On the Recovery of Core and Crustal Components of Geomagnetic Potential Fields

Laurent Baratchart, Christian Gerhards

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L. Baratchart[1], C. Gerhards[2]

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Abstract. In Geomagnetism it is of interest to separate the Earth's core magnetic field from the crustal magnetic field. However, measurements by satellites can only sense the sum of the two contributions. In practice, the measured magnetic field is expanded in spherical harmonics and separation into crust and core contribution is achieved empirically, by a sharp cutoff in the spectral domain. In this paper, we derive a mathematical setup in which the two contributions are modeled by harmonic potentials Φ_0 and Φ_1 generated on two different spheres $\mathbb{S}_{R_0}$ (crust) and $\mathbb{S}_{R_1}$ (core) with radii $R_1 < R_0$. Although it is not possible in general to recover Φ_0 and Φ_1 knowing their superposition $\Phi_0 + \Phi_1$ on a sphere $\mathbb{S}_{R_2}$ with radius $R_2 > R_0$, we show that it becomes possible if the magnetization $\mathbf{m}$ generating Φ_0 is localized in a strict subregion of $\mathbb{S}_{R_0}$. Beyond unique recoverability, we show in this case how to numerically reconstruct characteristic features of Φ_0 (e.g., spherical harmonic Fourier coefficients). An alternative way of phrasing the results is that knowledge of $\mathbf{m}$ on a nonempty open subset of $\mathbb{S}_{R_0}$ allows one to perform separation.

Keywords. Harmonic Potentials, Hardy-Hodge Decomposition, Separation of Sources, Geomagnetic Field, Extremal Problems

1 Introduction

The Earth's magnetic field $\mathbf{B}$, as measured by several satellite missions, is a superposition of various contributions, e.g., of iono-/magnetospheric fields, crustal magnetic field, and of the core/main magnetic field, see [25, 23, 33] for an overview and [27, 31, 36, 41] for some recent geomagnetic field models. While iono-/magnetospheric contributions can to a certain extent be filtered out due to their temporal variations, the separation of the core/main field $\mathbf{B}_{core}$ and the crustal field $\mathbf{B}_{crust}$ is typically based on the empirical observation that the power spectra of Earth magnetic field models have a sharp knee at spherical harmonic degree 15 (see, e.g., [26, 33]). However, under this spectral separation, large-scale contributions (i.e., spherical harmonic degrees smaller than 15) are entirely neglected in crustal magnetic field models. In [22], a Bayesian approach has been proposed that addresses the separation of geomagnetic sources based on their correlation structure. The correlation of certain components, e.g., internally and externally produced magnetic fields, can (to some extent) be obtained from the underlying geophysical equations. But this approach does not address the problem that some of the involved separation problems, e.g., the separation into crustal and core magnetic field contributions, are generally not unique for the given data situation. The goal of this paper is to derive conditions under which a rigorous separation of the contributions $\mathbf{B}_{crust}$ and $\mathbf{B}_{core}$ is possible, as well as to formulate extremal problems whose solutions lead to approximations of these contributions or certain features thereof. The main assumption that we make for our approach to work is that the magnetization generating $\mathbf{B}_{crust}$ is localized in a strict subregion

[1]INRIA, Project APICS, 2004 route de Lucioles, BP 93, Sophia-Antipolis F-06902 Cedex, France, e-mail: laurent.baratchart@sophia.inria.fr

[2]University of Vienna, Computational Science Center, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria, e-mail: christian.gerhards@univie.ac.at

of the crust. By linearity, this is equivalent to assuming that this magnetization is known on a spherical cap that may, in principle, be arbitrary small. For applications, this is interesting in as much as that the crustal magnetization may be estimated in certain places of the Earth from local measurements. Thus, given such a local estimation, its contribution can be substracted from global magnetic field measurements to yield a crustal contribution that stems from magnetizations localized in a strict subregion of the Earth (namely the complement of those places where a local estimate of the magnetization has been performed), thereby allowing us to apply the separation approach indicated in this paper. Similarly, if one can identify places on the Earth which are only weakly magnetized as compared to others, the separation process that we will describe may reasonably be applied by neglecting magnetizations in such places.

We assume throughout that the overall magnetic field is of the form $\mathbf{B} = \mathbf{B}_{crust} + \mathbf{B}_{core}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$, where $\mathbb{B}_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$ denotes the ball of radius $R_0 > 0$ and overline indicates closure (here R_0 can be interpreted as the radius of the Earth). Since the sources of $\mathbf{B}_{crust}$ and $\mathbf{B}_{core}$ are located inside $\mathbb{B}_{R_0}$ (hence, the corresponding magnetic fields are curl-free and divergence-free in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$), there exist potential fields Φ, Φ_{crust}, Φ_{core} such that $\mathbf{B} = \nabla\Phi$, $\mathbf{B}_{crust} = \nabla\Phi_{crust}$, and $\mathbf{B}_{core} = \nabla\Phi_{core}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$. Therefore, from a mathematical point of view, the problem reduces to finding unique Φ_{crust}, Φ_{core} from the knowledge of Φ (but we should keep in mind that the actual measurements bear on the magnetic field $\mathbf{B}$).

It is known that $\mathbf{B}_{crust}$ is generated by a magnetization $\mathbf{M}$ confined in a thin spherical shell $\mathbb{B}_{R_0-d,R_0} = \{x \in \mathbb{R}^3 : R_0 - d < |x| < R_0\}$ of thickness $d > 0$ (for the Earth, $d \approx 30$km is typical), therefore the corresponding magnetic potential can be expressed as (see, e.g., [8, 19])

$$\Phi_{crust}(x) = \frac{1}{4\pi} \int_{\mathbb{B}_{R_0-d,R_0}} \mathbf{M}(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}\lambda(y), \quad x \in \mathbb{R}^3, \tag{1.1}$$

where the dot indicates Euclidean scalar product in $\mathbb{R}^3$ and λ the Lebesgue measure. Due to the thinness of the magnetized layer relative to the Earth's radius, it is reasonable to substitute the volumetric $\mathbf{M}$ by a spherical magnetization $\mathbf{m}$ (i.e., $\mathbf{M} = \mathbf{m} \otimes \delta_{\mathbb{S}_{R_0}}$ in a distributional sense). Then, the magnetic potential (1.1) becomes

$$\Phi_{crust}(x) = \frac{1}{4\pi} \int_{\mathbb{S}_{R_0}} \mathbf{m}(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}\omega_{R_0}(y), \quad x \in \mathbb{R}^3 \setminus \mathbb{S}_{R_0}, \tag{1.2}$$

where $\mathbb{S}_{R_0} = \{x \in \mathbb{R}^3 : |x| = R_0\}$ denotes the sphere of radius $R_0 > 0$ and $\mathrm{d}\omega_{R_0}$ the corresponding surface element. When interested in reconstructing the actual magnetization $\mathbf{M}$, substituting a spherical magnetization $\mathbf{m}$ is of course a significant restriction (however, one that is fairly frequent in Geomagnetism). But since our main focus is on $\mathbf{B}_{crust}$ and the corresponding potential Φ_{crust} rather than the magnetization itself, this restriction actually involves no loss of information: in Section 3 we show that, under mild summability assumptions, any potential Φ_{crust} produced by a volumetric magnetization $\mathbf{M}$ in $\mathbb{B}_{R_0-d,R_0}$ can also be generated by a spherical magnetization $\mathbf{m}$ on $\mathbb{S}_{R_0}$.

The core/main contribution $\mathbf{B}_{core}$ is governed by the Maxwell equations (see, e.g., [5])

$$\nabla \times \mathbf{B}_{core} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}_{core}), \tag{1.3}$$
$$\nabla \cdot \mathbf{B}_{core} = 0, \tag{1.4}$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}_{core}, \tag{1.5}$$
$$\nabla \cdot \mathbf{E} = \rho, \tag{1.6}$$

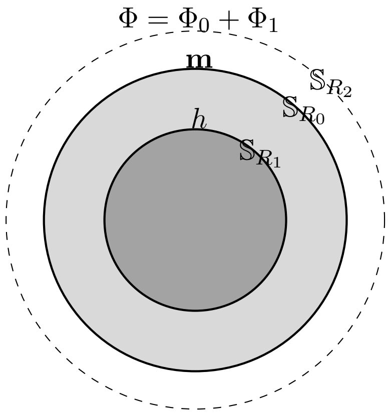

Figure 1: Illustration of the setup of Problem 1.1.

where σ denotes the conductivity, ρ the charge density, and $\mathbf{u}$ the fluid velocity in the Earth's outer core (the constant permeability μ_0 and permittivity ε_0 have been set to 1). The conductivity σ is assumed to be zero outside a sphere $\mathbb{S}_{R_1}$ of radius $0 < R_1 < R_0$. The condition $R_1 < R_0$ is crucial to the forthcoming arguments and is justified by common geophysical practice and results (see, e.g., [6, 34]). In particular it implies that $\nabla \times \mathbf{B}_{core} = 0$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$, therefore, $\mathbf{B}_{core} = \nabla \Phi_{core}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ for some harmonic potential Φ_{core}. Although the geophysical processes in the Earth's outer core can be extremely complex, of importance to us is only that Φ_{core} can be expressed in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ as a Poisson transform:

$$\Phi_{core}(x) = \frac{1}{4\pi R_1} \int_{\mathbb{S}_{R_1}} h(y) \frac{|x|^2 - R_1^2}{|x-y|^3} \mathrm{d}\omega_{R_1}(y), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}, \tag{1.7}$$

for some scalar valued auxiliary function h on $\mathbb{S}_{R_1}$; this follows from previous considerations which imply that Φ_{core} is harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ and continuous in $\mathbb{R}^3 \setminus \mathbb{B}_{R_1}$. Summarizing, the problem we treat in this paper is the following (the setup is illustrated in Figure 1):

Problem 1.1. Let $\Phi \in L^2(\mathbb{S}_{R_2})$ be given on a sphere $\mathbb{S}_{R_2} \subset \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$ of radius $R_2 > R_0$. Assume Φ is decomposable into $\Phi = \Phi_0 + \Phi_1$ on $\mathbb{S}_{R_2}$, where $\Phi_0 = \Phi_0[\mathbf{m}]$ is of the form (1.2), with $\mathbf{m} \in L^2(\mathbb{S}_{R_0}, \mathbb{R}^3)$, and $\Phi_1 = \Phi_1[h]$ is of the form (1.7), with $h \in L^2(\mathbb{S}_{R_1})$ and $R_1 < R_0$. Are Φ_0 and Φ_1 uniquely determined by the knowledge of Φ on $\mathbb{S}_{R_2}$, and if yes can they be reconstructed efficiently?

The answer to the uniqueness issue in Problem 1.1 is generally negative. But under the additional assumption that $\mathrm{supp}(\mathbf{m}) \subset \Gamma_{R_0}$ for a strict subregion $\Gamma_{R_0} \subset \mathbb{S}_{R_0}$ (i.e. $\overline{\Gamma_{R_0}} \neq \mathbb{S}_{R_0}$), uniqueness is guaranteed. This follows from results in [7, 28] and their formulation on the sphere in [17], to be reviewed in greater detail in Section 4. In fact, we show in this case that h and the curl-free contribution of $\mathbf{m}$ can be reconstructed uniquely from the knowledge of Φ. Additionally, we provide a means of approximating $\langle \Phi_0, g \rangle_{L^2(\mathbb{S}_{R_2})}$ knowing Φ on $\mathbb{S}_{R_2}$, where g is some appropriate test function (e.g., a spherical harmonic). This allows one to separate the crustal and the core contributions to the Geomagnetic potential if, e.g., the crustal magnetization can be estimated over a small subregion on Earth by other means.

Throughout the paper, we call Φ_0 the crustal contribution and Φ_1 the core contribution. We should point out that the examples we provide at the end of the paper are not based on real Geomagnetic field data but they reflect some of the main properties of realistic scenarios (e.g., the domination of the core contribution at low spherical harmonic degrees). In Section 3, we take a closer look at harmonic potentials of the form (1.1) and (1.2) and show that the balayage onto $\mathbb{S}_{R_0}$ of a volumetric potential supported in $\mathbb{B}_{R_0-d,R_0}$ preserves divergence form. More precisely, if $\mathbf{M}$ is supported in $\mathbb{B}_{R_0-d,R_0}$ and its restriction to $\mathbb{S}_R$ is uniformly square-summable for $R \in (R_0-d, R_0)$, then there exists a spherical magnetization $\mathbf{m}$ supported on $\mathbb{S}_{R_0}$, which is square summable and generates the same potential as $\mathbf{M}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$. The latter property justifies the above-described modeling of the crustal magnetic field. Auxiliary material on geometry, spherical decomposition of vector fields as well as Sobolev and Hardy spaces is recapitulated in Section 2. Eventually, in Section 5 we provide some initial examples of numerical approximation of Φ_0 and $\langle \Phi_0, g\rangle_{L^2(\mathbb{S}_{R_2})}$, followed by a brief conclusion in Section 6. Some technical results on potentials of distributions, gradients, and divergence-free vector fields are gathered in the appendix.

2 Auxiliary Notations and Results

For $R>0$, the sphere $\mathbb{S}_R$ is a smooth, compact oriented surface embedded in $\mathbb{R}^3$. That is, $\mathbb{S}_R$ can be covered with finitely many open sets $U_1, \cdots, U_N$ (i.e., open for the topology induced by $\mathbb{R}^3$), each of which is homeomorphic to a planar domain via a chart $\psi_\ell : U_\ell \to V_\ell \subset \mathbb{R}^2$, in such a way that changes of charts $\psi_k \circ \psi_\ell^{-1} : \psi_\ell(U_\ell \cap U_k) \to V_k$ are C^∞-smooth with positive Jacobian determinant and each map $\psi_\ell^{-1} : V_\ell \to \mathbb{R}^3$ is C^∞-smooth with injective derivative. Though we do not rely on a specific system of charts in what follows, we may for example pick $N=4$ and $V_1 = (-\pi/3, \pi/3) \times (-2\pi/3, 2\pi/3)$ with $\psi_1^{-1}(y_1,y_2) = (R\cos y_1 \cos y_2, R\cos y_1 \sin y_2, R\sin y_1)^T$, $V_2 = (-\pi/3,\pi/3)\times(\pi/3, 5\pi/3)$ with ψ_2^{-1} given by the same formula as ψ_1^{-1}, and $V_3 = V_1$ as well as $V_4 = V_2$ along with $\psi_3^{-1} = P \circ \psi_1^{-1}$ and $\psi_4^{-1} = P \circ \psi_2^{-1}$ where the rotation $P : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $P(x_1,x_2,x_3)^T = (x_3,x_1,x_2)^T$; here and below, a superscript "T" means "transpose".

We put for simplicity $\phi_\ell = \psi_\ell^{-1}$ so that $\phi_\ell(V_\ell) = U_\ell$, and if $x \in U_\ell$ then the tangent space T_x to $\mathbb{S}_R$ at x is the image of the derivative $\mathrm{D}\phi_\ell(\psi_\ell(x)) : \mathbb{R}^2 \to \mathbb{R}^3$. The latter may be described intrinsically as $T_x = \{y \in \mathbb{R}^3 : x \cdot y = 0\}$. A differentiable or C^k-smooth function $f : \mathbb{S}_R \to \mathbb{R}$ is a function such that $f \circ \phi_\ell$ is differentiable or has continuous partial derivatives up to order k on the Euclidean open set V_ℓ, for each ℓ. Due to the simple geometry of the sphere $\mathbb{S}_R$, it is in fact equivalent to require that the radial extension $\bar{f}(x) = f(R\frac{x}{|x|})$ of f has the corresponding regularity on $\mathbb{R}^3 \setminus \{0\}$. We simply say that f is smooth if it is C^∞-smooth. The differential of f at $x \in U_\ell$ is the linear map $\mathrm{d}f(x) : T_x \to \mathbb{R}$ given at $v \in T_x$ by $\mathrm{d}f(x)(v) = \mathrm{d}(f \circ \phi_\ell)(y)(w)$, where $y = \psi_\ell(x)$ and $w \in \mathbb{R}^2$ is such that $v = \mathrm{d}\phi_\ell(y)(w)$. Here, the Euclidean differential $d(f \circ \phi_\ell)(y)$ is defined as usual:

$$\mathrm{d}(f\circ\phi_\ell)(y)(w) = \partial_{y_1}(f\circ\phi_\ell)(y)w_1 + \partial_{y_1}(f\circ\phi_\ell)(y)w_2, \quad y=(y_1,y_2)^T, \quad w=(w_1,w_2)^T, \tag{2.1}$$

where ∂_{y_j} indicates partial derivative with respect to y_j. The definition of smoothness and the computation of derivatives in local coordinates extend componentwise to functions $\mathbb{S}_R \to \mathbb{R}^k$.

By restriction, the Euclidean scalar product in $\mathbb{R}^3$ defines a scalar product on T_x whose coefficients, when expressed in local coordinates, vary smoothly with x. More precisely, if $x \in U_\ell$ and $\psi_\ell(x) = (y_1,y_2)^T = y$, then T_x gets parametrized by $u_1\partial_{y_1}\phi_\ell(y) + u_2\partial_{y_2}\phi_\ell(y)$ where u_1, u_2 range over $\mathbb{R}$, and the bilinear map $(v,\bar{v}) \mapsto v \cdot \bar{v}$ induced on $T_x \times T_x$ by the Euclidean scalar product has metric tensor $\mathbf{G} = (g_{i,j})_{i,j=1,2} = (\partial_{y_i}\phi_\ell(y) \cdot \partial_{y_j}\phi_\ell(y))_{i,j=1,2}$ in the coordinates u_1, u_2. This makes $\mathbb{S}_R$ into a Riemannian manifold with area measure ω_R such that, whenever $E \subset V_\ell$ is a Borel set,

$$\omega_R(\phi_i(E)) = \int_E \sqrt{g}\, \mathrm{d}\lambda \tag{2.2}$$

where $g = \det(\mathbf{G})$ and λ indicates Lebesgue measure in $\mathbb{R}^2$; the change of variable formula shows that ω_R does not depend on the coordinate system, and clearly measurability of f with respect to ω_R is equivalent to measurability of $f \circ \phi_\ell$ with respect to λ for each ℓ. One can then define Lebesgue spaces of (equivalence classes of ω_R-a.e. coinciding) functions with integrable p-th power on $\mathbb{S}_R$, in particular we let $L^2(\mathbb{S}_R)$ be the space of square integrable scalar valued spherical functions $f : \mathbb{S}_R \to \mathbb{R}$, while $L^2(\mathbb{S}_R, \mathbb{R}^3)$ denotes the space of square integrable vector valued spherical functions $\mathbf{f} : \mathbb{S}_R \to \mathbb{R}^3$, equipped with the inner products $\langle f, h\rangle_{L^2(\mathbb{S}_R)} = \int_{\mathbb{S}_R} f(y)h(y)\mathrm{d}\omega_R(y)$ and $\langle \mathbf{f}, \mathbf{h}\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = \int_{\mathbb{S}_R} \mathbf{f}(y) \cdot \mathbf{h}(y)\mathrm{d}\omega_R(y)$, respectively. A vector field $\mathbf{f} : \mathbb{S}_R \to \mathbb{R}^3$ is said to be tangential if $\mathbf{f}(x) \in T_x$ for all $x \in \mathbb{S}_R$. The subspace of all tangential vector fields in $L^2(\mathbb{S}_R, \mathbb{R}^3)$ is denoted by $\mathcal{T}_R$. Note that smooth vector fields are dense in $\mathcal{T}_R$, for using a partition of unity this amounts to the density of smooth functions in the weighted spaces $L^2_{\sqrt{g}}(V_\ell, \mathbb{R}^3)$ (weighted by $\sqrt{g}$, that is).

