Non-local Conservation Law from Stochastic Particle Systems

Marielle Simon, Christian Olivera

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Abstract

We consider an interacting particle system in $\mathbb{R}^d$ modelled as a system of N stochastic differential equations. The limiting behaviour as the size N grows to infinity is achieved as a law of large numbers for the empirical density process associated with the interacting particle system.

Key words and phrases: *Stochastic differential equations; Fractal conservation law, Lévy process; Particle systems; Semi-group approach.*

MSC2010 subject classification: 60H20, 60H10, 60F99.

1 Introduction

1.1 Context

There is a vast and growing interest in modeling systems of large (though still finite) population of individuals subject to mutual interaction and random dispersal (due to, for instance, the environment). We refer the reader to [6] for a recent textbook on the subject. More precisely, the behavior of such systems is often described as the limit of the number of individuals tends to infinity. While at the *microscopic* scale, the population is well modeled by stochastic differential equations (SDEs), the *macroscopic* description of the population densities is provided by partial differential equations (PDEs), which can be of different types, for instance linear PDEs for Black-Scholes models, or non-linear PDEs for density-dependent diffusions. All these systems may characterize the collective behavior of individuals in biology models, but also agents in economics and finance. The range of application of this area is huge.

In the present paper the limit processes that we want to obtain belong to the family of non-local PDEs, which in our case are related to anomalous diffusions. For that purpose, we study the asymptotic behaviour of a system of particles which interact *moderately*, i.e. an

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intermediate situation between weak and strong interaction, and which are submitted to random scattering. It is well-known that in the case where the particles interact with only few others (which is often the most realistic case), the results can be qualitatively different, but also mathematically more challenging. Nowadays, there are few rigorous results at disposal, starting from the seminal paper [26].

The Lagrangian description of our dynamics of the moderately-interacting particles is given via a system of stochastic differential equations. Suppose that for each $i \in \mathbb{N}$, the process $X_t^{i,N}$ satisfies the system of coupled stochastic differential equations in $\mathbb{R}^d$

$$dX_t^{i,N} = F\bigg(X_t^{i,N}, \frac{1}{N}\sum_{k=1}^{N} V^N\big(X_t^{i,N} - X_t^{k,N}\big)\bigg)\, dt + dL_t^i, \tag{1}$$

where $\{L_t^i\}_{i\in\mathbb{N}}$ are independent $\mathbb{R}^d$-valued symmetric α-stable Lévy processes on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and the functions $V^N : \mathbb{R}^d \to \mathbb{R}_+$ and $F : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ are continuous and will be specified in the next paragraph.

Thus we are assuming that the system of N particles is subject to random dispersal, modelled as α-stable Lévy processes. In this model randomness may be due to external sources, for instance unpredictable irregularities of the environment (like obstacles, changeable soils, varying visibility). The moderate interaction is represented by the form of V^N which writes as

$$V^N(x) = N^\beta V(N^{\frac{\beta}{d}} x),$$

for some function $V : \mathbb{R}^d \to \mathbb{R}_+$ and $\beta \in (0, 1)$. The case $\beta = 0$ would correspond to the long range interaction, and the case $\beta = 1$ to the nearest neighbor interaction. Further hypotheses on V and β are given in Assumption 1 below.

We are interested in the bulk behaviour of the whole population of the particles, and therefore a natural object for mathematical investigation is the probability measure-valued empirical process, defined as follows: let

$$S_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$$

be the empirical process associated to $\{X_t^{i,N}\}_{i=1,\dots,N}$, where δ_a is the delta Dirac measure concentrated at a. The drift term which appears in (1) describes the interaction of the i-th particle located at $X_t^{i,N}$ with the random field S_t^N generated by the whole system of particles at time t.The usual procedure includes the following steps:

- show the convergence of the process S^N to a deterministic measure process S^∞;
- and then, identify S^∞ as the weak solution to some suitable PDE.

In our case the dynamics of the empirical measure is fully determined by Itô's formula: when we apply it to $\phi\big(X_t^{i,N}\big)$, for any test function $\phi \in C_0^\infty(\mathbb{R}^d)$ which is smooth and compactly supported, we obtain that the empirical measure S_t^N satisfies

$$\begin{aligned}\big\langle S_t^N, \phi \big\rangle &= \big\langle S_0^N, \phi \big\rangle + \int_0^t \Big\langle S_s^N, F\big(\cdot\,,\ (V^N * S_s^N)\big)\, \nabla\phi \Big\rangle\, ds \\ &\quad + \frac{1}{2}\int_0^t \big\langle S_s^N, \mathcal{L}\phi \big\rangle\, ds + \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d - \{0\}} \big(\phi(X_{s_-}^{i,N} + z) - \phi(X_{s_-}^{i,N})\big)\, d\mathcal{N}^i(dsdz),\end{aligned} \tag{2}$$

where $\mathcal{L}$ is the non-local operator corresponding to a symmetric α-stable Lévy process, defined as

$$\mathcal{L}\phi(x) = \int_{\mathbb{R}^d - \{0\}} \big(\phi(x+z) - \phi(x) - \mathbf{1}_{\{|z|\le 1\}} \nabla\phi(x)\cdot z\big)\, d\nu(z), \tag{3}$$

and $(\nu, \mathcal{N})$ are characteristics coming from the Lévy-Itô decomposition Theorem, which satisfy some conditions related to $\alpha \in (1,2)$, see Section 2.1 below for more details. As a example which satisfies the conditions of our main result, let us give the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$, for some $\alpha \in (1,2)$, which corresponds to (3) with

$$d\nu(z) = \frac{K_\alpha\, dz}{|z|^{1+\alpha}}.$$

Our interest lies in the investigation of the behaviour of the dynamics of the processes $t \mapsto S_t^N$ in the limit $N \to \infty$. This kind of problems was considered by Oelschläger [21], Jourdain and Méléard [16], Méléard and Roelly-Coppoletta [20], when the sytem of SDE is driven by standard Brownian motions (see also the introduction in [11]). Up to our knowledge, the case of Lévy processes driven dynamics has not been fully investigated in the literature. Let us now state the main result of this paper.

1.2 Assumptions and main result

In the following we denote by $\|\cdot\|_{\mathbb{L}^p}$ the usual $\mathbb{L}^p$-norm on $\mathbb{R}^d$, and by $\|\cdot\|_{\mathbb{L}^p\to\mathbb{L}^q}$ the usual operator norm. Moreover, for any $\varepsilon \in \mathbb{R}$, we denote by $\mathbb{H}^\varepsilon = \mathbb{H}^\varepsilon(\mathbb{R}^d)$ the usual Bessel space of all functions $u \in \mathbb{L}^2(\mathbb{R}^d)$ such that

$$\|u\|_{\mathbb{H}^\varepsilon}^2 := \Big\|\mathcal{F}^{-1}\big[(1+|\lambda|^2)^{\frac{\varepsilon}{2}}\mathcal{F}(u)(\lambda)\big]\Big\|_{\mathbb{L}^2}^2 < \infty$$

where $\mathcal{F}$ denotes the Fourier transform of u.

For every $\varepsilon \in \mathbb{R}$, the Sobolev spaces $\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)$ are well defined, see [25] for the material needed here. For positive ε the restriction of $f \in \mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)$ to a ball $\mathcal{B}(0,R) \subset \mathbb{R}^d$ is in $\mathbb{W}^{\varepsilon,2}(\mathcal{B}(0,R))$. The Sobolev spaces $\mathbb{H}^\varepsilon$ and $\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)$ have equivalent norms. Let us list below our assumptions which we use to derive the macroscopic limit:

Assumption 1 *Take* $\alpha \in (1,2)$*. We assume that there exists a continuous probability density* $V : \mathbb{R}^d \to \mathbb{R}_+$ *such that*

- V *is compactly supported and symmetric,*
- $V^N(x) = N^\beta V(N^{\frac{\beta}{d}} x)$ *for some* $\beta > 0$*,*
- $V \in \mathbb{H}^{\varepsilon+\delta}(\mathbb{R}^d)$*, for some* $\varepsilon, \delta > 0$*,*

There are further conditions on $(\beta, \varepsilon, \delta)$*. For a technical reason that we will appear later (more precisely in Lemma 13), we need to assume that* ε *satisfies*

$$\frac{d}{2} < \varepsilon < \frac{(1-\beta)d}{2\beta} - \left(1 - \frac{\alpha}{2}\right) \tag{4}$$

and that δ *satisfies*

$$1 - \frac{\alpha}{2} < \delta \leq \frac{(1-\beta)d}{2\beta} - \varepsilon. \tag{5}$$

This is our main technical assumption. The inequality which appears in (4) *gives an extra condition on* β *which reads as*

$$0 < \beta < \frac{1}{2 + \frac{2-\alpha}{d}}.$$

Let us note that the sequence $\{V^N\}_N$ *is a family of* mollifiers *which will allow us to introduce a mollified version of the empirical density (see below). Moreover we assume:*

- $F \in \mathbb{L}^\infty(\mathbb{R}^d \times \mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}^d \times \mathbb{R})$.
- *The sequence of measures* $\{S_0^N\}_N$ *converges weakly to* $u_0(\cdot)dx$ *in probability, as* $N \to \infty$*, where* $u_0 \in \mathbb{L}^1(\mathbb{R}^d)$*.*
- *The sequence of mollified initial measures* $\{V^N * S_0^N\}_N$ *is uniformly bounded in the Sobolev space* $\mathbb{H}^\varepsilon$*, namely:*

$$\sup_{N \in \mathbb{N}} \left\| V^N * S_0^N \right\|_{\mathbb{H}^\varepsilon} < \infty.$$

