Impedance transmission conditions for eddy current problems

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ABSTRACT. We present impedance conditions up to the second order of approximation for the time-harmonic eddy current problem with a thin conducting sheet. The conditions are derived asymptotically for vanishing sheet thickness ε where the skin depth is scaled like ε. The first order condition is the perfect electric conductor boundary condition. The second order condition turns out to be a generalized Poincaré-Steklov map between tangential components of the magnetic field and the electric field.

CONTENTS

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1. INTRODUCTION

Many electronic devices feature thin plates or sheets of metal supplying efficient electromagnetic shielding. Due to their high conductivity and large aspect ratio of the plates the shielding properties are reached with a minimum use of metal. Precisely their large aspect ratio makes the numerical simulation of such devices more of a challenge, especially if standard methods like the finite element method (FEM) shall be applied. If the sheet is thin and, if, moreover, the magnetic fields decay rapidly inside the layer due to high conductivities (the skin effect), then meshes with very small cells are required. The numerical modeling is much simplified if the thin sheet is replaced by transmission conditions on its mid-surface. Meshes with much larger cells can be used with the so called *impedance transmission conditions*, which relate the electric and magnetic fields on both sides of the interface.

The eddy current problem has become an important research topic worldwide since several decades [5, 20, 22, 6, 16, 1, 8, 29]. In this paper we address the issue of impedance transmission conditions (ITCs) for highly conducting and thin sheets for the eddy current problem in three dimensions. The concept of ITCs is rather classical in the context of electromagnetic wave propagation phenomena. Already in 1902 Levi-Civita introduced ITCs [21] (see also [3, 34]) for Maxwell's equations. He postulated that the electric field is continuous over the interface whereas the magnetic field has a discontinuity, which is proportional to the sheet thickness and conductivity. These conditions fit naturally with boundary integral formulations [19, 24] as well as FEMs [27, 28, 4, 18]. Schmidt and Tordeux [32] have shown for the eddy current model in 2D that these conditions, which they call ITC-1-0, appear as the asymptotic limit when the sheet thickness ε tends to zero while the conductivity tends to infinity like $1/\varepsilon$. In this case it has been shown [29] that these transmission conditions exhibit a robust linear error reduction with ε, i.e., independent of the conductivity or frequency. Recently, conditions in the framework of [32] have been derived for the axisymmetric setting and varying thickness [15].

Alternatively, the so called *thin layer boundary condition* [35, 20, 22, 17, 13] are derived taking into account the boundary layer behaviour of the solution. More precisely, the fields are for higher conductivities/frequencies hyberbolic functions inside the sheet. In this way, the thin layer boundary conditions exhibit jumps for the electromagnetic field and involve sheet thickness and conductivity as arguments of hyberbolic functions. The thin layer boundary conditions come with an increased complexity, but it has been shown that they do not lead to lower error levels than the simpler ITC-1-0 conditions [29].

ITCs with drastically reduced error levels can systematically been derived by an *asymptotic analysis* of the Maxwell's equations with conducting sheets where the sheet thickness ε tends to zero. For example, for the eddy current model in 2D two families of ITCs have been derived using asymptotic expansions, this is the family ITC-1-N [33] in which the conductivity is scaled like $1/\varepsilon$, and the family ITC-2-N [30], in which the conductivity is scaled like $1/\varepsilon^2$. For both families N corresponds to the order, where the convergence of the modelling error outside the sheet is like ε^{N+1} in their respective asymptotic regime. This convergence is not always robust in terms of the conductivity [29] but it turns out that the ITC-2-0 and ITC-2-1 conditions are robust and can be used from very low to very high frequencies. For the ITC-2-1 conditions a quadratic convergence of the modelling error in the sheet thickness has been observed numerically, which is uniform in the conductivity or frequency. The many different ITCs in two dimensions can be easily integrated in FEMs or boundary element methods [31].

Since most electromagnetic devices require the modeling in three dimensions we aim to derive ITCs for the time-harmonic eddy current problem in 3D. To obtain robust conditions as the ITC-2-1 in 2D we consider the asymptotic regime in which the conductivity is scaled like $1/\varepsilon^2$. We consider the general case of curved thin sheets where the magnetic permeability may take different values inside and on the two sides of the sheet. Here, we use techniques to derive transmission conditions for electromagnetic models in 3D including thin layers, see *e.g.* [10, 9, 11, 26].