If $f : \mathbb{S}_R \to \mathbb{R}$ is differentiable at $x \in \mathbb{S}_R$, the differential $\mathrm{d}f(x) : T_x \to \mathbb{R}$ is a linear form and thus may be represented as the scalar product with some unique vector in T_x, denoted by $\nabla_{\mathbb{S}_R} f(x)$ and called the gradient of f at x. Clearly if f is smooth, the map $\nabla_{\mathbb{S}_R} f : \mathbb{S}_R \to \mathbb{R}^3$ lies in $\mathcal{T}_R$. The Sobolev space $W^{1,2}(\mathbb{S}_R)$ may be defined as the completion of smooth functions with respect to the norm [20]:

$$\|f\|_{W^{1,2}(\mathbb{S}_R)} = \left(\|f\|^2_{L^2(\mathbb{S}_R)} + \|\nabla_{\mathbb{S}_R} f\|^2_{L^2(\mathbb{S}_R,\mathbb{R}^3)} \right)^{1/2}. \tag{2.3}$$

If the V_ℓ are bounded and g is bounded from above and below by strictly positive constants in every chart (as is the case with the system of charts we mentioned before), it holds that $f \in W^{1,2}(\mathbb{S}_R)$ if and only if, for each ℓ, the function $f \circ \phi_\ell$ lies in the Euclidean Sobolev space $W^{1,2}(V_\ell)$ of functions in $L^2(V_\ell)$ whose first distributional derivatives again lie in $L^2(V_\ell)$ [29]. The differential of f is then given a.e. by (2.1), where this time ∂_{y_j} indicates the distributional derivative, and the gradient $\nabla_{\mathbb{S}_R} f(x)$ is still defined via the representation $\mathrm{d}f(x)(v) = \nabla_{\mathbb{S}_R} f(x) \cdot v$ for $v \in T_x$. Note, however, that $\nabla_{\mathbb{S}_R} f(x)$ needs not be a pointwise derivative in the strong sense, see [39, Ch.VIII]. One intrinsic definition of $\mathrm{d}f$ as a weak differential uses the language of forms, for which one may consult [42, Ch. 4] (see a short account in Appendix B): if $f \in W^{1,2}(\mathbb{S}_R)$, then for every smooth 1-form θ on $\mathbb{S}_R$ it holds that

$$\int_{\mathbb{S}_R} f \mathrm{d}\theta = -\int_{\mathbb{S}_R} \mathrm{d}f \wedge \theta. \tag{2.4}$$

Let us put $\mathcal{G}_R = \{\nabla_{\mathbb{S}_R} f : f \in W^{1,2}(\mathbb{S}_R)\}$. We claim that $\mathcal{G}_R$ is closed in $L^2(\mathbb{S}_R, \mathbb{R}^3)$. Indeed, if $\nabla_{\mathbb{S}_R} f_n$ is a Cauchy sequence in $\mathcal{G}_R$, where $f_n \in W^{1,2}(\mathbb{S}_R)$ is defined up to an additive constant, we may pick f_n so that $\int_{\mathbb{S}_R} f_n \mathrm{d}\omega_R = 0$ and then it follows from the Hölder and the Poincaré inequalities [20, Prop. 3.9] that $\|f_n - f_m\|_{L^2(\mathbb{S}_R)} \leq C\|\nabla_{\mathbb{S}_R} f_n - \nabla_{\mathbb{S}_R} f_m\|_{L^2(\mathbb{S}_R,\mathbb{R}^3)}$ for some constant C. Hence f_n is a Cauchy sequence in $W^{1,2}(\mathbb{S}_R)$, therefore it converges to some f there and consequently $\nabla_{\mathbb{S}_R} f_n$ converges to $\nabla_{\mathbb{S}_R} f$ in $L^2(\mathbb{S}_R, \mathbb{R}^3)$, by (2.3). Thus, $\mathcal{G}_R$ is complete and therefore it is closed in $L^2(\mathbb{S}_R, \mathbb{R}^3)$, which proves the claim.

When $\mathbf{h}$ is a smooth tangential vector field on $\mathbb{S}_R$, its divergence $\operatorname{div}\mathbf{h}$ is the smooth real valued function such that

$$\int_{\mathbb{S}_R} f \operatorname{div}\mathbf{h}\, \mathrm{d}\omega_R = -\int_{\mathbb{S}_R} (\nabla_{\mathbb{S}_R} f) \cdot \mathbf{h}\, \mathrm{d}\omega_R, \qquad f \in C^\infty(\mathbb{S}_R). \tag{2.5}$$

In other words, the divergence operator is dual to differentiation [42, Sec. 4.10 & 6.2]. In this case, it holds in local coordinates that [24, Lem. 5.1.2]

$$\big(\operatorname{div}\mathbf{h}\big) \circ \phi_\ell(y) = \frac{1}{\sqrt{g}} \sum_{i=1}^{2} \partial_{y_i}\big(\sqrt{g}\, u_i(y)\big) \quad \text{if} \quad \mathbf{h} \circ \phi_\ell(y) = \sum_{i=1}^{2} u_i(y)\, \partial_{y_i}\phi_\ell(y). \tag{2.6}$$

In differential geometric terms, (2.6) expresses that $\operatorname{div}\mathbf{h}$ is the trace of the covariant differential of $\mathbf{h}$ [24, Sec. 4.16 & 4.1.7].

When $\mathbf{h} \in \mathcal{T}_R$ is not smooth, (2.5) and (2.6) must be interpreted in a weak sense, namely $\operatorname{div}\mathbf{h}$ is the distribution on $\mathbb{S}_R$ acting on smooth real-valued functions by

$$\langle f\,,\, \operatorname{div}\mathbf{h}\rangle = -\int_{\mathbb{S}_R} \nabla_{\mathbb{S}_R} f \cdot \mathbf{h}\, \mathrm{d}\omega_R, \qquad f \in C^\infty(\mathbb{S}_R). \tag{2.7}$$

Clearly (2.7) extends by density to a linear form on $W^{1,2}(\mathbb{S}_R)$, upon letting f converge to a Sobolev function. Then, it is apparent that $\mathcal{D}_R = \{\mathbf{h} \in \mathcal{T}_R : \operatorname{div}\mathbf{h} = 0\}$ is the orthogonal complement to $\mathcal{G}_R$ in $\mathcal{T}_R$. In particular,

$$\mathcal{T}_R = \mathcal{G}_R \oplus \mathcal{D}_R, \tag{2.8}$$

which is the so-called Helmholtz-Hodge decomposition. The particular geometry of $\mathbb{S}_R$ makes it easy to see that $\mathbf{f} \in \mathcal{D}_R$ if and only if its radial extension $\bar{\mathbf{f}}(x) = \mathbf{f}(R\frac{x}{|x|})$ is divergence free, as a $\mathbb{R}^3$-valued distribution on $\mathbb{R}^3 \setminus \{0\}$. Throughout the paper, we will typically use the notation $\nabla_{\mathbb{S}_R}\cdot$ for the divergence on the sphere.

Consider now the operator $J_x : T_x \to T_x$ given by $J_x(v) = \frac{x}{|x|} \times v$, $v \in T_x$, where $\times$ indicates the vector product in $\mathbb{R}^3$; that is, J_x is the rotation by $\pi/2$ in T_x. We define $J : \mathcal{T}_R \to \mathcal{T}_R$ to be the isometry acting pointwise as J_x on T_x, namely $(J\mathbf{f})(x) = J_x(\mathbf{f}(x))$ for $\mathbf{f} \in \mathcal{T}_R$. It turns out that $J(\mathcal{G}_R) = \mathcal{D}_R$. This fact, that we call the rotation lemma, holds on any simply connected surface; a proof seems not easy to find in the literature and we provide one in Appendix C (for the special case of continuously differentiable tangential vector fields on the sphere, the assertion essentially corresponds to [15, Thm. 2.10]).

The previous considerations are conveniently recaped by introducing the normalized surface gradient $\nabla_{\mathbb{S}}$ (with respect to the unit sphere $\mathbb{S} = \mathbb{S}_1$), to be interpreted as the tangential contribution of the Euclidean gradient ∇. That is, at a point $x \in \mathbb{R}^3 \setminus \{0\}$, the Euclidean gradient can be written as $\nabla = \frac{x}{|x|}\partial_\nu - \frac{1}{|x|}\nabla_{\mathbb{S}}$, with $\partial_\nu = \frac{x}{|x|} \cdot \nabla$ to mean the radial derivative. Thus, if $g(x) = f(R\frac{x}{|x|})$ is the radial extension of $f \in C^1(\mathbb{S}_R)$ to $\mathbb{R}^3 \setminus \{0\}$, then $\nabla g(x) = \frac{1}{R}\nabla_{\mathbb{S}} f(x)$ for $x \in \mathbb{S}_R$. So, $\nabla_{\mathbb{S}} f(x) = R\nabla_{\mathbb{S}_R} f(x)$ is the unique vector in T_x such that $\frac{v}{R} \cdot \nabla_{\mathbb{S}} f(x) = v \cdot \nabla g(x)$ for all $v \in T_x$. In another connection, the surface curl gradient $\mathrm{L}_{\mathbb{S}}$ is the tangential differential operator "orthogonal to $\nabla_{\mathbb{S}}$" defined as $\mathrm{L}_{\mathbb{S}} = \frac{x}{|x|} \times \nabla_{\mathbb{S}}$ in a point $x \in \mathbb{S}_R$. Since $\mathcal{D}_R = J\mathcal{G}_R$ we know that $\mathcal{D}_R = \{\mathrm{L}_{\mathbb{S}} f : f \in W^{1,2}(\mathbb{S}_R)\}$, and if we let $\mathcal{N}_R$ indicate the space of radial vector fields in $L^2(\mathbb{S}_R, \mathbb{R}^3)$ (those whose value at x is perpendicular to T_x for each $x \in \mathbb{S}_R$), we get from (2.8) the orthogonal decomposition

$$L^2(\mathbb{S}_R, \mathbb{R}^3) = \mathcal{N}_R \oplus \mathcal{G}_R \oplus \mathcal{D}_R. \tag{2.9}$$

Related to the latter but of more relevance to our problem is the Hardy-Hodge decomposition that we now explain. For that purpose, we require the following definition.

Definition 2.1. The Hardy space $\mathcal{H}^2_{+,R}$ of harmonic gradients in $\mathbb{B}_R$ is defined by

$$\mathcal{H}^2_{+,R} = \{\mathbf{g} = \nabla g \text{ with } g : \mathbb{B}_R \to \mathbb{R} \text{ such that } \Delta g = 0 \text{ in } \mathbb{B}_R \text{ and } \|\nabla g\|_{2,+} < \infty\},$$

where $\|\mathbf{g}\|_{2,+} = \left(\sup_{r\in[0,1)} \int_{\mathbb{S}_r} |\mathbf{g}(ry)|^2 \mathrm{d}\omega_r(y)\right)^{\frac{1}{2}}$ and Δ is the Euclidean Laplacian in $\mathbb{R}^3$. Likewise, the Hardy space $\mathcal{H}^2_{-,R}$ of harmonic gradients in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$ is defined by

$$\mathcal{H}^2_{-,R} = \left\{\mathbf{g} = \nabla g : \text{ with } g : \mathbb{R}^3 \setminus \overline{\mathbb{B}_R} \to \mathbb{R} \text{ such that } \Delta g = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\mathbb{B}_R} \text{ and } \|\nabla g\|_{2,-} < \infty\right\},$$

where $\|\mathbf{g}\|_{2,-} = \left(\sup_{r\in(1,\infty)} \int_{\mathbb{S}_r} |\mathbf{g}(ry)|^2 \mathrm{d}\omega_R(y)\right)^{\frac{1}{2}}$. Note that, by Weyl's lemma [11, Theorem 24.9], it makes no difference whether the Euclidean gradient and Laplacian are understood in the distributional or in the strong sense.

Members of $\mathcal{H}^2_{+,R}$ and $\mathcal{H}^2_{-,R}$ have non-tangential limits a.e. on $\mathbb{S}_R$, and if $\mathbf{g} \in \mathcal{H}^2_{\pm,R}$, its nontangential limit has $L^2(\mathbb{S}_R, \mathbb{R}^3)$-norm equal to $\|\mathbf{g}\|_{2,\pm}$, see [39, VII.3.1] and [40, VI.4]. We still write $\mathbf{g}$ for this non-tangential limit and we regard it as the trace of $\mathbf{g}$ on $\mathbb{S}_R$. This way Hardy spaces can be interpreted as function spaces on $\mathbb{S}_R$ as well as on $\mathbb{B}_R$ or $\mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$, but the context will make it clear if the Euclidean or the spherical interpretation is meant because the argument belongs to $\mathbb{R}^3 \setminus \mathbb{S}_R$ in the former case and to $\mathbb{S}_R$ in the latter. The Hardy-Hodge decomposition is the orthogonal sum

$$L^2(\mathbb{S}_R, \mathbb{R}^3) = \mathcal{H}^2_{+,R} \oplus \mathcal{H}^2_{-,R} \oplus \mathcal{D}_R. \tag{2.10}$$

Projecting (2.10) onto the tangent space $\mathcal{T}_R$ and grouping the first two summands into a single gradient vector field yields back the Hodge decomposition (2.8). The Hardy-Hodge decomposition drops out at once from [3] and (2.8). Its application to the study of inverse magnetization problems has been illustrated in [7, 17, 28]. Although not studied in mathematical detail, spherical versions of the Hardy-Hodge decomposition have previously been used to a various extent in Geomagnetic applications (see, e.g., [5, 16, 19, 32]).

By means of the reflection $\mathcal{R}_R(x) = \frac{R^2}{|x|^2}x$ across $\mathbb{S}_R$, we define the Kelvin transform $K_R[f]$ of a function f defined on an open set $\Omega \subset \mathbb{R}^3$ to be the function on $\mathcal{R}_R(\Omega)$ given by

$$K_R[f](x) = \frac{R}{|x|} f(\mathcal{R}_R(x)), \quad x \in \mathcal{R}_R(\Omega). \tag{2.11}$$

A function f is harmonic in Ω if and only if $K_R[f]$ is harmonic in $\mathcal{R}_R(\Omega)$ (e.g., [4, Thm. 4.7]).

Now, assume that $\mathbf{f} \in \mathcal{H}^2_{+,R}$ with $\mathbf{f} = \nabla f$ and $f(0) = 0$. Then $\nabla K_R[f] \in \mathcal{H}^2_{-,R}$. In fact, if for $\mathbf{f} \in \mathcal{H}^2_{+,R}$ (resp. $\mathbf{f} \in \mathcal{H}^2_{-,R}$) we let $\int \mathbf{f}$ indicate the harmonic function f in $\mathbb{B}_R$ (resp. in $\mathbb{R}^3 \setminus \overline{\mathbb{B}}_R$) whose gradient is $\mathbf{f}$, normalized so that $f(0) = 0$ (resp. $\lim_{|x|\to\infty} f(x) = 0$), then $\mathbf{f} \mapsto \nabla K_R \circ \int \mathbf{f}$ maps $\mathcal{H}^2_{+,R}$ continuously into $\mathcal{H}^2_{-,R}$ and back [3]. Moreover, in view of (2.11) we have that

$$\nabla K_R[f](x) = \frac{R^3 \nabla f(\mathcal{R}_R(x))}{|x|^3} - 2\, x \cdot \nabla f(\mathcal{R}_R(x))\, \frac{R^3 x}{|x|^5} - f(\mathcal{R}_R(x))\, \frac{Rx}{|x|^3}\,. \tag{2.12}$$

Clearly f and $K_R[f]$ coincide on $\mathbb{S}_R$, therefore the tangential components of ∇f and $\nabla K_R[f]$ agree on $\mathbb{S}_R$ (these are the spherical gradients $\nabla_{\mathbb{S}_R} f$ and $\nabla_{\mathbb{S}_R} K_R[f]$). The normal components $\partial_\nu f$ and $\partial_\nu K_R[f]$, though, are different. Indeed, we get from (2.12) that

$$\partial_\nu K_R[f](x) = -\partial_\nu f(x) - \frac{f(x)}{R}, \quad x \in \mathbb{S}_R. \tag{2.13}$$

We turn to some special systems of functions. First, let $\{Y_{n,k}\}_{n\in\mathbb{N}_0,\, k=1,\dots,2n+1}$ be an $L^2(\mathbb{S})$-orthonormal system of spherical harmonics of degrees n and orders k. A possible choice is

$$Y_{n,k}(x) = \begin{cases} \sqrt{\frac{2n+1}{2\pi}\frac{(k-1)!}{(2n+1-k)!}}\, P_{n,n+1-k}(\sin(\theta))\cos((n+1-k)\varphi) & k = 1,\dots,n, \\ \sqrt{\frac{2n+1}{4\pi}}\, P_{n,0}(t), & k = n+1, \\ \sqrt{\frac{2n+1}{2\pi}\frac{(2n+1-k)!}{(k-1)!}}\, P_{n,k-(n+1)}(\sin(\theta))\sin((k-(n+1))\varphi) & k = n+2,\dots,2n+1, \end{cases}$$

for $x = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))^T \in \mathbb{S}_1$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \varphi \in [0, 2\pi)$, and $P_{n,k}$ the associated Legendre polynomials of degree n and order k (see, e.g., [15, Ch. 3] for details; another common notation is to indicate the order of the spherical harmonics by $k = -n, \dots, n$ rather than $k = 1, \dots, 2n+1$). Then $H^R_{n,k}(x) = \left(\frac{|x|}{R}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right)$ is a homogeneous, harmonic polynomial of degree n in $\mathbb{R}^3$ (sometimes also called inner harmonic). In fact, every homogeneous harmonic polynomial in $\mathbb{R}^3$ can be expressed as a linear combination of inner harmonics. The Kelvin transform $H^R_{-n-1,k} = K_R[H^R_{n,k}]$ is a harmonic function in $\mathbb{R}^3 \setminus \{0\}$ with $\lim_{|x|\to\infty} H^R_{-n-1,k}(x) = 0$ (sometimes called outer harmonic). In [3, Lemma 4] the following result was shown.

Lemma 2.2. *The vector space* $\operatorname{span}\{\nabla H^R_{-n-1,k}\}_{n\in\mathbb{N}_0,\, k=1,\dots,2n+1}$ *is dense in* $\mathcal{H}^2_{-,R}$ *and the vector space* $\operatorname{span}\{\nabla H^R_{n,k}\}_{n\in\mathbb{N}_0,\, k=1,\dots,2n+1}$ *is dense in* $\mathcal{H}^2_{+,R}$.

For each fixed $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R$, the function $g_x(y) = \frac{1}{|x-y|}$ is harmonic in a neighborhood of $\overline{\mathbb{B}}_R$ and, therefore, its gradient

$$\mathbf{g}_x(y) = \nabla_x\, g_x(y) = \frac{x-y}{|x-y|^3}$$

lies in $\mathcal{H}^2_{+,R}$. As a consequence of Lemma 2.2, we shall prove the following density result.

Lemma 2.3. *The vector space* $\operatorname{span}\{\mathbf{g}_x : x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R\}$ *is dense in* $\mathcal{H}^2_{+,R}$ *and the vector space* $\operatorname{span}\{\mathbf{g}_x : x \in \mathbb{B}_R\}$ *is dense in* $\mathcal{H}^2_{-,R}$.

Proof. As $K_R[g_x] = \frac{1}{|x|} g_{x/|x|^2}$ and $\nabla K_R \circ \int$ is an isomorphism from $\mathcal{H}^2_{-,R}$ onto $\mathcal{H}^2_{+,R}$ (see discussion before (2.12)), we need only prove the second assertion. Define $g(y) = \frac{1}{|y|}$ as a function of $y \in \mathbb{R}^3 \setminus \{0\}$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = n$, the derivative $\partial_\alpha g(y) = \frac{\partial^n}{\partial^{\alpha_1} y_1 \partial^{\alpha_2} y_2 \partial^{\alpha_3} y_3} g(y)$ is of the form $\frac{H_\alpha(y)}{|y|^{1+2n}}$, where H_α is a homogeneous harmonic polynomial of degree n, and actually every homogeneous harmonic polynomial H_α is a scalar multiple of $|y|^{(1+2n)} \partial_\alpha g(y)$ for some α [4, Lemma 5.15]. The discussion before Lemma 2.2 now implies that $\partial_\alpha g$ is an element of $\operatorname{span}\{H^R_{-n-1,k}\}_{n\in\mathbb{N}_0,\, k=1,\dots,2n+1}$. Thus, by this lemma, we are done if we can show that whenever $\mathbf{f} \in \mathcal{H}^2_{-,R}$ is orthogonal in $L^2(\mathbb{S}_R, \mathbb{R}^3)$ to all $\mathbf{g}_x$, $x \in \mathbb{B}_R$, then it must be orthogonal to all $\nabla H^R_{-n-1,k}$. To this end, differentiating $\langle \mathbf{f}, \mathbf{g}_x \rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = 0$ with respect to x leads us to

$$0 = \left\langle \mathbf{f}\,,\, \nabla \frac{H_\alpha(.-x)}{|\cdot -x|^{1+2n}} \right\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} \tag{2.14}$$

for all $\alpha \in \mathbb{N}_0^3$ and $n = |\alpha|$. Setting $x = 0$ yields

$$0 = \left\langle \mathbf{f}\,,\, \nabla \frac{H_\alpha}{|\cdot|^{1+2n}} \right\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = R^{-2n-1} \left\langle \mathbf{f}\,,\, \nabla K_R[H_\alpha] \right\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} .$$

Since every inner harmonic $H^R_{n,k}$ can be expressed as a linear combination of the H_α, this relation and the considerations before Lemma 2.2 imply that $\langle \mathbf{f}\,,\, \nabla H^R_{-n-1,k} \rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = 0$ for all $n \in \mathbb{N}_0$, $k = 1, \dots, 2n+1$, which is the desired conclusion. $\square$

3 Harmonic Potentials in Divergence-Form

The potential of a measure μ on $\mathbb{R}^3$ is defined by

$$p_\mu(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \mathrm{d}\mu(y). \tag{3.1}$$

It is the solution of $\Delta\Phi = \mu$ in $\mathbb{R}^3$ which is "smallest" at infinity. If $\mu \geq 0$, the potential p_μ is a superharmonic function and therefore it is either finite quasi-everywhere or identically $-\infty$, see [2] for these properties and the definition of "quasi everywhere". Decomposing a signed measure into its positive and negative parts (the Hahn decomposition) yields that p_μ is finite quasi-everywhere if μ is finite and compactly supported (i.e., if $\mathrm{supp}(\mu)$, which is closed by definition, is also bounded). If $\mathrm{supp}(\mu) \subset \overline{\mathbb{B}}_R$, the Riesz representation theorem and the maximum principle for harmonic functions imply that there exists a unique measure $\hat{\mu}$ with $\mathrm{supp}(\hat{\mu}) \subset \mathbb{S}_R$ such that

$$\int g(y)\mathrm{d}\mu(y) = \int g(y)\mathrm{d}\hat{\mu}(y)$$

for every continuous function g in $\overline{\mathbb{B}}_R$ which is harmonic in $\mathbb{B}_R$. Since $y \mapsto 1/|x-y|$ is harmonic in a neighbourhood of $\overline{\mathbb{B}}_R$ when $x \notin \overline{\mathbb{B}}_R$, this entails that the potentials p_μ and $p_{\hat{\mu}}$ coincide in $\mathbb{R}^3 \setminus \overline{\mathbb{B}}_R$, i.e.,

$$p_\mu(x) = p_{\hat{\mu}}(x), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R.$$

The measure $\hat{\mu}$ is called the balayage of μ onto $\mathbb{S}_R$ (see, e.g., [2]). In fact, the potentials p_μ and $p_{\hat{\mu}}$ coincide quasi-everywhere on $\mathbb{S}_R$ as well. An expression for $\hat{\mu}$ easily follows from the Poisson representation of a function f which is continuous in $\overline{\mathbb{B}}_R$ and harmonic in $\mathbb{B}_R$:

$$f(x) = \frac{1}{4\pi R} \int_{\mathbb{S}_R} \frac{R^2-|x|^2}{|x-y|^3} f(y)\, \mathrm{d}\omega_R(y), \quad x \in \mathbb{B}_R. \tag{3.2}$$

Clearly Equation (3.2), Fubini's theorem and the definition of balayage imply that

$$d\hat{\mu}(x) = \mathrm{d}\mu_{|\mathbb{S}_R}(x) + \left(\frac{1}{4\pi R} \int_{\mathbb{B}_R} \frac{R^2-|y|^2}{|x-y|^3} \mathrm{d}\mu(y) \right) \mathrm{d}\omega_R(x). \tag{3.3}$$

Lemma 3.1. *Let the measure μ be supported in $\overline{\mathbb{B}_R}$. Furthermore, assume that μ is absolutely continuous in $\mathbb{B}_R$ with a density h (i.e., $\mathrm{d}\mu(y) = h(y)dy$) that satisfies the Hardy condition*

$$\operatorname*{ess.\,sup}_{0\le r<1} \int_{\mathbb{S}_r} |h(y)|^2 \, \mathrm{d}\omega_r(y) < \infty. \tag{3.4}$$

Then the balayage $\hat{\mu}$ of μ on $\mathbb{S}_R$ is absolutely continuous with respect to ω_R (i.e., $d\hat{\mu}(y) = \hat{h}(y)\mathrm{d}\omega_R(y)$) and it has a density $\hat{h} \in L^2(\mathbb{S}_R)$.

