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► **To cite this version:**

Nicolas Keriven, Rémi Gribonval, Gilles Blanchard, Yann Traonmilin. Random Moments for Sketched Mixture Learning. SPARS2017 - Signal Processing with Adaptive Sparse Structured Representations workshop, Jun 2017, Lisbon, Portugal. <hal-01494045>

HAL Id: hal-01494045

<https://hal.inria.fr/hal-01494045>

Submitted on 22 Mar 2017

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Random Moments for Sketched Mixture Learning

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Abstract—We present a method to solve large-scale mixture learning tasks from a *sketch* of the data, formed by random generalized empirical moments. We give empirical and theoretical results on k -means and Gaussian Mixture Model estimation problems.

I. INTRODUCTION

Consider samples $z_i \in \mathbb{R}^d$, $1 \leq i \leq n$, drawn *i.i.d.* from a distribution π . Given a class of *hypotheses* \mathcal{H} and a *loss function* $\ell: \mathbb{R}^d \times \mathcal{H} \rightarrow \mathbb{R}$, statistical learning consists in finding the hypothesis $h^* \in \mathcal{H}$ that minimizes the *expected risk* $\mathcal{R}(h) = \mathbb{E}_\pi \ell(z, h)$. Since the distribution π is not directly available, usual learning procedures minimize the empirical risk instead: $\hat{\mathcal{R}}_n(h) = \sum_i \ell(z_i, h)/n$.

This traditional approach is however challenged when samples z are high-dimensional (large d) or in great number (large n). The first case has been dealt with using random projections [1] or feature selection [2], while the second gave birth to online learning [3] or coresets [4]. We advocate here that when n is large, some learning tasks can be done using only a collection of generalized empirical moments, referred to as *sketch*, as a (highly) compressed representation of the database. A simple example is Principal Component Analysis (PCA), which can be done with only the empirical covariance. Such sketches can be computed online, and/or in a distributed/parallel manner, and do not require the database to be stored on one single device.

We present here a method to perform k -means or Gaussian Mixture Model (GMM) estimation with identity covariance from a sketch formed by a (weighted) random sampling of the characteristic function. Such inverse problems bear similarities with sparse recovery in continuous spaces [5]. Define the sketching operator:

$$\mathcal{A}\pi = \frac{1}{\sqrt{m}} \left[\mathbb{E}_{z \sim \pi} \exp(-i\omega_j^T z) / c_{\omega_j} \right]_{j=1}^m \quad (1)$$

where $c_{\omega_j} > 0$ are some weights and frequencies $\omega_j \in \mathbb{R}^d$ are drawn *i.i.d.* from a weighted Gaussian distribution $\Lambda(\omega) \propto c_\omega^2 \mathcal{N}(0, \sigma^2 \mathbf{I})$. The empirical sketch used in practice is denoted $\mathbf{y} = \frac{1}{n\sqrt{m}} \left[\sum_{i=1}^n \exp(-i\omega_j^T z_i) / c_{\omega_j} \right]_{j=1}^m$.

II. MAIN RESULTS

We now present our main results on k -means and GMM estimation. In each case, c_ω and σ^2 are appropriately chosen and not detailed in this abstract. Leveraging tools from kernel embeddings of distributions [6] and Random Fourier features [7], our analysis is inspired by Compressive Sensing results [8], [9], adapted to the proposed infinite-dimensional framework.

A. k -means

In the k -means problem, each hypothesis is a set of centroids $h = \{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ and the loss function is $\ell(z, h) = \min_l \|z - \mathbf{c}_l\|_2^2$.

Assumptions. We restrict to a family of hypotheses where centroids are 2ε -separated from each other and contained in a ball of radius M , and denote $\mathcal{H}_{k,\varepsilon,M}$ the corresponding class of hypotheses.

Result. Denote $h^* \in \mathcal{H}_{k,\varepsilon,M}$ the true minimizer of the expected risk \mathcal{R} and \hat{h} the hypothesis recovered from the sketch by

$$\hat{h} = \operatorname{argmin}_{h \in \mathcal{H}_{k,\varepsilon,M}} \min_{\alpha_1, \dots, \alpha_k} \left\| \mathbf{y} - \mathcal{A} \left(\sum_{l=1}^k \alpha_l \delta_{\mathbf{c}_l} \right) \right\|_2 \quad (2)$$

where $\alpha_l \geq 0$ and $\sum_{l=1}^k \alpha_l = 1$.

If $m \geq \mathcal{O}(k^2 d^3 \operatorname{poly} \log(k, d) \log(1/\rho \cdot M/\varepsilon))$, then with joint probability $1 - \rho$ on the drawing of z_i and ω_j it holds that

$$\mathcal{R}(\hat{h}) \lesssim \mathcal{R}(h^*) + \mathcal{O}\left(\sqrt{kd^2/n}\right). \quad (3)$$

B. Gaussian mixture with identity covariance

In the GMM learning problem, a hypothesis is a set of means and weights $h = \{\mu_1, \dots, \mu_k, \alpha_1, \dots, \alpha_k\}$ and the loss function is $\ell(z, h) = -\log \pi_h(z)$, where $\pi_h = \sum_{l=1}^k \alpha_l \mathcal{N}(\mu_l, \mathbf{I})$ is a GMM.