There are several similarities in this work and in the work in Reference [26] in which Péron *et al.* have proposed ITCs of order 1 and 2 for the time-harmonic Maxwell equations in three dimensions. The transmission conditions are derived asymptotically for vanishing sheet thickness ε where the skin depth is kept proportional to ε. The condition of order 1 turns out to be the perfect electric conductor (PEC) boundary condition. The conditions of order 2 appear as generalized Poincaré-Steklov maps between tangential components of the magnetic field and the electric field, and they are of Wentzell type involving second order surface differential operators. However the conditions of order 2 are not adapted for the eddy current problem.

In this paper we derive ITCs up to the second order of approximation for the eddy current problem in 3D. The condition of order 1 is the PEC boundary condition. The conditions of order 2 turn out to be Poincaré-Steklov maps between tangential components of the magnetic field and the electric field. A main contribution of this work is the proof of well-posedness and stability results for the second order asymptotic model. Those conditions of order 2 are simpler than the second order conditions for the time-harmonic Maxwell equations [26] since the new Poincaré-Steklov map is the scalar part of the Poincaré-Steklov operator of Wentzell type [26].

This paper is organized as follows. Section 2 presents the model with a formulation in terms of the magnetic field and a formal derivation of impedance conditions based on an asymptotic expansion in the thickness parameter ε. Then, in Section 3 as main results the equivalent model of order 1, which satisfies PEC boundary conditions, and the equivalent model of order 2 are given. The derivation of the equivalent models is based on an asymptotic expansion which is presented in detail and order by order in Section 4. The equivalent model of order 2 involves as an impedance transmission condition a generalized Poincaré-Steklov map (tangential components of magnetic field to tangential components of electric field), whose structure simplifies for a symmetric configuration of material constants. We prove stability results for this model of order 2. We introduce at the end of the Section 3 a regularized variational formulation for the second order model. Finally we introduce variational formulations for the exact electric field and for the limit model $\mathbf{E}_0$, and we prove stability and convergence results in Appendix A.

2. The mathematical model and equivalent models with transmission conditions

After the introduction of notations in Sec. 2.1 and the mathematical model for the electric and magnetic field in Sec. 2.2 we present the magnetic field formulation in Sec. 2.3. Then we present a guideline on the derivation of impedance conditions (section 2.4), where impedance conditions for the considered model will be given up to order 2 in the Sec. 3.

2.1. **Notations.** For any orientable and closed surface Γ of $\mathbb{R}^3$ the unit normal vector $\mathbf{n}$ on Γ is outwardly oriented from the interior domain enclosed by Γ towards the outer domain, see e.g. Fig. 1. Let $\mathbf{v}$ a vector field on Γ, then we denote by

$$\mathbf{v}_{\mathbf{T}} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) \ ,$$

the vector field of its tangent components and the space of L^2-integrable tangent vector fields by $\mathbf{L}^2_t(\Gamma) := \{\mathbf{v} \in (\mathrm{L}^2(\Gamma))^3,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}$.

We denote by $\mathbf{curl}_\Gamma$ the tangential rotational operator (which applies to functions defined on Γ) and by curl_Γ the surface rotational operator (which applies to vector fields) [25, 11, 26] :

$$\forall\, f \in C^\infty(\Gamma), \quad \mathbf{curl}_\Gamma\, f = (\nabla_\Gamma f) \times \mathbf{n}\ ,$$
$$\forall\, \mathbf{v} \in (C^\infty(\Gamma))^3, \quad \mathrm{curl}_\Gamma\, \mathbf{v} = \mathrm{div}_\Gamma\,(\mathbf{v}\times\mathbf{n})\ ,$$

where ∇_Γ and div_Γ are respectively the tangential gradient and the surface divergence on Γ. This allows us to define the space of tangent vector fields of the operator curl_Γ [25, 11, 26]:

$$\mathsf{TH}(\mathrm{curl}_\Gamma, \Gamma) = \{\mathbf{v} \in \mathbf{L}^2_t(\Gamma),\ \mathrm{curl}_\Gamma\, \mathbf{v} \in \mathrm{L}^2(\Gamma)\,\}\ ,$$

which is, equipped with the graph norm of curl_Γ, a Hilbert space.

