Trajectory Estimation for Exponential Parameterization and Different Samplings

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Abstract. This paper discusses the issue of fitting reduced data $Q_m = \{q_i\}_{i=0}^m$ with piecewise-quadratics to estimate an unknown curve γ in Euclidean space. The interpolation knots $\{t_i\}_{i=0}^m$ with $\gamma(t_i) = q_i$ are assumed to be unknown. Such *non-parametric interpolation* commonly appears in computer graphics and vision, engineering and physics [1]. We analyze a special scheme aimed to supply the missing knots $\{\hat{t}_i^\lambda\}_{i=0}^m \approx \{t_i\}_{i=0}^m$ (with $\lambda \in [0, 1]$) - the so-called *exponential parameterization* used in computer graphics for curve modeling. *A blind uniform guess*, for $\lambda = 0$ coupled with more-or-less uniform samplings yields a linear convergence order in trajectory estimation. In addition, for ε-uniform samplings ($\varepsilon \geq 0$) and $\lambda = 0$ an extra acceleration $\alpha_\varepsilon(0) = \min\{3, 1+2\varepsilon\}$ follows [2]. On the other hand, for $\lambda = 1$ *cumulative chords* render a cubic convergence order $\alpha(1) = 3$ within a general class of admissible samplings [3]. A recent theoretical result [4] is that for $\lambda \in [0, 1)$ and more-or-less uniform samplings, sharp orders $\alpha(\lambda) = 1$ eventuate. Thus no acceleration in $\alpha(\lambda) < \alpha(1) = 3$ prevails while $\lambda \in [0, 1)$. Finally, another recent result [5] proves that for all $\lambda \in [0, 1)$ and ε-uniform samplings, the respective accelerated orders $\alpha_\varepsilon(\lambda) = \min\{3, 1 + 2\varepsilon\}$ are independent of λ. The latter extends the case of $\alpha_\varepsilon(\lambda = 0) = 1 + 2\varepsilon$ to all $\lambda \in [0, 1)$. We revisit here [4] and [5] and verify their sharpness experimentally.

Keywords: Interpolation, numerical analysis, computer graphics and vision

1 Introduction

The sampled data points $Q_m = \{q_i\}_{i=0}^m$ with $\gamma(t_i) = q_i \in \mathbb{R}^n$ define the pair $(\{t_i\}_{i=0}^m, Q_m)$ commonly coined as *non-reduced data*. We also require here that $t_i < t_{i+1}$ and $q_i \neq q_{i+1}$ hold. Moreover, assume that $\gamma : [0, T] \to \mathbb{R}^n$ (with $0 < T < \infty$) is sufficiently smooth (specified later) and that it defines a regular curve $\dot{\gamma}(t) \neq \mathbf{0}$. In order to estimate the unknown curve γ with an arbitrary interpolant $\bar{\gamma} : [0, T] \to \mathbb{R}^n$ it is necessary to assume that $\{t_i\}_{i=0}^m \in V_G^m$, i.e. that the following *admissibility condition* is satisfied:

$$\lim_{m\to\infty} \delta_m = 0, \quad \text{where} \quad \delta_m = \max_{0\leq i\leq m-1}(t_{i+1} - t_i). \tag{1}$$

We omit here the subscript m in δ_m by setting $\delta = \delta_m$. In this paper, two substantial subfamilies of V_G^m are discussed.

The *first one* $V_{mol}^m \subset V_G^m$ includes *more-or-less uniform samplings* [6], [7]:

$$\beta\delta \leq t_{i+1} - t_i \leq \delta, \tag{2}$$

for some $\beta \in (0, 1]$. The left inequality in (2) excludes samplings with distance between consecutive knots smaller then $\beta\delta$. The right inequality follows from (1). Condition (2), as shown in [6], can be replaced by the equivalent condition (3) holding for each $i = 0, 1, \dots m-1$ and some constants $0 < K_1 \leq K_2$:

$$\frac{K_1}{m} \leq t_{i+1} - t_i \leq \frac{K_2}{m}. \tag{3}$$

The *second subfamily* $V_\varepsilon^m \subset V_G^m$ is that of ε*-uniform samplings* [2]:

$$t_i = \phi(\frac{iT}{m}) + O(\frac{1}{m^{1+\varepsilon}}), \tag{4}$$

where $\varepsilon \geqslant 0$, $\phi : [0, T] \to [0, T]$ is smooth and $\dot{\phi} > 0$ (so that $t_i < t_{i+1}$). Clearly, the smaller ε gets, the bigger distortion of uniform distribution occurs (modulo ϕ). The case when $\varepsilon = 0$ needs a special attention so that the inequality $t_i < t_{i+1}$ holds. However, the latter is asymptotically guaranteed for all ε positive. Note that each ε-uniform sampling with $\varepsilon > 0$ is also more-or-less uniform [6].

