Sliding Mode Control Design for Linear Evolution Equations with Uncertain Measurements and Exogenous Perturbations

Sergiy Zhuk, Andrey Polyakov, Olexander Nakonechnyi

Sliding Mode Control Design for Linear Evolution Equations with Uncertain Measurements and Exogenous Perturbations

Sergiy Zhuk * **Andrey Polyakov** ** **Olexander Nakonechnyi** ***

* *IBM Research, Dublin, Ireland (e-mail: sergiy.zhuk@ie.ibm.com).*
** *Inria Lille-Nord Europe, Villeneuve d'Ascq, France(e-mail: andrey.polyakov@inria.fr)*
*** *Taras Shevchenko National University, Kyiv, Ukraine*

Abstract: The paper studies the problem of sliding mode control design for linear evolution equations with incomplete and noisy measurements of the output and additive exogenous disturbances. The key result of the paper is an algorithm, generating an output-based feedback, which steers the state as close as possible to a given sliding hyperplane in a finite time. The optimality of the designed feedbacks is proven. The efficacy of the proposed algorithm is illustrated by a numerical example.

Keywords: Robust Control; Sliding Mode Control; Distributed Parameter System

1. INTRODUCTION

Robust output-based feedback control algorithms are required for many practical applications. The output-based sliding mode control design methodology is well-developed for finite dimensional systems (see, for example, Edwards, Akoachere, and Spurgeon (2001), Edwards and Spurgeon (1998), Utkin, Guldner, and Shi (2009), Shtessel, Edwards, Fridman, and Levant (2014) and references therein). Infinite dimensional (distributed parameter) systems are widely used in practise, e.g., to model flexible robots, controlled turbulent flows, combustion and other chemical processes. The sliding mode methodology can also be used to design controllers for such complicated systems Orlov and Utkin (1982), Orlov (1983). We refer the reader to Levaggi (2002a, 0, 1), Orlov (2008), Pisano and Orlov (2012), Orlov, Pisano, and Usai (2013) for an extensive overview of the recent achievements in this field.

We stress that, in practice, it is quite difficult to apply the state of the art sliding mode methods in the case of noisy measurements (see, Poznyak (2004); Utkin (1992)) and/or mismatched disturbances (see, Castanos and Fridman (2006); Edwards and Spurgeon (1995); Polyakov and Poznyak (2011)). The aim of this paper is to propose a mathematically sound extension of the sliding mode control methodology allowing one to deal with the aforementioned cases efficiently. Specifically, we consider conventional (first order) sliding mode control principles and study the problem of observer-based sliding mode control design for a plant described by a linear evolution equation in a Hilbert space with additive exogenous disturbances and L^2-bounded deterministic measurement noises. Note that, in this case, the solution of the classical sliding model control problem does not exists, i.e., it is impossible to ensure the ideal/exact sliding mode (even in the finite dimensional case) due to the noise in the measurements. Following Zhuk and Polyakov (2014, 1, 1); Zhuk, Polyakov, and Nakonechniy (2017) we propose to generalize the notion of the solution of the classical sliding model control problem for linear evolution equations, i.e., to construct a control law u providing the state's motion as close as possible (in the minimax sense) to the selected sliding surface. To design such u we first provide a dual description of the reachability set for a linear evolution equation, and then solve the following minimax control problem: find a feedback control u steering the minimax center of the reachability set towards the sliding surface. The dual description of the reachability set relies upon the minimax framework Chernousko (1994); Kurzhanski and Valyi (1997); Milanese and Tempo (1985); Poznyak, Fridman, and Bejarano (2004) and duality argument Zhuk (2009, 1).

The paper is organized as follows. The next section presents the problem statement and basic assumptions. The minimax observer for linear systems is discussed in Section III. The problem of control design is studied in Section IV. Next the numerical simulation results and conclusions are provided. The proofs of the propositions are given in the appendix.

Throughout the paper the following notations are used: H, H_u, H_d, H_y are abstract Hilbert spaces, $\langle x, y\rangle_H$ denotes the canonical inner product of H, $\|x\|_H^2 := \langle x, x\rangle_H$, $\mathscr{L}(H, H)$ denotes the space of linear continuous operators from H to H, $A^\star$ denotes the adjoint of a linear operator A, $\mathscr{D}(A)$ denotes the domain of A, I denotes the identity operator of the corresponding space, $L^2(0, T, H)$ denotes the space of squared integrable functions on $(0, T)$ with values in H.