Proof. Starting from (3.3) and the assumption that μ is absolutely continuous, we find that the density $\hat{h}$ of $\hat{\mu}$ is

$$\hat{h}(x) = \frac{1}{4\pi R} \int_{\mathbb{B}_R} \frac{R^2 - |y|^2}{|x-y|^3} h(y) \mathrm{d}\lambda(y), \qquad x \in \mathbb{S}_R.$$

Using Fubini's theorem and the identity

$$\left| \frac{x}{|x|} - |x|y \right| = \left| \frac{y}{|y|} - |y|x \right|, \qquad x, y \in \mathbb{R}^3 \setminus \{0\},$$

together with the changes of variable $\eta = \frac{\xi}{r}$, $y = \frac{rx}{R^2}$, we are led to

$$\begin{aligned}
\|\hat{h}\|^2_{L^2(\mathbb{S}_R)} =& \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_{\mathbb{B}_R} \frac{R^2 - |y|^2}{|x-y|^3} h(y)\, \mathrm{d}\lambda(y) \right)^2 \mathrm{d}\omega_R(x) \\
=& \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{R^2 - |\xi|^2}{|x-\xi|^3} h(\xi)\, \mathrm{d}\omega_r(\xi) \right) dr \right)^2 \mathrm{d}\omega_R(x) \\
\le& \frac{R}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{R^2 - |\xi|^2}{|x-\xi|^3} h(\xi)\, \mathrm{d}\omega_r(\xi) \right)^2 dr \right) \mathrm{d}\omega_R(x) \\
=& \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{1 - (\frac{r}{R})^2}{|\frac{x}{R} - \frac{\xi}{R}|^3} h(\xi)\, \mathrm{d}\omega_r(\xi) \right)^2 dr \right) \mathrm{d}\omega_R(x) \\
=& \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{1 - \left|\frac{rx}{R^2}\right|^2}{|\frac{\xi}{r} - \frac{rx}{R^2}|^3} h(\xi)\, \mathrm{d}\omega_r(\xi) \right)^2 dr \right) \mathrm{d}\omega_R(x) \\
=& \frac{1}{(4\pi R)^2} \int_0^R r^4 \left(\int_{\mathbb{S}_R} \left(\int_{\mathbb{S}_1} \frac{1 - \left|\frac{rx}{R^2}\right|^2}{|\eta - \frac{rx}{R^2}|^3} h(r\eta)\, \mathrm{d}\omega_1(\eta) \right)^2 \mathrm{d}\omega_R(x) \right) dr \\
=& \int_0^R r^4 \left(\frac{1}{4\pi (\frac{r}{R})^2} \int_{\mathbb{S}_{\frac{r}{R}}} \left(\frac{1}{4\pi} \int_{\mathbb{S}_1} \frac{1 - |y|^2}{|\eta - y|^3} h(r\eta)\, \mathrm{d}\omega_1(\eta) \right)^2 \mathrm{d}\omega_{\frac{r}{R}}(y) \right) dr.
\end{aligned} \tag{3.5}$$

Now, the function

$$f(y) = \frac{1}{4\pi} \int_{\mathbb{S}_1} \frac{1 - |y|^2}{|\eta - y|^3} h(r\eta)\, \mathrm{d}\omega_1(\eta)$$

is the Poisson integral of $h(r\cdot)$ over the unit sphere $\mathbb{S}_1$ (and represents the middle integral on the right hand side of (3.5)). Thus, f is harmonic in $\mathbb{B}_1$ and its square $|f|^2$ is subharmonic there. The latter implies that the mean of $|f|^2$ over the sphere $\mathbb{S}_{\frac{r}{R}}$, $r < R$, is not greater than its mean over $\mathbb{S}_1$, i.e.,

$$\frac{1}{4\pi(\frac{r}{R})^2}\int_{\mathbb{S}_{\frac{r}{R}}} |f(y)|^2 \mathrm{d}\omega_{\frac{r}{R}}(y) \leq \lim_{\frac{s}{R}\to 1-} \frac{1}{4\pi(\frac{s}{R})^2}\int_{\mathbb{S}_{\frac{s}{R}}} |f(y)|^2 \mathrm{d}\omega_{\frac{s}{R}}(y) = \frac{1}{4\pi}\int_{\mathbb{S}_1} |h(r\eta)|^2\,\mathrm{d}\omega_1(\eta)$$
$$= \frac{1}{4\pi r^2}\int_{\mathbb{S}_r} |h(y)|^2\,\mathrm{d}\omega_r(y) \leq \frac{M}{4\pi r^2},$$

where the constant $M > 0$ comes from the Hardy condition (3.4). Together with (3.5), we find that

$$\|\hat{h}\|^2_{L^2(\mathbb{S}_R)} \leq \frac{MR^3}{12\pi},$$

eventually showing that $\hat{h} \in L^2(\mathbb{S}_R)$ and that $\hat{\mu}$ is absolutely continuous with respect to ω_R with density $\hat{h}$. $\square$

More generally, an arbitrary distribution D with compact support has a potential p_D given outside of $\mathrm{supp}(D)$ by

$$p_D(x) = D\left(-\frac{1}{4\pi}\frac{1}{|x-\cdot|}\right), \quad x \in \mathbb{R}^3 \setminus \mathrm{supp}D. \tag{3.6}$$

Compactness of $\mathrm{supp}(D)$ easily implies that D indeed acts on $-1/(4\pi|x-\cdot|)$ when $x \notin \mathrm{supp}D$ so that P_D is well-defined, see Appendix A for details. If D is supported in $\overline{\mathbb{B}}_R$ (in particular, if it is supported in some shell $\overline{\mathbb{B}_{R-d,R}}$), we define the balayage of D onto $\mathbb{S}_R$ to be the distribution $\hat{D}$ on $\mathbb{S}_R$ that satisfies

$$p_{\hat{D}}(x) = p_D(x), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R.$$

Strictly speaking, $\hat{D}$ is a distribution on $\mathbb{S}_R$ so that $p_{\hat{D}}$ makes no sense, and we should rather write $p_{\hat{D}\otimes\delta_{\mathbb{S}_R}}$ where $\hat{D}\otimes\delta_{\mathbb{S}_R}$ is the distribution on $\mathbb{R}^3$ which is the tensor product of $\hat{D}$ with the measure $\delta_{\mathbb{S}_R}$ corresponding in spherical coordinates to a Dirac mass at $r = R$, see [38]. Nevertheless, to alleviate notation, we do write $p_{\hat{D}}$. Thus, what is meant in (3.6) when $D = \hat{D}$ is that $\hat{D}$ is applied to the restriction to $\mathbb{S}_R$ of $-1/(4\pi|x-\cdot|)$.

We briefly comment on the existence and uniqueness of such a balayage in Appendix A. If D is (associated with) a measure μ, then (3.6) coincides with (3.1) and the balayage was given in (3.3). The main difference between the case of a finite compactly supported measure μ and the case of a general compactly supported distribution D is that usually $p_D(x)$ cannot be assigned a meaning when $x \in \mathrm{supp}(D)$ whereas p_μ is well-defined quasi everywhere on $\mathrm{supp}(\mu)$. We say that D is in divergence form if

$$D = \nabla\cdot\mathbf{M}, \tag{3.7}$$

where $\nabla\cdot$ is to be understood as the distributional divergence and $\mathbf{M}$ is a $\mathbb{R}^3$-valued distribution. If, e.g., $\mathbf{M} \in L^2(\mathbb{B}_{R-d,R}, \mathbb{R}^3)$ and $\mathrm{supp}(\mathbf{M}) \subset \overline{\mathbb{B}_{R-d,R}}$, then the corresponding potential p_D coincides with Φ_{crust} in (1.1). Now we can formulate the main result of this section, namely, that balayage preserves divergence form for those $\mathbf{M}$ satisfying a Hardy condition.

Lemma 3.2. *Let $D = \nabla \cdot \mathbf{M}$, where $\mathbf{M} \in L^2(\mathbb{B}_R, \mathbb{R}^3)$ satisfies the Hardy condition*

$$\operatorname*{ess.\,sup}_{0\leq r<1} \int_{\mathbb{S}_r} |\mathbf{M}(y)|^2 \,\mathrm{d}\omega_r(y) < \infty.$$

Then there exists $\mathbf{m} \in L^2(\mathbb{S}_R, \mathbb{R}^3)$ such that $\hat{D} = \nabla \cdot (\mathbf{m} \otimes \delta_{\mathbb{S}_R})$ is the balayage of D onto $\mathbb{S}_R$.

Proof. Let $\mathbf{M} = (M_1, M_2, M_3)^T$ denote the components of $\mathbf{M}$. The definition of p_D yields

$$\begin{aligned} p_D(x) &= \frac{1}{4\pi} \int_{\mathbb{B}_R} \mathbf{M}(y) \cdot \frac{x-y}{|x-y|^3} \,\mathrm{d}\lambda(y) \\ &= \frac{1}{4\pi} \sum_{j=1}^{3} \int_{\mathbb{B}_R} M_j(y) \frac{x_j - y_j}{|x-y|^3} \,\mathrm{d}\lambda(y), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R. \end{aligned} \tag{3.8}$$

If we choose the measure μ_j such that $\mathrm{d}\mu_j(y) = M_j(y)dy$, we get from Lemma 3.1 and the Hardy condition on $\mathbf{M}$ that there exists a $m_j \in L^2(\mathbb{S}_R)$ such that balayage of μ_j onto $\mathbb{S}_R$ is given by the measure $\hat{\mu}_j$ with $d\hat{\mu}_j = m_j \mathrm{d}\omega_R$, $j = 1, 2, 3$. Setting $\mathbf{m} = (m_1, m_2, m_3)^T$ and observing that $g_{x,j}(y) = \frac{x_j - y_j}{|x-y|^3} = -\partial_{x_j} \frac{1}{|x-y|}$ is harmonic in $\mathbb{B}_R$ and continuous in $\overline{\mathbb{B}_R}$, for fixed $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R$, then the definition of balayage yields together with (3.8) that

$$\begin{aligned} p_D(x) &= \frac{1}{4\pi} \sum_{j=1}^{3} \int_{\mathbb{S}_R} m_j(t) \frac{x_j - y_j}{|x-y|^3} \,\mathrm{d}\omega_R(y) \\ &= \frac{1}{4\pi} \int_{\mathbb{S}_R} \mathbf{m}(y) \cdot \frac{x-y}{|x-y|^3} \,\mathrm{d}\omega_R(y) = p_{\hat{D}}(x), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R. \end{aligned} \tag{3.9}$$

The latter implies that $\hat{D} = \nabla \cdot (\mathbf{m} \otimes \delta_{\mathbb{S}_R})$, as announced. $\square$

Remark 3.3. Lemma 3.2 eventually justifies the statement made in the introduction that, to every square summable volumetric magnetization $\mathbf{M}$ in the Earth's crust $\mathbb{B}_{R-d,R}$ that satisfies the Hardy condition, there exists a spherical magnetization $\mathbf{m}$ on $\mathbb{S}_R$ that produces the same magnetic potential and therefore also the same magnetic field in the exterior of the Earth.

4 Separation of Potentials

We are now in a position to approach Problem 1.1. For this we study the nullspace of the potential operator Φ^{R_1,R_0,R_2} (cf. Definition 4.1), mapping a magnetization $\mathbf{m}$ on $\mathbb{S}_{R_0}$ and an auxiliary function $h \in L^2(\mathbb{S}_{R_1})$ to the sum of the potentials (1.2) and (1.7) on $\mathbb{S}_{R_2}$. First, we show in Section 4.1 that uniqueness holds in Problem 1.1 if $\operatorname{supp} \mathbf{m} \neq \mathbb{S}_{R_0}$. Similar results hold for the magnetic field operator $\mathbf{B}^{R_1,R_0,R_2} = \nabla \Phi^{R_1,R_0,R_2}$ (cf. Theorem 4.5), and also for a modified potential field operator Ψ^{R_1,R_0,R_2} (cf. Definition 4.8 and Theorem 4.9). The operator Ψ^{R_1,R_0,R_2} reflects the potential of two magnetizations $\mathbf{m}$ and $\mathfrak{m}$ supported on two different spheres (i.e., at different depths), therefore it does not apply to the separation of the crustal and core contributions since the latter does not arise from a magnetization (cf.(1.3)). Still it is of interest on its own, moreover we get it at no extra cost. In Section 4.2, we discuss how the previous results can be used to approximate quantities like the Fourier coefficients $\langle \Phi_0, Y_{n,k} \rangle_{L^2(\mathbb{S}_{R_2})}$ of Φ_0. Finally, in Section 4.3, we show that $\Phi = \Phi_0 + \Phi_1$ may well vanish though $\Phi_0, \Phi_1 \neq 0$. This follows from Lemma 4.19 and answers the uniqueness issue of Problem 1.1 in the negative when $\operatorname{supp} \mathbf{m} = \mathbb{S}_{R_0}$.

4.1 Uniqueness Issues

In accordance with the notation from Problem 1.1, we define two operators: one mapping a spherical magnetization $\mathbf{m}$ to the potential $p_{\hat{D}}$ with $\hat{D} = \nabla \cdot (\mathbf{m} \otimes \delta_{\mathbb{S}_{R_0}})$, and the other mapping an auxiliary function $h \in L^2(\mathbb{S}_{R_1})$ to its Poisson integral, both evaluated on $\mathbb{S}_{R_2}$.

Definition 4.1. Let $0 < R_1 < R_0 < R_2$ be fixed radii and Γ_{R_0} a closed subset of $\mathbb{S}_{R_0}$. Let

$$\Phi_0^{R_0,R_2} : L^2(\Gamma_{R_0}, \mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}), \quad \mathbf{m} \mapsto \frac{1}{4\pi} \int_{\Gamma_{R_0}} \mathbf{m}(y) \cdot \frac{x-y}{|x-y|^3} \mathrm{d}\omega_{R_0}(y), \quad x \in \mathbb{S}_{R_2},$$

and

$$\Phi_1^{R_1,R_2} : L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2}), \quad h \mapsto \frac{1}{4\pi R_1} \int_{\mathbb{S}_{R_1}} h(y) \frac{|x|^2 - R_1^2}{|x-y|^3} \mathrm{d}\omega_{R_1}(y), \quad x \in \mathbb{S}_{R_2}.$$

The superposition of the two operators above is denoted by

$$\Phi^{R_1,R_0,R_2} : L^2(\Gamma_{R_0}, \mathbb{R}^3) \times L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2}), \quad (\mathbf{m}, h) \mapsto \Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h].$$

We start by characterizing the potentials $p_{\hat{D}}$, with $\hat{D}$ in divergence-form, which are zero in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$.

Lemma 4.2. *Let $\mathbf{m} \in L^2(\mathbb{S}_R, \mathbb{R}^3)$ and $\hat{D} = \nabla \cdot (\mathbf{m} \otimes \delta_{\mathbb{S}_R})$ be in divergence-form. Let further $\mathbf{m} = \mathbf{m}_+ + \mathbf{m}_- + \mathbf{d}$ be the Hardy-Hodge decomposition of $\mathbf{m}$, i.e., $\mathbf{m}_+ \in \mathcal{H}^2_{+,R}$, $\mathbf{m}_- \in \mathcal{H}^2_{-,R}$, and $\mathbf{d} \in \mathcal{D}_R$. Then $p_{\hat{D}}(x) = 0$, for all $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$, if and only if $\mathbf{m}_+ \equiv 0$. Analogously, $p_{\hat{D}}(x) = 0$, for all $x \in \mathbb{B}_R$, if and only if $\mathbf{m}_- \equiv 0$.*

Proof. We already know that $\mathbf{g}_x(y) = \frac{x-y}{|x-y|^3}$ lies in $\mathcal{H}^2_{+,R}$ for every fixed $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$. The orthogonality of the Hardy-Hodge decomposition and the representation (3.9) of $p_{\hat{D}}$ yield that $\mathbf{m}_-$ and $\mathbf{d}$ do not change $p_{\hat{D}}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$. Conversely, if $p_{\hat{D}}(x) = 0$ for all $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$, then

$$p_{\hat{D}}(x) = \langle \mathbf{g}_x, \mathbf{m} \rangle_{L^2(\mathbb{S}_R, \mathbb{R}^3)} = \langle \mathbf{g}_x, \mathbf{m}_+ \rangle_{L^2(\mathbb{S}_R, \mathbb{R}^3)} = 0, \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_R}.$$

Since Lemma 2.3 asserts that $\operatorname{span}\{\mathbf{g}_x : x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R\}$ is dense in $\mathcal{H}^2_{+,R}$, the above relation implies $\mathbf{m}_+ \equiv 0$. The assertion for the case where $p_{\hat{D}}(x) = 0$, for all $x \in \mathbb{B}_R$ likewise follows by observing that $\mathbf{g}_x(y) = \frac{x-y}{|x-y|^3}$ lies in $\mathcal{H}^2_{-,R}$ for fixed $x \in \mathbb{B}_R$. $\square$

Since $\Phi_0^{R_0,R_2}[\mathbf{m}] = p_{\hat{D}}$, we may use Lemma 4.2 to characterize the nullspace of $\Phi_0^{R_0,R_2}$ (extending the magnetization $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ by zero on $\mathbb{S}_{R_0} \setminus \Gamma_{R_0}$ if the latter is nonempty). As to $\Phi_1^{R_1,R_2}$, we know its nullspace reduces to zero because the Poisson integral (1.7) yields the unique harmonic extension of $h \in L^2(\mathbb{S}_{R_1})$ to $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ which is zero at infinity (i.e. h is the nontangential limit of its Poisson extension a.e. on $\mathbb{S}_{R_1}$, see [4, Thm. 6.13]). This motivates the following statement on the nullspace $N(\Phi^{R_1,R_0,R_2})$ of Φ^{R_1,R_0,R_2}.

