a) rewrites as: for all $i \in [\![1; n-1]\!]$, $(\forall v \rhd^{\blacklozenge_{\star,i}} w)(\exists u \rhd^{\blacklozenge_{\star,i+1}} v)(w = w)$;
- Condition 3.(b) rewrites as: for all $i \in [\![2; n]\!]$, $(\forall v \rhd^{\blacklozenge^-_{\star,i}} w)(\exists u \rhd^{\blacklozenge^-_{\star,i+1}} v)(w = w)$.

This entails that these conditions can be axiomatized by special inference rules for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$, because of Theorem 12. $\square$

**Proposition 7.** *$\mathsf{D}_\blacklozenge$ is sound and complete for the tense Gaggle logic $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}_{\mathbb{C}_\blacklozenge}, \models\right)$.*

*Proof.* We want to prove that $\mathsf{D}_\blacklozenge$ is also sound and complete for the tense Gaggle logic $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}_{\mathbb{C}_\blacklozenge}, \models\right)$, where $\mathcal{M}_{\mathbb{C}_\blacklozenge}$ is the class of tense $\mathbb{C}_\blacklozenge$–models whose relation $\sqsubseteq$ is not only reflexive and transitive but also antisymmetric.

Soundness is proved without difficulty by Lemma 13, because $\mathcal{M}_{\mathbb{C}_\blacklozenge} \subseteq \mathcal{M}^-_{\mathbb{C}_\blacklozenge}$. For completeness, because of Lemma 6, we only need to consider consecutions of the form $\tau_1(X \vdash Y)$ from $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$, where $X \vdash Y$ are *protoanalytic* consecutions of $[\mathcal{S}]_{\mathsf{C}}$. Let $X \vdash Y$ be a protoanalytic consecution of $[\mathcal{S}]_{\mathsf{C}}$ valid in $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}_{\mathbb{C}_\blacklozenge}, \models\right)$, that is, such that for all $(M_\blacklozenge, w_\blacklozenge) \in \mathcal{M}_{\mathbb{C}_\blacklozenge}$ we have that $(M_\blacklozenge, w_\blacklozenge) \models \tau_1(X \vdash Y)$ $(*)$. Assume towards a contradiction that it is not valid on $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}^-_{\mathbb{C}_\blacklozenge}, \models\right)$. Then, there is $(M'_\blacklozenge, w'_\blacklozenge) \in \mathcal{M}^-_{\mathbb{C}_\blacklozenge}$ such that it is not the case that $(M'_\blacklozenge, w'_\blacklozenge) \models \tau_1(X \vdash Y)$. Then, following the same methodology as in the previous Section 14.1, we can transform $(M'_\blacklozenge, w'_\blacklozenge)$ into a $\mathsf{C}_\blacklozenge$–model $(M''_\blacklozenge, w''_\blacklozenge)$ of $\mathcal{M}_{\mathbb{C}_\blacklozenge}$ (where $\sqsubseteq_\blacklozenge$ is therefore antisymmetric) such that it is not the case that $(M''_\blacklozenge, w''_\blacklozenge) \models \tau_1(X \vdash Y)$ (because of Lemmas 11 and 12). This contradicts our assumption $(*)$. Hence, for all protoanalytic consecutions $X \vdash Y$ of $[\mathcal{S}]_{\mathsf{C}}$, if for all $(M_\blacklozenge, w_\blacklozenge) \in \mathcal{M}_{\mathbb{C}_\blacklozenge}$ we have that $(M_\blacklozenge, w_\blacklozenge) \models \tau_1(X \vdash Y)$ then for all $(M_\blacklozenge, w_\blacklozenge) \in \mathcal{M}^-_{\mathbb{C}_\blacklozenge}$ we also have that $(M_\blacklozenge, w_\blacklozenge) \models \tau_1(X \vdash Y)$. Thus, by the completeness result of Lemma 13, we also have that $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge$. Since every consecution provable in $\mathsf{D}_\blacklozenge$ is of the form $\tau_1(X \vdash Y)$ (for some protoanalytic consecution $X \vdash Y$), again because of Lemma 6, we obtain our result. $\square$