Let Ω_- and Ω_+ be Lipschitz domains with a common interface $\Gamma := \partial\Omega_- \cap \partial\Omega_+$, which is a closed set, and let $\mathbf{n}$ on Γ be the unit normal vector directed into Ω_+ (see Fig. 2). Then, for functions $f \in C^\infty(\Omega_\pm)$, which are possibly discontinuous over the interface Γ, we denote by $[f]_\Gamma$ the jump of f across Γ:

$$[f]_\Gamma = f|_{\Gamma^+} - f|_{\Gamma^-}$$

where for any $\mathbf{x}_\Gamma \in \Gamma$ the one-sided traces are defined by

$$f|_{\Gamma^\pm}(\mathbf{x}_\Gamma) := \lim_{s\to 0^\pm} f(\mathbf{x}_\Gamma + s\,\mathbf{n}).$$

Furthermore, we denote by $\{f\}_\Gamma$ the mean value of f across Γ:

$$\{f\}_\Gamma = \frac{1}{2}(f|_{\Gamma^+} + f|_{\Gamma^-})\ .$$

We use the same definition for vector fields $\mathbf{v} \in (C^\infty(\Omega_\pm))^3$, and with an abuse of notation, for the tangential traces:

$$\{\mathbf{v}\times\mathbf{n}\}_\Gamma := \{\mathbf{v}\}_\Gamma \times \mathbf{n}, \qquad [\mathbf{v}\times\mathbf{n}]_\Gamma := [\mathbf{v}]_\Gamma \times \mathbf{n},$$
$$\{\mathbf{v}_\mathbf{T}\}_\Gamma := (\{\mathbf{v}\}_\Gamma)_\mathbf{T}\,, \qquad [\mathbf{v}_\mathbf{T}]_\Gamma := ([\mathbf{v}]_\Gamma)_\mathbf{T}\,.$$

Finally, we define by $\mathbf{H}(\mathrm{curl}, \Omega_\pm)$ the completion of the space $(C^\infty(\overline{\Omega}_\pm))^3$ with respect to the natural graph norm of curl, which is a Hilbert space as well. Then, for vector fields $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega_\pm)$ both the jump $[\mathbf{v}\times\mathbf{n}]_\Gamma$ and the mean value $\{\mathbf{v}\times\mathbf{n}\}_\Gamma$ are in the Hilbert space

$$\mathsf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{v} \in (H^{-\frac{1}{2}}(\Gamma))^3,\ \mathrm{div}_\Gamma\, \mathbf{v} \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)\,\}\ ,$$

which is, equipped with the graph norm $(\|\mathbf{v}\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|\,\mathrm{div}_\Gamma\, \mathbf{v}\|^2_{H^{-\frac{1}{2}}(\Gamma)})^{\frac{1}{2}}$ of the operator div_Γ, the dual of $\mathsf{TH}(\mathrm{curl}_\Gamma, \Gamma)$.

2.2. **The time-harmonic eddy current problem.** Throughout the paper we denote by $\Omega \subset \mathbb{R}^3$ the domain of interest, which is composed of three subdomains (see Figure 1) as

$$\Omega = \Omega^\varepsilon_- \cup \overline{\Omega^\varepsilon_o} \cup \Omega^\varepsilon_+$$

corresponding to different linear materials. The subdomain Ω^ε_o is a thin layer of constant thickness ε surrounding the subdomain Ω^ε_-. The boundary of the subdomain Ω^ε_- is the smooth surface denoted by Γ^ε_- while Γ^ε_+ is the boundary of the subdomain $\overline{\Omega^\varepsilon_-} \cup \Omega^\varepsilon_o$. The mid-surface of the thin layer Ω^ε_o is denoted by Γ. In all that follows, unless specified, all the considered domains are smooth domains in $\mathbb{R}^3$.

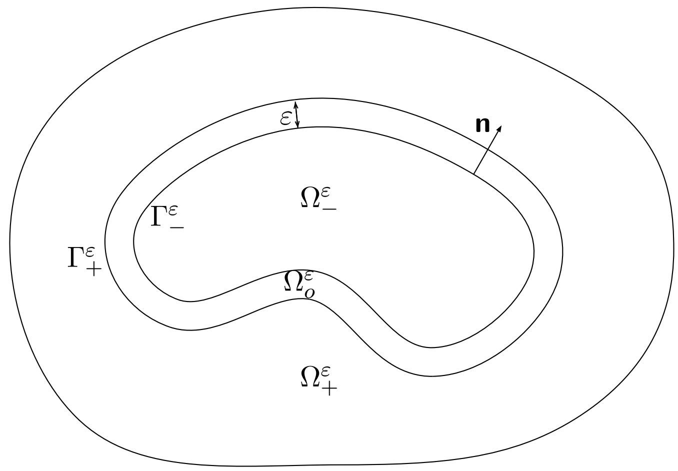

FIGURE 1. A cross-section of the domain Ω and the subdomains Ω_-^ε, Ω_o^ε, Ω_+^ε.