2. PROBLEM STATEMENT

Assume that $A : H \to H$ generates a strongly continuous semigroup and $t \mapsto x(t) \in H$ is the mild solution of the following linear evolution equation:

$$\frac{dx}{dt} = Ax(t) + Bu(t) + Dd(t)\,, x(0) = x_0\,, \qquad (1)$$

where $x_0 \in H$ – initial condition, $u \in L^2(0, T, H_u)$ – control function, $d \in L^2(0, T, H_d)$ – uncertain disturbance, and $B \in \mathscr{L}(H_u, H)$, $D \in \mathscr{L}(H_d, H)$ are given operators. For the convenience of the reader we recall the concepts of strongly continous semigroups of linear operators and mild solutions at the beginning of appendix.

The output of (1), $y(t) \in H_y$ is measured in the following form:

$$y(t) = Cx(t) + w(t)\,, t \in [0, T)\,, \qquad (2)$$

where $C \in \mathscr{L}(H, H_y)$ is an observation operator, which represents a mathematical model of a gauge, and $w \in L^2(0, T, H_y)$ is unknown deterministic measurement noise.

We further assume that x_0, d, w are uncertain and belong to the following bounding set:

$$\mathscr{E}(T) := \{(x_0, d, w) : \rho_T(x_0, d, w; S, Q, R) \le 1\} \subset H \times L^2(0, T, H_d) \times L^2(0, T, H_y)\,, \qquad (3)$$

where

$$\rho_t(x, d, w; S, Q, R) := \langle Sx, x\rangle_H + \int_0^t \langle Qd(s), d(s)\rangle_{H_d} ds + \int_0^t \langle Rw(s), w(s)\rangle_{H_y} ds\,,$$

and S, Q, R are given self-adjoint positive definite bounded linear operators in H, H_d and H_y respectively. Clearly, ρ_T defines a new norm in the space $H \times L^2(0, T, H_d) \times L^2(0, T, H_y)$, and $\mathscr{E}(T)$ represents the unit ball of this space w.r.t. to ρ. In what follows we suppose that H_u and H_y are abstract Hilbert spaces. In particular case, they can be finite dimensional.

The aim of this paper is to generalize the result of Zhuk and Polyakov (2014, 1, 1); Zhuk et al. (2017) on the sliding mode control design for uncertain finite-dimensional LTI systems to the case of abstract evolution equations. Namely, for a fixed $T < +\infty$ and a finite-rank linear operator $F : H \to H_u$ such that $FB : H_u \to H_u$ is a linear bounded invertible operator, we propose a control law $u \in L^2(0, T, H_u)$ in the form of a function of the output, which solves the minimax version of the classical Mayer optimal control problem:

$$\sup_{(x_0,d,w)\in\mathscr{E}(T)} \|Fx(T)\|_{H_u} \to \min_u \quad \text{s.t. (1) - (2)} \qquad (4)$$

We recall that the classical sliding mode control problem is (see, Orlov (2008); Utkin et al. (2009)) to find a feedback control law u which (i) steers the state of (1) towards a given linear hyperplane $Fx = 0$, and (ii) guarantees that the state does not leave this plane, provided FB is a linear bounded invertible operator. It is worth noting Edwards and Spurgeon (1998); Utkin et al. (2009) that the latter condition (in finite dimensional case) is necessary for existence of a control law, which ensures sliding mode on the surface $Fx = 0$. The problem of selection of the operator F is considered in Orlov (2008). Here we assume that a proper F has been selected.

3. DUAL DESCRIPTION OF THE REACHABILITY SET

According to the classical methodology of the sliding mode control design, the precise knowledge of the so-called sliding variable $\sigma(t) := Fx(t)$ is required in order to ensure the motion of the system (1) on the surface $Fx = 0$. In the considered case this information is not available as the output $y(t)$ is incomplete and noisy and state equation is subject to uncertain disturbances. In the following proposition we construct the a priori reachability set of the evolution equation (1), i.e., the set of all the states of (1) which are compatible with all possible outputs y and uncertainty description $\mathscr{E}(T)$. This representation is then used to solve (4).