## 14.3 Other Expressiveness Results

**Proposition 8** (Kracht [35])**.** *Let $R$ be a special inference rule for $[\mathcal{S}]_{\mathbb{C}_\blacklozenge}$, let $\varphi$ be a primitive formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$ and let $(F, w)$ be a pointed $\mathsf{C}$–frame. Then, we have $(F, w) \models R$ if, and only if, $(F, w) \models \tau_2(R)$, and we have $(F, w) \models \varphi$ if, and only if, $(F, w) \models \tau_2^-(\varphi)$. Moreover,*

$$\models \tau_2\left(\tau_2^-(\varphi)\right) \leftrightarrow \varphi \qquad\qquad \tau_2^-\left(\tau_2(R)\right) = R$$

**Proposition 9** (Sahlqvist [51], Kracht [34]). *Let $\varphi$ be a protoanalytic formula of $\mathcal{L}_{\mathbb{C}_\blacklozenge}$ and let $\Theta(x)$ be a protoanalytic formula of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$. Then,*

$$\models \tau_3\left(\tau_3^-\left(\Theta(x)\right)\right) \leftrightarrow \Theta(x) \qquad\qquad \models \tau_3^-\left(\tau_3\left(\varphi\right)\right) \leftrightarrow \varphi$$

**Proposition 10.** *Let $\mathsf{C} \subseteq \mathbb{C}$. For all $\varphi \in \mathcal{L}^{\mathrm{FOL},G}_{\mathsf{C}_\blacklozenge}$ and all pointed $(M, w) \in \mathcal{M}_{\mathsf{C}}$, we have that $(M, w) \models \varphi$ if, and only if, $(M, w) \models \tau_4(\varphi)$, and, vice versa, for all $\psi \in \mathcal{L}^{\mathrm{FOL},G}_{\mathsf{C}}$, we have $(M, w) \models \psi$ if, and only if, $(M, w) \models \tau_4^-(\psi)$. Hence,*

$$\models \tau_4\left(\tau_4^-\left(\psi\right)\right) \leftrightarrow \psi \qquad\qquad \models \tau_4^-\left(\tau_4\left(\varphi\right)\right) \leftrightarrow \varphi$$

*Proof.* By induction on the number of restricted quantifiers in $\varphi$ or $\psi$. □

## 14.4 Proof of our Characterization Theorem

**Theorem 15.** *Let $R$ be a special inference rule in $\mathcal{S}_{\mathsf{C}_\blacklozenge}$. Then, $R$ and $\tau_4\left(\tau_3\left(\tau_2\left(R\right)\right)\right)$ are local $\mathsf{C}$–frame correspondents.*

*Proof.* It follows from Theorem 11 and Propositions 8 and 10. □

**Lemma 14.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $\mathcal{F}_0$ be a class of pointed $\mathsf{C}_\blacklozenge$–frames. There exists a finite set of special inference rules $\Sigma_\blacklozenge$ for $[\mathcal{L}]_{\mathsf{C}_\blacklozenge}$–consecutions such that $\mathsf{D}_\blacklozenge + \Sigma_\blacklozenge$ is sound and complete for the logic $\left([\mathcal{S}]_{\mathsf{C}_\blacklozenge}, \mathcal{F}_0, \models\right)$ if, and only if, there exists a finite set $\Theta(x)$ of protoanalytic formulas of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$ that defines $\mathcal{F}_0$. Moreover, $\Sigma_\blacklozenge$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma_\blacklozenge$, as follows: $\Theta(x) \triangleq \tau_4\left(\tau_3\left(\tau_2\left(\Sigma_\blacklozenge\right)\right)\right)$ and $\Sigma_\blacklozenge \triangleq \tau_2^-\left(\tau_3^-\left(\tau_4^-\left(\Theta(x)\right)\right)\right)$.*