Theorem 4.3. *Let the setup be as in Definition 4.1 and assume that $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then the nullspace of Φ^{R_1,R_0,R_2} is given by*

$$N(\Phi^{R_1,R_0,R_2}) = \{(\mathbf{d}, 0) : \mathbf{d} \in \mathcal{D}_{R_0}, \ \operatorname{supp}(\mathbf{d}) \subset \Gamma_{R_0}\}.$$

Proof. Clearly $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ is harmonic in $\mathbb{R}^3 \setminus \{\Gamma_{R_0} \cup \mathbb{S}_{R_1}\}$ and vanishes at infinity. If $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0$ for $x \in \mathbb{S}_{R_2}$, then it follows from the maximum principle that $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0$ for all $x \in \mathbb{R}^3 \setminus \mathbb{B}_{R_2}$. Subsequently, by real analyticity, $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ must vanish identically in $\mathbb{R}^3 \setminus \{\Gamma_{R_0} \cup \overline{\mathbb{B}_{R_1}}\}$ which is connected because $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Thus, $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ extends harmonically (by the zero function) across Γ_{R_0}:

$$\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}. \tag{4.1}$$

Since $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)] = \Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h]$, where $\Phi_1^{R_1,R_2}[h]$ is harmonic on $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$, we find that $\Phi_0^{R_0,R_2}[\mathbf{m}]$ in turn extends harmonically across Γ_{R_0}, therefore it is harmonic in all of $\mathbb{R}^3$. Additionally $\Phi_0^{R_0,R_2}[\mathbf{m}]$ vanishes at infinity, hence $\Phi_0^{R_0,R_2}[\mathbf{m}](x) = 0$ for all $x \in \mathbb{R}^3$ by Liouville's theorem. Since $\Phi_0^{R_0,R_2}[\mathbf{m}] = p_{\hat{D}}$ for $\hat{D} = \nabla \cdot (\mathbf{m} \otimes \delta_{\mathbb{S}_{R_0}})$, Lemma 4.2 now implies that $\mathbf{m} = \mathbf{d} \in \mathcal{D}_{R_0}$ with $\operatorname{supp} \mathbf{d} \subset \Gamma_{R_0}$. Next, as $\Phi_0^{R_0,R_2}[\mathbf{m}]$ vanishes identically on $\mathbb{R}^3$, we get from (4.1) that $\Phi_1^{R_1,R_2}[h](x) = 0$ for all $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$. Then, injectivity of the Poisson transform entails that $h \equiv 0$, hence $N(\Phi^{R_1,R_0,R_2}) \subset \{(\mathbf{m}, 0) : \mathbf{m} \in \mathcal{D}_{R_0} \operatorname{supp}(\mathbf{m}) \subset \Gamma_{R_0}\}$.

The reverse inclusion $N(\Phi^{R_1,R_0,R_2}) \supset \{(\mathbf{m},0) : \mathbf{m} \in \mathcal{D}_{R_0}, \operatorname{supp}(\mathbf{m}) \subset \Gamma_{R_0}\}$ is clear because Lemma 4.2 yields that $\Phi^{R_1,R_0,R_2}[(\mathbf{m},0)](x) = \Phi_0^{R_0,R_2}[\mathbf{m}](x) = 0$, for all $x \in \mathbb{R}^3 \setminus \Gamma_{R_0}$ if $\mathbf{m} \in \mathcal{D}_{R_0}$. $\square$

Corollary 4.4. *Notation being as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, let $\Phi = \Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ for some $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and some $h \in L^2(\mathbb{S}_{R_1})$. Then, a pair of potentials of the form $\bar{\Phi}_0 = \Phi_0^{R_0,R_2}[\bar{\mathbf{m}}]$ and $\bar{\Phi}_1 = \Phi_1^{R_1,R_2}[\bar{h}]$, with $\bar{\mathbf{m}} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $\bar{h} \in L^2(\mathbb{S}_{R_1})$, is uniquely determined by the condition $\Phi(x) = \bar{\Phi}_0(x) + \bar{\Phi}_1(x)$, $x \in \mathbb{S}_{R_2}$.*

Proof. From Theorem 4.3 we get that h is uniquely determined by the values of Φ on $\mathbb{S}_{R_2}$, and also that the components $\mathbf{m}_+ \in \mathcal{H}^2_{+,R_0}$ and $\mathbf{m}_- \in \mathcal{H}^2_{-,R_0}$ of the Hardy-Hodge decomposition of $\mathbf{m}$ are uniquely determined. The former implies $\bar{h} \equiv h$ and the latter $\bar{\mathbf{m}} \equiv \mathbf{m} + \bar{\mathbf{d}}$, for some $\bar{\mathbf{d}} \in \mathcal{D}_{R_0}$. By Lemma 4.2 we have that $\Phi_0^{R_0,R_2}[\mathbf{m}](x) = \Phi_0^{R_0,R_2}[\mathbf{m} + \bar{\mathbf{d}}](x)$ for $x \in \mathbb{R}^3 \setminus \mathbb{S}_{R_0}$, so we eventually find that $\bar{\Phi}_0$ and $\bar{\Phi}_1$ are uniquely determined. $\square$

Corollary 4.4 answers the uniqueness issue of Problem 1.1 in the positive provided that $\operatorname{supp}(\mathbf{m}) \neq \mathbb{S}_{R_0}$. In other words, assuming a locally supported magnetization, it is possible to separate the contribution of the Earth's crust from the contribution of the Earth's core if only the superposition of both magnetic potentials is known on some external orbit $\mathbb{S}_{R_2}$. Of course, in Geomagnetism, it is the magnetic field $\mathbf{B} = \nabla \Phi$ which is measured rather than the magnetic potential Φ. However, the result carries over at once to this setting. More in fact is true: if $\operatorname{supp}(\mathbf{m}) \neq \mathbb{S}_{R_0}$, separation is possible if only the normal component of $\mathbf{B}$ is known on $\mathbb{S}_{R_2}$. Indeed, we have the following theorem.

Theorem 4.5. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, and consider the operator*

$$\mathbf{B}^{R_1,R_0,R_2} : L^2(\Gamma_{R_0}, \mathbb{R}^3) \times L^2(\mathbb{S}_{R_1}, \mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}, \mathbb{R}^3), \quad (\mathbf{m}, h) \mapsto \nabla \Phi_0^{R_0,R_2}[\mathbf{m}] + \nabla \Phi_1^{R_1,R_2}[h].$$

Define further the normal operator:

$$\mathbf{B}_\nu^{R_1,R_0,R_2} : L^2(\Gamma_{R_0}, \mathbb{R}^3) \times L^2(\mathbb{S}_{R_1}, \mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}), \quad (\mathbf{m}, h) \mapsto \partial_\nu \left(\Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h] \right).$$

Then the nullspaces of $\mathbf{B}^{R_1,R_0,R_2}$ and $\mathbf{B}_\nu^{R_1,R_0,R_2}$ are all given by

$$N(\mathbf{B}^{R_1,R_0,R_2}) = N(\mathbf{B}_\nu^{R_1,R_0,R_2}) = \{(\mathbf{d}, 0) : \mathbf{d} \in \mathcal{D}_{R_0}, \ \operatorname{supp}(\mathbf{d}) \subset \Gamma_{R_0}\}.$$

Proof. Let $\mathbf{B}_\nu^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0$ for $x \in \mathbb{S}_{R_2}$. Then $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ has vanishing normal derivative on $\mathbb{S}_{R_2}$, and is otherwise harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_2}}$. Note that $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ is even harmonic across $\mathbb{S}_{R_2}$ onto a slightly larger open set, hence there is no issue of smoothness to define derivatives everywhere on $\mathbb{S}_{R_2}$. Since $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ vanishes at infinity, its Kelvin transform $u = K_{R_2}[\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]]$ is harmonic in $\mathbb{B}_{R_2}$ with $u(0) = 0$ [4, Thm. 4.8], and by (2.13) it holds that $\partial_\nu u(x) + u(x)/R_2 = 0$ for $x \in \mathbb{S}_{R_2}$. Now, if u is nonconstant and x is a maximum place for u on $\mathbb{S}_{R_2}$, then $\partial_\nu u(x) > 0$ by the Hopf lemma [4, Ch. 1, Ex. 25]. Hence $u(x) < 0$, implying that $u < 0$ on $\mathbb{B}_{R_2}$, which contradicts the maximum principle because $u(0) = 0$. Therefore u vanishes identically and so does $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ on $\mathbb{S}_{R_2}$. Appealing to Theorem 4.3 now achieves the proof. □

The next corollary follows in the exact same manner as Corollary 4.4. To state it, we indicate with a subscript ν the normal component of a field in $L^2(\mathbb{S}_{R_2},\mathbb{R}^3)$ while a subscript τ denotes the tangential component.

Corollary 4.6. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, and let the operator $\mathbf{B}^{R_1,R_0,R_2}$, be as in Theorem 4.5. Define further the operators*

$$\mathbf{B}_0^{R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \to L^2(\mathbb{S}_{R_2},\mathbb{R}^3), \quad \mathbf{m} \mapsto \nabla\Phi_0^{R_0,R_2}[\mathbf{m}],$$

and

$$\mathbf{B}_1^{R_1,R_2} : L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2},\mathbb{R}^3), \quad h \mapsto \nabla\Phi_1^{R_1,R_2}[h].$$

Let further $\mathbf{B} = \mathbf{B}^{R_1,R_0,R_2}[(\mathbf{m},h)]$, with $\mathbf{m} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$. A pair of fields of the form $\bar{\mathbf{B}}_0 = \mathbf{B}_0^{R_0,R_2}[\bar{\mathbf{m}}]$ and $\bar{\mathbf{B}}_1 = \mathbf{B}_1^{R_1,R_2}[\bar{h}]$, with $\bar{\mathbf{m}} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and $\bar{h} \in L^2(\mathbb{S}_{R_1})$, is uniquely determined by the condition $\mathbf{B}_\nu(x) = (\bar{\mathbf{B}}_0)_\nu(x) + (\bar{\mathbf{B}}_1)_\nu(x)$ and thus, a fortiori, by the condition $\mathbf{B}(x) = \bar{\mathbf{B}}_0(x) + \bar{\mathbf{B}}_1(x)$ for $x \in \mathbb{S}_{R_2}$.

Remark 4.7. Opposed to the normal component, it does not suffice to know the tangential component $\mathbf{B}_\tau$ on $\mathbb{S}_{R_2}$ in order to obtain uniqueness of $\mathbf{B}_0$ and $\mathbf{B}_1$. Namely, letting $\mathbf{m} \equiv 0$ and h be any nonzero constant function on $\mathbb{S}_{R_1}$, then $\mathbf{B}_\tau(x) = (\mathbf{B}_0)_\tau(x) + (\mathbf{B}_1)_\tau(x) = \nabla_{\mathbb{S}_{R_2}}\Phi_0^{R_0,R_2}[\mathbf{m}](x) + \nabla_{\mathbb{S}_{R_2}}\Phi_1^{R_1,R_2}[h](x) = 0$ and $\mathbf{B}_0(x) = \nabla\Phi_0^{R_0,R_2}[\mathbf{m}](x) = 0$ but $\mathbf{B}_1(x) = \nabla\Phi_1^{R_1,R_2}[h](x) \neq 0$ for $x \in \mathbb{S}_{R_2}$.

Analogously to the previous considerations, one can separate two potentials produced by two magnetizations located on two distinct spheres of radii $R_1 < R_0$ (of which the outer magnetization again has to be supported on a strict subset of $\mathbb{S}_{R_0}$). We need only slightly change the setup of Definition 4.1:

Definition 4.8. Let $0 < R_1 < R_0 < R_2$ be fixed radii and $\Gamma_{R_0} \subset \mathbb{S}_{R_0}$ a closed subset. We define

$$\Psi_0^{R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}), \quad \mathbf{m} \mapsto \frac{1}{4\pi}\int_{\Gamma_{R_0}} \mathbf{m}(y) \cdot \frac{x-y}{|x-y|^3}\, d\omega_{R_0}(y), \quad x \in \mathbb{S}_{R_2},$$

and

$$\Psi_1^{R_1,R_2} : L^2(\mathbb{S}_{R_1},\mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}), \quad \mathbf{m} \mapsto \frac{1}{4\pi}\int_{\mathbb{S}_{R_1}} \mathbf{m}(y) \cdot \frac{x-y}{|x-y|^3}\, d\omega_{R_0}(y), \quad x \in \mathbb{S}_{R_2}.$$

The superposition of these two operators is denoted by

$$\Psi^{R_1,R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \times L^2(\mathbb{S}_{R_1},\mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}), \quad (\mathbf{m},\mathfrak{m}) \mapsto \Psi_0^{R_0,R_2}[\mathbf{m}] + \Psi_1^{R_1,R_2}[\mathfrak{m}].$$

Theorem 4.9. *Let the setup be as in Definition 4.8 and assume that $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then the nullspace of Ψ^{R_1,R_0,R_2} is given by*

$$N(\Psi^{R_1,R_0,R_2}) = \{(\mathbf{d}, \mathfrak{m}_- + \mathfrak{d}) : \mathfrak{m}_- \in \mathcal{H}^2_{-,R_1},\ \mathbf{d} \in \mathcal{D}_{R_0},\ \mathrm{supp}(\mathbf{d}) \subset \Gamma_{R_0},\ \mathfrak{d} \in \mathcal{D}_{R_1}\}. \tag{4.2}$$

Proof. Let $\Psi^{R_1,R_0,R_2}[(\mathbf{m}, \mathfrak{m})](x) = 0$ for all $x \in \mathbb{S}_{R_2}$. The same argument as in the proof of Theorem 4.3 then leads us to $\Psi_0^{R_0,R_2}[\mathbf{m}](x) = 0$, $x \in \mathbb{R}^3$, and $\Psi_1^{R_1,R_2}[\mathfrak{m}](x) = 0$, $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$. The former yields $\mathbf{m} = \mathbf{d} \in \mathcal{D}_{R_0}$, like in Theorem 4.3. As to the latter, we observe that $\Psi_1^{R_1,R_2}[\mathfrak{m}] = p_{\hat{D}}$ with $\hat{D} = \nabla \cdot (\mathfrak{m} \otimes \delta_{\mathbb{S}_{R_1}})$, so Lemma 4.2 yields that $\mathfrak{m} = \mathfrak{m}_- + \mathfrak{d}$, where $\mathfrak{m}_- \in \mathcal{H}^2_{-,R_1}$ and $\mathfrak{d} \in \mathcal{D}_{R_1}$. Thus, the left hand side of (4.2) is included in the right hand side. The reverse inclusion is a direct consequence of Lemma 4.2. □

Corollary 4.10. *Let the setup be as in Definition 4.8 and assume that $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Let further $\Psi = \Psi^{R_1,R_0,R_2}[(\mathbf{m}, \mathfrak{m})]$, with $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $\mathfrak{m} \in L^2(\mathbb{S}_{R_1}, \mathbb{R}^3)$. A pair of potentials of the form $\bar{\Psi}_0 = \Psi_0^{R_0,R_2}[\bar{\mathbf{m}}]$ and $\bar{\Psi}_1 = \Psi_1^{R_1,R_2}[\bar{\mathfrak{m}}]$, with $\bar{\mathbf{m}} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $\bar{\mathfrak{m}} \in L^2(\mathbb{S}_{R_1}, \mathbb{R}^3)$, is uniquely determined by the condition $\Psi(x) = \bar{\Psi}_0(x) + \bar{\Psi}_1(x)$, $x \in \mathbb{S}_{R_2}$.*

Proof. From Theorem 4.3 we get that the components $\mathbf{m}_+ \in \mathcal{H}^2_{+,R_0}$ and $\mathbf{m}_- \in \mathcal{H}^2_{-,R_0}$ of the Hardy-Hodge decomposition of $\mathbf{m}$ are uniquely determined by the knowledge of Ψ on $\mathbb{S}_{R_2}$, while for $\mathfrak{m}$ only the component $\mathfrak{m}_+ \in \mathcal{H}^2_{+,R_1}$ is uniquely determined. The former implies $\bar{\mathbf{m}} \equiv \mathbf{m} + \bar{\mathbf{d}}$, for some $\bar{\mathbf{d}} \in \mathcal{D}_{R_0}$, and the latter yields $\bar{\mathfrak{m}} \equiv \mathfrak{m} + \bar{\mathfrak{m}}_- + \bar{\mathfrak{d}}$, for some $\bar{\mathfrak{m}}_- \in \mathcal{H}^2_{-,R_1}$ and $\bar{\mathfrak{d}} \in \mathcal{D}_{R_1}$. Since Lemma 4.2 yields that $\Psi_0^{R_0,R_2}[\mathbf{m}](x) = \Psi_0^{R_0,R_2}[\mathbf{m} + \mathbf{d}](x)$ and $\Psi_1^{R_1,R_2}[\mathfrak{m}](x) = \Psi_1^{R_1,R_2}[\mathfrak{m} + \bar{\mathfrak{m}}_- + \bar{\mathfrak{d}}](x)$ for $x \in \mathbb{S}_{R_2}$, we eventually find that $\bar{\Psi}_0$ and $\bar{\Psi}_1$ are uniquely determined. □

Remark 4.11. Analogs of Theorems 4.3, 4.9 and Corollaries 4.10, 4.10 are easily seen to hold for the case of finitely many magnetizations $\mathbf{m}_1, \ldots, \mathbf{m}_n$, and $\mathfrak{m}$ supported respectively on spheres $\mathbb{S}_{R_{0,1}}, \ldots, \mathbb{S}_{R_{0,n}}$ and $\mathbb{S}_{R_1}$ of radii $R_1 < R_{0,1} < \ldots < R_{0,n} < R_2$, under the localization assumptions that $\mathrm{supp}(\mathbf{m}_i)$ is a strict subset of $\mathbb{S}_{R_{0,i}}$, $i = 1, \ldots, n$. The corresponding separation properties may be of interest when investigating the depth profile of crustal magnetizations.

4.2 Reconstruction Issues

In this section, we discuss how quantities such as the Fourier coefficients $\langle \Phi_0, Y_{n,k} \rangle_{L^2(\mathbb{S}_{R_2})}$ of Φ_0 can be approximated knowing Φ, without having to reconstruct Φ_0 itself. Such Fourier coefficients are of interest, e.g., when looking at the power spectra of Φ and Φ_0 (cf. the empirical way of separating the crustal and the core magnetic fields mentioned in the introduction). As an extra piece of notation, given $\Gamma_R \subset \mathbb{S}_R$ and $f : \mathbb{S}_R \to \mathbb{R}^k$, we let $f_{|\Gamma_R} : \Gamma_R \to \mathbb{R}^k$ designate the restriction of f to Γ_R.

Theorem 4.12. *Let the setup be as in Definition 4.1 and assume that $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then, for every $\varepsilon > 0$ and every function $\mathbf{g} \in \mathcal{H}^2_{+,R_0} \oplus \mathcal{H}^2_{-,R_0}$, there exists $f \in L^2(\mathbb{S}_{R_2})$ (depending on ε and $\mathbf{g}$) such that*

$$\left| \langle \Phi^{R_1,R_0,R_2}[\mathbf{m}, h], f \rangle_{L^2(\mathbb{S}_{R_2})} - \langle \mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}} \rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \right| \leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3) \times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$.

Proof. According to Theorem 4.3 and the orthogonality of the Hardy-Hodge decomposition, $(\mathbf{g}_{|\Gamma_{R_0}}, 0)$ is orthogonal to the nullspace $N(\Phi^{R_1,R_0,R_2})$ of Φ^{R_1,R_0,R_2}, for if $\operatorname{supp}\mathbf{d} \subset \Gamma_{R_0}$ then $\langle \mathbf{g}_{|\Gamma_{R_0}}, \mathbf{d}\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} = \langle \mathbf{g}, \mathbf{d}\rangle_{L^2(\mathbb{S}_{R_0},\mathbb{R}^3)} = 0$. Therefore, $(\mathbf{g}_{|\Gamma_{R_0}}, 0)$ lies in the closure of the range of the adjoint operator $\left(\Phi^{R_1,R_0,R_2}\right)^*$, i.e., to each $\varepsilon > 0$ there is $f \in L^2(\mathbb{S}_{R_2})$ with

$$\left\| \left(\Phi^{R_1,R_0,R_2}\right)^* [f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0) \right\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon. \tag{4.3}$$

Taking the scalar product with $(\mathbf{m}, h)$, we get from (4.3) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
&\left| \langle \Phi^{R_1,R_0,R_2}[\mathbf{m}, h], f\rangle_{L^2(\mathbb{S}_{R_2})} - \langle \mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}}\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \right| \\
&= \left| \left\langle (\mathbf{m}, h), \left(\Phi^{R_1,R_0,R_2}\right)^* [f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0) \right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_2})} \right| \\
&\leq \left\| \left(\Phi^{R_1,R_0,R_2}\right)^* [f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0) \right\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \\
&\leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},
\end{aligned}$$

which is the desired result. □

Corollary 4.13. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then, for every $\varepsilon > 0$ and every function $g \in L^2(\mathbb{S}_{R_2})$, there exists $f \in L^2(\mathbb{S}_{R_2})$ (depending on ε and g) such that*

$$\left| \langle \Phi^{R_1,R_0,R_2}[\mathbf{m}, h], f\rangle_{L^2(\mathbb{S}_{R_2})} - \langle \Phi_0^{R_0,R_2}[\mathbf{m}], g\rangle_{L^2(\mathbb{S}_{R_2})} \right| \leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$.