Proposition 1. Assume that x is a mild solution of (1) for some $(x_0, d, w) \in \mathscr{E}(T)$. Then, for any $t^* \in [0, T]$ the following estimate holds true:

$$\sup_{(x_0,d,w)\in\mathscr{E}(t^*)} |\langle l, x(t^*) - \hat{x}(t^*)\rangle_H| = \langle l, P(t^*)l\rangle_H^{\frac{1}{2}}\,, \forall l \in H\,, \qquad (5)$$

where

- *the linear bounded self adjoint positive definite operator P is the unique solution of the following infinite-dimensional differential Riccati equation:*

$$\frac{d}{dt}\langle P(t)v, q\rangle_H = \langle P(t)A^\star v, q\rangle_H + \langle P(t)v, A^\star q\rangle_H + \langle DQ^{-1}D^\star v, q\rangle_H - \langle P(t)C^\star RCP(t)v, q\rangle_H, P(0) = S^{-1}, \qquad (6)$$

- *$\hat{x}$ is the unique mild solution of the following evolution equation:*

$$\begin{cases} \dfrac{d\hat{x}(t)}{dt} = A\hat{x}(t) + P(t)C^\star R(y(t) - C\hat{x}(t)) + Bu(t), \\ \hat{x}(0) = 0\,, \end{cases} \qquad (7)$$

The proofs of the propositions are omitted for shortness. Using (Balakrishnan, 1981, p.339, Th.6.8.3) the next corollary can also be proven.

Corollary 1. Assume that A, D and $A^\star, C^\star$ are exponentially stabilizable. Then

$$\lim_{t\to+\infty} |\langle l, x(t) - \hat{x}(t)\rangle_H| \le \langle l, P^\infty l\rangle_H^{\frac{1}{2}}\,, \forall l \in H\,, \qquad (8)$$

where P^∞ is the unique self adjoint solution of the algebraic Riccati equation:

$$\langle P^\infty v, A^\star v\rangle_H + \langle A^\star v, P^\infty v\rangle_H + \langle Q^{-1}D^\star v, D^\star v\rangle_H - \langle RCP^\infty v, CP^\infty v\rangle_H\,. \qquad (9)$$

In addition, $A - P^\infty C^\star RC$ generates an exponentially stable semigroup.

It is worth noting that (5) is describing an ellipsoid, which is centered at vector $\hat{x}(T)$ with axes defined by eigenfunctions of $P(T)$. This ellipsoid is, in fact, the worst-case realisation of the reachability set of (1), i.e., it takes into account all $(x_0, d, w) \in \mathscr{E}(T)$. The estimate (8) describes an ellipsoid which contains all the states of (1) in the limit $t \to \infty$. Finally, we stress that P does not depend on the control parameter u. This suggests to design the controller u as a function of the center of the ellipsoid, $\hat{x}$.

4. CONTROL DESIGN

Denoting the sliding variable by $\sigma = Fx$ we derive

$$\sigma(T) = Fx(T) = \hat{\sigma}(T) + Fe(T),$$
$$|\langle l, e(T)\rangle_H| \le \langle l, P(T)l\rangle_H^{\frac{1}{2}}, \quad \forall l \in H,$$

where $\hat{\sigma}(T) = F\hat{x}(T)$, and $\hat{x}$ satisfies (7).

Proposition 2. If the control u verifies the following equality:

$$\hat{\sigma}(T) = 0 \tag{10}$$

then it solves the minimax control problem (4).

Usually (see, e.g. Orlov (2008), Levaggi (2013)), additional technical considerations are required in order to apply a discontinuous sliding mode control and prove the existence of solutions in this case. The latter proposition allows us to construct a continuous feedback control which verifies the condition (10): indeed, define

$$u_{eq}(t) = -(FB)^{-1}F\left[A\hat{x}(t) + P(t)C^\star R(y(t) - C\hat{x}(t))\right], \tag{11}$$

We claim that u_{eq} solves the problem (4). Indeed, $\forall v \in H_u$ this feedback provides

$$\langle F\hat{x}(t), v\rangle_{H_u} =$$
$$\int_0^t \langle FA\hat{x}(s) + FP(s)C^\star R(y(s) - C\hat{x}(s)) + FBu(s), v\rangle_{H_u} ds =$$
$$\int_0^t \langle (F - FB(FB)^{-1}F)[A\hat{x}(s) + P(s)C^\star R(y(s) - C\hat{x}(s))], v\rangle_{H_u} ds$$
$$= 0.$$