*Proof.* Let $\Sigma_\blacklozenge$ be a finite set of special inference rules for $\mathcal{L}_{\mathsf{C}_\blacklozenge}$–consecutions. Then, by Theorem 12, $\mathsf{D}_\blacklozenge + \Sigma_\blacklozenge$ is sound and complete for $\left([\mathcal{S}]_{\mathsf{C}_\blacklozenge}, \mathcal{F}_0, \models\right)$ if, and only if, $\mathcal{F}_0$ can be defined by some finite set $\Theta_0(x)$ of primitive formulas of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}_\blacklozenge}$ computable as follows: $\Theta_0(x) \triangleq \tau_3\left(\tau_2(\Sigma_\blacklozenge)\right)$. However, by Proposition 10, $\Theta_0(x)$ is equivalent on the class of pointed $\mathsf{C}$–frames to a finite set of protoanalytic formulas $\Theta(x) \triangleq \tau_4\left(\Theta_0(x)\right)$ of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$. Hence, $\mathsf{D}_\blacklozenge + \Sigma_\blacklozenge$ is sound and complete for $\left(\mathcal{S}_{\mathsf{C}_\blacklozenge}, \mathcal{F}_0, \models\right)$ if, and only if, $\mathcal{F}_0$ can be defined by some finite set $\Theta(x)$ of protoanalytic formulas of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$. Moreover, again using Proposition 10, $\Sigma_\blacklozenge$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma_\blacklozenge$, as follows: $\Theta(x) \triangleq \tau_4\left(\tau_3\left(\tau_2\left(\Sigma_\blacklozenge\right)\right)\right)$ and $\Sigma_\blacklozenge \triangleq \tau_2^-\left(\tau_3^-\left(\tau_4^-\left(\Theta(x)\right)\right)\right)$. □

**Lemma 15.** *Let $\Sigma$ be a set of special inference rules for $[\mathcal{S}]_{\mathbb{C}}$ and let $X \vdash Y \in [\mathcal{S}]_{\mathbb{C}}$. Then, $X \vdash Y$ is provable in $\mathsf{Galog} + \Sigma$ if, and only if, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$. Moreover, the proof of $\tau_1(X \vdash Y)$ in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$ is effectively computable from the proof of $X \vdash Y$ in $\mathsf{Galog} + \Sigma$.*

*Proof.* We first prove the left to right direction. If $X \vdash Y$ is provable in $\mathsf{Galog} + \Sigma$, then $X \vdash Y$ is valid in $\left([\mathcal{S}]_{\mathbb{C}}, \mathcal{F}_0, \Vdash\right)$, where $\mathcal{F}_0$ is the class of pointed $\mathsf{C}$–frames defined by the inference rules of $\Sigma$. Therefore, by Proposition 6, $\tau_1(X \vdash Y)$ is valid in $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{F}_0, \models\right)$, where $\mathcal{F}_0$ is also the class of pointed $\mathsf{C}$–frames defined by the inference rules of $\tau_1(\Sigma)$. Moreover, by Lemma 6, $\tau_1(\Sigma)$ is a set of special inference rules and, by Theorem 15, it locally corresponds to the set of protoanalytic formulas $\tau_4\left(\tau_3\left(\tau_2\left(\tau_1(\Sigma)\right)\right)\right)$ of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$. So, $\mathcal{F}_0$ is also the class of pointed $\mathsf{C}$–frames defined by the set of protoanalytic formulas $\tau_4\left(\tau_3\left(\tau_2\left(\tau_1(\Sigma)\right)\right)\right)$ of $\mathcal{L}^{\mathrm{FOL}}_{\mathsf{C}}$. Hence, by Lemma 14, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$.