Let us introduce the *mollified empirical measure* (the theoretical analogue of the numerical method of kernel smoothing) g_t^N defined as

$$g_t^N(x) = \left(V^N * S_t^N\right)(x), \qquad x \in \mathbb{R}^d.$$

We are now ready to state our main result:

Theorem 2 *We assume Assumption 1. Then, for every* $\eta \in (\frac{d}{2}, \varepsilon)$*, the sequence of processes* $\{(g_t^N)_{t\in[0,T]}\}_N$ *converges in probability with respect to the*

- weak star topology of $\mathbb{L}^\infty\left([0,T]\,;\,\mathbb{L}^2(\mathbb{R}^d)\right)$,
- weak topology of $\mathbb{L}^2\left([0,T]\,;\,\mathbb{H}^\varepsilon(\mathbb{R}^d)\right)$
- strong topology of $\mathbb{L}^2\left([0,T]\,;\,\mathbb{H}^\eta_{\mathrm{loc}}(\mathbb{R}^d)\right)$

as $N\to\infty$, to the unique weak solution of the non-local PDE

$$\partial_t u(t,x)+\operatorname{div}(F(x,u)u)-\mathcal{L}u(t,x)=0\,,\qquad u|_{t=0}=u_0, \tag{6}$$

where $\mathcal{L}$ has been defined in (3) and is the operator of a symmetric α-stable Lévy process. Namely, for all $\phi\in C_0^\infty(\mathbb{R}^d)$ it holds

$$\langle u(t,\cdot),\phi\rangle=\langle u_0,\phi\rangle+\int_0^t\langle u,F(\cdot,u)\nabla\phi\rangle\,ds+\frac{1}{2}\int_0^t\langle u,\mathcal{L}\phi\rangle\,ds. \tag{7}$$

The literature concerning the type of equations mentioned above is immense. We will only give a partial and incomplete survey of some parts that we feel more relevant for this paper. For a more complete discussion and many more references, we refer the reader to the nice works [1, 2, 3, 9, 27]. A large variety of phenomena in physics and finance are modelled by linear anomalous diffusion equations, see [27]. Fractional conservation laws are generalizations of convection-diffusion equations and appear in some physical models for over-driven detonation in gases [7] and semiconductor growth [29], and in areas like dislocation dynamics, hydrodynamics, and molecular biology.

About the propagation of chaos phenomena, let us also mention that it has recently been studied in the context well beyond that of the Brownian motion, namely, in the situation where the driving Brownian motions have been replaced by Lévy processes and anomalous diffusions. We mention the works [5, 14, 15]. In [14] the authors consider a singular fractal conservation and they construct a McKean-style non-linear process and then use it to develop an interacting particle system whose empirical measure strongly converges to the solution. In [5] a weak result of this type has been obtained. In [15] the authors deal with an interacting particle system whose empirical measure strongly converges to the solution of a one-dimensional fractional non-local conservation law via the non-linear martingale problem associated to the PDE.

The main result of this paper is to generalize the propagation of chaos Theorem given by Oelschläger [21] for systems of stochastic differential equations driven by Lévy noise, which include non-linear terms as

$$\int_0^t\Big\langle S_s^N,F\big(\cdot\,,\,(V^N*S_s^N)\big)\,\nabla\phi\Big\rangle ds$$

where ϕ is a smooth test function. Since S_t^N converges only weakly, it is required that $V_N*S_t^N$ *converges uniformly*, in the space variable, in order to pass to the limit. Maybe in special cases one can perform special tricks but the question of uniform convergence is a natural one in this problem and it is also of independent interest, hence we investigate when it holds true. Notice that the moderate interaction assumption in [21] reads as $\beta \in (0, \frac{d}{d+2})$ whereas here we obtain $\beta \in (0, \frac{d}{2-\alpha+2d})$, where α is one of the main characteristics of our Lévy process. The case $\beta = 1$ is much more challenging, and up to our knowledge, not solved for the time being.

Finally, we mention that our source of inspiration was the paper [11] where the authors use a semi-group approach in order to study the propagation of chaos for a system of Brownian particles with proliferation, and there they obtain the condition $\beta \in (0, \frac{1}{2})$, which already improved the one of [21]. The differences mainly rely on martingale estimates which, in the present case, are of Lévy type, and therefore contain jumps. We keep the semi-group approach exposed in [11], but all our main technical lemmas involve new analytic tools.

Here follows an outline of the paper: in Section 2 below we gather some well-known results that we use in the paper (with precise references for all the proofs) concerning stable Lévy processes, semi-group properties and criteria of convergence. In Section 3 we prove Theorem 2, by following three main steps: first, obtain uniform bounds for the mollified empirical measure; second, find compact embeddings to extract convergent subsequences; and third, pass to the limit and use a uniqueness result for the solution to (6).

2 Preliminaries

2.1 Stable Lévy processes

We list a collection of definitions and classical results that can be found in any textbook or monography on Lévy processes. We refer to *Applebaum (2009)* [4], *Kunita (2004)* [18] and *Sato (2013)* [23] where all the results and definitions presented in this section are treated.

Definition 3 (Lévy process) *A process $L = (L_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a* Lévy process *if the following conditions are fulfilled:*

1. *L starts at 0 $\mathbb{P}-$a.s., i.e. $\mathbb{P}(L_0 = 0) = 1$;*
2. *L has independent increments, i.e. for $k \in \mathbb{N}$ and $0 \leq t_0 < \cdots < t_k$,*
$$L_{t_1} - L_{t_0}, \ldots, L_{t_k} - L_{t_{k-1}} \quad \textit{are independent} \;;$$
3. *L has stationary increments, i.e., for $0 \leq s \leq t$, $L_t - L_s$ is equal in distribution to L_{t-s} ;*
4. *L is stochastically continuous, i.e. for all $t \geq 0$ and $\varepsilon > 0$*
$$\lim_{s\to t} \mathbb{P}(|L_t - L_s| > \varepsilon) = 0.$$

The reader can find the proof of the following result in [4, Theorem 2.1.8]:

Proposition 4 *Every Lévy process has a càd-làg modification that is itself a Lévy process.*

Due to this fact, we assume moreover that every Lévy process has almost surely càd-làg paths. Following [4, Section 2.4] and [23, Chapter 4] we state the *Lévy-Itô decomposition theorem* which characterizes the paths of a Lévy process in the following way.

Theorem 5 (Lévy-Itô decomposition Theorem) *Consider $b \in \mathbb{R}^d$, σ a positive definite matrix of $\mathbb{R}^{d\times d}$ and ν a measure defined on the Borelians of $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < \infty$.*

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes exist, L^1, L^2, L^3 and L^4 with the following properties:

- *$L^1_t = bt$, for all $t \geq 0$ is called a constant drift ;*
- *L^2 is a Brownian motion with covariance $\sqrt{\sigma}$;*
- *L^3 is a compound Poisson process ;*
- *L^4 is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on every finite time interval.*

Hence, for $L = L^1 + L^2 + L^3 + L^4$ there exists a probability space on which $(L_t)_{t\geq 0}$ is a Lévy process such that $\mathbb{E}\big[e^{i<\xi,L_t>}\big] = \exp(-t\psi(\xi))$ where the characteristic exponent $\psi(\xi)$ is given by

$$\psi(\xi) = i\langle b, \xi\rangle - \frac{1}{2}\langle \sigma\xi, \xi\rangle + \int_{\mathbb{R}^d} \big(e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle \mathbf{1}_{\{|z|<1\}}\big)\nu(dz), \quad \xi \in \mathbb{R}^d.$$

Conversely, given a Lévy process defined on a probability space, there exists $b \in \mathbb{R}^d$, a Wiener process $(B_t)_{t\geq 0}$, a covariance matrix $\sqrt{\sigma} \in \mathbb{R}^{d\times d}$ and an independent Poisson random measure $\mathcal{N}$ defined on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with intensity measure ν such that, for all $t \geq 0$,

$$L_t = bt + \sqrt{\sigma}B_t + \int_0^t \int_{0<|z|<1} z\, \widetilde{\mathcal{N}}(dsdz) + \int_0^t \int_{|z|>1} z\, \mathcal{N}(dsdz). \tag{8}$$

More precisely, the Poisson random measure $\mathcal{N}$ is defined by

$$\mathcal{N}((0,t] \times U) = \sum_{s\in(0,t]} \mathbf{1}_U(L_s - L_{s_-}) \quad \textit{for any } U \in \mathcal{B}(\mathbb{R}^d - \{0\}),\ t > 0,$$

and the compensated Poisson random measure is given by

$$\widetilde{\mathcal{N}}((0,t] \times U) = \mathcal{N}((0,t] \times U) - t\nu(U).$$

Throughout this paper we consider furthermore that $(L_t)_{t\geq 0}$ is a symmetric α-stable process for some $\alpha \in (1,2)$. We recall some facts about *symmetric α-stable processes.* These can be completely defined via their characteristic function, which is given by (see [23] for instance)

$$\mathbb{E}\big[e^{i<\xi,L_t>}\big] = e^{-t\psi(\xi)},$$

where

$$\psi(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi,z\rangle} + i\langle \xi, z\rangle \mathbf{1}_{\{|z|\leq 1\}}\right) d\nu(z), \tag{9}$$

and the Lévy measure ν with $\nu(\{0\}) = 0$ is given by

$$\nu(U) = \int_{S^{d-1}} \int_0^\infty \frac{\mathbf{1}_U(r\theta)}{r^{\alpha+d}}\, dr\, d\mu(\theta),$$

where μ is some symmetric finite measure concentrated on the unit sphere S^{d-1}, called *spectral measure* of the stable process L_t.