2 Problem Formulation and Motivation

We say that the family $F_\delta : [0, T] \to \mathbb{R}^n$ satisfies $F_\delta = O(\delta^\alpha)$ if $\|F_\delta\| = O(\delta^\alpha)$, where $\|\cdot\|$ denotes the Euclidean norm. Another words there are constants $K > 0$ and $\delta_0 > 0$ such that $\|F_\delta\| \leq K\delta^\alpha$, for all $\delta \in (0, \delta_0)$ and $t \in [0, T]$ - [8].

A standard result for *non-reduced data* $(\{t_i\}_{i=0}^m, Q_m)$ for piecewise r-degree polynomial $\bar{\gamma} = \tilde{\gamma}_r$ reads [6], [9]:

Theorem 1. *Let $\gamma \in C^{r+1}$ be a regular curve $\gamma : [0, T] \to \mathbb{R}^n$ with knot parameters $\{t_i\}_{i=0}^m \in V_G^m$ given. Then a piecewise r-degree Lagrange polynomial interpolation $\tilde{\gamma}_r$ used with $\{t_i\}_{i=0}^m$ known, yields a sharp estimate:*

$$\tilde{\gamma}_r = \gamma + O(\delta^{r+1}). \tag{5}$$

By (5) piecewise-quadratics (-cubics) $\tilde{\gamma}_2$ ($\tilde{\gamma}_3$) yield *cubic* (*quartic*) order error terms.

In many applications in computer graphics and computer vision, engineering or physics, the so-called *reduced data* Q_m are encountered (see e.g. [1], [10], [11], or [12]), where the knots $\{t_i\}_{i=0}^m$ are unknown and have to be first guessed somehow. A family of the so-called *exponential parameterization* $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$ is commonly used for curve modeling [11], [13]:

$$\hat{t}_0 = 0, \qquad \hat{t}_{i+1} = \hat{t}_i + \|q_{i+1} - q_i\|^\lambda, \tag{6}$$

where $0 \leq \lambda \leq 1$ and $i = 0, 1, \ldots, m-1$. The cases when $\lambda \in \{1, 0.5, 0\}$, yield *cumulative chords*, *centripetal* or *blind uniform parameterizations*, respectively.

We call a piecewise r-degree polynomial based on (6) and Q_m as $\hat{\gamma} = \hat{\gamma}_r : [0, \hat{T}] \to \mathbb{R}^n$, where $\hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|^\lambda$. Note that in case of any reduced data Q_m for asymptotics estimation of γ by $\hat{\gamma}_r$, *a re-parameterization* $\psi : [0, T] \to [0, \hat{T}]$ synchronizing both domains of γ and $\hat{\gamma}_r$, needs to be defined (see e.g. [6]).

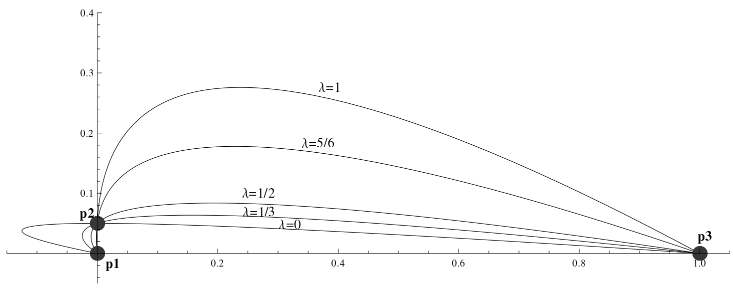

Fig. 1. Interpolating three points $Q_2 = \{(0, 0), (0, 0.05), (1, 0)\}$ with $\hat{\gamma}_2$, for $\lambda = 0, 1/3, 1/2, 5/6, 1 \in [0, 1]$

Example 1. Figure 1 shows different $\hat{\gamma}_2$ passing through Q_2 with various $\lambda \in \{0, 1/3, 1/2, 5/6, 1\}$ set in (6). Such curves' fluctuation given different knots and interpolation schemes is commonly exploited for sparse data in 2D and 3D computer graphics in the context of curve modeling - see [10], [11], or [12]. □

Example 2. Another application, elucidating the influence of knots selection on interpolation stems from the computer vision field. Figure 2 shows the image of the same knee joint section. The goal is to isolate from such image the kneecap and to find its area, amounting here to $A = 5237$ pixels. The interpolation points Q_m positioned on the boundary are selected e.g. by the physician (here $m = 5$). Of course, the internal parametrization of the kneecap boundary (i.e. some curve γ) remains unknown. Upon invoking $\hat{\gamma}_2$ (with three quadratic segments) coupled with guessed knots in accordance with (6) we obtain different estimates of γ by $\hat{\gamma}_2$ and consequently various kneecap area A_λ approximations. Namely for $\lambda \in \{0, 1/4, 1/2, 3/4, 1\}$ the following $A \approx A_\lambda \in \{5197, 5209, 5234, 5293, 5376\}$ (in pixels) hold, respectively. The centripetal parameterization (i.e. for $\lambda = 1/2$) on this specific sparse data Q_m yields the best result. □