Second, we prove the right to left direction. Assume that $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$. By Lemma 7 and Proposition 7, $\tau_1(\mathsf{Galog})$ and $\mathsf{D}_\blacklozenge$ are both sound and complete for the tense Gaggle logic $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{M}_{\mathbb{C}_\blacklozenge}, \Vdash\right)$. Thus, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$ if, and only if, $\tau_1(X \vdash Y)$ is provable in $\tau_1(\mathsf{Galog}) + \tau_1(\Sigma)$. Hence, $\tau_1(X \vdash Y)$ is provable in $\tau_1(\mathsf{Galog}) + \tau_1(\Sigma)$. Then, again by Proposition 7 and also because of Proposition 6, $X \vdash Y$ is provable in $\mathsf{Galog} + \Sigma$. $\square$

**Theorem 16** (Correspondence for inference rules). *Let $\Sigma$ be a finite set of special inference rules in $[\mathcal{S}]_{\mathbb{C}}$. Then, $\Sigma$ and $\Theta(x) \triangleq \tau_3(\tau_2(\tau_1(\Sigma)))$ are local frame correspondents.*

*Proof.* It follows directly from Theorem 13 and Proposition 6. $\square$

**Theorem 17** (Canonicity for inference rules). *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $\Sigma$ be a finite set of special inference rules in $[\mathcal{S}]_{\mathsf{C}}$. Then, $\mathsf{Galog}(\mathsf{C}) + \Sigma$ is sound and complete for the logic $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \Vdash\right)$, where $\mathcal{F}_0$ is the class of pointed $\mathsf{C}$–frames defined by the set of protoanalytic formulas $\Theta(x) \triangleq \tau_4(\tau_3(\tau_2(\tau_1(\Sigma))))$ of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$.*

*Proof.* We first prove completeness. Let $X \vdash Y \in [\mathcal{S}]_{\mathsf{C}}$ and assume that $X \vdash Y$ is valid in $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \Vdash\right)$. Then, $\tau_1(X \vdash Y)$ is valid in $\left([\mathcal{S}]_{\mathsf{C}_\blacklozenge}, \mathcal{F}_\blacklozenge, \models\right)$ by Proposition 6, where $\mathcal{F}_\blacklozenge$ is the class of $\mathsf{C}_\blacklozenge$–frames associated to the $\mathsf{C}$–frames of $\mathcal{F}_0$. Moreover, $\tau_1(\Sigma)$ is a set of special inference rules by Lemma 6. So, by Lemma 14, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$. Thus, $X \vdash Y$ is provable in $\mathsf{Galog} + \Sigma$ by Lemma 15, and therefore also in $\mathsf{Galog}(\mathsf{C}) + \Sigma$ (we recall that $\mathsf{Galog} + \Sigma$ admits the cut rule and therefore any proof of $X \vdash Y$ in $\mathsf{Galog} + \Sigma$ will resort to logical rules where only the connectives of $X \vdash Y$ occur). Now, we prove soundness. Assume that $X \vdash Y$ is provable in $\mathsf{Galog}(\mathsf{C}) + \Sigma$. Then, by Lemma 15, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\blacklozenge + \tau_1(\Sigma)$. Moreover, $\tau_1(\Sigma)$ is a set of special inference rules by Lemma 6. So, by Lemma 14, $\tau_1(X \vdash Y)$ is valid in $\left([\mathcal{S}]_{\mathbb{C}_\blacklozenge}, \mathcal{F}_\blacklozenge, \models\right)$. And finally, $X \vdash Y$ is valid in $\left([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \Vdash\right)$ by Proposition 6. $\square$

**Theorem 18.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $\mathcal{F}_0$ be a class of pointed $\mathsf{C}$–frames defined by a set $\Theta(x)$ of protoanalytic formulas of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$ with one free variable $x$. Then, the logic $\left([\mathcal{S}]_{\mathbb{C}}, \mathcal{F}_0, \Vdash\right)$ is properly protodisplay. Moreover, the set of special inference rules $\Sigma$ of the proper display calculus which is sound and complete for $([\mathcal{S}]_{\mathbb{C}}, \mathcal{F}_0, \Vdash)$ is effectively computable from $\Theta(x)$ as follows: $\Sigma \triangleq \tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)$.*