Proof. First observe that

$$\left\langle \Phi_0^{R_0,R_2}[\mathbf{m}], g \right\rangle_{L^2(\mathbb{S}_{R_2})} = \left\langle \mathbf{m}, \left(\Phi_0^{R_0,R_2}\right)^* [g] \right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)}, \tag{4.4}$$

where the adjoint operator of $\Phi_0^{R_0,R_2}$ is given by

$$\begin{aligned}
&\left(\Phi_0^{R_0,R_2}\right)^* : L^2(\mathbb{S}_{R_2}) \to L^2(\Gamma_{R_0}, \mathbb{R}^3), \quad g \mapsto \mathbf{H}[g]_{|\Gamma_{R_0}}, \\
&\mathbf{H}[g](x) = -\frac{1}{4\pi} \int_{\mathbb{S}_{R_2}} g(y) \frac{x - y}{|x - y|^3} \mathrm{d}\omega_{R_2}(y), \quad x \in \mathbb{S}_{R_0}.
\end{aligned} \tag{4.5}$$

Clearly $\mathbf{H}[g] \in \mathcal{H}^2_{+,R_0}$ whenever $g \in L^2(\mathbb{S}_{R_2})$, therefore, (4.4) together with Theorem 4.12 yield the desired result. □

Remark 4.14. The interest of Corollary 4.13 from the Geophysical viewpoint lies with the fact that $\Phi^{R_1,R_0,R_2}[\mathbf{m}, h]$ (more specifically: its gradient) corresponds to the measurements on $\mathbb{S}_{R_2}$ of the superposition of the core and crustal contributions, whereas $\Phi_0^{R_0,R_2}[\mathbf{m}]$ corresponds to the crustal contribution alone. Thus, if we can compute f knowing g, we shall in principle be able to get information on the crustal contribution up to arbitrary small error. Note also that $(\mathbf{g}, 0) \notin \operatorname{Ran}\left(\Phi^{R_1,R_0,R_2}\right)^*$ unless $\mathbf{g} \equiv 0$, due to the injectivity of the adjoint of the Poisson transform (which is again a Poisson transform). Therefore we can only hope for an approximation of $\langle \Phi_0^{R_0,R_2}[\mathbf{m}], g\rangle_{L^2(\mathbb{S}_{R_2})}$ in Corollary 4.13, up to a relative error of $\varepsilon > 0$, but not for an exact reconstruction.

Results analogous to Theorem 4.12 and Corollary 4.13 mechanically hold in the setup of Theorem 4.5 and Corollary 4.6 (i.e., separation of the crustal and core magnetic fields $\mathbf{B}_0$ and $\mathbf{B}_1$ instead of the potentials) and in the setup of Theorem 4.9 and Corollary 4.10 (i.e., separation of the potentials Ψ_0 and Ψ_1 due to magnetizations on $\mathbb{S}_{R_0}$ and $\mathbb{S}_{R_1}$). Below we state the corresponding results but we omit the proofs for they are similar to the previous ones.

Theorem 4.15. *Let the setup be as in Theorem 4.5. Then, for every $\varepsilon > 0$ and every field $\mathbf{g} \in \mathcal{H}^2_{+,R_0} \oplus \mathcal{H}^2_{-,R_0}$, there exists $\mathbf{f} \in L^2(\mathbb{S}_{R_2}, \mathbb{R}^3)$ (depending on ε and $\mathbf{g}$) such that*

$$\left| \langle \mathbf{B}^{R_1,R_0,R_2}[\mathbf{m}, h], \mathbf{f} \rangle_{L^2(\mathbb{S}_{R_2},\mathbb{R}^3)} - \langle \mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}} \rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \right| \leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$. The same holds if $\mathbf{B}^{R_1,R_0,R_2}[\mathbf{m}, h]$ gets replaced by $\mathbf{B}_\nu^{R_1,R_0,R_2}[\mathbf{m}, h]$ (resp. $\mathbf{B}_\tau^{R_1,R_0,R_2}[\mathbf{m}, h]$), this time with $\mathbf{f} \in L^2(\mathbb{S}_{R_2})$ (resp $\mathbf{f} \in \mathcal{T}_{R_2}$).

Corollary 4.16. *Let the setup be as in Theorem 4.5 and Corollary 4.6. Then, for every $\varepsilon > 0$ and every field $\mathbf{g} \in L^2(\mathbb{S}_{R_2}, \mathbb{R}^3)$, there exists $\mathbf{f} \in L^2(\mathbb{S}_{R_2}, \mathbb{R}^3)$ (depending on ε and $\mathbf{g}$) such that*

$$\left| \langle \mathbf{B}^{R_1,R_0,R_2}[\mathbf{m}, h], \mathbf{f} \rangle_{L^2(\mathbb{S}_{R_2},\mathbb{R}^3)} - \langle \mathbf{B}_0^{R_0,R_2}[\mathbf{m}], \mathbf{g} \rangle_{L^2(\mathbb{S}_{R_2},\mathbb{R}^3)} \right| \leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$. The same holds if $\mathbf{B}^{R_1,R_0,R_2}[\mathbf{m}, h]$ gets replaced by $\mathbf{B}_\nu^{R_1,R_0,R_2}[\mathbf{m}, h]$ (resp. $\mathbf{B}_\tau^{R_1,R_0,R_2}[\mathbf{m}, h]$), this time with $\mathbf{f} \in L^2(\mathbb{S}_{R_2})$ (resp $\mathbf{f} \in \mathcal{T}_{R_2}$).

Theorem 4.17. *Let the setup be as in Definition 4.8 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then, for every $\varepsilon > 0$ and every $\mathbf{g} \in \mathcal{H}^2_{+,R_0} \oplus \mathcal{H}^2_{-,R_0}$, there exists $f \in L^2(\mathbb{S}_{R_2})$ (depending on ε and $\mathbf{g}$) such that*

$$\left| \langle \Psi^{R_1,R_0,R_2}[\mathbf{m}, \mathfrak{m}], f \rangle_{L^2(\mathbb{S}_{R_2})} - \langle \mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}} \rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \right| \leq \varepsilon \|(\mathbf{m}, \mathfrak{m})\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1},\mathbb{R}^3)},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $\mathfrak{m} \in L^2(\mathbb{S}_{R_1}, \mathbb{R}^3)$.

Corollary 4.18. *Let the setup be as in Definition 4.8 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then, for every $\varepsilon > 0$ and every function $g \in L^2(\mathbb{S}_{R_2})$, there exists $f \in L^2(\mathbb{S}_{R_2})$ (depending on ε and g) such that*

$$\left| \langle \Psi^{R_1,R_0,R_2}[\mathbf{m}, \mathfrak{m}], f \rangle_{L^2(\mathbb{S}_{R_2})} - \langle \Psi_0^{R_0,R_2}[\mathbf{m}], g \rangle_{L^2(\mathbb{S}_{R_2})} \right| \leq \varepsilon \|(\mathbf{m}, \mathfrak{m})\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1},\mathbb{R}^3)},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $\mathfrak{m} \in L^2(\mathbb{S}_{R_1}, \mathbb{R}^3)$.

4.3 The Case $\Gamma_{R_0} = \mathbb{S}_{R_0}$

We turn to the case where $\Gamma_{R_0} = \mathbb{S}_{R_0}$. Then, uniqueness no longer holds in Problem 1.1, but one can obtain the singular value decomposition of Φ^{R_1,R_0,R_2} fairly explicitly and thereby quantify non-uniqueness. Indeed basic computations using spherical harmonics yield:

$$
\begin{aligned}
&(\Phi_0^{R_0,R_2})^*[Y_{n,k}](x) \\
&= \frac{1}{4\pi}\int_{\mathbb{S}_{R_2}} Y_{n,k}\left(\frac{y}{|y|}\right)\nabla_x \frac{1}{|x-y|}\mathrm{d}\omega_{R_2}(y) \\
&= \frac{1}{4\pi}\sum_{m=0}^{\infty}\nabla_x\int_{\mathbb{S}_{R_2}}\frac{1}{|y|}\left(\frac{|x|}{|y|}\right)^m Y_{n,k}\left(\frac{y}{|y|}\right)P_m\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \\
&= \frac{1}{4\pi}\sum_{m=0}^{\infty}\sum_{l=1}^{2m+1}\frac{4\pi}{2m+1}\frac{1}{R_2^{m+1}}\nabla_x\left(|x|^m Y_{m,l}\left(\frac{x}{|x|}\right)\right)\int_{\mathbb{S}_{R_2}} Y_{n,k}\left(\frac{y}{|y|}\right)Y_{m,l}\left(\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \\
&= \frac{R_2}{2n+1}\nabla H_{n,k}^{R_2}(x) = \frac{R_2}{2n+1}\left(\frac{R_0}{R_2}\right)^n \nabla H_{n,k}^{R_0}(x), \quad x\in\mathbb{S}_{R_0},
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
&(\Phi_1^{R_1,R_2})^*[Y_{n,k}](x) \\
&= \frac{1}{4\pi R_1}\int_{\mathbb{S}_{R_2}} Y_{n,k}\left(\frac{y}{|y|}\right)\frac{|y|^2-R_1^2}{|x-y|^3}\mathrm{d}\omega_{R_2}(y) \\
&= \frac{1}{4\pi R_1}\sum_{m=0}^{\infty}(2m+1)\int_{\mathbb{S}_{R_2}}\frac{1}{|y|}\left(\frac{|x|}{|y|}\right)^m P_m\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)Y_{n,k}\left(\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \\
&= \frac{1}{R_1R_2}\sum_{m=0}^{\infty}\sum_{l=1}^{2m+1}\left(\frac{R_1}{R_2}\right)^m Y_{m,l}\left(\frac{x}{|x|}\right)\int_{\mathbb{S}_{R_2}} Y_{n,k}\left(\frac{y}{|y|}\right)Y_{m,l}\left(\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \\
&= \left(\frac{R_1}{R_2}\right)^{n-1} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x\in\mathbb{S}_{R_1},
\end{aligned}
\tag{4.7}
$$

where $H_{n,k}^{R_2}$, $H_{n,k}^{R_0}$ are the inner harmonic from Section 2 and P_m the Legendre polynomial of degree m (see, e.g., [13, 15, Ch. 3] for details). So, we get for the adjoint operator $(\Phi^{R_1,R_0,R_2})^*$ that

$$
(\Phi^{R_1,R_0,R_2})^*[Y_{n,k}] = \left(\frac{R_2}{2n+1}\left(\frac{R_0}{R_2}\right)^n \nabla H_{n,k}^{R_0}, \left(\frac{R_1}{R_2}\right)^{n-1} Y_{n,k}\right)^T. \tag{4.8}
$$

Similar calculations also yield that

$$
\Phi_0^{R_0,R_2}[\nabla H_{n,k}^{R_0}](x) = \frac{n}{R_2}\left(\frac{R_0}{R_2}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right), \quad x\in\mathbb{S}_{R_2},
$$

and

$$
\Phi_1^{R_1,R_2}[Y_{n,k}](x) = \left(\frac{R_1}{R_2}\right)^{n+1} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x\in\mathbb{S}_{R_2},
$$

so we obtain for Φ^{R_1,R_0,R_2} that

$$
\Phi^{R_1,R_0,R_2}[\alpha\nabla H_{n,k}^{R_0}, \beta Y_{m,l}] = \alpha\frac{n}{R_2}\left(\frac{R_0}{R_2}\right)^n Y_{n,k} + \beta\left(\frac{R_1}{R_2}\right)^{m+1} Y_{m,l}, \tag{4.9}
$$

with $\alpha, \beta \in \mathbb{R}$. Based on the representations (4.8) and (4.9), further computation leads us to a characterization of the nullspace of Φ^{R_1,R_0,R_2} in Lemma 4.19. Note that $\Phi^{R_1,R_0,R_2}: L^2(\Gamma_{R_0},\mathbb{R}^3) \times L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2})$ is a compact operator, being the sum of two compact operators (for $\Phi_0^{R_0,R_2}$ and $\Phi_1^{R_1,R_2}$ have continuous kernels).

Lemma 4.19. *Let $\Gamma_{R_0} = \mathbb{S}_{R_0}$, then the nullspace of Φ^{R_1,R_0,R_2} is given by*

$$\begin{aligned} N(\Phi^{R_1,R_0,R_2}) =& \{(\mathbf{m}_- + \mathbf{d}, 0) : \mathbf{m}_- \in \mathcal{H}^2_{-,R_0}, \mathbf{d} \in \mathcal{D}_{R_0}\} \\ &\cup \overline{\operatorname{span}\left\{\left(\nabla H_{n,k}^{R_0}, -\frac{n}{R_1}\left(\frac{R_0}{R_1}\right)^n Y_{n,k}\right)^T : n \in \mathbb{N}, k = 1, \ldots, 2n+1\right\}}, \end{aligned}$$

while the orthogonal complement reads

$$N(\Phi^{R_1,R_0,R_2})^\perp = \overline{\operatorname{span}\left\{\left(\nabla H_{n,k}^{R_0}, \frac{2n+1}{R_1}\left(\frac{R_1}{R_0}\right)^n Y_{n,k}\right)^T : n \in \mathbb{N}, k = 1, \ldots, 2n+1\right\}}.$$

All non-zero eigenvalues values of $(\Phi^{R_1,R_0,R_2})^*\Phi^{R_1,R_0,R_2}$ are of the form

$$\sigma_n = \frac{n}{2n+1}\left(\frac{R_0}{R_2}\right)^{2n} + \left(\frac{R_1}{R_2}\right)^{2n}, \quad n \in \mathbb{N},$$

and the corresponding eigenvectors in $L^2(\mathbb{S}_{R_0},\mathbb{R}^3) \times L^2(\mathbb{S}_{R_1})$ are

$$\left(\nabla H_{n,k}^{R_0}, \frac{2n+1}{R_1}\left(\frac{R_1}{R_0}\right)^n Y_{n,k}\right)^T, \quad n \in \mathbb{N},\ k = 1, \ldots, 2n+1.$$

Lemma 4.19 entails that the nullspace of Φ^{R_1,R_0,R_2} contains elements of the form $(\mathbf{m}, h)$ with $h \neq 0$, hence $\Phi^{R_1,R_0,R_2}[(\mathbf{m}, h)]$ may well vanish on $\mathbb{S}_{R_2}$ even though $\Phi_1^{R_1,R_2}[h]$ is nonzero there, by injectivity of the Poisson representation. In other words, separation of the potentials $\Phi_0^{R_0,R_2}$ and $\Phi_1^{R_1,R_2}$ knowing their sum on $\mathbb{S}_{R_2}$ is no longer possible in general if $\Gamma_{R_0} = \mathbb{S}_{R_0}$.

5 Extremal Problems and Numerical Examples

In this section, we provide some first approaches on how the results from the previous sections can be used to approximate the Fourier coefficients of Φ_0 (cf. Section 5.1), as well as Φ_0 itself via the reconstruction of $\mathbf{m}$ and h (cf. Section 5.2). For brevity, we treat only separation of the crustal and core magnetic potentials (underlying operator Φ^{R_1,R_0,R_2}) and not the separation of the crustal and core magnetic fields (underlying operator $\mathbf{B}^{R_1,R_0,R_2}$) nor the separation of potentials generated by two magnetizations on different spheres (underlying operator Ψ^{R_1,R_0,R_2}). The procedure in such cases is of course similar.

5.1 Reconstruction of Fourier Coefficients of Φ_0

To get a feeling of how functions f in Corollary 4.13 behave, let us derive some of their basic properties. Recall they where isentified to be those $f \in L^2(\mathbb{S}_{R_2})$ satisfying (4.3) with $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[g]$.

Lemma 5.1. *Let* $0 \neq g \in L^2(\mathbb{S}_{R_2})$ *and set* $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[g]$. *To each* $\varepsilon > 0$, *let* $f_\varepsilon \in L^2(\mathbb{S}_{R_2})$ *satisfy* $\|(\Phi^{R_1,R_0,R_2})^*[f_\varepsilon] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$. *Then:*

(a) $\lim_{\varepsilon\to 0} \|f_\varepsilon\|_{L^2(\mathbb{S}_{R_2})} = \infty$,

(b) $\lim_{\varepsilon\to 0} \|(\Phi_1^{R_1,R_2})^*[f_\varepsilon]\|_{L^2(\mathbb{S}_{R_1})} = 0$,

(c) $\lim_{\varepsilon\to 0} \langle f_\varepsilon, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})} = 0$, *for fixed* $n \in \mathbb{N}_0$, $k = 1, \dots, n$.

Proof. From the considerations in Remark 4.14 we know that $(\mathbf{g}_{|\Gamma_{R_0}}, 0) \in \overline{\mathrm{Ran}\big((\Phi^{R_1,R_0,R_2})^*\big)}$ but $(\mathbf{g}_{|\Gamma_{R_0}}, 0) \notin \mathrm{Ran}\big((\Phi^{R_1,R_0,R_2})^*\big)$. Thus, $\|f_\varepsilon\|_{L^2(\mathbb{S}_{R_2})}$ cannot remain bounded as $\varepsilon \to 0$, otherwise a weak limit point $f_0 \in L^2(\mathbb{S}_{R_2})$ would meet $(\Phi^{R_1,R_0,R_2})^*[f_0] = (\mathbf{g}_{|\Gamma_{R_0}}, 0)$, a contradiction which proves (a). Next, the relation

$$\begin{aligned}
&\|(\Phi^{R_1,R_0,R_2})^*[f_\varepsilon] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|^2_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \\
&= \|(\Phi_0^{R_0,R_2})^*[f_\varepsilon] - \mathbf{g}_{|\Gamma_{R_0}}\|^2_{L^2(\Gamma_{R_0},\mathbb{R}^3)} + \|(\Phi_1^{R_1,R_2})^*[f_\varepsilon]\|^2_{L^2(\mathbb{S}_{R_1})} \leq \varepsilon^2
\end{aligned}$$

immediately implies that $\lim_{\varepsilon\to 0} \|(\Phi_1^{R_1,R_2})^*[f_\varepsilon]\|_{L^2(\mathbb{S}_{R_1})} = 0$ which is (b). Finally, expanding f_ε in spherical harmonics, one readily verifies that (4.7) together with (b) yields part (c). □

Next, we give a quantitative appraisal of the fact that the Fourier coefficients of $\Phi_0^{R_0,R_2}$ on $\mathbb{S}_{R_2}$, to be estimated up to relative precision ε by choosing $g = Y_{p,q}$ in Corollary 4.13, can be approximated directly by those of Φ^{R_1,R_0,R_2} (i.e., neglecting entirely the core contribution) when $\frac{R_1}{R_2}$ is small enough (i.e., the core is far from the measurement orbit) and the degree p is large enough. We also give a quantitative version of Lemma 5.1 point (c). This provides us with bounds on the validity of the separation technique consisting merely of a sharp cutoff in the frequency domain.

Lemma 5.2. *Let* $\varepsilon > 0$ *and set* $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$ *for some* $p \in \mathbb{N}_0$ *and* $q \in \{1, \dots 2p+1\}$.

(a) If $R_1^2\big(\frac{R_1}{R_2}\big)^{p-1} \leq \varepsilon$, *then* $f = Y_{p,q}$ *satisfies*

$$\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon. \tag{5.1}$$

(b) If $f \in L^2(\mathbb{S}_{R_2})$ *satisfies* $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$, *then*

$$|\langle f, Y_{p,q}\rangle_{L^2(\mathbb{S}_{R_2})}| \leq \varepsilon \frac{R_2^{p-1}}{R_1^{p+1}}. \tag{5.2}$$

Proof. To prove (a), note that $(\Phi^{R_1,R_0,R_2})^* = \Big((\Phi_0^{R_0,R_2})^*, (\Phi_1^{R_1,R_2})^*\Big)$ and by (4.7) that

$$\|(\Phi_1^{R_1,R_2})^*[f]\|_{L^2(\mathbb{S}_{R_1})} = R_1^2 \left(\frac{R_1}{R_2}\right)^{p-1} \leq \varepsilon,$$

while $\|(\Phi_0^{R_0,R_2})^*[f] - \mathbf{g}_{|\Gamma_{R_0}}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} = 0$ if $f = Y_{p,q}$. Hence (5.1) holds.

As to (b), any $f \in L^2(\mathbb{S}_{R_2})$ satisfying $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g},0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$ satisfies in particular, in view of (4.7):

$$\|(\Phi_1^{R_1,R_2})^*[f]\|^2_{L^2(\mathbb{S}_{R_1})} = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} R_1^4 \left(\frac{R_1}{R_2}\right)^{2(n-1)} |\langle f, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}|^2 \leq \varepsilon^2,$$

from which (5.2) follows at once. □

We turn to the computation of a function f as in Corollary 4.13, regardless of assumptions on $\frac{R_1}{R_2}$ or on the degree of a spherical harmonics $Y_{n,k}$ for which we want to estinate $\langle\Phi_0^{R_0,R_2}, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}$. One way is to solve the following extremal problem. Note that finding f requires no data on the potential Φ that we eventually want to separate into $\Phi_0 + \Phi_1$.