Assumptions. We restrict to a class of hypotheses where means are separated from each other and contained in a ball of radius M , and denote $\mathcal{H}_{k,M}$ the corresponding class of hypotheses. Unlike k -means, the separation between means cannot be as small as desired, and there is a trade-off between the required separation and the required number of measurements m . A few values are given in Table I.

Result. Denote $h^* \in \mathcal{H}_{k,M}$ the true minimizer of the expected risk \mathcal{R} and \hat{h} the hypothesis recovered from the sketch by solving

$$\hat{h} = \operatorname{argmin}_{h \in \mathcal{H}_{k,M}} \|\mathbf{y} - \mathcal{A}\pi_h\|_2. \quad (4)$$

If the number of measurements m is large enough (see Tab. I), with joint probability $1 - \rho$ on the drawing of z_i and ω_j it holds that

$$\mathcal{R}(\hat{h}) - \mathcal{R}(h^*) \lesssim \inf_{h \in \mathcal{H}_{k,M}} \|\pi - \pi_h\|_{\text{TV}} + \mathcal{O}\left(\sqrt{1/n}\right) \quad (5)$$

where the \mathcal{O} hides some dependencies in k, d (roughly behaving like m in Tab. I). The bound also involves the best approximation of π by a GMM for the TV norm (L^1 norm for densities).

III. EXPERIMENTAL RESULTS

The optimization problems (2) and (4) are non-convex and seem hard to solve exactly. Heuristically, a greedy algorithm inspired by sparse recovery referred to as Compressive Learning OMP (CLOMP) [10]–[12] has been previously shown to perform well. We compare a Matlab implementation of CLOMP available at [13] with Matlab's `kmeans` function and VLFeat's [14] `gmm` function.

In Fig. 1, the sketched approach is seen to lead to tremendous savings in time of execution and memory consumption when the number of items n is large, while achieving the same precision as the corresponding traditional approach for a limited number of measurements $m \approx \mathcal{O}(kd)$. Fig. 2 further confirms that $m \approx \mathcal{O}(kd)$ is empirically sufficient, hence the theoretical guarantees for $m \gtrsim \mathcal{O}(k^2 d^2)$ are probably pessimistic.

Further work will combine the sketching technique with dimensionality-reduction methods to treat *both* large d and n .

TABLE I

TRADE-OFF BETWEEN REQUIRED SEPARATION OF MEANS AND NUMBER OF MEASUREMENTS IN THE GMM LEARNING PROBLEM.

Separation of means	Number of measurements
$\mathcal{O}(\sqrt{d \log k})$	$m \geq \mathcal{O}(k^2 d^2 \text{polylog}(k, d) \log(M/\rho))$
$\mathcal{O}(\sqrt{d + \log k})$	$m \geq \mathcal{O}(k^3 d^2 \text{polylog}(k, d) \log(M/\rho))$
$\mathcal{O}(\sqrt{\log k})$	$m \geq \mathcal{O}(k^2 d^2 e^d \text{polylog}(k, d) \log(M/\rho))$

ACKNOWLEDGMENT

This work was supported in part by the European Research Council, PLEASE project (ERC-StG- 2011-277906).