Moreover, we assume the following additional property: for some constant $C_\alpha > 0$,

$$\psi(\xi) \geq C_\alpha |\xi|^\alpha, \qquad \text{for any } \ \xi \in \mathbb{R}^d. \tag{10}$$

We remark that the above condition is equivalent to the fact that the support of the spectral measure μ is not contained in the proper linear subspace of $\mathbb{R}^d$, see [22]. Finally, note that the Lévy-Itô decomposition now reads as

$$L_t = \int_0^t \int_{|z|>1} z\, \mathcal{N}(dsdz) + \int_0^t \int_{0<|z|<1} z\, \widetilde{\mathcal{N}}(dsdz), \tag{11}$$

which corresponds to (8) with $b = 0$ and $\sigma = 0$. Now, we recall the following well-known properties about the symmetric α-stable processes (see for instance [23, Proposition 2.5] and [22, Section 3]).

Proposition 6 *Let μ_t be the law of the symmetric α-stable process L_t. Then*

1. (Scaling property). *For any $\lambda > 0$, L_t and $\lambda^{-\frac{1}{\alpha}} L_{\lambda t}$ have the same finite dimensional law. In particular, for any $t > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$, $\mu_t(A) = \mu_1(t^{-\frac{1}{\alpha}} A)$.*

2. (Existence of smooth density). *For any $t > 0$, μ_t has a smooth density ρ_t with respect to the Lebesgue measure, which is given by*

$$\rho_t(x) = \frac{1}{(2\pi)^d} \int e^{-i\langle x,\xi\rangle}\ e^{-t\psi(\xi)}\ d\xi.$$

Moreover $\rho_t(x) = \rho_t(-x)$ and for any $k \in \mathbb{N}$, $\nabla^k \rho_t \in \mathbb{L}^1(\mathbb{R}^d)$.

3. (Moments). *For any $t > 0$, if $\beta < \alpha$, then $\mathbb{E}\big[|L_t|^\beta\big] < \infty$, and if $\beta \geq \alpha$ then $\mathbb{E}\big[|L_t|^\beta\big] = \infty$.*

2.2 Semi-group and Sobolev spaces

From now on, we denote $\mathbb{L}^p = \mathbb{L}^p(\mathbb{R}^d)$ and $\mathbb{H}^\varepsilon = \mathbb{H}^\varepsilon(\mathbb{R}^d)$. The family of operators, for $t \geq 0$,

$$\left(e^{t\mathcal{L}} f\right)(x) = \int_{\mathbb{R}^d} p_t(y-x) f(y)\, dy$$

defines a Markov semi-group in each space $\mathbb{H}^\varepsilon$; with little abuse of notation, we write $e^{t\mathcal{L}}$ for each value of ε.

We consider the operator $A : D(A) \subset \mathbb{L}^2 \to \mathbb{L}^2$ defined as $Af = \Delta f$. For $\varepsilon \geq 0$, the fractional powers $(I-A)^\varepsilon$ are well defined for every $\varepsilon \in \mathbb{R}$ and $\|(I-A)^{\varepsilon/2} f\|_{\mathbb{L}^2}$ is equivalent to the norm of $\mathbb{H}^\varepsilon$. We recall also

Proposition 7 *For every $\varepsilon \geq 0$ and $\alpha \in (1,2)$, and given $T > 0$, there is a constant $C_{\varepsilon,\alpha,T}$ such that, for any $t \in (0,T]$,*

$$\left\| (I-A)^\varepsilon\, e^{t\mathcal{L}} \right\|_{\mathbb{L}^2 \to \mathbb{L}^2} \leq \frac{C_{\varepsilon,\alpha,T}}{t^{2\varepsilon/\alpha}}. \tag{12}$$

Proof. We only sketch the proof, which is standard.

Step 1: Using the scaling property $\rho_t(x) = t^{-\frac{d}{\alpha}} \rho_1(t^{-\frac{1}{\alpha}} x)$, we arrive at

$$\left|\nabla^{k+m} \rho_t * f\right|(x) \leq \frac{t^{-\frac{k}{\alpha}}}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} \left|\nabla^m f(z)\right| \left|(\nabla^k \rho_1)(t^{-\frac{1}{\alpha}} z - t^{-\frac{1}{\alpha}} x)\right| dz.$$

Thus

$$\left\|\nabla^{k+m} \rho_t * f\right\|_{\mathbb{L}^2} \leq t^{-\frac{k}{\alpha}} \left\|\nabla^m f\right\|_{\mathbb{L}^2} \left\|\nabla^k \rho_1\right\|_{\mathbb{L}^1}$$

It follows that

$$\left\|\nabla^k e^{t\mathcal{L}}\right\|_{\mathbb{H}^{m+k} \to \mathbb{H}^m} \leq C t^{-\frac{k}{\alpha}}.$$

Step 2: The sub-Markovian of $e^{t\mathcal{L}}$ (see [4]) implies that

$$\|e^{t\mathcal{L}} f\|_{\mathbb{L}^2} \leq \|f\|_{\mathbb{L}^2}.$$

Step 3: Finally, by a standard interpolation inequality we conclude the proposition for any ε. ∎

2.3 Positive operator

Lemma 8 *We assume that $f \in \mathbb{L}^1 \cap \mathbb{L}^2$ and $f \geq 0$. Then for any $t \geq 0$, $(I-A)^{\varepsilon/2}\, e^{t\mathcal{L}} f \geq 0$.*

Proof. We observe that $g := (I - A)^{\varepsilon/2} e^{t\mathcal{L}} f \in \mathbb{L}^1$. In fact,

$$\|g\|_{\mathbb{L}^1} = \left\| (I - A)^{\varepsilon/2} e^{t\mathcal{L}} f \right\|_{\mathbb{L}^1} \leq \|f\|_{\mathbb{L}^1} \ \ \left\| (I - A)^{\varepsilon/2} e^{t\mathcal{L}} \right\|_{\mathbb{L}^1} \ < \ \infty.$$

In order to prove that the function $g := (I - A)^{\varepsilon/2} e^{t\mathcal{L}} f$ is non-negative, by Bochner's Theorem it is sufficient to prove that its Fourier transform $\widehat{g}$ is definite positive, namely $\mathrm{Re}\left[\sum_{i,j=1}^{n} \widehat{g}\,(\lambda_i - \lambda_j)\, \xi_i \overline{\xi}_j\right] \geq 0$ for every $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}^d$ and $\xi_i \in \mathbb{C}$, $i = 1, ..., n$. We have

$$\widehat{g}\,(\lambda) = \left(1 + |\lambda|^2\right)^{\varepsilon/2} e^{-t\psi(\lambda)} \widehat{f}\,(\lambda)$$

where ψ has been defined in (9). Thus we have to prove that, given $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}^d$ and $\xi_i \in \mathbb{C}$, $i = 1, ..., n$, one has

$$\mathrm{Re}\left[\sum_{i,j=1}^{n} \left(1 + |\lambda_i - \lambda_j|^2\right)^{\varepsilon/2} e^{-t\psi(\lambda_i - \lambda_j)} \widehat{f}\,(\lambda_i - \lambda_j)\, \xi_i \overline{\xi}_j\right] \geq 0$$

namely

$$\begin{aligned}
&\sum_{i=1}^{n} \mathrm{Re}\left[e^{-t\psi(0)} \widehat{f}\,(0)\, \xi_i \overline{\xi}_i\right] \\
&+ \sum_{i<j} \left(1 + |\lambda_i - \lambda_j|^2\right)^{\varepsilon/2} e^{-t\psi(\lambda_i - \lambda_j)} \Big(\mathrm{Re}\left[\widehat{f}\,(\lambda_i - \lambda_j)\, \xi_i \overline{\xi}_j\right] + \mathrm{Re}\left[\widehat{f}\,(\lambda_j - \lambda_i)\, \xi_j \overline{\xi}_i\right]\Big) \geq 0.
\end{aligned}$$

We observe

$$\sum_{i=1}^{n} \mathrm{Re}\left[e^{-t\psi(0)} \widehat{f}\,(0)\, \xi_i \overline{\xi}_i\right] = |\xi|^2 e^{-t\psi(0)}\, \mathrm{Re}\left[\widehat{f}(0)\right] = |\xi|^2 e^{-t\psi(0)} \int f(x)\ dx \geq 0.$$

Since f is non-negative, for $i \neq j$ we obtain

$$\mathrm{Re}\left[\widehat{f}\,(\lambda_i - \lambda_j)\, \xi_i \overline{\xi}_j\right] + \mathrm{Re}\left[\widehat{f}\,(\lambda_j - \lambda_i)\, \xi_j \overline{\xi}_i\right] \geq 0.$$

Using these two facts above we get the result. ■

2.4 Maximal function

Let f be a locally integrable function on $\mathbb{R}^d$. The *Hardy-Littlewood maximal function* is defined by

$$\mathbb{M}f(x) = \sup_{0<r<\infty} \left\{ \frac{1}{|\mathcal{B}_r|} \int_{\mathcal{B}_r} f(x+y)\ dy \right\},$$

where $\mathcal{B}_r = \left\{x \in \mathbb{R}^d \ : |x| < r\right\}$. The following results can be found in [24].