Problem 5.3. Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Fix $g \in L^2(\mathbb{S}_{R_2})$ as well as $\varepsilon > 0$, and set $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[g]$. Then, find $f \in W^{1,2}(\mathbb{S}_{R_2})$ such that

$$\|f\|_{W^{1,2}(\mathbb{S}_{R_2})} = \inf_{\substack{\bar f \in W^{1,2}(\mathbb{S}_{R_2}),\\ \|(\Phi^{R_1,R_0,R_2})^*[\bar f] - (\mathbf{g}_{|\Gamma_{R_0}},0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})}\leq\varepsilon}} \|\bar f\|_{W^{1,2}(\mathbb{S}_{R_2})}. \tag{5.3}$$

It may look strange to seek $f \in W^{1,2}(\mathbb{S}_{R_2})$ whereas Corollary 4.13 merely deals with scalar products in $L^2(\mathbb{S}_{R_2})$. This extra-smoothness requirement, though, helps regularizing the problem.

Lemma 5.4. *Let the setup be as in Problem 5.3 and $g \in L^2(\mathbb{S}_{R_2})$ with $\|\mathbf{g}_{|\Gamma_{R_0}}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} > \varepsilon$. Then, there exists a unique solution $0 \not\equiv f \in W^{1,2}(\mathbb{S}_{R_2})$ to Problem 5.3. Moreover, the constraint in (5.3) is saturated, i.e. $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}},0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} = \varepsilon$.*

Proof. Since $\mathbf{H}[g]$ given by (4.5) lies in $\mathcal{H}^2_{+,R_0}$, the same argument as in the proof of Theorem 4.12 and the density of $W^{1,2}(\mathbb{S}_{R_2})$ in $L^2(\mathbb{S}_{R_2})$ together imply the existence of $\bar f \in W^{1,2}(\mathbb{S}_{R_2})$ such that $\|(\Phi^{R_1,R_0,R_2})^*[\bar f] - (\mathbf{g}_{|\Gamma_{R_0}},0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$ is satisfied, which ensures that the closed convex subset of $W^{1,2}(\mathbb{S}_{R_2})$ defined by

$$\mathcal{C}_\varepsilon = \left\{\bar f \in W^{1,2}(\mathbb{S}_{R_2}) : \|(\Phi^{R_1,R_0,R_2})^*[\bar f] - (\mathbf{g}_{|\Gamma_{R_0}},0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon\right\}$$

is non-empty. Existence and uniqueness of a minimizer f now follows from that of a projection of minimum norm on any nonempty convex set in a Hilbert space. From the assumption that $\|\mathbf{g}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} > \varepsilon$, we get that $f \not\equiv 0$ because $0 \notin \mathcal{C}_\varepsilon$. If the constraint is not saturated, then there is $\delta > 0$ such that, for every $\bar f \in W^{1,2}(\mathbb{S}_{R_2})$ with $\|\bar f\|_{W^{1,2}(\mathbb{S}_{R_2})} \leq 1$, also $f + t\bar f$ satisfies the constraint $\|(\Phi^{R_1,R_0,R_2})^*[f + t\bar f] - (\mathbf{g},0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$ for $t \in (-\delta,\delta)$. Since f is a minimizer, this implies

$$0 = \partial_t \|f + t\bar f\|^2_{W^{1,2}(\mathbb{S}_{R_2})}\Big|_{t=0} = 2\left\langle f, \bar f\right\rangle_{W^{1,2}(\mathbb{S}_{R_2})},$$

for every $\bar f \in W^{1,2}(\mathbb{S}_{R_2})$ with $\|\bar f\|_{W^{1,2}(\mathbb{S}_{R_2})} \leq 1$. Thus $f \equiv 0$, contradicting what precedes. □

Remark 5.5. Lemma 5.1 and the exponential decay of the eigenvalues of $(\Phi_1^{R_1,R_2})^*$ in (4.7) suggest that most of the relevant information regarding a solution $f \in W^{1,2}(\mathbb{S}_{R_2})$ of Problem 5.3 must be contained in Fourier coefficients $\langle f, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}$ of increasingly high degrees n as $\varepsilon \to 0$. Lemma 5.2 provides a hint at the range of accuracies ε for which numerical solutions of Problem 5.3 with $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$ behave differently for small and large p.

Discretization

For the actual solution of Problem 5.3, we assume that $\|\mathbf{g}_{|\Gamma_{R_0}}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} > \varepsilon$, hence the constraint is saturated by Lemma 5.4, and we use a Lagrangian formulation and obtain from [9, Thm. 2.1] that $f \in W^{1,2}(\mathbb{S}_{R_2})$ solves for

$$\left(\mathrm{Id} + \lambda \left(\Phi^{R_1,R_0,R_2}\right)^{**} \left(\Phi^{R_1,R_0,R_2}\right)^{*}\right)[f] = \lambda \left(\Phi^{R_1,R_0,R_2}\right)^{**}[(\mathbf{g}_{|\Gamma_{R_0}}, 0)], \tag{5.4}$$

where $\lambda > 0$ is such that $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} = \varepsilon$. Here, the operator $\left(\Phi^{R_1,R_0,R_2}\right)^{**}$ stands for the adjoint of the restriction of $\left(\Phi^{R_1,R_0,R_2}\right)^{*}$ to the domain $W^{1,2}(\mathbb{S}_{R_2})$. In order to avoid computing $\left(\Phi^{R_1,R_0,R_2}\right)^{**}$, we rewrite (5.4) in variational form: to

$$\begin{aligned} &\langle f, \varphi\rangle_{W^{1,2}(\mathbb{S}_{R_2})} + \lambda \left\langle \left(\Phi^{R_1,R_0,R_2}\right)^{*}[f], \left(\Phi^{R_1,R_0,R_2}\right)^{*}[\varphi]\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \\ &= \lambda \left\langle (\mathbf{g}_{|\Gamma_{R_0}}, 0), \left(\Phi^{R_1,R_0,R_2}\right)^{*}[\varphi]\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})}, \end{aligned} \tag{5.5}$$

for all $\varphi \in W^{1,2}(\mathbb{S}_{R_2})$. Remark 5.5 indicates that a discretization of f in terms of finitely many spherical harmonics is generally not advisable. As a remedy, we use a discretization in terms of the Abel-Poisson kernels

$$K_\gamma(t) = \frac{1}{4\pi}\frac{1-\gamma^2}{(1+\gamma^2-2\gamma t)^{\frac{3}{2}}}, \quad t \in [-1,1]. \tag{5.6}$$

More precisely, we expand f as

$$f(x) = \sum_{m=1}^{M} \alpha_m K_{\gamma,m}(x) = \sum_{m=1}^{M} \alpha_m \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \gamma^n Y_{n,k}\left(\frac{x}{|x|}\right) Y_{n,k}(x_m), \quad x \in \mathbb{S}_{R_2}, \tag{5.7}$$

where $K_{\gamma,m}(x) = K_\gamma(\frac{x}{|x|}\cdot x_m)$. The parameter $\gamma \in (0,1)$ is fixed and controls the spatial localization of $K_{\gamma,m}$ (a parameter γ close to one means a strong localization) while $x_m \in \mathbb{S}_1$, $m = 1,\dots M$, denote the spatial centers of the kernels $K_{\gamma,m}$. Furthermore, one can see from (5.7) that γ relates to the influence of higher spherical harmonic degrees in the discretization of f. Some general properties of the Abel-Poisson kernel K_γ can be found, e.g., in [13, Ch. 5]. Computations based on the representations in Section 4.3 yield

$$\begin{aligned} &(\Phi^{R_1,R_0,R_2})^*[K_{\gamma,m}] \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} Y_{n,k}(x_m)\gamma^n \left(\frac{R_2}{2p+1}\left(\frac{R_0}{R_2}\right)^n \nabla H_{n,k}^{R_0}, \left(\frac{R_1}{R_2}\right)^{n-1} Y_{n,k}\right)^T \\ &= \left(\nabla \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \gamma^n \frac{R_2}{2p+1}\left(\frac{R_0}{R_2}\right)^n \left(\frac{|\cdot|}{R_0}\right)^n Y_{n,k}(x_m) Y_{n,k}\left(\frac{\cdot}{|\cdot|}\right), \left(\frac{R_2}{R_1}\right) K_{\frac{\gamma R_1}{R_2},m}\right)^T \\ &= \left(\frac{R_2}{4\pi}\nabla F_{\frac{\gamma|\cdot|}{R_2},m}, \left(\frac{R_2}{R_1}\right) K_{\frac{\gamma R_1}{R_2},m}\right)^T, \end{aligned} \tag{5.8}$$

where $F_{\gamma,m}(x) = F_\gamma(\frac{x}{|x|}\cdot x_m)$, with $F_\gamma(t) = (1+\gamma^2-2\gamma t)^{-\frac{1}{2}}$ for $t \in [-1,1]$. Inserting (5.7) and (5.8) into (5.5), fixing $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$ and choosing $\varphi = K_{\gamma,n}$ for $n = 1,\dots,M$, as test functions, we are lead to the following system of linear equations

$$\mathbf{M}\boldsymbol{\alpha} = \mathbf{d}, \tag{5.9}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\lambda}\left\langle K_{\gamma,m}, K_{\gamma,n}\right\rangle_{W^{1,2}(\mathbb{S}_{R_2})} + \left(\frac{R_2}{4\pi}\right)^2 \left\langle \nabla F_{\frac{\gamma|\cdot|}{R_2},m}, \nabla F_{\frac{\gamma|\cdot|}{R_2},n}\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \\ + \left(\frac{R_2}{R_1}\right)^2 \left\langle K_{\frac{\gamma R_1}{R_2},m}, K_{\frac{\gamma R_1}{R_2},n}\right\rangle_{L^2(\mathbb{S}_{R_1})} \end{pmatrix}_{n,m=1,\ldots,M},$$

$$\boldsymbol{\alpha} = \left(\alpha_m\right)_{m=1,\ldots,M},$$

$$\mathbf{d} = \left(\frac{R_2^2}{4\pi(2p+1)}\left(\frac{R_0}{R_2}\right)^p \langle \nabla H_{p,q}^{R_0}, \nabla F_n\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)}\right)_{n=1,\ldots,M}.$$

A function f of the form (5.7), determined by coefficients α_m, $m = 1, \ldots, M$, which solve (5.9) will from now on be denoted as $f_{p,q}$. We use $f_{p,q}$ as an approximation of the solution to (5.5) for the choice $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$.

A Numerical Example

In order to generate input data $\Phi = \Phi^{R_1,R_0,R_2}[\mathbf{m}, h] = \Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h]$ for a test example, we choose

$$\mathbf{m}(x) = b_1 \frac{x}{|x|} L_{\gamma_1}\left(\frac{x}{|x|}\cdot y_1\right) + b_2 \frac{x}{|x|} L_{\gamma_2}\left(\frac{x}{|x|}\cdot y_2\right), \quad b_1 = 15,\ b_2 = 10,$$

$$h(x) = \sum_{n=0}^{5}\sum_{k=1}^{2n+1} a_{n,k} Y_{n,k}\left(\frac{x}{|x|}\right), \quad a_{0,1} = a_{1,1} = 2^5,\ a_{2,5} = a_{3,5} = a_{4,5} = 2^4,\ a_{5,5} = 2^3,$$

$$a_{n,k} = 0 \text{ else}, \tag{5.10}$$

with $y_1 = (0, 0, -1)^T$ and $y_2 = (0, \frac{1}{2}, -\frac{\sqrt{3}}{2})^T$. The functions L_{γ_i} are chosen as follows:

$$L_{\gamma_i}(t) = \begin{cases} 0, & t \in [-1, \gamma_i), \\ \frac{(t-\gamma_i)^k}{(1-\gamma_i)^k}, & t \in [\gamma_i, 1], \end{cases} \tag{5.11}$$

for $k = 3$. These functions have been studied in more detail in [37] and are suited for our purposes since they are compactly supported and allow a recursive computation of the Fourier coefficients of $\mathbf{m}$. The parameters $\gamma_i \in (-1, 1)$ reflect the localization of L_{γ_i} (a parameter γ_i close to one means a strong localization). In our test examples, we investigate the two setups $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ and $\gamma_1 = \frac{3}{5}, \gamma_2 = \frac{3}{5}$, where latter reflects a slightly stronger localization of the underlying magnetization. The (unknown) crustal contribution is then denoted by $\Phi_0 = \Phi_0^{R_0,R_2}[\mathbf{m}]$ and the (unknown) core contribution by $\Phi_1 = \Phi_1^{R_1,R_2}[h]$. For the involved radii, we choose $R_0 = 1$ and $R_2 = 1.06$ (at scales of the Earth, the latter indicates a realistic satellite altitude of about 380km above the Earth's surface) and $R_1 = 0.5$ (at scales of the Earth, this is a rough approximation of the radius of the outer core). The subregion $\Gamma_{R_0} = \{x \in \mathbb{S}_{R_0} : x \cdot (0, 0, 1)^T \leq 0\}$ is set to be the Southern hemisphere and the chosen magnetizations of the form (5.10) satisfy $\mathrm{supp}(\mathbf{m}) \subset \Gamma_{R_0}$. For our computations, we use the localization parameter $\gamma = 0.95$ and choose $M = 8,499$ uniformly distributed centers

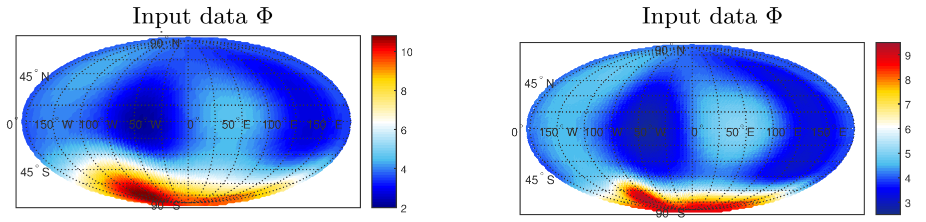

Figure 2: Spatial plot of the input data Φ with parameters $\gamma_1 = \frac{1}{20}$, $\gamma_2 = \frac{1}{2}$ (*left*) and $\gamma_1 = \frac{3}{5}$, $\gamma_2 = \frac{3}{5}$ (*right*) for the magnetization $\mathbf{m}$ from (5.10).

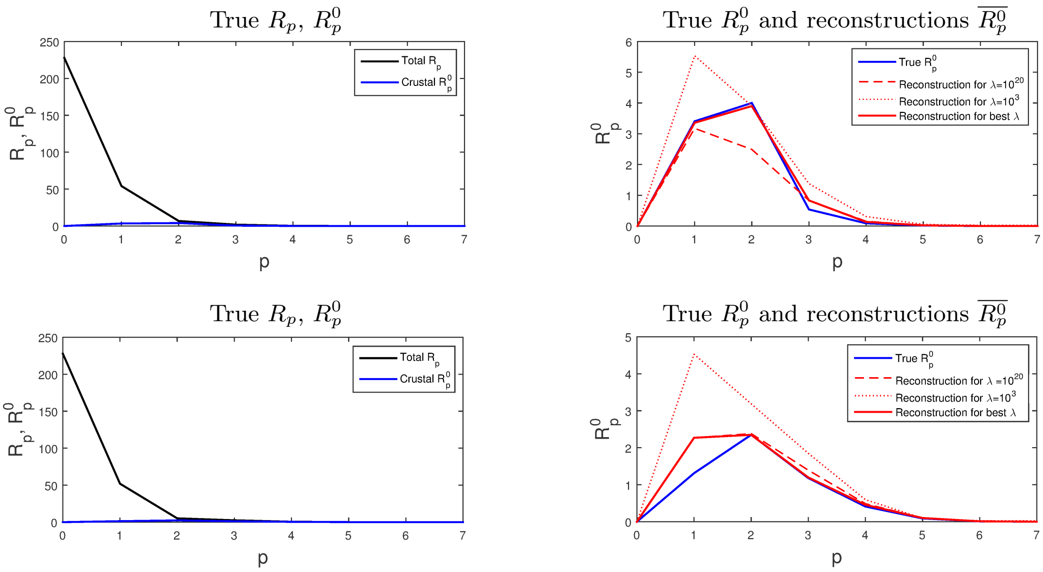

Figure 3: *Left*: Power spectrum R_p of the input data Φ and power spectrum R_p^0 of the crustal contribution Φ_0. *Right*: True crustal power spectrum R_p^0 (blue) and reconstructed power spectrum $\overline{R_p^0}$ (red) for different parameters λ. The *top row* shows the results for the parameters $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ and the *bottom row* for $\gamma_1 = \frac{3}{5}, \gamma_2 = \frac{3}{5}$.

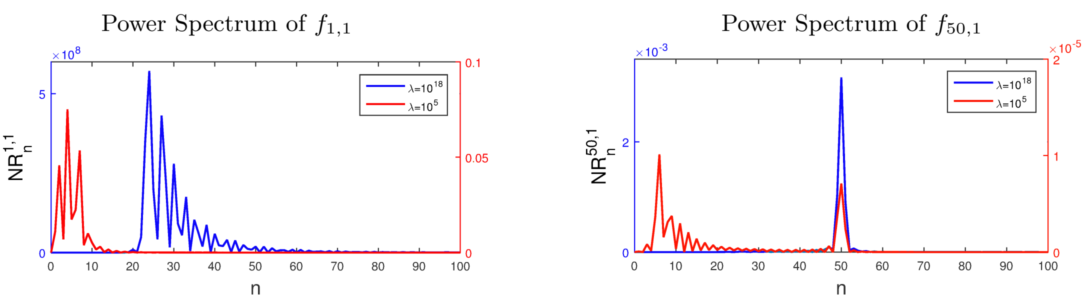

Figure 4: Scaled power spectrum $NR_n^{p,q}$ for $p = 1$, $q = 1$ (*left*) and $p = 50$, $q = 1$ (*right*).

$x_m \in \mathbb{S}_1$, $m = 1, \ldots, M$, for the kernels $K_{\gamma,m}$. All numerical integrations necessary during the procedure are performed via the methods of [10] (when the integration region comprises the entire sphere $\mathbb{S}_{R_0}$, $\mathbb{S}_{R_1}$, or $\mathbb{S}_{R_2}$, respectively) and [21] (when the integration is only performed over the spherical cap $\mathbb{S}_{R_0} \setminus \Gamma_{R_0}$). The input data for the two different setups associated with γ_1, γ_2 are shown in Figure 2. These setups are not based on real geomagnetic data but they reflect a typical geomagnetic situation in the sense that the core contribution clearly dominates the crustal contribution at low spherical harmonic degrees. Figure 3 shows that an empirical separation by a sharp cut-off at degree $p = 2$ or $p = 3$ would neglect relevant information in the crustal contribution.

According to Corollary 4.13, an approximation of the Fourier coefficient $\langle \Phi_0, Y_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})}$ of the crustal contribution Φ_0 is now given by $\langle \Phi, f_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})}$, with $f_{p,q}$ of the form described in the previous subsection. We do this for various degrees p and orders q and we illustrate the results in terms of power spectra: The crustal power spectrum is defined as

$$R_p^0 = R_p[\Phi_0] = \sum_{q=1}^{2p+1} \left| \langle \Phi_0, Y_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})} \right|^2, \quad p \in \mathbb{N}_0.$$

Our approximated power spectrum is then of the form

$$\overline{R_p^0} = \sum_{q=1}^{2p+1} \left| \langle \Phi, f_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})} \right|^2, \quad p \in \mathbb{N}_0.$$

The power spectrum of the input signal Φ (i.e., the superposition of the crustal and core contribution) is analogously defined by $R_p = R_p[\Phi] = \sum_{q=1}^{2p+1} |\langle \Phi, Y_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})}|^2$.

Figure 3 shows the reconstructed power spectra and we see that they yield good results (for a well-chosen parameter λ), in both setups under investigation. Stronger deviations mainly occur at lower spherical harmonic degrees p. The solid red spectrum in Figure 3 indicated as 'Reconstruction for best λ' does not reflect the result for a single choice of λ but rather for (possibly different) best λ in each degree p of the spectrum. The setup for magnetizations $\mathbf{m}$ with parameters $\gamma_1 = \frac{3}{5}, \gamma_2 = \frac{3}{5}$ was chosen to investigate magnetizations with a slightly stronger localization, meaning that the corresponding potential Φ_0 has slightly stronger contributions at higher spherical harmonic degrees than for the setup $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ (compare the right hand images in Figure 3). In Figure 4, we illustrate the effects mentioned in Remark 5.5 by observing the scaled power spectrum $N_n^{p,q} = NR_n[f_{p,q}] = \frac{1}{2n+1} R_n[f_{p,q}] = \frac{1}{2n+1} \sum_{k=1}^{2n+1} |\langle f_{p,q}, Y_{n,k} \rangle_{L^2(\mathbb{S}_{R_2})}|^2$, $n \in \mathbb{N}_0$, for $p = 1$, $q = 1$, and $p = 50$, $q = 1$ (we scaled by a factor $\frac{1}{2n+1}$ solely to get a better idea of the average strength of the Fourier coefficients $|\langle f_{p,q}, Y_{n,k} \rangle_{L^2(\mathbb{S}_{R_2})}|$, $k = 1, \ldots, 2n+1$, for fixed degree n). As expected from Remark 5.5, larger Lagrange parameters λ (which correspond to smaller ε) result in a shift of the major contributions of the power spectrum towards higher spherical harmonic degrees. However, for $p = 50$, $q = 1$, the major spike around $n = 50$ remains, somewhat motivating a different behaviour of the Fourier coefficients of $f_{p,q}$ for larger degrees p compared to smaller p.