REFERENCES

- [1] C. Boutsidis, A. Zouzias, and P. Drineas, "Random Projections for k-means Clustering," in *Advances in Neural Information and Processing Systems (NIPS)*, 2010, pp. 298–306.
- [2] J. Altschuler, A. Bhaskara, Gang, Fu, V. Mirrokni, A. Rostamizadeh, and M. Zadimoghaddam, "Greedy Column Subset Selection: New Bounds and Distributed Algorithms," in *Proceedings of The 33rd International Conference on Machine Learning*, 2016.
- [3] O. Cappé and E. Moulines, "Online EM Algorithm for Latent Data Models," *Journal of the Royal Statistical Society*, vol. 71, no. 3, pp. 593–613, 2009.
- [4] D. Feldman and M. Langberg, "A unified framework for approximating and clustering data," *Proceedings of the forty-third annual ACM symposium on Theory of computing*, no. 46109, pp. 569–578, 2011.
- [5] E. J. Candès and C. Fernandez-Granda, "Super-resolution from noisy data," *Journal of Fourier Analysis and Applications*, vol. 19, no. 6, pp. 1229–1254, 2013.
- [6] A. J. Smola, A. Gretton, L. Song, and B. Schölkopf, "A Hilbert Space Embedding for Distributions," in *International Conference on Algorithmic Learning Theory*, 2007, pp. 13–31.
- [7] A. Rahimi and B. Recht, "Random Features for Large Scale Kernel Machines," *Advances in Neural Information Processing Systems (NIPS)*, no. 1, pp. 1–8, 2007.
- [8] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A simple proof of the restricted isometry property for random matrices," *Constructive Approximation*, vol. 28, no. 3, pp. 253–263, 2008.
- [9] A. Bourrier, M. E. Davies, and T. Peleg, "Fundamental performance limits for ideal decoders in high-dimensional linear inverse problems," *IEEE Transactions on Information Theory*, vol. 60, no. 12, pp. 7928–7946, 2014.
- [10] N. Keriven, A. Bourrier, R. Gribonval, and P. Pèrèz, "Sketching for Large-Scale Learning of Mixture Models," in *IEEE International Conference on Acoustic, Speech and Signal Processing (ICASSP)*, 2016.
- [11] N. Keriven, N. Tremblay, Y. Traonmilin, and R. Gribonval, "Compressive K-means," in *IEEE International Conference on Acoustic, Speech and Signal Processing (ICASSP)*, 2017.
- [12] N. Keriven, A. Bourrier, R. Gribonval, and P. Pèrèz, "Sketching for Large-Scale Learning of Mixture Models," *arXiv preprint arXiv:1606.02838*, pp. 1–50, 2016.
- [13] N. Keriven, N. Tremblay, and R. Gribonval, "SketchMLbox : a Matlab toolbox for large-scale learning of mixture models," 2016. [Online]. Available: sketchml.gforge.inria.fr
- [14] A. Vedaldi and B. Fulkerson, "VLFeat - An open and portable library of computer vision algorithms," Tech. Rep., 2010. [Online]. Available: www.vlfeat.org

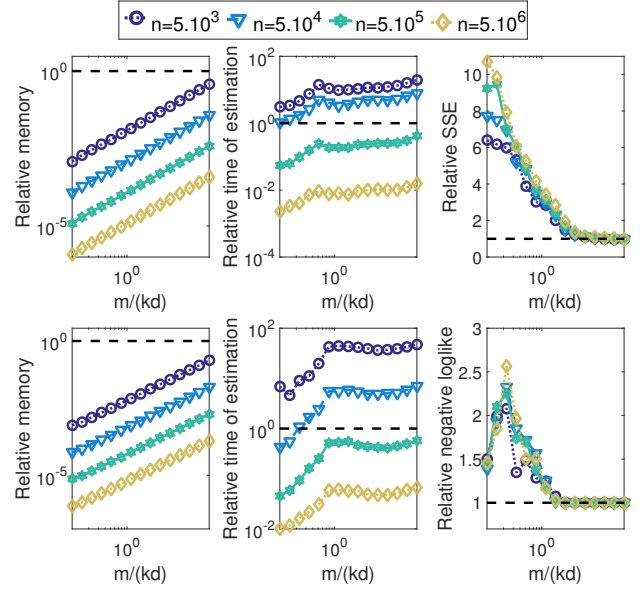


Fig. 1. Relative memory consumption (left), time of estimation (center) and precision (right) for compressive k -means (top) and GMM estimation (bottom) with $k = 10$ components in dimension $d = 10$, compared to Matlab's `kmeans` and VLFeat's `gmm` functions (dotted black lines).

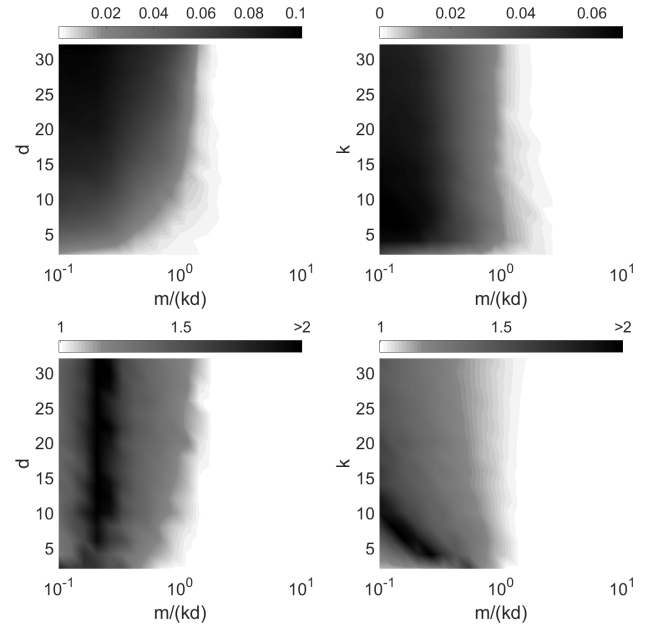


Fig. 2. Relative precision for k -means (top) and GMM estimation (bottom) with respect to the relative number of measurements $m/(kd)$. On the left $k = 10$ and on the right $d = 10$.