More *real data examples* emphasizing the importance of the knots' selection for a given interpolation scheme in *computer graphics* (light-source motion estimation or image rendering), *computer vision* (image segmentation or video compression), *geometry* (trajectory, curvature or area estimation) or in *engineering and physics* (fast particles' motion estimation) can be found e.g. in [1].

a) b) c)

Fig. 2. Isolating the kneecap with $\hat{\gamma}_2$, for a) $\lambda = 0$, b) $\lambda = 0.5$, c) $\lambda = 1$

2.1 Uniform Parameterization - $\lambda = 0$

The case when $\lambda = 0$, transforms (6) into to blind *uniform* knots' guesses $\hat{t}_i = i$. For $r = 2$ and $\lambda = 0$ in (6) the following holds [2]:

Theorem 2. *Let the unknown* $\{t_i\}_{i=0}^m$ *be sampled* ε*-uniformly, where* $\varepsilon > 0$ *and* $\gamma \in C^4$*. Then there is a uniform piecewise-quadratic Lagrange interpolant* $\hat{\gamma}_2 : [0, \hat{T} = m] \to \mathbb{R}^n$*, calculable in terms of* Q_m *(with* $\hat{t}_i = i$*) and piecewise* C^∞ *re-parameterization* $\psi : [0, T] \to [0, \hat{T}]$ *such that sharp estimates hold:*

$$\hat{\gamma}_2 \circ \psi = \gamma + O(\delta^{\min\{3,1+2\varepsilon\}}). \tag{7}$$

Th. 2 with $\alpha_{\varepsilon>0}(0) = \min\{3, 1 + 2\varepsilon\}$ extends to $\varepsilon = 0$ provided $\{t_i\}_{i=0}^m$ satisfies $t_i < t_{i+1}$ and falls also into more-or-less uniformity (2). The latter renders linear convergence order $\alpha_{\varepsilon=0}(0) = 1$ - see [6]. Evidently, for ε-uniform samplings there is an acceleration from $\alpha_{\varepsilon=0}(0) = 1$ via $\alpha_{0<\varepsilon<1}(0) = 1 + 2\varepsilon$ to $\alpha_{\varepsilon\geq 1}(0) = 3$.

2.2 Cumulative Chords - $\lambda = 1$

The opposite case when $\lambda = 1$ in (6) renders *cumulative chords* [11], [12]. This choice of $\{\hat{t}_i\}_{i=0}^m$ uses the geometry of Q_m and gives better trajectory estimation (at least for $r = 2, 3$) as opposed to $\lambda = 0$ and (7) [3]:

Theorem 3. *Let* γ *be a regular* C^k *curve in* $\mathbb{R}^n$*, where* $k \geqslant r + 1$ *and* $r = 2, 3$ *sampled according to* (1)*. Let* $\hat{\gamma}_r : [0, \hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|] \to \mathbb{R}^n$ *be the cumulative chord piecewise-quadratic(-cubic) interpolant defined by* Q_m *and* $\lambda = 1$ *in* (6)*. Then there is a piecewise-*C^r *re-parameterization* $\psi : [0, T] \to [0, \hat{T}]$*, with*

$$\hat{\gamma}_r \circ \psi = \gamma + O(\delta^{r+1}). \tag{8}$$

The asymptotics from Th. 2 and Th. 3 are sharp - see [3] and [6]. For $r = 2$ and $\lambda = 1$, formula (8) yields the cubic order $\alpha(1) = 3$ which not only improves (7) but also matches the non-reduced data case (5) (with $r = 2, 3$).

2.3 Exponential Parameterization - $\lambda \in [0,1]$

Recent research by [4] extends the results from Th. 2 (where $\lambda = 0$) and Th. 3 (where $\lambda = 1$ with $r = 2$) to the remaining cases of exponential parameterization (6) i.e. to $\lambda \in [0,1]$. As proved in [4], for more-or-less uniform samplings (2), (6) and $r = 2$ any choice of $\lambda \in [0,1)$ does not improve the asymptotics for γ approximation. In fact, for all $\lambda \in [0,1)$ we have $\alpha(\lambda) = 1$. Indeed we obtain [4]:

Theorem 4. *Suppose γ is a regular C^3 curve in $\mathbb{R}^n$ sampled more-or-less uniformly* (2)*. Let $\hat{\gamma}_2 : [0, \hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|^\lambda] \to \mathbb{R}^n$ be the piecewise-quadratic interpolant defined by Q_m and* (6) *(with $\lambda \in [0,1]$). Then there is a piecewise-C^∞ re-parameterization $\psi : [0,T] \to [0,\hat{T}]$, such that for $\lambda \in [0,1)$ we have:*

$$\hat{\gamma}_2 \circ \psi = \gamma + O(\delta). \tag{9}$$

In addition, for either $\{t_i\}_{i=0}^m$ uniform or $\lambda = 1$ used with samplings (1) *the following holds:*

$$\hat{\gamma}_2 \circ \psi = \gamma + O(\delta^3). \tag{10}$$

Both (9) and (10) are sharp (proved analytically). Th. 4 underlines discontinuity of $\alpha(\lambda)$ at $\lambda = 1$ with the jump by 2 in respective convergence orders. Another unexpected fact comes from the proof of Th. 4. Namely, a natural candidate for a re-parameterization, i.e. a Lagrange quadratic $\psi_i : [t_i, t_{i+2}] \to [\hat{t}_i, \hat{t}_{i+2}]$ satisfying $\psi_i(t_{i+j}) = \hat{t}_{i+j}$ (for $j = 0,1,2$) can be a non-injective function [4].

The most recent result [5] shows that for ε-uniform samplings (4) (with $\varepsilon > 0$) the asymptotics established in Th. 4 improves from $\alpha(\lambda) = 1$ to $\alpha_{\varepsilon>0}(\lambda) = \min\{3, 1+2\varepsilon\}$, for each $\lambda \in [0,1)$. Indeed the following holds [5]:

Theorem 5. *Suppose γ is a regular C^4 curve in $\mathbb{R}^n$ sampled according to the ε-uniformity condition* (4) *with $\varepsilon > 0$. Let $\hat{\gamma}_2 : [0, \hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|^\lambda] \to \mathbb{R}^n$ be the piecewise-quadratic interpolant defined by Q_m and* (6) *(with $\lambda \in [0,1)$). Then there is a piecewise-C^∞ re-parameterization $\psi : [0,T] \to [0,\hat{T}]$, such that:*

$$\hat{\gamma}_2 \circ \psi = \gamma + O(\delta^{\min\{3, 1+2\varepsilon\}}). \tag{11}$$

By Th. 4*, formula* (11) *extends to $\varepsilon = 0$ (with $\lambda \in [0,1)$) if extra condition* (2) *on* 0*-uniform sampling is imposed. The case $\lambda = 1$ by Th.* 4 *yields $\hat{\gamma}_2 \circ \psi = \gamma + O(\delta^3)$.*

Again (11) is proved analytically to be sharp. Clearly, by Th. 5 an extra acceleration (11) in convergence rates for all $\lambda \in [0,1]$ and (4) with $\varepsilon > 0$ occurs. The latter coincides with Th. 2 holding for $\lambda = 0$. The formula (11) is only dependent on ε, not on λ. It should be pointed out that by [5], for each $\varepsilon > 0$ the quadratic ψ_i defines a genuine re-parameterization of $[t_i, t_{i+2}]$ into $[\hat{t}_i, \hat{t}_{i+2}]$.

2.4 Aim of this Research

In this paper we verify experimentally *the sharpness* of asymptotics for trajectory estimation claimed by Th. 4 and Th. 5. By *sharpness* we understand the existence of at least one curve $\gamma \in C^r([0,T])$ (with r set accordingly) sampled with some admissible samplings $\{t_i\}_{i=0}^m \in V_G^m$ for which the asymptotic estimates in question are exactly matched. The tests conducted herein are confined merely to the planar and spatial curves. It should, however be emphasized that all quoted herein Theorems 1-5 admit regular curves in $\mathbb{R}^n$. Some motivation standing behind the applications of interpolating reduced data is also here presented. More examples of real reduced n-dimensional data Q_m which can be fitted with piecewise-quadratics $\hat{\gamma}_2$ or any other interpolation schemes based on exponential parameterization (6) can be found e.g. in [1].

3 Experiments

All tests presented in this paper are performed in *Mathematica 9.0* [14] on a 2.4GHZ Intel Core 2 Duo computer with 8GB RAM. Note that since $T = \sum_{i=1}^{m}(t_{i+1} - t_i) \leq m\delta$ the following holds $m^{-\alpha} = O(\delta^\alpha)$, for $\alpha > 0$. Hence, the verification of any asymptotics expressed in terms of $O(\delta^\alpha)$ can be performed by examining the claims of Th. 4 or Th. 5 in terms of $O(1/m^\alpha)$ asymptotics.