Lemma 9 *For all $f \in \mathbb{W}^{1,1}(\mathbb{R}^d)$ there exists a constant $C_d > 0$ and a Lebesgue zero set $E \subset \mathbb{R}^d$ such that*

$$|f(x) - f(y)| \leq C_d \, |x-y| \, \Big(\mathbb{M}|\nabla f|(x) + \mathbb{M}|\nabla f|(y) \Big) \qquad \text{for any } x, y \in \mathbb{R}^d \backslash E.$$

Moreover, for all $p > 1$ there exists a constant $C_{d,p} > 0$ such that for all $f \in \mathbb{L}^p(\mathbb{R}^d)$

$$\|\mathbb{M} f\|_{\mathbb{L}^p} \leq C_{d,p} \, \|f\|_{\mathbb{L}^p}.$$

2.5 Criterion of convergence in probability

Lemma 10 (Gyongy-Krylov [12]) *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random elements in a Polish space Ψ equipped with the Borel σ-algebra. Then X_n converges in probability to a Ψ-valued random element if, and only if, for each pair $\{X_\ell, X_m\}$ of subsequences, there exists a subsequence $\{v_k\}$,*

$$v_k = (X_{\ell(k)}, X_{m(k)}),$$

converging weakly to a random element v supported on the diagonal set

$$\big\{ (x,y) \in \Psi \times \Psi : x = y \big\}.$$

3 Proof of Theorem 2

The strategy is as follows:

1. Recall that we already defined

$$g_t^N(x) = (V^N * S_t^N)(x) = \int_{\mathbb{R}^d} V^N(x-y) dS_t^N(y) \,. \tag{13}$$

   Using a mild (semi-group) formulation of the identity satisfied by g_t^N (Section 3.1), we prove uniform bounds in Section 3.2 (Lemma 11, Lemma 13 and Lemma 14).

2. Then we apply compactness arguments and Sobolev embeddings to have subsequences which converge so as to pass to the limit (Sections 3.3 and 3.4).

3. Finally, we use previous works to obtain that the weak solution to (6) is unique in our class of convergence (Section 3.5).

3.1 The equation for g_t^N in mild form

We want to deduce an identity for $g_t^N(x)$ from (2). For $h>0$, let us consider the regularized function $e^{h\mathcal{L}}V^N$. For every given $x\in\mathbb{R}^d$ let us take, in identity (2), the test function $\phi_x(y)=\left(e^{h\mathcal{L}}V^N\right)(x-y)$. We get

$$
\begin{aligned}
\left(e^{h\mathcal{L}}g_t^N\right)(x) &= \left(e^{h\mathcal{L}}g_0^N\right)(x)+\int_0^t\left\langle S_s^N, F(\cdot\ ,\ g_s^N)\nabla\left(e^{h\mathcal{L}}V^N\right)(x-\cdot)\right\rangle ds \\
&+\frac{1}{2}\int_0^t\mathcal{L}\left(e^{h\mathcal{L}}g_s^N\right)(x)ds \\
&+\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\left\{\left(e^{h\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}+z\right)-\left(e^{h\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}\right)\right\}d\widetilde{\mathcal{N}}^i(dsdz).
\end{aligned}
$$

Let us write, in the sequel,

$$
\left\langle S_s^N, F(\cdot\ ,\ g_s^N)\nabla\left(e^{h\mathcal{L}}V^N\right)(x-\cdot)\right\rangle =: \left(\nabla\left(e^{h\mathcal{L}}V^N\right)*\left(F(\cdot\ ,\ g_s^N)S_s^N\right)\right)(x)\,.
$$

and similarly for similar expressions. Following a standard procedure, used for instance by [8], we may rewrite this equation in mild form:

$$
\begin{aligned}
e^{h\mathcal{L}}g_t^N &= e^{t\mathcal{L}}\left(e^{h\mathcal{L}}g_0^N\right)+\int_0^t e^{(t-s)\mathcal{L}}\left(\nabla e^{h\mathcal{L}}V^N*\left(F(\cdot\ ,\ g_s^N)S_s^N\right)\right)ds \\
&+\frac{1}{N}\sum_{i=1}^N\int_0^t e^{(t-s)\mathcal{L}}\int_{\mathbb{R}^d-\{0\}}\left\{\left(e^{h\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}+z\right)-\left(e^{h\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}\right)\right\}d\widetilde{\mathcal{N}}^i(dsdz).
\end{aligned}
$$

By inspection of the convolution explicit formula for $e^{(t-s)\mathcal{L}}$, we see that $e^{(t-s)\mathcal{L}}\nabla f=\nabla e^{(t-s)\mathcal{L}}f$, and we can also use the semi-group property, hence we may also write

$$
\begin{aligned}
e^{h\mathcal{L}}g_t^N &= e^{(t+h)\mathcal{L}}g_0^N+\int_0^t\nabla e^{(t+h-s)\mathcal{L}}\left(V^N*\left(F(\cdot\ ,\ g_s^N)S_s^N\right)\right)ds \\
&+\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\Big\{\left(e^{(t-s+h)\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}+z\right) \\
&\qquad-\left(e^{(t-s+h)\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}\right)\Big\}d\widetilde{\mathcal{N}}^i(dsdz). \qquad (14)
\end{aligned}
$$

This is the identity which we use below. We can also pass to the limit as $h\to 0$ and deduce

$$
\begin{aligned}
g_t^N &= e^{t\mathcal{L}}g_0^N+\int_0^t\nabla e^{(t-s)\mathcal{L}}\left(V^N*\left(F(\cdot\ ,\ g_t^N)S_s^N\right)\right)ds \\
&+\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\Big\{\left(e^{(t-s)\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}+z\right) \\
&\qquad-\left(e^{(t-s)\mathcal{L}}V^N\right)\left(x-X_{s_-}^{i,N}\right)\Big\}d\widetilde{\mathcal{N}}^i(dsdz). \qquad (15)
\end{aligned}
$$

In what follows we denote by M_t^N the martingale

$$M_t^N := \frac{1}{N}\sum_{i=1}^{N}\int_0^t \int_{\mathbb{R}^d-\{0\}} \Big\{\big(e^{(t-s+h)\mathcal{L}}V^N\big)\big(x - X_{s_-}^{i,N} + z\big) - \big(e^{(t-s+h)\mathcal{L}}V^N\big)\big(x - X_{s_-}^{i,N}\big)\Big\} d\widetilde{\mathcal{N}}^i(dsdz). \quad (16)$$

3.2 Uniform $\mathbb{L}^2$ bounds

3.2.1 First estimate on g^N

Lemma 11 *For any $t \in [0,T]$ and $N \in \mathbb{N}$, it holds*

$$\mathbb{E}\Big[\big\| (I-A)^{\varepsilon/2}\, g_t^N \big\|_{\mathbb{L}^2}\Big] \le C_\varepsilon.$$

Proof. STEP 1. Let us use the $\mathbb{L}^2$-norm $\|\cdot\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)}$ in the product space $\Omega \times \mathbb{R}^d$ with respect to the product measure. From (14) we obtain

$$\begin{aligned}
&\big\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} g_t^N \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \\
&\qquad \le \big\| (I-A)^{\varepsilon/2}\, e^{(t+h)\mathcal{L}} g_0^N \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} && (17)\\
&\qquad + \int_0^t \big\| (I-A)^{\varepsilon/2}\, \nabla e^{(t+h-s)\mathcal{L}} \left(V^N * \left(F(\cdot\ ,\ g_s^N) S_s^N\right)\right) \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} ds && (18)\\
&\qquad + \big\| (I-A)^{\varepsilon/2}\, M_t^N \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} && (19)
\end{aligned}$$

where M_t^N has been defined in (16).

STEP 2. The first term (17) can be estimated by

$$\big\| (I-A)^{\varepsilon/2}\, e^{(t+h)\mathcal{L}} g_0^N \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \le \big\|(I-A)^{\varepsilon/2} g_0^N \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \le C$$

where from now on we denote generically by $C > 0$ any constant independent of N. The boundedness of $\|(I-A)^{\varepsilon/2} g_0^N\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)}$ is assumed in Assumption 1.