The electromagnetic properties in Ω are given by the piecewise-constant functions $\underline{\mu}^\varepsilon$, and $\underline{\sigma}^\varepsilon$ corresponding to the respective magnetic permeability, and conductivity of the possibly different materials in the three subdomains. They are given by

$$\underline{\mu}^\varepsilon = \begin{cases} \mu_-, & \text{in } \Omega_-^\varepsilon, \\ \mu_o, & \text{in } \Omega_o^\varepsilon, \\ \mu_+, & \text{in } \Omega_+^\varepsilon, \end{cases} \qquad \underline{\sigma}^\varepsilon = \begin{cases} 0, & \text{in } \Omega_-^\varepsilon, \\ \sigma_o^\varepsilon = \varepsilon^{-2}\tilde{\sigma}, & \text{in } \Omega_o^\varepsilon, \\ 0, & \text{in } \Omega_+^\varepsilon. \end{cases}$$

We consider ε as a parameter, on which $\underline{\mu}^\varepsilon$ depend through the definition of the subdomains, where in $\underline{\sigma}^\varepsilon$ in addition we assume an explicit dependence of the layer conductivity σ_o^ε on ε. With this correlation the thinner is the layer, the larger is the conductivity in the layer. The dependence like ε^{-2} corresponds for $\varepsilon \to 0$ to asymptotically constant ratio of skin depth $d_{\text{skin}} = \sqrt{2/(\omega\mu_o\sigma_o^\varepsilon)}$ and thickness ε [29, 26], i.e., they behave the same for $\varepsilon \to 0$.

Let us denote by $\mathbf{j}$ the time-harmonic current source (with time convention $\exp(-\mathrm{i}\omega t)$) and let $\omega > 0$ be the angular frequency. For the sake of simplicity, we assume that $\mathbf{j}$ is smooth enough, $\mathbf{j}$ is divergence free ($\operatorname{div}\mathbf{j} = 0$ in Ω) and the support of $\mathbf{j}$ does not meet the layer Ω_o^ε, and we write $\mathbf{j}_\pm = \mathbf{j}$ in $\Omega_\pm^\varepsilon$. The time-harmonic eddy current problem is [6, 16, 8] :

$$\operatorname{curl}\mathbf{E}^\varepsilon - i\omega\underline{\mu}^\varepsilon\mathbf{H}^\varepsilon = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H}^\varepsilon - \underline{\sigma}^\varepsilon\mathbf{E}^\varepsilon = \mathbf{j} \quad \text{in} \quad \Omega\,, \tag{2.1a}$$

$$\operatorname{div}\mathbf{E}_\pm^\varepsilon = 0 \quad \text{in} \quad \Omega_-^\varepsilon \cup \Omega_+^\varepsilon \quad \text{and} \quad \int_{\Gamma_\pm^\varepsilon} \mathbf{E}_\pm^\varepsilon \cdot \mathbf{n}\,\mathrm{d}S = 0\,. \tag{2.1b}$$

The equations (2.1a) are Faraday's and Ampère's laws without displacement current. These equations link the electric field $\mathbf{E}^\varepsilon$ and the magnetic field $\mathbf{H}^\varepsilon$. The equations (2.1b) are *gauge conditions* for the electric field, we refer the reader to Remark 3 in [8] for a justification of such conditions. We complement this problem with a perfect electric conductor (PEC) boundary condition

$$\mathbf{E}^\varepsilon \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}^\varepsilon \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega\,.$$