Note that for a parametric regular curve $\gamma : [0,T] \to \mathbb{R}^n$, $\lambda \in [0,1]$ and m varying between $m_{min} \leq m \leq m_{max}$ the i-th component of the error for γ estimation is defined here as follows:

$$E_m^i = \sup_{t\in[t_i,t_{i+2}]} \|(\hat{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t)\| = max_{t\in[t_i,t_{i+2}]}\|(\hat{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t)\|,$$

as $\tilde{E}_m^i(t) = \|(\hat{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t)\| \geq 0$ is continuous over each compact subinterval $[t_i, t_{i+2}] \subset [0,T]$. The maximal value E_m of $\tilde{E}_m(t)$ (the track-sum of $\tilde{E}_m^i(t)$), for each $m = 2k$ (here $k = 1, 2, 3, \ldots, m/2$) is found by using *Mathematica* optimization built-in functions: *Maximize* or *FindMinimum* (the latter applied to $-\tilde{E}_m(t)$). From the set of *absolute errors* $\{E_m\}_{m=m_{min}}^{m_{max}}$ the numerical estimate $\bar{\alpha}(\lambda)$ of genuine order $\alpha(\lambda)$ is subsequently computed by using *a linear regression* applied to the pair of points $(\log(m), -\log(E_m))$ (see also [6]). Since piecewisely $deg(\hat{\gamma}_2) = 2$ the number of interpolation points $\{q_i\}_{i=0}^m$ is odd i.e. $m = 2k$ as indexing runs over $0 \leq i \leq m$. The *Mathematica* built-in functions *LinearModelFit* renders the coefficient $\bar{\alpha}(\lambda)$ from the computed regression line $y(x) = \bar{\alpha}(\lambda)x + b$ based on pairs of points $\{(\log(m), -\log(E_m))\}_{m=m_{min}}^{m_{max}}$. Finally, recall that as justified in Th. 5 any ε-uniform sampling with $\varepsilon > 0$ gives asymptotically ψ_i as re-parameterization of $[t_i, t_{i+2}]$ into $[\hat{t}_i, \hat{t}_{i+2}]$. Once $\varepsilon = 0$, one may apply a simple computational test (for $m = m_{max}$) by verifying whether either $\psi_i^{(1)}(t_i) \geq 0$ and $\psi_i^{(1)}(t_{i+2}) > 0$ or $\psi_i^{(1)}(t_i) > 0$ and $\psi_i^{(1)}(t_{i+2}) \geq 0$ hold over each subinterval $[t_i, t_{i+2}]$. The latter combined with the linearity of $\psi_i^{(1)}$ guarantees that ψ_i is a re-parameterization. More discussion on the issue of enforcing ψ_i to be a re-parameterization can be found in [5].

3.1 Fitting Reduced Data for Planar Curves

The testing commences with the simplest possible curve, i.e. a straight line.

Example 3. Consider *a regular straight line*: $\gamma_l(t) = \left(t/\sqrt{5}, 2t/\sqrt{5}\right) \subset \mathbb{R}^2$ for $t \in [0,1]$, sampled with $\delta_1 = i/m$ according to ε-uniform sampling (4) (with $\varepsilon > 0$):

$$t_{i+1} - t_i = \delta_1(1+\delta_1^\varepsilon) \quad \text{and} \quad t_{i+2} - t_i = \delta_1(1-\delta_1^\varepsilon), \tag{12}$$

where $t_0 = 0$ and $t_m = 1$. The plot of γ_l sampled by (12), with $\varepsilon = 0.5$ and $m = 12$ is shown in Figure 3. Recalling (1), note that here $\delta = \delta_1(1+\delta_1^\varepsilon)$. As

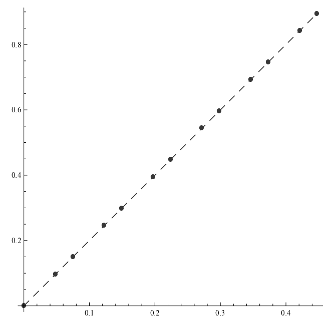

Fig. 3. The plot of the line γ_l sampled as in (12), for $m = 12$ and $\varepsilon = 0.5$

$\varepsilon > 0$, the quadratic ψ_i is a re-parameterization [5]. The linear regression applied to $m_{min} = 100 \leq m \leq m_{max} = 120$ yields the estimates for $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ which are presented in Table 1. An inspection of Table 1 confirms *the sharpness or nearly sharpness* of Th. 5. □

λ	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1$
$\alpha_\varepsilon(\lambda)$	1.20	1.66	2.00	2.40	2.80	3.00
0.00	1.47	1.80	2.10	2.46	2.85	3.04
0.10	1.45	1.80	2.10	2.46	2.85	3.04
0.33	1.42	1.80	2.10	2.46	2.85	3.04
0.50	1.39	1.80	2.10	2.46	2.85	3.04
0.70	1.37	1.79	2.10	2.47	2.85	3.04
0.90	1.36	1.79	2.10	2.47	2.85	3.04

Table 1. Computed $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = 1+2\varepsilon$ for γ_l and sampling (12) interpolated by $\hat{\gamma}_2$ with some discrete values $\lambda \in [0,1)$ and $\varepsilon \in (0,1]$

We pass now to the next example involving a spiral curve in $\mathbb{R}^2$.