In fact, this feedback is an analog of "equivalent control" in a sliding mode control system, which can be found explicitly: indeed, u_{eq} depends on P and $\hat{x}$ which, *in the case of observer-based control design, can be computed* (numerically or even analytically in some cases considered in the following section). Moreover, $\hat{x}$ starts at the linear sliding hypersurface $F\hat{x} = 0$ as $\hat{x}(0) = 0$ so the minimax center of the reachability set stays on the hyperplane $F\hat{x} = 0$ and the actual state $x(t)$ fluctuates in the ellipsoid centered at $\hat{x}$, i.e.,

$$|\langle Fx(t), v\rangle_{H_u}| \le \langle F^\star v, P(t)F^\star v\rangle_H^{\frac{1}{2}}, \quad \forall v \in H_u.$$

The speed at which $x(t)$ approaches the sliding hyperplane is proportional to the speed of the decay of the corresponding eigen-values of P. In the infinite-horizon case, the actual state of the plant reaches the sliding surface exactly, provided $\langle F^\star v, P^\infty F^\star v\rangle_H^{\frac{1}{2}} = 0$ for any $v \in H_u$ (see (8)).

5. EXAMPLE

In the numerical example we will present the analytical solutions of (6)-(7) and derive the corresponding formula for the sliding mode control. To this end we assume that $H = H^\star$ and A is a linear symmetric operator such that $\langle -A\psi, \psi\rangle_H \ge c\|\psi\|_H^2$ for a given $c > 0$ and A has a complete orthogonal system of eigenvectors $\{e_k\}$, i.e., e_k is the unique solution of the following equation: $\langle -Ae_k, \psi\rangle_H = \lambda_k e_k$, $\forall\psi \in \mathscr{D}(A)$. We further assume that $B = I$, $D = I$, $S = I$, $Q = I$ where I denotes the identity operator in the corresponding space and $F : H \to \mathbb{R}^M$ is defined as follows

$$Fx(T) = [\,\langle e_1, x(T)\rangle_H \ \dots \ \langle e_M, x(T)\rangle_H\,].$$

We suppose that the observation operator, C is of finite rank, i.e.,

$$y_i(t) = \langle e_i, x(t)\rangle_H + w_i\,, i = 1\dots N$$

and set $R := I$ so that $H_y = \mathbb{R}^N$ and $C^\star RCx = \sum_{k=1}^N \langle e_k, x\rangle_H e_k$. For simplicity we assume that $N < M$.

In this setting, the Riccati equation reads as follows:

$$\frac{d}{dt}\langle P(t)e_i, e_j\rangle_H = -\lambda_i\langle P(t)e_i, e_j\rangle_H - \lambda_j\langle P(t)e_i, e_j\rangle_H$$
$$+\langle e_i, e_j\rangle_H - \sum_{k=1}^N \langle P(t)e_k, e_i\rangle_H\langle P(t)e_k, e_j\rangle_H\,, \langle P(0)e_i, e_j\rangle_H$$
$$= \langle e_i, e_j\rangle_H\,,$$

for all e_i and e_j. The above equation is the scalar Riccati differential equation, and since $\langle e_i, e_j\rangle_H = 0$ if $i \ne j$, it follows that $\langle P(t)e_i, e_j\rangle_H \equiv 0$ if $i \ne j$ and so we get that the Riccati operator P is diagonal, i.e.,

$$P(t)x = \sum_{j=1}^\infty P_i(t)\langle e_j, x\rangle_H e_j$$

where P_j solves an ODE of the following form:

$$\dot{P}_j = -2\lambda_j P_j + \langle e_j, e_j\rangle_H - P_j^2\,, P_j(0) = 1\,, j \le N\,,$$
$$\dot{P}_j = -2\lambda_j P_j + \langle e_j, e_j\rangle_H\,, P_j(0) = 1\,, j > N\,,$$

As a result, the equation (7) reads as follows:

$$\frac{d\langle\hat{x}(t), e_i\rangle_H}{dt} = -\lambda_i\langle\hat{x}(t), e_i\rangle - \sum_{k=1}^N \langle e_k, \hat{x}\rangle_H\langle P(t)e_k, e_i\rangle_H +$$
$$\sum_k y_k(t)\langle P(t)e_k, e_i\rangle_H + \langle u(t), e_i\rangle_H = -\lambda_i\langle\hat{x}(t), e_i\rangle +$$
$$P_i(t)(y_i - \langle e_i, \hat{x}\rangle_H) + \langle u(t), e_i\rangle_H\,, \langle\hat{x}(0), e_i\rangle_H = 0\,, i \le N\,,$$
$$\tfrac{d\langle\hat{x}(t), e_i\rangle_H}{dt} = -\lambda_i\langle\hat{x}(t), e_i\rangle_H + \langle u(t), e_i\rangle_H\,,$$
$$\langle\hat{x}(0), e_i\rangle_H = 0\,, i > N\,.$$