*Proof.* By Lemma 14, $\mathsf{D}_\blacklozenge + \Sigma_\blacklozenge$ is sound and complete for $([\mathcal{S}]_{\mathsf{C}_\blacklozenge}, \mathcal{F}_0, \models)$, where $\Sigma_\blacklozenge \triangleq \tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)$. Now, $\Sigma_\blacklozenge = \tau_1\left(\tau_1^-(\Sigma_\blacklozenge)\right)$ by Lemma 6, and $\tau_1^-(\Sigma_\blacklozenge)$ is a set of special inference rules in $[\mathcal{S}]_{\mathbb{C}}$. So, by Theorem 17, $\mathsf{Galog} + \tau_1^-(\Sigma_\blacklozenge)$ is sound and complete for the logic $\left([\mathcal{S}]_{\mathbb{C}}, \mathcal{F}_0, \Vdash\right)$, where $\mathcal{F}_0$ is the class of pointed $\mathsf{C}$–frames defined by the following set of protoanalytic formulas of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$:

$$\tau_4\left(\tau_3\left(\tau_2\left(\tau_1\left(\tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)\right)\right)\right)\right)$$

which is equivalent to $\Theta(x)$ on the class of pointed $\mathsf{C}$–frames, by Propositions 6, 8, 9, 10. $\square$

**Theorem 19** (Characterization). *Let $\mathsf{C} \subseteq \mathbb{C}$. A logic $([\mathcal{S}]_{\mathsf{C}}, \mathcal{F}_0, \Vdash)$ based on a class $\mathcal{F}_0$ of pointed $\mathsf{C}$–frames is protodisplay by a set of protoanalytic inference rules $\Sigma$ if, and only if, $\mathcal{F}_0$ is defined by some finite set $\Theta(x)$ of protoanalytic formulas of $\mathcal{L}_{\mathsf{C}}^{\mathrm{FOL}}$ with one free variable. Moreover, $\Sigma$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma$, as follows: $\Theta(x) \triangleq \tau_4(\tau_3(\tau_2(\tau_1(\Sigma))))$ and $\Sigma \triangleq \tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)$ (see Figure 9).*

*Proof.* The result follows straightforwardly from Theorems 17 and 18. □

## 15 Display Calculus $\mathbf{DLM}_{\mathtt{t}}$

The display calculus $\mathbf{DLM}_{\mathtt{t}}$, taken from [35], is recalled in Figures 16 and 17.

$$\begin{array}{c} X \vdash \bullet Y \\ \hline\hline \bullet X \vdash Y \end{array} \qquad \frac{X \vdash \varphi \quad \varphi \vdash Y}{X \vdash Y}\ cut$$

$$\begin{array}{c} X, Y \vdash Z \\ \hline\hline X \vdash Z, *Y \end{array} \qquad \begin{array}{c} X, Y \vdash Z \\ \hline\hline Y \vdash *X, Z \end{array}$$

$$\begin{array}{c} X \vdash Y, Z \\ \hline\hline X, *Z \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y, Z \\ \hline\hline *Y, X \vdash Z \end{array}$$

$$\begin{array}{c} *X \vdash Y \\ \hline\hline *Y \vdash X \end{array} \qquad \begin{array}{c} Y \vdash *X \\ \hline\hline X \vdash *Y \end{array}$$

$$\begin{array}{c} **X \vdash Y \\ \hline\hline X \vdash Y \end{array}\ N_1 l \qquad \begin{array}{c} X \vdash **Y \\ \hline\hline X \vdash Y \end{array}\ N_1 r$$

$$\begin{array}{c} X \vdash Z \\ \hline\hline \mathbf{I}, X \vdash Z \end{array}\ \mathbf{I}l \qquad \begin{array}{c} X \vdash Z \\ \hline\hline X \vdash \mathbf{I}, Z \end{array}\ \mathbf{I}r$$