Example 4. Consider now the following regular *spiral curve* $\gamma_{sp1} : [0,1] \to \mathbb{R}^2$: $\gamma_{sp1}(t) = ((t+0.2)\cos(\pi(1-t)), (t+0.2)\sin(\pi(1-t)))$, sampled according to the following ε-uniform sampling (4):

$$t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{m^{1+\varepsilon}}, \tag{13}$$

with $t_0 = 0$ and $t_m = 1$. Figure 4 shows γ_{sp1} (a dashed line) and $\hat{\gamma}_2$ (a continuous line) sampled by (13) with $\varepsilon = 0.7$, $m = 12$ and $\lambda \in \{0, 1\}$. The difference between both $\hat{\gamma}_2$ and γ_{sp1} on sparse data Q_{12} is minor as they both overlap.

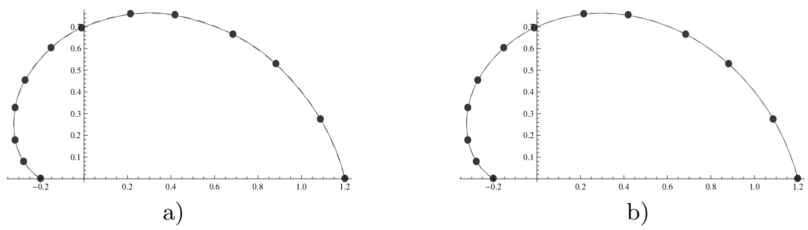

Fig. 4. The plot of the spiral γ_{sp1} sampled as in (13) (a dashed line) and interpolant $\hat{\gamma}_2$ (a continuous line), for $m = 12$ and $\varepsilon = 0.7$ with either a) $\lambda = 0$ or b) $\lambda = 1$

The Th. 5, for $\varepsilon > 0$ yields $\psi_i : [t_i, t_{i+2}] \to [\hat{t}_i, \hat{t}_{i+2}]$ as a re-parameterization. The case of $\varepsilon = 0$ renders (13) as more-or-less uniform (3) with $K_1 = 1/3$ and $K_2 = 5/3$. Sufficient conditions for ψ_i to be a re-parameterization are formulated in [5]. The latter enables to verify the validity of Th. 4 also for $\varepsilon = 0$. The linear regression applied to $m = 100 \leq m \leq m_{max} = 120$ renders computed $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1 + 2\varepsilon\}$ ($\varepsilon \geq 0$), which are listed in Table 2. Visibly, *the sharpness* or *nearly sharpness* of Th. 4 and Th. 5 is confirmed in Table 2. □

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$
$\alpha_\varepsilon(\lambda)$	1.000	1.200	1.660	2.000	2.400	2.800	3.000
0.00	0.981	1.286	1.716	2.023	2.419	2.96	3.004
0.10	0.983	1.282	1.718	2.029	2.435	2.97	3.005
0.33	0.985	1.277	1.726	2.051	2.496	2.93	3.016
0.50	0.988	1.276	1.740	2.081	2.584	3.01	3.017
0.70	0.995	1.283	1.778	2.178	2.782	2.94	3.030
0.90	1.036	1.354	2.051	2.271	3.005	2.89	3.031
$\alpha_\varepsilon(1)$	3.000	3.000	3.000	3.000	3.000	3.000	3.000
1.00	3.057	2.990	2.996	3.007	3.016	2.88	3.031

Table 2. Estimated $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = 1 + 2\varepsilon$ for γ_{sp1} and sampling (13) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0, 1]$ and $\varepsilon \in [0, 1]$

The last example involving curve in $\mathbb{R}^2$ refers to another spiral.

Example 5. Let *a planar regular convex spiral* $\gamma_{sp} : [0, 5\pi] \to \mathbb{R}^2$: $\gamma_{sp}(t) = ((6\pi - t)\cos(t), (6\pi - t)\sin(t))$ be sampled according to (13) (rescaled by factor

5π) with $t_0 = 0$ and $t_m = 5\pi$. Figure 5 illustrates γ_{sp} (a dashed line) and $\hat{\gamma}_2$ (a continuous line) coupled with (13), for $\varepsilon = 0.33$, $m = 22$ and $\lambda \in \{0, 1\}$. The difference between γ_{sp} and $\hat{\gamma}_2$ on reduced data Q_{22} is transparent (at least for $\lambda = 0$). As explained previously, the sampling (13) enforces ψ_i to be a re-