STEP 3. Let us come to the second term (18) above:

$$\begin{aligned}
&\int_0^t \big\| (I-A)^{\varepsilon/2}\, \nabla e^{(t+h-s)\mathcal{L}} \left(V^N * \left(F(\cdot\ ,\ g_s^N) S_s^N\right)\right) \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} ds \\
&\le C \int_0^t \big\|(I-A)^{1/2} e^{(t-s)\mathcal{L}}\big\|_{\mathbb{L}^2\to\mathbb{L}^2}\, \big\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} \left(V^N * \left(F(\cdot\ ,\ g_s^N) S_s^N\right)\right) \big\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} ds.
\end{aligned}$$

We have

$$\big\| (I-A)^{1/2}\, e^{(t-s)\mathcal{L}} \big\|_{\mathbb{L}^2\to\mathbb{L}^2} \le \frac{C}{(t-s)^{1/\alpha}},$$

from Proposition 7. On the other hand, for any $x \in \mathbb{R}^d$,

$$\left| \left(V^N * \left(F(\cdot\,,\, g_s^N) S_s^N\right)\right)(x) \right| \le \|F\|_\infty \left| V^N * S_s^N\,(x) \right| = \|F\|_\infty \; |g_s^N\,(x)|.$$

Then by Lemma 8 we have

$$\left\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} \Big[V^N * \left(F(\cdot\,,\, g_s^N) S_s^N\right) \Big] \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \le C(F)\, \left\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} g_s^N \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)}.$$

To summarize, we have

$$\int_0^t \left\| (I-A)^{\varepsilon/2}\, \nabla e^{(t+h-s)\mathcal{L}} \left(V^N * \left(F(\cdot\,,\, g_s^N) S_s^N\right)\right) \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} ds$$
$$\le \int_0^t \frac{C}{(t-s)^{1/\alpha}} \left\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} g_s^N \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} ds.$$

Step 4. The estimate of the third term (19) is quite tricky and we postpone it to Lemma 13 below, where we prove that $(I-A)^{\varepsilon/2}\, M_t^N$ is uniformly bounded in $\mathbb{L}^2(\Omega\times\mathbb{R}^d)$. Collecting the three bounds together, we have

$$\left\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} g_t^N \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \le C + \int_0^t \frac{C}{(t-s)^{1/\alpha}} \left\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} g_s^N \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} ds.$$

We may apply a generalized version of Gronwall lemma and conclude

$$\left\| (I-A)^{\varepsilon/2}\, e^{h\mathcal{L}} g_t^N \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \le C.$$

We may now take the limit as $h \to 0$. The proof is complete. ■

Remark 12 *Taking $\varepsilon = 0$ we get that g_t^N is also uniformly bounded: for any $t \in [0,T]$ and $N \in \mathbb{N}$,*

$$\mathbb{E}\left[\|g_t^N\|_{\mathbb{L}^2}\right] \le C. \tag{20}$$

3.2.2 Martingale estimate

Lemma 13 *For any $N \in \mathbb{N}$ it holds*

$$\left\| (I-A)^{\varepsilon/2}\, M_t^N \right\|_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \le C.$$

Proof. Step 1. We decompose M_t^N according to the sum of two integrals, over $|z| \le 1$ and $|z| \ge 1$, as follows:

$$M_t^N = I_1 + I_2$$
$$= \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{|z|\ge 1} \Big\{ \left(e^{(t-s+h)\mathcal{L}} V^N\right)\left(x - X_{s_-}^{i,N} + z\right) - \left(e^{(t-s+h)\mathcal{L}} V^N\right)\left(x - X_{s_-}^{i,N}\right) \Big\} d\widetilde{\mathcal{N}}^i(dsdz)$$
$$+ \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{|z|\le 1} \Big\{ \left(e^{(t-s+h)\mathcal{L}} V^N\right)\left(x - X_{s_-}^{i,N} + z\right) - \left(e^{(t-s+h)\mathcal{L}} V^N\right)\left(x - X_{s_-}^{i,N}\right) \Big\} d\widetilde{\mathcal{N}}^i(dsdz).$$

When $|z| \geq 1$ we can use the fact that $\int_{|z|\geq 1} d\nu(z) < \infty$. When $|z| \leq 1$ this is not true, but we know that $\int_{|z|\leq 1} |z|^2 d\nu(z) < \infty$. Therefore, for this second case, we use the tool of the maximal function introduced in Section 2.4. For the sake of clarity, along this proof we denote

$$F^N(s,x,z) := V^N\big(x - X^{i,N}_{s_-} + z\big) - V^N\big(x - X^{i,N}_{s_-}\big). \tag{21}$$

Step 2. Let us start with I_1. We have

$$\begin{aligned}
&\big\|(I-A)^{\varepsilon/2} I_1\big\|^2_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \\
&= \frac{2}{N^2}\int_{\mathbb{R}^d} \mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t \int_{|z|\geq 1} (I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z) d\widetilde{\mathcal{N}}^i(dsdz)\right|^2\right] dx \\
&= \frac{2}{N^2}\sum_{i=1}^N \int_{\mathbb{R}^d} \mathbb{E}\left[\int_0^t \int_{|z|\geq 1} \Big| (I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z)\Big|^2 d\nu(z) ds\right] dx \\
&= \frac{2}{N}\,\mathbb{E}\left[\int_0^t \int_{|z|\geq 1} \int_{\mathbb{R}^d} \Big| (I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z)\Big|^2 \, dx \; d\nu(z) ds\right].
\end{aligned}$$

We observe that

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big| (I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z)\Big|^2 \, dx &\leq 2\int_{\mathbb{R}^d} \Big| (I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} V^N(x)\Big|^2 \, dx \\
&\leq C\big\|V^N\big\|^2_{\mathbb{H}^\varepsilon}.
\end{aligned}$$

This implies that

$$\big\|\,(I-A)^{\varepsilon/2}\, I_1\big\|^2_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \leq \frac{C}{N}\big\|V^N\big\|^2_{\mathbb{H}^\varepsilon} \leq \frac{CN^{\beta+\frac{2\varepsilon\beta}{d}}\|V\|^2_{\mathbb{H}^\varepsilon}}{N} \leq C,$$

where in the last inequality we use Assumption 1.

Step 3. In the same way we obtain

$$\begin{aligned}
&\big\|(I-A)^{\varepsilon/2} I_2\big\|^2_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} \\
&= \frac{2}{N^2}\int_{\mathbb{R}^d} \mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t \int_{|z|\leq 1} (I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z) d\widetilde{\mathcal{N}}^i(dsdz)\right|^2\right] dx \\
&= \frac{2}{N^2}\sum_{i=1}^N \int_{\mathbb{R}^d} \mathbb{E}\left[\int_0^t \int_{|z|\leq 1} |\,(I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z)|^2 d\nu(z) ds\right] dx \\
&= \frac{2}{N^2}\sum_{i=1}^N \mathbb{E}\left[\int_0^t \int_{|z|\leq 1} \int_{\mathbb{R}^d} |\,(I-A)^{\varepsilon/2} e^{(t-s+h)\mathcal{L}} F^N(s,x,z)|^2 \; dx d\nu(z) ds\right]
\end{aligned}$$

where F^N has been defined in (21). We observe that

$$(I-A)^{\varepsilon/2}e^{(t-s+h)\mathcal{L}}F^N(s,x,z) \le C|z|\,\mathbb{M}\Big[\nabla(I-A)^{\varepsilon/2}\,e^{(t-s+h)\mathcal{L}}V^N\big(x-X^{i,N}_{s_-}+z\big)\Big] + C|z|\,\mathbb{M}\Big[\nabla(I-A)^{\varepsilon/2}\,e^{(t-s+h)\mathcal{L}}V^N\big(x-X^{i,N}_{s_-}\big)\Big].$$

From Lemma 9 we have

$$\Big\|\mathbb{M}\Big[\nabla(I-A)^{\varepsilon/2}e^{(t-s+h)\mathcal{L}}V^N\Big]\Big\|^2_{\mathbb{L}^2} \le C\Big\|\nabla(I-A)^{\varepsilon/2}e^{(t-s+h)\mathcal{L}}V^N\Big\|^2_{\mathbb{L}^2}.$$

This implies that

$$\Big\|(I-A)^{\varepsilon/2}e^{(t-s+h)\mathcal{L}}F^N(s,x,z)\Big\|^2_{\mathbb{L}^2} \le C\,|z|^2\,\Big\|\nabla(I-A)^{\varepsilon/2}e^{(t-s+h)\mathcal{L}}V^N\Big\|^2_{\mathbb{L}^2}.$$

Taking $\delta \in \big(1-\frac{\alpha}{2}, \frac{(1-\beta)d}{2\beta}-\varepsilon\big]$ (which exists by Assumption 1) we have

$$\begin{aligned}\big\|\,(I-A)^{\varepsilon/2}\,I_2\big\|^2_{\mathbb{L}^2(\Omega\times\mathbb{R}^d)} &\le \frac{C}{N}\int_0^t \big\|\nabla(I-A)^{\varepsilon/2}e^{(t-s+h)\mathcal{L}}V^N\big\|^2_{\mathbb{L}^2}ds \\ &\le \frac{C}{N}\big\|(I-A)^{(\varepsilon+\delta)/2}V^N\big\|^2_{\mathbb{L}^2}\int_0^t \frac{1}{(t+h-s)^{2(1-\delta)/\alpha}}ds \le C\end{aligned}$$

where we used that

$$\big\|V^N\big\|^2_{\mathbb{H}^{\varepsilon+\delta}} \le CN^{\beta+\frac{2(\varepsilon+\delta)\beta}{d}}\big\|V\big\|^2_{\mathbb{H}^{\varepsilon+\delta}} \le C$$

and $\|V\|_{\mathbb{H}^{\varepsilon+\delta}} < \infty$ from Assumption 1. ■

3.2.3 Second estimate on g^N

Lemma 14 *For any $\gamma \in (0,\frac{1}{2})$ and $N \in \mathbb{N}$, it holds*

$$\mathbb{E}\left[\int_0^T\int_0^T \frac{\big\|g^N_t - g^N_s\big\|^2_{\mathbb{H}^{-2}}}{|t-s|^{1+2\gamma}}\,ds\,dt\right] \le C.$$

Proof. In this proof we use the fact that $\mathbb{L}^2(\mathbb{R}^d) \subset \mathbb{W}^{-2,2}(\mathbb{R}^d)$ with continuous embedding, and that the linear operator Δ is bounded from $\mathbb{L}^2(\mathbb{R}^d)$ to $\mathbb{W}^{-2,2}(\mathbb{R}^d)$.