5.2 Approximate Reconstruction of Φ_0

While the previous section aimed at the reconstruction of the Fourier coefficients of Φ_0, we are now concerned with the reconstruction the magnetization $\mathbf{m}$ that generates Φ_0. Actually, the goal is still an approximation of Φ_0, but instead of solving multiple extremal problems like Problem 5.3 we rather solve a single least-squares problem to get an approximation $\bar{\mathbf{m}}$ of $\mathbf{m}$, and then we compute $\bar{\Phi}_0 = \Phi_0^{R_0,R_2}[\bar{\mathbf{m}}]$ to approximate Φ_0. Beyond the instrumental parametrizations from the previous section, the only input we retain from the rest of the paper is that, since we apply the technique on an example where $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, we know that separation of the core and crustal potentials is possible by Corollary 4.4. Still, we gather from Theorem 4.3 that $\mathbf{m}$ is not uniquely determined though Φ_0 is. So, in order to regularize the problem, we use standard penalization term to compute a candidate $\bar{\mathbf{m}}$ of small norm (weighted by α). More precisely, we consider the following extremal problem.

Problem 5.6. Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, and let $\Phi \in L^2(\mathbb{S}_{R_2})$ be given. Then, for fixed parameters $\alpha, \beta > 0$, find $\bar{\mathbf{m}} \in W^{2,2}(\mathbb{S}_{R_0}, \mathbb{R}^3)$ and $\bar{h} \in W^{2,2}(\mathbb{S}_{R_1})$ to minimize

$$\inf_{\substack{\bar{\mathbf{m}} \in W^{2,2}(\mathbb{S}_{R_0},\mathbb{R}^3),\\ \bar{h} \in W^{2,2}(\mathbb{S}_{R_1})}} \left\| \Phi - \Phi^{R_1,R_0,R_2}[\bar{\mathbf{m}}, \bar{h}] \right\|^2_{L^2(\mathbb{S}_{R_2})} + \alpha \|(\bar{\mathbf{m}}, \bar{h})\|^2_{W^{2,2}(\mathbb{S}_{R_0},\mathbb{R}^3)\times W^{2,2}(\mathbb{S}_{R_1})} + \beta \|\bar{\mathbf{m}}\|^2_{L^2(\mathbb{S}_{R_0}\setminus\Gamma_{R_0},\mathbb{R}^3)}.$$

Note that in this particular setup, the integration in the definition of the operator $\Phi_0^{R_0,R_2}$ is meant over the entire sphere $\mathbb{S}_{R_0}$ and not just over Γ_{R_0}.

Remark 5.7. Another (more natural) choice to obtain approximations of $\mathbf{m}$ and h would be to minimize

$$\inf_{\substack{\bar{\mathbf{m}} \in W^{2,2}(\Gamma_{R_0},\mathbb{R}^3),\\ \bar{h} \in W^{2,2}(\mathbb{S}_{R_1})}} \left\| \Phi - \Phi^{R_1,R_0,R_2}[\bar{\mathbf{m}}, \bar{h}] \right\|^2_{L^2(\mathbb{S}_{R_2})} + \alpha \|(\bar{\mathbf{m}}, \bar{h})\|^2_{W^{2,2}(\Gamma_{R_0},\mathbb{R}^3)\times W^{2,2}(\mathbb{S}_{R_1})}, \tag{5.12}$$

where this time the integration defining $\Phi_0^{R_0,R_2}$ is only over Γ_{R_0} (as always in this paper, with the exception of Problem 5.6 and Section 4.3). Solving (5.12) leads to magnetizations $\bar{\mathbf{m}}$ that are of class $W^{2,2}(\Gamma_{R_0}, \mathbb{R}^3)$, while solving Problem 5.6 leads to magnetizations $\bar{\mathbf{m}}$ that are of class $W^{2,2}(\mathbb{S}_{R_0}, \mathbb{R}^3)$ and localization in Γ_{R_0} has to be enforced by adding a penalty term (weighted by β). However, for the upcoming example, the minimization proposed in Problem 5.6 yielded slightly better results. Furthermore, it allowed an easier illustration of the effect of the localization constraint by simply dropping the penalty term (i.e., setting $\beta = 0$). Existence of minimizers is guaranteed in both cases by standard arguments. The typically difficult choice of parameters α, β will not be discussed here. In the provided examples, we simply chose those parameters that seemed to yield the best results when compared to the ground truth.

Discretization

In order to discretize Problem 5.6, we expand $\bar{\mathbf{m}}$ and $\bar{h}$ in terms of Abel-Poisson kernels the way indicated in Section 5.1:

$$\bar{\mathbf{m}}(x) = \sum_{i=1}^{3} \sum_{n=1}^{N} \bar{\alpha}_{i,n}\, o^{(i)} K_{\gamma,n}(x), \quad x \in \mathbb{S}_{R_0},$$

$$\bar{h}(x) = \sum_{n=1}^{N} \bar{\beta}_n\, K_{\gamma,n}(x), \quad x \in \mathbb{S}_{R_1}.$$

For brevity, the vectorial operators $o^{(i)}$ have been introduced to denote $o^{(1)} = \nu\,\mathrm{Id}$, $o^{(2)} = \nabla_{\mathbb{S}}$, and $o^{(3)} = \mathrm{L}_{\mathbb{S}}$ (with ν denoting the unit normal vector). Such localized kernels are suitable here since we know/assume in advance that the sought-after magnetization $\mathbf{m}$ is localized in some subregion Γ_{R_0}. Using this discretization, the minimization of Problem 5.6 reduces to solving the following set of linear equations for the coefficients $\bar{\alpha}_{i,n}$ and $\bar{\beta}_n$:

$$\mathbf{M}\boldsymbol{\gamma} = \mathbf{d}, \tag{5.13}$$

where

$$\mathbf{M} = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B}^T \\ \hline \mathbf{B} & \mathbf{C} \end{array}\right) \in \mathbb{R}^{4N\times 4N}, \quad \boldsymbol{\gamma} = (\overline{\boldsymbol{\beta}}|\overline{\boldsymbol{\alpha}}_j)^T_{j=1,2,3} \in \mathbb{R}^{4N}, \quad \mathbf{d} = (\mathbf{a}\,|\,\mathbf{b}_i)^T_{i=1,2,3} \in \mathbb{R}^{4N},$$

with

$$\begin{aligned}
\mathbf{A} &= \Big(\langle \Phi^1_n, \Phi^1_k\rangle_{L^2(\mathbb{S}_{R_2})} + \alpha\langle K_{\gamma,n}, K_{\gamma,k}\rangle_{W^{2,2}(\mathbb{S}_{R_1})}\Big)_{n,k=1,\ldots,N}, \\
\mathbf{B} &= (\mathbf{B}_i)_{i=1,2,3}, \quad \mathbf{B}_i = \Big(\langle \Phi^0_{i,n}, \Phi^1_k\rangle_{L^2(\mathbb{S}_{R_2})}\Big)_{n,k=1,\ldots,N}, \\
\mathbf{C} &= (\mathbf{C}_{i,j})_{i,j=1,2,3}, \\
\mathbf{C}_{i,j} &= \begin{pmatrix} \langle \Phi^0_{i,n}, \Phi^0_{j,k}\rangle_{L^2(\mathbb{S}_{R_2})} + \alpha\left\langle o^{(i)}K_{\gamma,n}, o^{(j)}K_{\gamma,k}\right\rangle_{W^{2,2}(\mathbb{S}_{R_0},\mathbb{R}^3)} \\ +\beta\left\langle o^{(i)}K_{\gamma,n}, o^{(j)}K_{\gamma,k}\right\rangle_{L^2(\mathbb{S}_{R_0}\setminus\Gamma_{R_0},\mathbb{R}^3)} \end{pmatrix}_{n,k=1,\ldots,N}, \\
\overline{\boldsymbol{\beta}} &= (\overline{\beta}_k)_{k=1,\ldots,N}, \quad \overline{\boldsymbol{\alpha}}_j = (\overline{\alpha}_{j,k})_{k=1,\ldots,N}, \\
\mathbf{a} &= \Big(\langle \Phi^1_n, \Phi\rangle_{L^2(\mathbb{S}_{R_2})}\Big)_{n=1,\ldots,N}, \quad \mathbf{b}_i = \Big(\langle \Phi^0_{i,n}, \Phi\rangle_{L^2(\mathbb{S}_{R_2})}\Big)_{n=1,\ldots,N},
\end{aligned}$$

and

$$\begin{aligned}
\Phi^0_{i,n}(x) &= \frac{1}{4\pi}\int_{\mathbb{S}_{R_0}} \Big(o^{(i)}K_{\gamma,n}(y)\Big)\cdot\frac{x-y}{|x-y|^3}\mathrm{d}\omega_{R_0}(y), \\
\Phi^1_n(x) &= \frac{1}{4\pi R_1}\int_{\mathbb{S}_{R_1}} K_{\gamma,n}(y)\frac{|x|^2-R_1^2}{|x-y|^3}\mathrm{d}\omega_{R_1}(y).
\end{aligned}$$

Again, all necessary numerical integrations are performed via the methods of [10] (when the integration region comprises the entire sphere $\mathbb{S}_{R_0}$, $\mathbb{S}_{R_1}$, or $\mathbb{S}_{R_2}$, respectively) and [21] (when the integration is only performed over the spherical cap $\mathbb{S}_{R_0}\setminus\Gamma_{R_0}$).

A Numerical Example

We use the same setup as in Section 5.1 (with parameters $\gamma_1 = \frac{1}{20}$, $\gamma_2 = \frac{1}{2}$) to generate $\Phi = \Phi^{R_1,R_0,R_2}[\mathbf{m}, h]$, $\Phi_0 = \Phi_0^{R_0,R_2}[\mathbf{m}]$, and $\Phi_1 = \Phi_1^{R_1,R_2}[h]$. In the discretization above, we choose $\gamma = 0.9$ and take $N = 10,235$ uniformly distributed centers $x_n \in \mathbb{S}_1$, $n = 1,\ldots,N$. As in the previous example, we choose radii $R_0 = 1$, $R_2 = 1.06$, and now additionally vary R_1 between 0.5 and 0.8. The subregion Γ_{R_0} is again the Southern hemisphere $\{x \in \mathbb{S}_{R_0} : x\cdot(0,0,1)^T < 0\}$. Approximations of $\bar{\mathbf{m}}$ and $\bar{h}$ are obtained by solving (5.13).

In Figure 5, we illustrate the potentials $\bar{\Phi}_0 = \Phi_0^{R_0,R_2}[\bar{\mathbf{m}}]$ and $\bar{\Phi}_1 = \Phi_1^{R_1,R_2}[\bar{h}]$ corresponding to the reconstructed $\bar{\mathbf{m}}$ and $\bar{h}$ for radius $R_1 = 0.5$, while in Figure 6 we set $R_1 = 0.8$. In the first case, we see that the reconstructions yield good approximations of the ground truths $\Phi_0 = \Phi_0^{R_0,R_2}[\mathbf{m}]$ and $\Phi_1 = \Phi_1^{R_1,R_2}[h]$. However, Figure 6 suggests that the reconstruction of

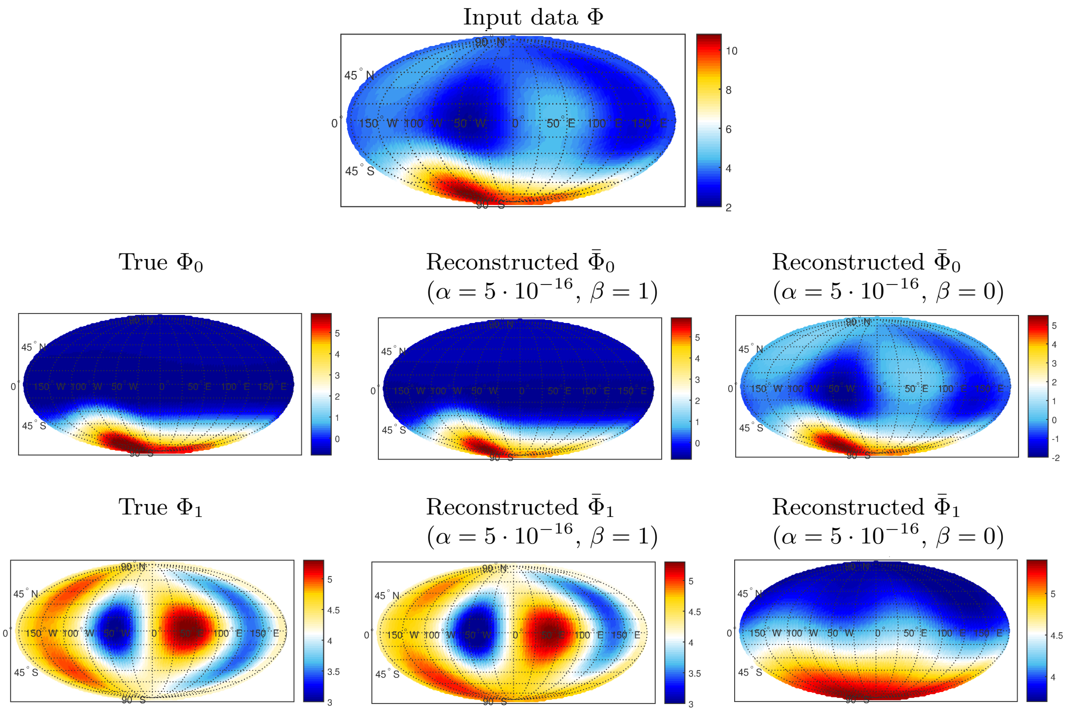

Figure 5: Results for radii $R_1 = 0.5$, $R_0 = 1$, and $R_2 = 1.06$: Input data $\Phi = \Phi_0 + \Phi_1$ (*top*), ground truth Φ_0, Φ_1 (*bottom left*), reconstructed $\bar{\Phi}_0$, $\bar{\Phi}_1$ with localization constraint (*bottom center*), and reconstructed $\bar{\Phi}_0$, $\bar{\Phi}_1$ without localization constraint (*bottom right*).

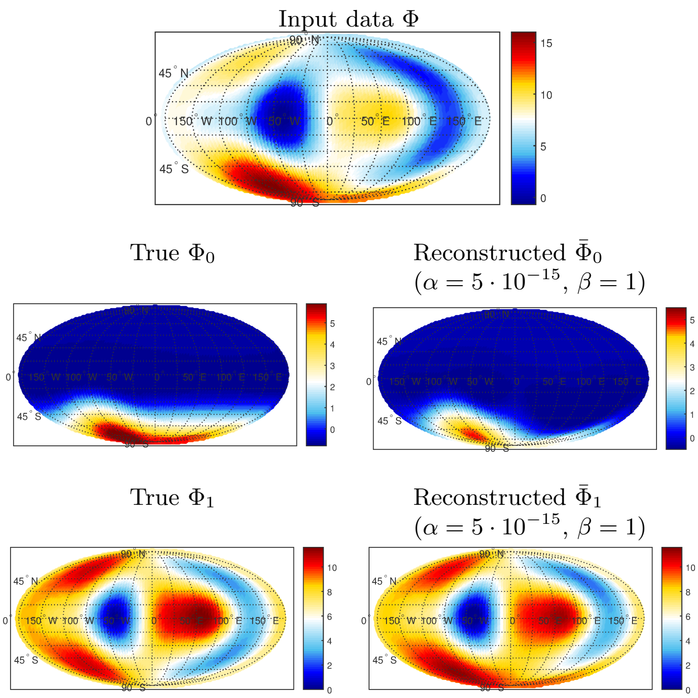

Figure 6: Results for radii $R_1 = 0.8$, $R_0 = 1$, and $R_2 = 1.06$: Input data $\Phi = \Phi_0 + \Phi_1$ (*top*), ground truth Φ_0, Φ_1 (*bottom left*), and reconstructed $\bar{\Phi}_0$, $\bar{\Phi}_1$ with localization constraint (*bottom right*).

the potential Φ_0 becomes numerically more critical as the spheres $\mathbb{S}_{R_1}$ and $\mathbb{S}_{R_0}$ get closer. The influence of the localization constraint on the reconstruction can be seen on the right set of images in Figure 5: neglecting the localization constraint (i.e., choosing $\beta = 0$) leads to a wrong separation of the contributions $\bar{\Phi}_0$ and $\bar{\Phi}_1$.

6 Conclusion

In this paper, we set up a geophysically reasonable model of the core and crustal magnetic field potentials Φ_1 and Φ_0 respectively, for which we showed that each single potential can be recovered uniquely if only the superposition $\Phi = \Phi_0 + \Phi_1$ is known on an external sphere $\mathbb{S}_{R_2}$. Furthermore, we supplied first approaches to the reconstruction of Φ_0 and of its Fourier coefficients. The latter is particularly interesting as it would allow a comparison with the empirical approach to separation based on a sharp cut-off in the power spectrum of Φ. Two main directions call for further study: (1) the geophysical post-processing of real geomagnetic data in order to back up (or deny) the assumption that $\mathbf{m}$ is supported in a subregion Γ_{R_0} of the Earth's surface; (2) improving numerical schemes allowing reconstruction of Φ_0 or its Fourier coefficients when the core contribution Φ_1 is clearly dominating (as is expected at lower spherical harmonic degrees in realistic geomagnetic field models) and when $\mathbb{S}_{R_1}$ is close to $\mathbb{S}_{R_0}$. The domination of the core contribution has been simulated to some extent in the presented examples but is expected to be stronger in real scenarios.

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A Appendix: balayage of distributions

Since potentials of distributions do not seem to be widely treated in the literature, let us briefly justify the statements made in Section 3. For any distribution D supported in a compact set $K \subset \mathbb{R}^3$, the corresponding potential p_D has been formally defined in (3.6) via

$$p_D(x) = D\left(-\frac{1}{4\pi}\frac{1}{|x-\cdot|}\right), \quad x \in \mathbb{R}^3 \setminus K. \tag{A.1}$$

Strictly speaking, this definition is not valid in that $-1/(4\pi|x-\cdot|)$ is neither smooth nor compactly supported in $\mathbb{R}^3$. However, for any compactly supported $\varphi_x \in C^\infty(\mathbb{R}^3)$ with $\varphi_x \equiv 1$ in a neighborhood of K and $\varphi_x \equiv 0$ in a neighborhood of x, the function $g_{\varphi_x}(y) = -\frac{1}{4\pi}\frac{1}{|x-y|}\varphi_x(y)$ is in $C^\infty(\mathbb{R}^3)$ and compactly supported. Clearly $D(g_{\varphi_x})$ is independent of the choice of φ_x, for if ψ_x is another function with the same properties then $g_{\psi_x} - g_{\varphi_x}$ is supported in $\mathbb{R}^3 \setminus K$ so that $D(g_{\psi_x} - g_{\varphi_x}) = 0$. Therefore (A.1) makes good sense if we understand the latter to mean $p_D(x) = D(g_{\varphi_x})$.

In what follows, we restrict ourselves to the case where K has smooth boundary ∂K. This is no loss of generality for the matter discussed in the paper, because we only consider situations where K is a closed ball and we want to define balayage onto the boundary sphere. The lemma below is a simple consequence of known density results for the fundamental solution of the Laplacian in $L^2(\partial K)$, $C^0(\partial K)$, and $W^{k,2}(\partial K)$, see, e.g., [12, 14, 18].

Proposition A.1. *Let $K \subset \mathbb{R}^3$ be a compact, simply connected set with C^∞-boundary ∂K and let $g_x(y) = \frac{1}{|x-y|}$. Then, the set of functions* $\operatorname{span}\{g_x : x \in \mathbb{R}^3 \setminus K\}$ *is dense in $C^k(\partial K)$, for any $k \in \mathbb{N}_0$.*

Proof. For every $f \in C^k(\partial K)$ and $\varepsilon > 0$, there exists $\bar{f} \in C^\infty(\partial K)$ with $\|f - \bar{f}\|_{C^k(\partial K)} < \varepsilon$. In particular, $\bar{f}$ is an element of the Sobolev space $W^{k+2,2}(\partial K)$. By [18, Thm. 8.8] we can find $N > 0$, coefficients $a_i \in \mathbb{R}$, and points $x_i \in \mathbb{R}^3 \setminus K$, $i = 1, \ldots, N$, such that

$$\left\|\bar{f} - \sum_{i=1}^{N} a_i \frac{1}{|x_i - \cdot|}\right\|_{W^{k+2,2}(\partial K)} < \varepsilon.$$

The Sobolev embedding theorem (see, e.g., [1]) now yields that $W^{k+2,2}(\partial K) \subset C^k(\partial K)$ and

$$\left\|\bar{f} - \sum_{i=1}^{N} a_i \frac{1}{|x_i - \cdot|}\right\|_{C^{(k)}(\partial K)} \leq M \left\|\bar{f} - \sum_{i=1}^{N} a_i \frac{1}{|x_i - \cdot|}\right\|_{W^{k+2,2}(\partial K)} < M\varepsilon$$

for some constant $M > 0$ depending only on k, which finishes the proof. □

Proposition A.2. *Let $K \subset \mathbb{R}^3$ be a compact, simply connected set with C^∞-boundary ∂K, and let D be a distribution with support in K. Then, there exists a unique distribution $\hat{D}$ on ∂K such that*

$$p_D(x) = p_{\hat{D}}(x), \quad x \in \mathbb{R}^3 \setminus K.$$

We call $\hat{D}$ the balayage of D onto ∂K.