$$\begin{array}{c} \mathbf{I} \vdash Y \\ \hline\hline *\mathbf{I} \vdash Y \end{array}\ Ql \qquad \frac{X \vdash \mathbf{I}}{X \vdash *\mathbf{I}}\ Qr$$

$$\frac{X \vdash Z}{Y, X \vdash Z}\ Wl \qquad \frac{X \vdash Z}{X, Y \vdash Z}\ Wr$$

$$\begin{array}{c} X_1, (X_2, X_3) \vdash Z \\ \hline\hline (X_1, X_2), X_3 \vdash Z \end{array}\ Al \qquad \begin{array}{c} Z \vdash X_1, (X_2, X_3) \\ \hline\hline Z \vdash (X_1, X_2), X_3 \end{array}\ Ar$$

$$\frac{X, Y \vdash Z}{Y, X \vdash Z}\ Pl \qquad \frac{Z \vdash X, Y}{Z \vdash Y, X}\ Pr$$

$$\frac{X, X \vdash Z}{X \vdash Z}\ Cl \qquad \frac{Z \vdash X, X}{Z \vdash X}\ Cr$$

$$\frac{\mathbf{I} \vdash Y}{\bullet\mathbf{I} \vdash Y}\ Ml \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bullet\mathbf{I}}\ Mr$$

Figure 16: Display calculus $\mathbf{DLM}_{\mathtt{t}}$: Structural Rules

$$\frac{}{p \vdash p}\ \mathsf{Id}$$

$$\frac{}{\mathbf{I} \vdash \top}\ \top_K \qquad \frac{\mathbf{I} \vdash X}{\top \vdash X}\ \top_A$$

$$\frac{X \vdash \mathbf{I}}{X \vdash \bot}\ \bot_K \qquad \frac{}{\bot \vdash \mathbf{I}}\ \bot_A$$

$$\frac{X \vdash * \varphi}{X \vdash \neg\varphi}\ \neg_K \qquad \frac{*\varphi \vdash X}{\neg\varphi \vdash X}\ \neg_A$$

$$\frac{X \vdash \varphi, \psi}{X \vdash \varphi \vee \psi}\ \vee_K \qquad \frac{\varphi \vdash X \quad \psi \vdash X}{\varphi \vee \psi \vdash X}\ \vee_A$$

$$\frac{X \vdash \varphi \quad X \vdash \psi}{X \vdash \varphi \wedge \psi}\ \wedge_K \qquad \frac{\varphi, \psi \vdash X}{\varphi \wedge \psi \vdash X}\ \wedge_A$$

$$\frac{X, \varphi \vdash \psi}{X \vdash \varphi \to \psi}\ \to_K \qquad \frac{X \vdash \varphi \quad \psi \vdash Y}{\varphi \to \psi \vdash * X, Y}\ \to_A$$

$$\frac{\bullet X \vdash \varphi}{X \vdash \Box\varphi}\ \Box_K \qquad \frac{\varphi \vdash X}{\Box\varphi \vdash \bullet X}\ \Box_A$$

$$\frac{X \vdash \varphi}{\bullet X \vdash \Diamond^{-}\varphi}\ \Diamond^{-}_K \qquad \frac{\varphi \vdash \bullet X}{\Diamond^{-}\varphi \vdash X}\ \Diamond^{-}_A$$

$$\frac{Y \vdash * \bullet * \varphi}{Y \vdash \Box^{-}\varphi}\ \Box^{-}_K \qquad \frac{\varphi \vdash X}{\Box^{-}\varphi \vdash * \bullet * X}\ \Box^{-}_A$$

$$\frac{X \vdash \varphi}{* \bullet * X \vdash \Diamond\varphi}\ \Diamond_K \qquad \frac{* \bullet * \varphi \vdash Y}{\Diamond\varphi \vdash Y}\ \Diamond_A$$

Figure 17: Display calculus $\mathbf{DLM}_{\mathtt{t}}$: Introduction Rules