Step 1. We recall the formula

$$\begin{aligned}\big(e^{h\mathcal{L}}g^N_t\big)(x) &= \big(e^{h\mathcal{L}}g^N_0\big)(x) + \int_0^t \Big\langle S^N_s, F(\cdot\,,\,g^N_s)\nabla\big(e^{h\mathcal{L}}V^N\big)\,(x-\cdot)\Big\rangle\,ds \\ &\quad + \frac{1}{2}\int_0^t \mathcal{L}\big(e^{h\mathcal{L}}g^N_s\big)(x)ds \\ &\quad + \frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\Big\{\big(e^{h\mathcal{L}}V^N\big)\big(x-X^{i,N}_{s_-}+z\big) - \big(e^{h\mathcal{L}}V^N\big)\big(x-X^{i,N}_{s_-}\big)\Big\}d\widetilde{\mathcal{N}}^i(dsdz).\end{aligned}$$

Then we have

$$\begin{aligned}
&\big(e^{h\mathcal{L}}g_t^N\big)(x)-\big(e^{h\mathcal{L}}g_s^N\big)(x)=\int_s^t\Big\langle S_r^N,F(\cdot\,,\,g_r^N)\nabla\big(e^{h\mathcal{L}}V^N\big)\,(x-\cdot)\Big\rangle\,dr\\
&\quad+\frac{1}{2}\int_s^t\mathcal{L}\big(e^{h\mathcal{L}}g_r^N\big)(x)dr\\
&\quad+\frac{1}{N}\sum_{i=1}^N\int_s^t\int_{\mathbb{R}^d-\{0\}}\Big\{\big(e^{h\mathcal{L}}V^N\big)\big(x-X_{r_-}^{i,N}+z\big)-\big(e^{h\mathcal{L}}V^N\big)\big(x-X_{r_-}^{i,N}\big)\Big\}d\widetilde{\mathcal{N}}^i(drdz).
\end{aligned}$$

Thus by letting $h\to 0$ we obtain, for $s<t$,

$$\begin{aligned}
&\mathbb{E}\Big[\|g_t^N-g_s^N\|_{\mathbb{H}^{-2}}^2\Big]\\
&\quad\le(t-s)\int_s^t\mathbb{E}\Big[\|\nabla\left(V^N*\big(F(\cdot\,,\,g_r^N)S_r^N\big)\right)\|_{\mathbb{H}^{-2}}^2\Big]dr+\frac{t-s}{2}\int_s^t\mathbb{E}\Big[\|\mathcal{L}g_r^N\|_{\mathbb{H}^{-2}}^2\Big]dr\\
&\quad+\mathbb{E}\Bigg[\Bigg\|\frac{1}{N}\sum_{i=1}^N\int_s^t\int_{\mathbb{R}^d-\{0\}}\Big\{(V^N)\big(x-X_{r_-}^{i,N}+z\big)-(V^N)\big(x-X_{r_-}^{i,N}\big)\Big\}d\widetilde{\mathcal{N}}^i(drdz)\Bigg\|_{\mathbb{H}^{-2}}^2\Bigg].
\end{aligned}$$

Step 2. We observe that

$$\begin{aligned}
\int_s^t\mathbb{E}\Big[\Big\|\nabla\left(V^N*\big(F(\cdot\,,\,g_r^N)S_r^N\big)\right)\Big\|_{\mathbb{H}^{-2}}^2\Big]dr&\le\int_s^t\mathbb{E}\Big[\Big\|\left(V^N*\big(F(\cdot\,,\,g_r^N)S_r^N\big)\right)\Big\|_{\mathbb{H}^{-1}}^2\Big]dr\\
&\le\int_s^t\Big\|\left(V^N*\big(F(\cdot\,,\,g_r^N)S_r^N\big)\right)\Big\|_{\mathbb{L}^2(\Omega\times\mathbb{R})}^2dr\\
&\le\int_s^t\|g_r^N\|_{\mathbb{L}^2(\Omega\times\mathbb{R})}^2dr\le C(t-s),
\end{aligned}$$

where in last step we used Lemma 11 (see (20)).

Step 3. Moreover

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg\|\frac{1}{N}\sum_{i=1}^N\int_s^t\int_{\mathbb{R}^d-\{0\}}\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}+z\big)-\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}\big)d\widetilde{\mathcal{N}}^i(drdz)\Bigg\|_{\mathbb{H}^{-2}}^2\Bigg]\\
&\qquad=\frac{1}{N^2}\sum_{i=1}^N\int_s^t\int_{\mathbb{R}^d-\{0\}}\big\|\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}+z\big)-\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}\big)\big\|_{\mathbb{H}^{-2}}^2\,drd\nu(z)\\
&\qquad=\frac{1}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\ge 1}\big\|\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}+z\big)-\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}\big)\big\|_{\mathbb{H}^{-2}}^2\,drd\nu(z)\\
&\qquad+\frac{1}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\le 1}\big\|\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}+z\big)-\big(V^N\big)\big(\,\cdot-X_{r_-}^{i,N}\big)\big\|_{\mathbb{H}^{-2}}^2\,drd\nu(z).
\end{aligned}$$

Step 4. We observe

$$\frac{1}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\geq 1}\big\|\big(V^N\big)\big(\cdot - X^{i,N}_{r_-}+z\big)-\big(V^N\big)\big(\cdot - X^{i,N}_{r_-}\big)\big\|^2_{\mathbb{H}^{-2}}\,drd\nu(z)$$
$$\leq \frac{2}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\geq 1}\big\|V^N\big\|^2_{\mathbb{H}^{-2}}\,drd\nu(z)\leq \frac{C}{N}\int_s^t\big\|V^N\big\|^2_{\mathbb{L}^2}\,dr\leq C(t-s),$$

where in the last step we use the fact that $\|V^N\|^2_{\mathbb{L}^2}\leq N^\beta\|V\|^2_{\mathbb{L}^2}$.

Step 5. Let us write

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\leq 1}\big\|\big(V^N\big)\big(\cdot - X^{i,N}_{r_-}+z\big)-\big(V^N\big)\big(\cdot - X^{i,N}_{r_-}\big)\big\|^2_{\mathbb{H}^{-2}}\,drd\nu(z)\\
&\leq \frac{C}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\leq 1}\big\|\big(V^N\big)\big(\cdot - X^{i,N}_{r_-}+z\big)-\big(V^N\big)\big(\cdot - X^{i,N}_{r_-}\big)\big\|^2_{\mathbb{H}^{-2}}\,drd\nu(z)\\
&\leq \frac{C}{N^2}\sum_{i=1}^N\int_s^t\int_{|z|\leq 1}|z|^2\,\big\|\mathbb{M}\big(V^N\big)\big\|^2_{\mathbb{H}^{-2}}\,drd\nu(z)\\
&\leq \frac{C}{N}\int_s^t\big\|\nabla\big(V^N\big)\big\|^2_{\mathbb{H}^{-2}}\,dr\leq\frac{C}{N}\int_s^t\big\|V^N\big\|^2_{\mathbb{L}^2}\,dr\leq C(t-s).
\end{aligned}$$

Step 6. Finally we claim that

$$\frac{1}{2}\int_s^t\mathbb{E}\Big[\big\|\mathcal{L}g^N_r\big\|^2_{\mathbb{H}^{-2}}\Big]dr\leq\frac{C}{2}\int_s^t\mathbb{E}\Big[\big\|g^N_r\big\|^2_{\mathbb{L}^2}\Big]dr\leq C(t-s).$$

The first inequality above can be proved as follows:

$$\begin{aligned}
\big\|\mathcal{L}g^N_r\big\|_{\mathbb{H}^{-2}} &= \sup_{\|f\|_{\mathbb{H}^2}\leq 1}|<\mathcal{L}g^N_r,f>|\\
&= \sup_{\|f\|_{\mathbb{H}^2}\leq 1}|<g^N_r,\mathcal{L}f>|\leq\sup_{\|f\|_{\mathbb{H}^2}\leq 1}\Big\{\big\|g^N_r\big\|_{\mathbb{L}^2}\,\big\|\mathcal{L}f\big\|_{\mathbb{L}^2}\Big\}\\
&\leq C\sup_{\|f\|_{\mathbb{H}^2}\leq 1}\Big\{\big\|g^N_r\big\|_2\big(\big\|\Delta f\big\|_{\mathbb{L}^2}+\big\|\nabla f\big\|_{\mathbb{L}^2}\big)\Big\} \qquad (22)\\
&\leq C\,\big\|g^N_r\big\|_{\mathbb{L}^2},
\end{aligned}$$

where we used [28, Lemma 2.4] to obtain (22). ■

3.3 Criterion of compactness

In this subsection we follow the arguments of [11, Section 6.1]. We start by constructing one space on which the sequence of the laws of $g^N_\cdot$ will be tight.