Now, if we set

$$\hat{u}_i(t) := \lambda_i\langle\hat{x}(t), e_i\rangle - P_i(t)(y_i - \langle e_i, \hat{x}\rangle_H)\,, i \le N\,,$$
$$\hat{u}_i(t) := \lambda_i\langle\hat{x}(t), e_i\rangle\,, N < i \le M\,,$$

then $\hat{u} = \sum_{i=1}^M \frac{\hat{u}_i}{\|e_i\|_H^2}e_i$ realizes the desired sliding mode control which steers $\hat{x}$ to $F\hat{x} = 0$ and keeps it there. Note that in this case the control becomes $\hat{u} = -\sum_{i=1}^N \frac{P_i(t)y_i}{\|e_i\|_H^2}e_i$.

For simulation we select $A = -\frac{\partial^2}{\partial z^2}$, $\mathscr{D}(A) = \{v \in L^2(0,1,\mathbb{R}) : v(0) = v(1) = 0\}$. In this case, $\lambda_k = (\pi k)^2, e_k = \sqrt{2}\sin(k\pi z), z \in [0,1]$. Let $N = 1$ and $M = 2$ then

$$\dot{P}_1(t) = -2\pi^2 P_1(t) + 1 - P_1^2, \quad P_1(0) = 1,$$

and

$$u(t,z) = -\sqrt{2}P_1(t)y_1(t)\sin(\pi z).$$

The simulation has been done in MATLAB using **pdepe** command and grid method for $0 \le t \le 0.5$ and $0 \le z \le 1$. The perturbations have been selected as follows $w_1(t) = \mathrm{sign}(\sin(5t))/\sqrt{3}$ and $(d(t))(z) = z(1-z)/\sqrt{(3)}$. The initial condition is selected as $x(0) = z(1-z)$. The result of simulation is presented at Fig. 1. The simulation has also been done for the uncontrolled case $u = 0$ showing that the designed control reduces the effect of perturbations almost twice: $\|x(0.5)\|_{L_2} = 0.0045$ provided u_{eq} has been applied, and $\|x(0.5)\|_{L_2} = 0.0083$ without control application.

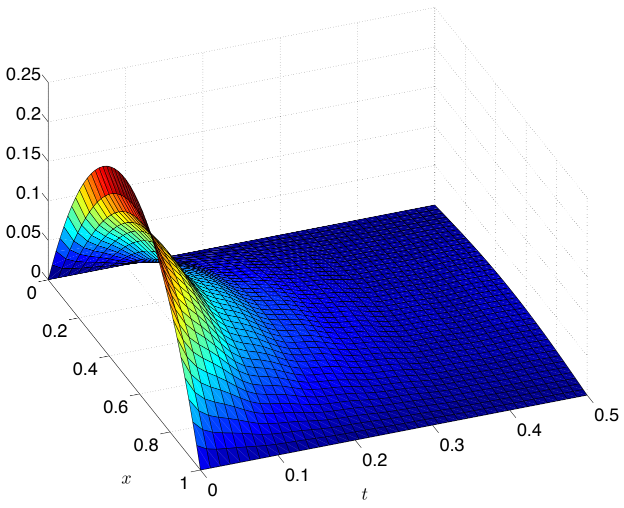

Fig. 1. The simulation results for controlled heat equation

6. CONCLUSION

The minimax sliding mode control which solves (4) generalizes the conventional sliding mode control: it steers the state of (1) towards the hyperplane $Fx(T) = 0$ as close as possible (in the minimax sense) as the exact reaching $Fx(T) = 0$, required in the definition of the conventional sliding mode control, cannot be guaranteed due to *unknown measurement noises* and *uncertain model disturbances*. We conjecture that the exact reaching may be guaranteed provided the model disturbance and measurement noise "disappear" after a given time instant T^*. This latter question will require a modification of the differential Riccati equation and is left for the future research. We stress that the exact "numerical" reaching, i.e., the distance between the actual state x and the sliding hyperplane is negligible, is possible, provided the eigen-values of the Riccati operator $P(t)$ rapidly decay to zero (see (5)), and the null-space of the algebraic Riccati operator P^∞ contains the sliding hyperplane.

7. ACKNOWLEDGMENTS

This research was partially supported by ANR Finite4SoS Project (ANR 15 CE23 0007).

REFERENCES

Balakrishnan, A. (1981). *Applied Functional Analysis*. Springer Verlag.