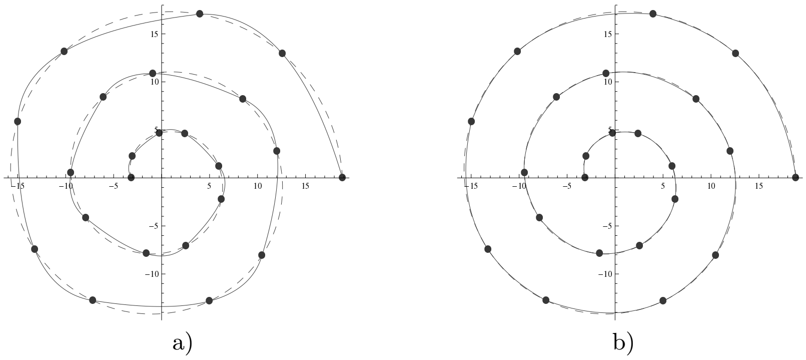

Fig. 5. The plot of the spiral γ_{sp} sampled as in (13) (a dashed line) and interpolant $\hat{\gamma}_2$ (a continuous line), for $m = 22$ and $\varepsilon = 0.33$ with either a) $\lambda = 0$ or b) $\lambda = 1$

parameterization for $\varepsilon > 0$. For $\varepsilon = 0$ one needs to resort to the sufficient conditions for $\psi_i^{(1)} > 0$ to hold (see [5]). In order to estimate the relevant coefficients $\alpha_\varepsilon(\lambda)$ a linear regression is again applied to $100 = m_{min} \leq m \leq m_{max} = 120$. The numerical results are listed in Table 3. Some computed $\alpha_\varepsilon(\lambda)$ from Table 3 exceed convergence orders claimed by Th. 5. However, the first column in Table 3 shows *the sharpness* of Th. 4. $\square$

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$
$\alpha_\varepsilon(\lambda)$	1.000	1.200	1.660	2.000	2.400	2.800	3.000
0.00	0.990	1.303	1.799	2.223	2.851	2.986	3.008
0.10	0.990	1.299	1.812	2.342	2.911	2.986	3.007
0.33	0.991	1.296	1.872	2.252	2.966	3.011	3.024
0.50	0,995	1.303	1.989	2.711	2.995	3.020	3.022
0.70	1.013	1.355	2.377	2.930	3.020	3.024	3.023
0.90	1.291	2.092	2.986	2.043	3.033	3.033	3.024
$\alpha_\varepsilon(1)$	3.000	3.000	3.000	3.000	3.000	3.000	3.000
1.00	3.000	2.866	2.901	2.938	2.976	2.995	3.000

Table 3. Estimated $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = 1 + 2\varepsilon$ for γ_{sp} and sampling (13) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0, 1]$ and $\varepsilon \in [0, 1]$

3.2 Fitting Reduced Data for Spatial Curves

The next example deals with the reduced data Q_m obtained by sampling the regular spatial curve in $\mathbb{R}^3$.

Example 6. We verify now the sharpness of Th. 4 and Th. 5 for *a quadratic elliptical helix*: $\gamma_h(t) = (2\cos(t), \sin(t), t^2)$, with $t \in [0, 2\pi]$ and sampled ε-uniformly (4) (here $\phi = id$) according to:

$$t_i = \begin{cases} \frac{2\pi i}{m} & \text{if } i \text{ even,} \\ \frac{2\pi i}{m} + \frac{2\pi}{2m^{1+\varepsilon}} & \text{if } i = 4k+1, \\ \frac{2\pi i}{m} - \frac{2\pi}{2m^{1+\varepsilon}} & \text{if } i = 4k+3. \end{cases} \tag{14}$$

The last knot t_m is set to 2π. Figure 6 illustrates the curve γ_h sampled in accordance with (14) for $\varepsilon = 0.5$ and $m = 22$. For $\varepsilon > 0$, by Th. 5 each quadratic

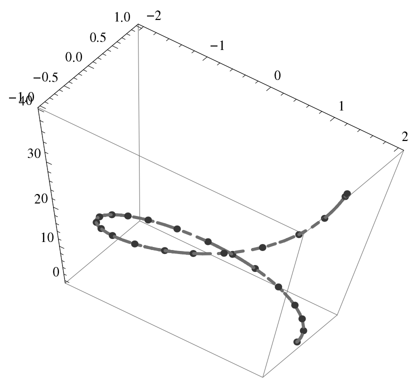

Fig. 6. The plot of the helix γ_h sampled as in (14), for $m = 22$ and $\varepsilon = 0.5$

$\psi_i : [t_i, t_{i+2}] \to [\hat{t}_i, \hat{t}_{i+2}]$ is a re-parameterization. Note that $\varepsilon = 0$ in (14) yields also more-or-less uniform sampling (3) with $K_1 = \pi$ and $K_2 = 3\pi$. The latter is stipulated by Th. 4. The sufficient conditions for $\{t_i\}_{i=0}^m$ to yield ψ_i as re-parameterization are specified in [5]. The linear regression is used here for $m_{min} = 100 \leq m \leq m_{max} = 120$. The corresponding computed estimates $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1 + 2\varepsilon\}$ are shown in Table 4. The experiments are consistent with the asymptotics from Th. 5. Thus the sharpness of (11) is also generically herein confirmed. Note also that the estimated convergence orders $\bar{\alpha}_{\varepsilon=0}(\lambda)$ are substantially faster than those claimed by Th. 4. □

The linear regression is used to estimate the asymptotic convergence rates $\alpha_\varepsilon(\lambda)$ for sufficiently large m. The estimates may sometimes be misleading when m is not sufficiently large.