A version of the Aubin-Lions Lemma, see [10, 19], states that when $E_0 \subset E \subset E_1$ are three Banach spaces with continuous dense embeddings, with E_0, E_1 reflexive, and E_0 compactly embedded into E, given $p, q \in (1,\infty)$ and $\gamma \in (0,1)$, the space $\mathbb{L}^q([0,T]\,;\,E_0) \cap \mathbb{W}^{\gamma,p}([0,T]\,;\,E_1)$ is compactly embedded into $\mathbb{L}^q([0,T]\,;\,E)$.

We use this lemma with $E = \mathbb{W}^{\eta,2}(D)$, $E_0 = \mathbb{W}^{\varepsilon,2}(D)$ with $\frac{d}{2} < \eta < \varepsilon$, and $E_1 = \mathbb{W}^{-2,2}(\mathbb{R}^d)$ where D is a regular bounded domain. We also choose $0 < \gamma < \frac{1}{2}$ in order to apply Lemma 14 (see below). The Aubin-Lions Lemma states that

$$\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\varepsilon,2}(D)\big) \cap \mathbb{W}^{\gamma,2}\big([0,T]\,;\,\mathbb{W}^{-2,2}(\mathbb{R}^d)\big)$$

is compactly embedded into $\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\eta,2}(D)\big)$. Now, consider the space

$$Y_0 := \mathbb{L}^\infty\big([0,T]\,;\,\mathbb{L}^2(\mathbb{R}^d)\big) \cap \mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)\big) \cap \mathbb{W}^{\gamma,2}\big([0,T]\,;\,\mathbb{W}^{-2,2}(\mathbb{R}^d)\big).$$

Using the Fréchet topology on $\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\eta,2}_{\mathrm{loc}}(\mathbb{R}^d)\big)$ defined as

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n}\left(1 \wedge \int_0^T \|f(t,\cdot)\|^2_{\mathbb{W}^{\eta,2}(\mathcal{B}(0,n))}dt\right)$$

one has that $\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)\big) \cap \mathbb{W}^{\gamma,2}\big([0,T]\,;\,\mathbb{W}^{-2,2}(\mathbb{R}^d)\big)$ is compactly embedded[1] into $\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\eta,2}_{\mathrm{loc}}(\mathbb{R}^d)\big)$.

Let us denote respectively by $\mathbb{L}^\infty_{w*}$ and $\mathbb{L}^2_w$ the spaces $\mathbb{L}^\infty$ and $\mathbb{L}^2$ endowed respectively with the weak star and weak topology. We have that Y_0 is compactly embedded into

$$Y := \mathbb{L}^\infty_{w*}\big([0,T]\,;\,\mathbb{L}^2(\mathbb{R}^d)\big) \cap \mathbb{L}^2_w\big([0,T]\,;\,\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)\big) \cap \mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\eta,2}_{\mathrm{loc}}(\mathbb{R}^d)\big). \tag{23}$$

Note that

$$\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\eta,2}_{\mathrm{loc}}(\mathbb{R}^d)\big) \subset \mathbb{L}^2\big([0,T]\,;\,C(D)\big)$$

for every regular bounded domain $D \subset \mathbb{R}^d$.

Let us now go back to the sequence of processes $\{g^N_\cdot\}_N$, for which we have proved several estimates. The Chebyshev inequality ensures that

$$\mathbb{P}\big(\|g^N_\cdot\|^2_{Y_0} > R\big) \le \frac{\mathbb{E}\big[\|g^N_s\|^2_{Y_0}\big]}{R}, \qquad \text{for any } R > 0.$$

Thus by Lemma 11 and Lemma 14 we obtain

$$\mathbb{P}\big(\|g^N_\cdot\|^2_{Y_0} > R\big) \le \frac{C}{R}, \qquad \text{for any } R > 0, N \in \mathbb{N}.$$

[1] The proof is elementary, using the fact that if a set is compact in $\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\eta,2}_{\mathrm{loc}}(\mathcal{B}(0,n))\big)$ for every n then it is compact in $\mathbb{L}^2\big([0,T]\,;\,\mathbb{W}^{\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^d)\big)$ with this topology.

The process $(g_t^N)_{t\in[0,T]}$ defines a probability $\mathbf{P}_N$ on Y. Last inequality implies that there exists a bounded set $B_\varepsilon \in Y_0$ such that $\mathbf{P}_N(B_\varepsilon) < 1-\varepsilon$ for all N, and therefore there exists a compact set $K_\varepsilon \in Y$ such that $\mathbf{P}_N(K_\varepsilon) < 1-\varepsilon$.

Denote by $\{\mathtt{L}^N\}_{N\in\mathbb{N}}$ the laws of the processes $\{g^N\}_{N\in\mathbb{N}}$ on Y_0, we have proved that $\{\mathtt{L}^N\}_{N\in\mathbb{N}}$ is tight in Y, hence relatively compact, by Prohorov's Theorem. From every subsequence of $\{\mathtt{L}^N\}_{N\in\mathbb{N}}$ it is possible to extract a further subsequence which converges to a probability measure $\mathtt{L}$ on Y. Moreover by a Theorem of Skorokhod (see [13, Theorem 2.7]), we are allowed, eventually after choosing a suitable probability space where all our random variables can be defined, to assume

$$g^N \to u \quad \text{in } Y, \qquad \text{a.s.}$$

where the law of u is $\mathtt{L}$.

3.4 Passing to the limit

In this paragraph we show that the limit u of g^N satisfies the weak formulation (7) of the non-local conservation equation (6).

STEP 1. By Itô's formula we have, for any test function ϕ, that

$$\begin{aligned}\int g_t^N(x)\phi(x)\,dx = &\int g_0^N(x)\phi(x)\,dx + \int_0^t \Big\langle S_s^N, F(\cdot\,,\, g_s^N)\nabla(V^N*\phi)(\cdot)\Big\rangle\,ds\\ &+\frac{1}{2}\int_0^t\int g_s^N(x)\mathcal{L}\phi(x)\,dxds\\ &+\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\Big\{(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big\}\,d\widetilde{\mathcal{N}}^i(dsdz).\end{aligned}$$

Passing to the limit we obtain

$$\begin{aligned}\int u(t,x)\phi(x)\,dx = &\int u_0(x)\phi(x)\,dx + \lim_{N\to\infty}\int_0^t \Big\langle S_s^N, F(\cdot\,,\, g_s^N)\nabla(V^N*\phi)(\cdot)\Big\rangle ds\\ &+\frac{1}{2}\int_0^t\int u(s,x)\mathcal{L}\phi(x)\,dxds\\ &+\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\Big\{(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big\}\,d\widetilde{\mathcal{N}}^i(dsdz).\end{aligned}$$

STEP 2. We claim that

$$\lim_{N\to\infty}\int_0^t \Big\langle S_s^N, F(\cdot\,,\, g_s^N)\nabla(V^N*\phi)(\cdot)\Big\rangle ds = \int_0^t\int u(s,x)F(x,u)\nabla\phi(x)\,dxds. \tag{24}$$

In order to prove the claim, we first observe, using the symmetry of V, that

$$\Big|\Big\langle S_s^N, F(\cdot\,,\ g_s^N)\nabla(V^N*\phi)\,(\cdot)\Big\rangle - \Big\langle g_s^N, F(\cdot\,,\ g_s^N)\nabla(V^N*\phi)\,(\cdot)\Big\rangle\Big|$$
$$\leq \sup_{x\in\mathbb{R}^d}\Big|F(x,\ g_s^N(x))\nabla(V^N*\phi)(x) - \big((F(\cdot\,,\ g_s^N)\nabla(V^N*\phi)\,(\cdot))*V^N\big)(x)\Big|,$$

We can control the last term, using the fact that

- V is a density (denoted below by ($\int V=1$)),
- F is Lipschitz and bounded (denoted below by ($F\in\mathrm{Lip}\cap L^\infty$)),
- V is compactly supported (denoted below by (V is c.s.)),
- and ϕ is compactly supported and smooth,

as follows:

$$\Big|F(x,\ g_s^N(x))\nabla(V^N*\phi)(x) - \big((F(\cdot\,,\ g_s^N)\nabla(V^N*\phi)\,(\cdot))*V^N\big)(x)\Big|$$
$$\stackrel{(\int V=1)}{\leq} \int V(y)\,\big|\nabla(V^N*\phi)(x)\big|\,\Big|F\big(x,g_s^N(x)\big) - F\Big(x-\frac{y}{N^{\frac{\beta}{d}}},g_s^N\Big(x-\frac{y}{N^{\frac{\beta}{d}}}\Big)\Big)\Big|dy$$
$$+\int V(y)\,\Big|\nabla(V^N*\phi)(x) - \nabla(V^N*\phi)\Big(x-\frac{y}{N^{\frac{\beta}{d}}}\Big)\Big|\,\big|F(x,g_s^N(x))\big|dy$$
$$\stackrel{(F\in\mathrm{Lip}\cap L^\infty)}{\leq} C\int V(y)\,\big|\nabla(V^N*\phi)(x)\big|\,\Big|g_s^N(x) - g_s^N\Big(x-\frac{y}{N^{\frac{\beta}{d}}}\Big)\Big|dy$$
$$+\frac{C}{N^{\frac{\beta}{d}}}\int V(y)|y|dy$$
$$\stackrel{(V\text{ is c.s.})}{\leq} \frac{C}{N^{\frac{\widetilde{\eta}\beta}{d}}}\sup_{x,y\in K}\frac{\big|g_s^N(x)-g_s^N(y)\big|}{|x-y|^{\widetilde{\eta}}}\int V(y)|y|^{\widetilde{\eta}}\,dy$$
$$+\frac{C}{N^{\frac{\beta}{d}}}\int V(y)|y|dy,$$

where K is a compact set and $\widetilde{\eta}=\eta-\frac{d}{2}$, where η has been defined in Section 3.3. Therefore we have obtained

$$\mathbb{E}\Big[\Big|F(x,\ g_s^N(x))\nabla(V^N*\phi)(x) - \big((F(\cdot\,,\ g_s^N)\nabla(V^N*\phi)\,(\cdot))*V^N\big)(x)\Big|\Big] \leq \frac{C}{N^{\frac{\widetilde{\eta}\beta}{d}}}.$$