Castanos, F. and Fridman, L. (2006). Analysis and design of integral sliding manifolds for systems with unmatched perturbations. *IEEE Transactions on Automatic Control*, 51(5), 853–858.

Chernousko, F. (1994). *State Estimation for Dynamic Systems*. Boca Raton, FL: CRC.

Edwards, C., Akoachere, A., and Spurgeon, S.K. (2001). Sliding-mode output feedback controller design using linear matrix inequalities. *IEEE Transactions on Automatic Control*, 46(1), 115–119.

Edwards, C. and Spurgeon, S. (1998). *Sliding Mode Control: Theory And Applications*. CRC Press.

Edwards, C. and Spurgeon, S.K. (1995). Sliding mode stabilization of uncertain systems using only output information. *Int. J. of Control*, 62(5), 1129–1144.

Kurzhanski, A. and Valyi, I. (1997). *Ellipsoidal calculus for estimation and control*. Birkhauser Boston Inc., Boston, MA.

Levaggi, L. (2002a). Infinite dimensional systems sliding motions. *European Journal of Control*, 8, 508–518.

Levaggi, L. (2002b). Sliding modes in banach spaces. *Differential and Integral Equations*, 15, 167–189.

Levaggi, L. (2013). Existence of sliding motions for nonlinear evolution equations in banach space. *Discrete and Continuous Dynamical Systems*, special issue, 477–487.

Milanese, M. and Tempo, R. (1985). Optimal algorithms theory for robust estimation and prediction. *IEEE Tran. Autom. Cont.*, 30(8), 730–738.

Orlov, Y. (1983). Application of lyapunov method in distributed systems. *Automation and Remote Control*, 44, 426–430.

Orlov, Y. (2008). *Discontinuous systems: Lyapunov analysis and robust synthesis under uncertainty conditions*. Springer.

Orlov, Y., Pisano, A., and Usai, E. (2013). Boundary control and observer design for an uncertain wave process by second-order sliding-mode technique. In *Proceeding of IEEE Conference on Decision and Control*.

Orlov, Y. and Utkin, V.I. (1982). Use of sliding modes in distributed system control problems. *Automation and Remote Control*, 43, 1127–1135.

Pisano, A. and Orlov, Y. (2012). Boundary second-order sliding-mode control of an uncertain heat process with unbounded matched perturbation. *Automatica*, 48, 1768–1775.

Polyakov, A. and Poznyak, A. (2011). Invariant ellipsoid method for minimization of unmatched disturbance effects in sliding mode control. *Automatica*, 47(7), 1450–1454.

Poznyak, A., Fridman, L., and Bejarano, F. (2004). Minimax integral sliding-mode control for multimodel linear uncertain systems. *IEEE Transactions on Automatic Control*, 49(1), 97–102.

Poznyak, A. (2004). *Variable Structure Systems: from Principles to Implementation*, volume 66 of *IEE Control Series*, chapter Deterministic output noise effects in sliding mode observation, 45–78. The IET, London, UK.

Shtessel, Y., Edwards, C., Fridman, L., and Levant, A. (2014). *Sliding Mode Control and Observation*. Birkhauser.

Utkin, V.I. (1992). *Sliding Modes in Control and Optimization*. Springer Verlag, Berlin.

Utkin, V., Guldner, J., and Shi, J. (2009). *Sliding Mode Control in Electro-Mechanical Systems*. CRC Press.

Zhuk, S. (2009). Estimation of the states of a dynamical system described by linear equations with unknown parameters. *Ukrainian Math. J.*, 61(2), 214–235.

Zhuk, S. (2013). Kalman duality principle for a class of ill-posed minimax control problems with linear differential-algebraic constraints. *Applied Mathematics and Optimisation*, 68(2), 289–309.

Zhuk, S. and Polyakov, A. (2014). On output-based sliding mode control design using minimax observer. In *13th Workshop on Variable Structure Systems, Nantes, France.*

Zhuk, S. and Polyakov, A. (2015). On output-based sliding mode control design using minimax observer. In *European Control Conference, Linz, Austria.*

Zhuk, S. and Polyakov, A. (2016). *Variable Structure Systems: from principles to implementation*, chapter Minimax Observer for Sliding Mode Control Design. IET.

Zhuk, S., Polyakov, A., and Nakonechniy, O. (2017). Note on minimax sliding mode control design for linear systems. *IEEE Transaction on Automatic Control.* DOI: 10.1109/TAC.2016.2612058.