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$
$\alpha_\varepsilon(\lambda)$	1.00	1.20	1.66	2.00	2.40	2.80	3.00
0.10	2.99	1.26	1.74	2.09	2.54	2.97	3.01
0.33	2.85	1.24	1.72	2.07	2.93	2.93	2.95
0.50	3.24	1.23	1.70	2.06	3.01	3.01	3.04
0.70	3.21	1.20	1.64	2.94	2.94	2.94	3.19
0.90	3.21	1.15	2.89	2.89	2.89	2.89	3.22
$\alpha_\varepsilon(1)$	3.00	3.00	3.00	3.00	3.00	3.00	3.00
1.00	3.21	2.89	2.91	2.92	2.93	2.88	3.21

Table 4. Estimated $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = 1 + 2\varepsilon$ for γ_h and sampling (14) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in [0,1]$

4 Conclusions

In this paper we discussed the problem of trajectory estimation via piecewise-quadratic interpolation based on reduced data Q_m. In particular, the exponential parameterization (6) which depends on parameter $\lambda \in [0,1]$ is herein examined. The latter is commonly used in computer graphics for curve modeling - see e.g. [9], [10], [11] or [12]. Special cases of (6) with $\lambda = 0$ (see e.g. [2]) or $\lambda = 1$ (see e.g. [3] or [6]) have been studied in the literature. Recent results from [4] and [5] with full mathematical proofs analyze the asymptotics in question for the remaining cases of $\lambda \in (0,1)$.

Th. 4 claims that the if $\{t_i\}_{i=0}^m$ is more-or-less uniform (2) there is no acceleration in trajectory estimation once λ varies within the interval $[0,1)$. The convergence orders are constant, i.e. $\alpha(\lambda) = 1$ for all $\lambda \in [0,1)$. In addition, there is a discontinuity at $\lambda = 1$ with a jump to $\alpha(1) = 3$ (valid for the general class of admissible samplings (1)).

It is known [2] that further acceleration can be achieved for ε-uniform samplings (4) (for $\varepsilon > 0$) and $\lambda = 0$ by reaching $\alpha_\varepsilon(0) = \min\{3, 1+2\varepsilon\}$. The most recent result by [5] extends the latter to arbitrary $\lambda \in [0,1)$ and $\varepsilon \geq 0$. The case $\varepsilon = 0$ is also admitted provided the curve γ is sampled according to (2). As established in Th. 5 the acceleration amounting to $\alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ is merely dependent on ε (not on $\lambda \in [0,1]$). Visibly the discontinuity in $\alpha_\varepsilon(\lambda)$ at $\lambda = 1$ is removed for $\varepsilon \geq 1$. It should also be emphasized that the proof of Th. 5 shows that the Lagrange quadratic $\psi_i : [t_i, t_{i+2}] \to [\hat{t}_i, \hat{t}_{i+2}]$ satisfying $\psi_i(t_{i+j}) = \hat{t}_{i+j}$ (for $j = 0,1$) forms a genuine re-parameterization in case of ε-uniform samplings (with $\varepsilon > 0$). The above theorem also formulates sufficient conditions for admissible samplings (1) (including the case $\varepsilon = 0$) guaranteeing ψ_i to render a re-parameterization.

We experimentally verify here *the sharpness* of the asymptotics established in [4] and [5]. Various numerical tests conducted in this paper confirm, at least for the examined curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ and samplings (12), (13) or (14) the sharpness of asymptotics claimed by both Th. 4 (see (9) and (10)) and Th. 5 (see (11)). Though all discussed herein results refer to the dense reduced data Q_m, high convergence orders yield in practice satisfactory approximation on sparse data. Thus as $\alpha(\lambda) = 3$ for either $\lambda = 1$ or $\varepsilon \geq 1$ and $\lambda \in [0, 1]$, we should expect for sufficiently sparse Q_m (but not too dense) a good performance in $\gamma \approx \hat{\gamma}_2$.

A possible extension of this work is to study other smooth interpolation schemes [9] combined with reduced data Q_m and exponential parameterization (6) - see [11]. Certain clues may be given in [15], where complete C^2 splines are dealt with for $\lambda = 1$, to obtain the fourth orders of convergence in length estimation. The analysis of C^1 interpolation for reduced data with cumulative chords (i.e. again with $\lambda = 1$) can additionally be found in [6] or [16]. More discussion on *applications* (including *real data* examples - see [1]) and *theory* of *non-parametric interpolation* can be found e.g. in [6], [10], [11] or [12].

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