Proof. First, we deal with the existence of a balayage. Since D is compactly supported, it is known that there are finitely many compactly supported continuous functions Φ_j and multi-indices $\alpha_j \in \mathbb{N}_0^3$, $j = 1, \ldots, m$, such that $D = \sum_{j=1}^{m} \partial_{\alpha_j} \Phi_j$ (see, e.g., [35]). Due to this representation, D acts on compactly supported functions $g \in C^M(\mathbb{R}^3)$, with $M = \max_{i=1,\ldots,N} |\alpha_i|$. Let f be a function in $C^\infty(\partial K)$ and h its unique harmonic continuation to the interior of K with $h = f$ on ∂K [29, Ch. 2]. A compactly supported function $g_f \in C^M(\mathbb{R}^3)$ satisfying $g_f = h$ in K can be computed as follows. The smoothness of ∂K implies there is an open cover $\{U_i\}_{i\in\mathbb{N}}$ of ∂K by open sets in $\mathbb{R}^3$ and diffeomorphisms $\Psi_i \in C^\infty(U_i, \mathbb{B}_1)$ that satisfy $\Psi_i(U_i \cap \partial K) \subset \mathbb{R}^2 \times \{0\}$, $\Psi_i(U_i \cap K) \subset \mathbb{R}^3_-$, and $\Psi_i(U_i \cap (\mathbb{R}^3 \setminus K)) \subset \mathbb{R}^3_+$. Here, $\mathbb{R}^3_\pm$ refer to upper and lower half spaces. Let $\{\varphi_i\}_{i\in\mathbb{N}} \subset C^\infty(\mathbb{R}^3)$ be a partition of unity subordinated to the cover $\{U_i\}_{i\in\mathbb{N}}$. According to the construction in [29, (2.21)], there exist functions $\bar{g}_i \in C^M(\mathbb{R}^3)$, $1 \le i \le N$, compactly supported in $\mathbb{B}_1$, with $\bar{g}_i = (h\varphi_i) \circ \Psi_i^{-1}$ on $\mathbb{R}^3_- \cap \mathbb{B}_1$ for every i. The function $g_f = \sum_{i=1}^{\infty} \bar{g}_i \circ \Psi_i$ gives us the desired extension of h. We now define $\hat{D}$ for any $f \in C^\infty(\partial K)$ by

$$\hat{D}(f) = D(g_f). \tag{A.2}$$

Since any two C^M-smooth extensions of h have the same derivatives of order less than or equal to M on K, we see that $\hat{D}$ does not depend on the particular extension of h that we use. Thus, it holds that

$$p_D(x) = p_{\hat{D}}(x), \quad x \in \mathbb{R}^3 \setminus K,$$

because when $x \notin K$, then $g_x(y)$ is a harmonic function of y in a neighborhood of K. Uniqueness of $\hat{D}$ is a direct consequence of the requirement $\hat{D}(g_x) = p_{\hat{D}}(x) = p_D(x) = D(g_x)$ for $x \in \mathbb{R}^3 \setminus K$, and of Lemma A.1 which guarantees the density of $\{g_x : x \in \mathbb{R}^3 \setminus K\}$ in $C^k(\partial K)$ for all $k \in \mathbb{N}_0$. □

B Appendix: differential forms and Hodge theory

Below we gather some basic definitions and facts from Hodge theory on a smooth simply connected surface $\mathcal{M}$ embedded in $\mathbb{R}^3$, that will be used to prove the rotation lemma in Appendix C. A detailed and more general treatment can be found, e.g., in [42, Ch. 6].

Tangent spaces, smooth functions, vector fields, metric tensor, area measure and Lebesgue spaces are defined as in Section 2. Note that $\mathcal{M}$ must be a finite union of topological spheres, as follows from the classification theorem for surfaces [30] and the fact that g-holed tori are not simply connected while projective planes cannot embed in $\mathbb{R}^3$. In particular $\mathcal{M}$ is orientable.

For $\mathcal{V}$ a real vector space of dimension 2, let $\mathcal{V}^*$ indicate its dual and $\mathcal{A}_2\mathcal{V}$ the bilinear alternating forms on $\mathcal{V}$. If (v_1, v_2) is a basis of $\mathcal{V}$, the linear maps $v_1^*, v_2^* : \mathcal{V} \to \mathbb{R}$ such that $v_j^*(v_k) = \delta_{jk}$ form a basis of $\mathcal{V}^*$, dual to (v_1, v_2). The bilinear alternating form $v_1^* \wedge v_2^*$ defined by

$$v_1^* \wedge v_2^*(w_1, w_2) = \det\big((v_j^*(w_k))_{j,k=1,2}\big) = v_1^*(w_1)v_2^*(w_2) - v_1^*(w_2)v_2^*(w_1)$$

is a basis of the 1-dimensional space $\mathcal{A}_2\mathcal{V}$. Hereafter we put

$$\mathcal{E}\mathcal{V} = \mathbb{R} \oplus \mathcal{V}^* \oplus \mathcal{A}_2\mathcal{V}.$$

If (w_1, w_2) is another basis of $\mathcal{V}$, we say that (w_1, w_2) has the same orientation as (v_1, v_2) if $v_1^* \wedge v_2^*(w_1, w_2) > 0$, the opposite orientation if $v_1^* \wedge v_2^*(w_1, w_2) < 0$. We orient $\mathcal{V}$ by choosing one of the two equivalence classes of bases with the same orientation. If $\mathcal{V}$ is equipped with a Euclidean scalar product $\langle\cdot,\cdot\rangle$, then each $L \in \mathcal{V}^*$ is of the form $L(v) = \langle w, v\rangle$ for some unique $w \in \mathcal{V}$. This way we identify $\mathcal{V}^*$ with $\mathcal{V}$ and $\mathcal{A}_2\mathcal{V}$ with the exterior product $\mathcal{V}\wedge\mathcal{V}$ (the tensor product $\mathcal{V}\otimes\mathcal{V}$ quotiented by all relations $v\otimes v = 0$). Under this identification, given a positively oriented orthonormal basis (e_1, e_2) of $\mathcal{V}$, we define the star operator $\mathcal{E}\mathcal{V} \to \mathcal{E}\mathcal{V}$ to be the linear map such that $*1 = e_1 \wedge e_2$, $*(e_1) = e_2$, $*(e_2) = -e_1$, $*(e_1 \wedge e_2) = 1$. The star operator does not depend on the positively oriented orthonormal basis we use to define it. Clearly, $** = \mathrm{id}$ on $\mathbb{R} \oplus \mathcal{A}_2\mathcal{V}$ and $** = -\mathrm{id}$ on $\mathcal{V}^*$.

We now introduce differential forms on $\mathcal{M}$. A 0-form is a function $\mathcal{M} \to \mathbb{R}$, a 1-form is a map associating to each $x \in \mathcal{M}$ a member of T_x^*, a 2-form is a map associating to x a member of $\mathcal{A}_2 T_x$; here and below, T_x indicates the tangent space to $\mathcal{M}$ at x. Given a k-form ω and a chart (U, ψ) on $\mathcal{M}$ with $\psi(U) = V \subset \mathbb{R}^2$, one can define a k-form $\tilde{\omega}$ on V by the rule

$$\tilde{\omega}(y)(v_1, \cdots, v_k) = \omega(\psi^{-1}(y))(\mathrm{D}\psi^{-1}(v_1), \cdots, \mathrm{D}\psi^{-1}(v_k)), \quad y \in V, \quad v_1, \cdots, v_k \in \mathbb{R}^2, \quad \text{(B.1)}$$

which represents ω in local coordinates using the isomorphism $\mathrm{D}\psi^{-1}(y) : \mathbb{R}^2 \to T_{\psi^{-1}(y)}$. This way a form on $\mathcal{M}$ may be regarded as a collection of forms on images of charts which define the same form ω on overlaps via (B.1). Hence if we use a superscript prime to denote another system of local coordinates and if we set $h = \psi' \circ \psi^{-1}$ for the corresponding change of charts, we have if $k = 2$ that

$$\tilde{\omega}(y)(v_1, v_2) = (\det(\mathrm{D}h(y)))\, \tilde{\omega}'(h(y))(v_1, v_2), \quad y \in V \cap h^{-1}(V'). \quad \text{(B.2)}$$

A 1-form ω can be written in local coordinates as $\tilde{\omega}(y) = a(y)\mathrm{d}y_1 + b(y)\mathrm{d}y_2$, where a, b are real functions of $y \in \psi(U)$ and $\mathrm{d}y_1$, $\mathrm{d}y_2$ is the basis of $(\mathbb{R}^2)^*$ dual to the canonical basis of $\mathbb{R}^2$. If ω is a 2-form, then $\tilde{\omega}(y) = c(y)\mathrm{d}y_1 \wedge \mathrm{d}y_2$ where c is real-valued on V. The wedge product is an associative binary operation on forms, bilinear over functions, that associates to a k_1-form ω_1 and a k_2-form ω_2 a $k_1 + k_2$-form $\omega_1 \wedge \omega_2$ such that, in local coordinates, $*(\mathrm{d}y_1) \wedge *(\mathrm{d}y_2) = \mathrm{d}y_1 \wedge \mathrm{d}y_2 = - * (\mathrm{d}y_2) \wedge *(\mathrm{d}y_1)$ and $*(\mathrm{d}y_1) \wedge *(\mathrm{d}y_1) = *(\mathrm{d}y_2) \wedge *(\mathrm{d}y_2) = 0$. Note that k-forms with $k > 2$ (mapping $x \in \mathcal{M}$ to a k-linear alternating map on $(T_x)^k$) are identically zero for T_x has dimension 2. The wedge product is independent of the chart used to compute a local representative. We say that a 1-form or a 2-form is smooth if its coefficients a, b or c are smooth functions in every chart. We write $\Lambda^k\mathcal{M}$ for the space of smooth forms of degree k on $\mathcal{M}$, and we let $\Lambda\mathcal{M} = \oplus_{k=0}^2 \Lambda^k\mathcal{M}$ for the direct sum.

A smooth 2-form ω can be integrated over a Borel set $E \subset \mathcal{M}$: if (U, ψ) is a chart with $\psi(U) = V$ and $\tilde{\omega} = c(y)\mathrm{d}y_1 \wedge \mathrm{d}y_2$, and if moreover $E \subset U$, we set $\int_E \omega = \int_{\psi(E)} c(y)\mathrm{d}\lambda(y)$ where λ indicates Lebesgue measure. In the general case we cover E with finitely many domains of charts and we use a partition of unity; relation (B.2) and the change of variable formula ensure that the definition does not depend on which charts or partition we use.

The exterior differential $\mathrm{d} : \Lambda^k\mathcal{M} \to \Lambda^{k+1}\mathcal{M}$ is defined as follows. If g is a function, then $\mathrm{d}g$ is the usual differential, namely in local coordinates $\widetilde{\mathrm{d}g} = \partial\tilde{g}/\partial y_1 \mathrm{d}y_1 + \partial\tilde{g}/\partial y_2 \mathrm{d}y_2$. If $\tilde{\omega} = a\mathrm{d}y_1 + b\mathrm{d}y_2$ is a 1-form in local coordinates, then $\widetilde{\mathrm{d}\omega} = (\partial b/\partial y_1 - \partial a/\partial y_2)\mathrm{d}y_1 \wedge \mathrm{d}y_2$. The differential of a 2-form is zero. Differentiation is meaningful in that it is independent of the chart used to compute its local representative. Moreover it holds that $\mathrm{d} \circ \mathrm{d} = 0$. If $\mathrm{d}\omega = 0$, we say that ω is closed, and if $\omega = \mathrm{d}\nu$ for some ν we say that ω is exact. Exact forms are closed, and the quotient space of closed k-forms by exact k-forms is called the k-th (de Rham) cohomology group $H^k(\mathcal{M})$. The simple connectedness of $\mathcal{M}$ means that $H^1(\mathcal{M}) = 0$, i.e. every closed 1-form on $\mathcal{M}$ is exact [42, Ch. 5].

The Hodge-star operator maps $\Lambda^k\mathcal{M}$ to $\Lambda^{2-k}\mathcal{M}$ for $0 \leq k \leq 2$, by acting pointwise as the star operator on $\mathcal{E}T_x$ for each $x \in \mathcal{M}$. If we identify a 1-form ω with the tangent vector field $\mathbf{v}_\omega$ such that $\omega(x)(w) = \mathbf{v}_\omega(x) \cdot w$ for $w \in T_x$, then the Hodge star operator merely rotates $\mathbf{v}_\omega$ by $\pi/2$ in the tangent space at each point. To check that it maps smooth forms to smooth forms, we need only produce in a neighborhood of each $x_0 \in \mathcal{M}$ a positively oriented orthonormal basis $(e_1(x), e_2(x))$ of T_x that varies smoothly with x. If (U, ψ) is a chart with $x_0 \in U$ and $V = \psi(U)$, we may choose $e_j(\psi^{-1}(y)) = D\psi^{-1}(y)\mathbf{G}(y)^{-1/2}\kappa_j$ for $y \in V$, where $\mathbf{G}$ is the metric tensor and κ_1, κ_2 the canonical basis of $\mathbb{R}^2$. We denote the action of the Hodge star operator on a form ω by $*\omega$, as no confusion should arise with the star operator acting on $\mathcal{E}T_x$ for fixed x. Next, one defines a pairing on $\Lambda^k\mathcal{M}$ by letting

$$\langle \omega_1, \omega_2 \rangle = \int_{\mathcal{M}} \omega_1 \wedge *\omega_2. \tag{B.3}$$

Identifying T_x^* and T_x via the scalar product in $\mathbb{R}^3$, it follows from the definitions, with the notation of (B.2), that in local coordinates $\widetilde{e_1 \wedge *e_2} = \widetilde{e_2 \wedge *e_1} = 0$ and, in addition,

$$\widetilde{1 \wedge *1} = \widetilde{e_1 \wedge *e_1} = \widetilde{e_2 \wedge *e_2} = \widetilde{(e_1 \wedge e_2) \wedge *(e_1 \wedge e_2)} = \sqrt{g}\,\mathrm{d}y_1 \wedge \mathrm{d}y_2.$$

Hence (B.3) is symmetric and positive definite, moreover we have that

$$\langle f, f \rangle = \|f\|^2_{L^2(\mathcal{M})} \quad \text{and} \quad \langle \omega, \omega \rangle = \|\mathbf{v}_\omega\|^2_{L^2(\mathcal{M},\mathbb{R}^3)}, \qquad f \in \Lambda^0\mathcal{M},\ \omega \in \Lambda^1\mathcal{M}. \tag{B.4}$$

One extends $\langle \cdot, \cdot \rangle$ to a scalar product on $\Lambda\mathcal{M}$ by requiring that forms of different degree are orthogonal. Let $\delta : \Lambda^k\mathcal{M} \to \Lambda^{k-1}\mathcal{M}$ be the operator defined by $\delta(\omega) = (-1)^{k(2-k)} * \mathrm{d}(*\omega)$. Since $** \omega = (-1)^k\omega$ when $\omega \in \Lambda^k\mathcal{M}$, it holds if $\omega_1 \in \Lambda^{k-1}\mathcal{M}$ and $\omega_2 \in \Lambda^k\mathcal{M}$ that

$$\mathrm{d}(\omega_1 \wedge *\omega_2) = \mathrm{d}\omega_1 \wedge *\omega_2 + (-1)^{k-1}\omega_1 \wedge \mathrm{d}(*\omega_2) = \mathrm{d}\omega_1 \wedge *\omega_2 - \omega_1 \wedge *\delta(\omega_2),$$

and since the left hand side integrates to 0 over $\mathcal{M}$ by Stoke's theorem it implies that δ is the adjoint of d in $\Lambda\mathcal{M}$ equipped with (B.3). In particular, we see from (B.4) that δ must coincide with the divergence operator on $\Lambda^1\mathcal{M}$ when the latter is identified with smooth tangent vector fields. The operator $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ which maps $\Lambda^k\mathcal{M}$ into itself is the Laplace Beltrami operator on $\Lambda\mathcal{M}$. The kernel of Δ in $\Lambda^k\mathcal{M}$ is the space of harmonic k-forms, denoted by $\mathcal{H}^k$. Now, a fundamental result in Hodge theory [42, Thm. 6.8] is the existence of an orthogonal sum:

$$\Lambda^k\mathcal{M} = \mathrm{d}(\Lambda^{k-1}\mathcal{M}) \oplus \delta(\Lambda^{k+1}\mathcal{M}) \oplus \mathcal{H}^k, \qquad k = 0, 1, 2, \tag{B.5}$$

where orthogonality holds with respect to (B.3) (by convention $\Lambda^{-1}\mathcal{M} = \{0\}$). Using (B.5) and elliptic regularity theory, one can further show that each equivalence class in the cohomology group $H^k(\mathcal{M})$ has a unique harmonic representative [42, Thm. 6.11]. Since $H^1(\mathcal{M}) = \{0\}$ we deduce that $\mathcal{H}^1 = 0$, hence the orthogonal decomposition (B.5) specializes in our case to

$$\Lambda^1\mathcal{M} = \mathrm{d}(\Lambda^0\mathcal{M}) \oplus \delta(\Lambda^2\mathcal{M}). \tag{B.6}$$

Moreover, since $*$ is obviously surjective $\Lambda^2\mathcal{M} \to \Lambda^0\mathcal{M}$ (for the inverse image of a smooth function f is $f\mathrm{d}e_1 \wedge \mathrm{d}e_2$), we get that

$$\mathrm{Im}(\delta : \Lambda^2\mathcal{M} \to \Lambda^1\mathcal{M}) = \mathrm{Im}(*\mathrm{d} : \Lambda^0 \to \Lambda^1\mathcal{M}). \tag{B.7}$$

C Appendix: the rotation lemma

In the notation of Section 2, we prove below that the operator $J : \mathcal{T}_R \to \mathcal{T}_R$, which rotates a tangent vector field by $\pi/2$ at every point in the positively oriented tangent plane, isometrically maps tangential gradients to divergence free vector fields and vice-versa. This we call the rotation lemma. The result actually holds on any smooth simply connected compact surface $\mathcal{M}$ embedded in $\mathbb{R}^3$, and we deal below with this more general version but restricting ourselves to the sphere would not simplify the proof.

Gradients, Sobolev spaces, tangent and divergence-free vector fields are defined as in Section 2. Thus, letting $\mathcal{T}$, $\mathcal{G}$ and $\mathcal{D}$ indicate respectively tangent, gradient, and divergence free vector fields in $L^2(\mathcal{M}, \mathbb{R}^3)$, we have the orthogonal decomposition:

$$\mathcal{T} = \mathcal{G} \oplus \mathcal{D}. \tag{C.1}$$

As pointed out in Appendix B, $\mathcal{M}$ is orientable, which makes it possible to define J as rotation of a tangent vector field pointwise by $\pi/2$ in the positively oriented tangent plane.

Lemma C.1. *For $\mathcal{M}$ a compact simply connected surface embedded in $\mathbb{R}^3$, the map $J : \mathcal{T} \to \mathcal{T}$ isometrically maps $\mathcal{G}$ onto $\mathcal{D}$ and conversely.*

Proof. That J is isometric is obvious for it preserves length pointwise. Moreover, since $J^2 = -I$, it suffices to establish that $J(\mathcal{G}) = \mathcal{D}$. By (C.1) this amounts to prove that $\mathcal{T} = \mathcal{G} \oplus J(\mathcal{G})$, and since smooth vector fields and smooth functions are dense in $\mathcal{T}$ and $W^{1,2}(\mathcal{M})$ respectively, it is enough by the isometric character of J to show that

$$\mathcal{T}_S = \mathcal{G}_S \oplus J(\mathcal{G}_S), \tag{C.2}$$

where the subscript "S" indicates the smooth elements of the corresponding space. Now, representing a 1-form ω as the pointwise Euclidean scalar product with a tangent vector field $\mathbf{v}_\omega$ as we did in Appendix B, we have for any smooth function $f : \mathcal{M} \to \mathbb{R}$ that $\mathbf{v}_{\mathrm{d}f}$ is just the gradient $\nabla_{\mathcal{M}} f$ and, since we observed in the latter appendix that the Hodge star operator coincides with J on $\mathbf{v}_\omega$, the decomposition (C.2) follows immediately from (B.6) and (B.7). $\square$