Thus,

$$\lim_{N\to\infty}\int_0^t \Big\langle S_s^N, F(\cdot\ ,\ g_s^N)\ \nabla(V^N*\phi)\,(\cdot)\ \Big\rangle ds$$
$$= \lim_{N\to\infty}\int_0^t \Big\langle g_s^N, F(\cdot\ ,\ g_s^N)\nabla(V^N*\phi)\,(\cdot)\ \Big\rangle ds.$$
$$= \lim_{N\to\infty}\int_0^t \int g_s^N(x)F(x,g_s^N)\nabla(V^N*\phi)\,(x)\ dxds$$
$$= \int u(s,x)F(x,u(s,x))\nabla\phi(x)\ dxds$$

where in the last equality we used that $g_s^N \to u$ strongly in $\mathbb{L}^2\left([0,T]\ ;\ C(D)\right)$.

Step 3. We claim that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d-\{0\}}\Big\{(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big\}d\widetilde{\mathcal{N}}^i(dsdz)=0. \quad (25)$$

First we observe that

$$\lim_{N\to\infty}\mathbb{E}\Bigg[\bigg|\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{|z|\geq 1}\Big\{(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big\}d\widetilde{\mathcal{N}}^i(dsdz)\bigg|^2\Bigg]$$
$$= \lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^N\int_0^t\int_{|z|\geq 1}\Big|(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big|^2\ d\nu(z)ds$$
$$\leq \lim_{N\to\infty}\frac{C}{N}=0. \quad (26)$$

On other hand we have

$$\lim_{N\to\infty}\mathbb{E}\Bigg[\bigg|\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{|z|\leq 1}\Big\{(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big\}d\widetilde{\mathcal{N}}^i(dsdz)\bigg|^2\Bigg]$$
$$= \lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^N\int_0^t\int_{|z|\leq 1}\Big|(V^N*\phi)\big(-X_{s_-}^{i,N}+z\big)-(V^N*\phi)\big(-X_{s_-}^{i,N}\big)\Big|^2\ d\nu(z)ds$$
$$\leq C\lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^N\int_0^t\int_{|z|\leq 1}|z|^2\ d\nu(z)ds \leq C\lim_{N\to\infty}\frac{1}{N}=0. \quad (27)$$

From (26) and (27) we conclude (25). Summarizing, we have proved (7).

3.5 Uniqueness of PDE

In order to make the paper self-contained we present the uniqueness result for the PDE (6). We also refer the reader to [1, 9, 17].

Theorem 15 . *There is at most one weak solution of equation* (6) *in* $\mathbb{L}^2\big([0,T]\ ;\ \mathbb{H}^\eta\big)$ *with* $\eta > \frac{d}{2}$.

Proof. Let u^1, u^2 be two weak solutions of the equation (6) with the same initial condition u_0. Let $\{\rho_\varepsilon(x)\}_\varepsilon$ be a family of standard symmetric mollifiers. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we can use $\rho_\varepsilon(x-\cdot)$ as test function in the equation (7). Set $u^i_\varepsilon(t,x) = u^i(t,\cdot) * \rho_\varepsilon(\cdot)(x)$ for $i = 1,2$. Then we have

$$u^i_\varepsilon(t,x) = (u_0 * \rho_\varepsilon)(x) + \int_0^t \mathcal{L}u^i_\varepsilon(s,x)\,ds + \int_0^t \big(\nabla\rho_\varepsilon * u^i F(\cdot,u^i)\big)(s,x)\,ds.$$

Writing this identity in mild form we obtain (we write $u^i\,(t)$ for the function $u^i\,(s,\cdot)$ and $S(t)$ for e^{tL})

$$u^i_\varepsilon(t) = S(t)(u_0 * \rho_\varepsilon) + \int_0^t S(t-s)\ \big(\nabla\rho_\varepsilon * u^i F(\cdot,u^i)\big)ds.$$

The function $U = u^1 - u^2$ satisfies

$$\rho_\varepsilon * U(t) = \int_0^t \nabla S(t-s)\big(\rho_\varepsilon * \big[u^1F(\cdot,u^1) - u^2F(\cdot,u^2)\big]\big)\,ds.$$

Thus we obtain

$$\|\rho_\varepsilon * U(t)\|_{\mathbb{L}^2} \le \int_0^t \Big\|\nabla S(t-s)\big(\rho_\varepsilon * \big[u^1F(\cdot,u^1) - u^2F(\cdot,u^2)\big]\big)\Big\|_{\mathbb{L}^2}\,ds.$$

Using the proposition 7 we have

$$\|\rho_\varepsilon * U(t)\|_{\mathbb{L}^2} \le \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\Big\|\rho_\varepsilon * \big[u^1F(\cdot,u^1) - u^2F(\cdot,u^2)\big]\Big\|_{\mathbb{L}^2}\,ds.$$

Taking the limit as $\varepsilon \to 0$ we arrive

$$\|U(t)\|_{\mathbb{L}^2} \le \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\Big\|\big[u^1F(\cdot,u^1) - u^2F(\cdot,u^2)\big]\Big\|_{\mathbb{L}^2}\,ds.$$

By easy calculation we have

$$\|U(t)\|_{\mathbb{L}^2} \le \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\big\|UF(\cdot,u^1)\big\|_{\mathbb{L}^2} + \big\|u^2(F(\cdot,u^1) - F(\cdot,u^1))\big\|_{\mathbb{L}^2}\,ds.$$

Notice that the function F is globally Lipschitz and bounded. It follows

$$\|U(t)\|_{\mathbb{L}^2} \leq \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \big(C\|U\|_{\mathbb{L}^2} + \|u^2 U\|_{\mathbb{L}^2} \big)\, ds.$$

By hypothesis $u^2 \in \mathbb{L}^2([0,T]\ ;\ \mathbb{H}^\eta)$ with $\eta > \frac{d}{2}$. Then by the Sobolev embeddings (see [25, Section 2.8.1]), we have $u^2 \in \mathbb{L}^2\big([0,T]\ ;\ C_b(\mathbb{R}^d)\big)$. It follows that

$$\|U(t)\|_{\mathbb{L}^2} \leq \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \big(C\|U\|_{\mathbb{L}^2} + \|u^2\|_{\mathbb{L}^\infty}\, \|U\|_{\mathbb{L}^2} \big)\, ds.$$

By Gronwall's Lemma we conclude $U = 0$. ■

3.6 Convergence in probability

Corollary 16 *The sequence $\{g^N\}_{N\in\mathbb{N}}$ converges in probability to u.*

Proof. We denote the joint law of (g^N, g^M) by $\nu^{N,M}$. Similarly to the proof of tightness for g^N we have that the family $\{\nu^{N,M}\}$ is tight in $Y \times Y$.

Let us take any subsequence ν^{N_k,M_k}. By Prohorov's theorem, it is relatively weakly compact hence it contains a weakly convergent subsequence. Without loss of generality we may assume that the original sequence $\{\nu^{N,M}\}$ itself converges weakly to a measure ν. According to the Skorokhod immersion theorem, we infer the existence of a probability space $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\big)$ with a sequence of random variables $(\bar{g}^N, \bar{g}^M)$ converging almost surely in $Y \times Y$ to random variable $(\bar{u}, \check{u})$ and the laws of $(\bar{g}^N, \bar{g}^M)$ and $(\bar{u}, \check{u})$ under $\bar{\mathbb{P}}$ coincide with $\nu^{N,M}$ and ν, respectively.

Analogously, it can be applied to both $\bar{g}^N$ and $\bar{g}^M$ in order to show that $\bar{u}$ and $\check{u}$ are two solutions of the PDE (7). By the uniqueness property of the solutions to (7) we have $\bar{u} = \check{u}$. Therefore

$$\nu\big((x,y) \in Y \times Y\ :\ x = y\big) = \bar{\mathbb{P}}(\bar{u} = \check{u}) = 1.$$

Now, we have all in hands to apply Gyongy-Krylov's characterization of convergence in probability (Lemma 10). It implies that the original sequence is defined on the initial probability space converges in probability in the topology of Y to a random variable μ. ■

Acknowledgements

C.O. is partially supported by CNPq through the grant 460713/2014-0 and FAPESP by the grants 2015/04723-2 and 2015/07278-0. This work benefited from the support of the project EDNHS ANR-14-CE25-0011 of the French National Research Agency (ANR). The work of M.S. was also supported by the Labex CEMPI (ANR-11-LABX-0007-01).

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