2.3. **Magnetic field formulation.** The system of first order partial differential equations (2.1) can be reduced to a system of second order partial differential equations for the magnetic field by eliminating the electric field [5, 1]

$$\operatorname{curl}\operatorname{curl}\mathbf{H}_o^\varepsilon - \mathrm{i}\omega\mu_o\sigma_o^\varepsilon\mathbf{H}_o^\varepsilon = 0, \quad \text{in} \quad \Omega_o^\varepsilon\,, \tag{2.2a}$$

$$\operatorname{curl}\mathbf{H}_\pm^\varepsilon = \mathbf{j}_\pm, \quad \text{in} \quad \Omega_-^\varepsilon \cup \Omega_+^\varepsilon\,, \tag{2.2b}$$

$$\operatorname{div}\mathbf{H}_\pm^\varepsilon = 0, \quad \text{in} \quad \Omega_-^\varepsilon \cup \Omega_+^\varepsilon\,, \tag{2.2c}$$

with the transmission conditions across the two conductor surfaces Γ_+^ε and Γ_-^ε

$$\mathbf{H}_\pm^\varepsilon \times \mathbf{n} = \mathbf{H}_o^\varepsilon \times \mathbf{n}, \quad \text{on} \quad \Gamma_\pm^\varepsilon\,, \tag{2.2d}$$

$$\mu_\pm\mathbf{H}_\pm^\varepsilon \cdot \mathbf{n} = \mu_o\mathbf{H}_o^\varepsilon \cdot \mathbf{n}, \quad \text{on} \quad \Gamma_\pm^\varepsilon\,, \tag{2.2e}$$

and with the perfect magnetic conductor (PMC) boundary condition

$$\mathbf{H}^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.2f}$$

Here $\mathbf{H}_\dagger^\varepsilon$, $\dagger = -, o, +$ denote the restrictions of $\mathbf{H}^\varepsilon$ to the respective subdomain $\Omega_\dagger^\varepsilon$.

2.4. **Guideline on the derivation of impedance conditions.** In this section we give a guideline on the derivation of impedance conditions for the magnetic field (2.2), which is based on an asymptotic expansion in the thickness parameter ε in Sect. 4. We will then propose two equivalent models $\mathbf{H}_0$ (in Sec. 3.1) and $\mathbf{H}_\varepsilon^1$ (in Sec. 3.2) for the magnetic field. The first model $\mathbf{H}_0$ is of order 1 , i.e., it satisfies at least formally $\mathbf{H}^\varepsilon - \mathbf{H}_0 = \mathcal{O}(\varepsilon)$ and the second model $\mathbf{H}_\varepsilon^1$ is of order 2, i.e. it satisfies at least formally $\mathbf{H}^\varepsilon - \mathbf{H}_\varepsilon^1 = \mathcal{O}(\varepsilon^2)$. These models are defined in ε-independent domains Ω_-, Ω_+, where Ω_- denotes the domain Ω_-^ε in the limit $\varepsilon \to 0$ and Ω_+ the domain Ω_+^ε for $\varepsilon \to 0$, *i.e.* $\Omega_+ = \Omega \setminus \overline{\Omega_-}$ (see Figure 2).

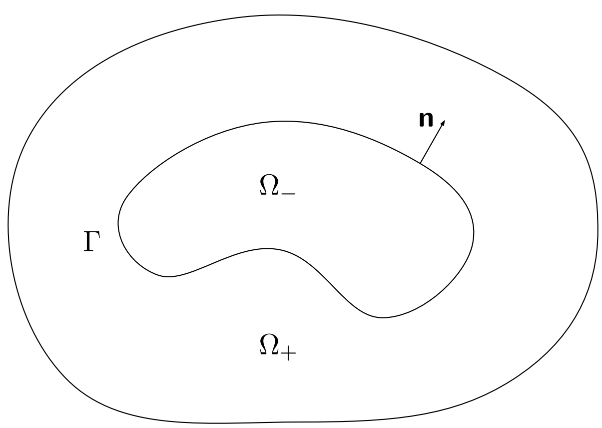

FIGURE 2. A cross-section of the domain Ω and the subdomains Ω_-, Ω_+.

To define these equivalent models it is convenient to introduce the electromagnetic properties of the "background" problem by simple extension of the values of $\underline{\mu}^\varepsilon$ outside the sheet in the extended domains Ω_- and Ω_+:

$$\mu = \begin{cases} \mu_-, & \text{in } \Omega_-, \\ \mu_+, & \text{in } \Omega_+. \end{cases}$$

In the following we present briefly a formal derivation of impedance conditions. We summarize this process in two steps.

First step : a multiscale expansion. The first step consists in deriving a multiscale expansion for the solution $\mathbf{H}^\varepsilon$ of the model problem (2.2) : it possesses an asymptotic expansion in power series of the small parameter ε

$$\mathbf{H}^\varepsilon(\mathbf{x}) \approx \mathbf{H}_0(\mathbf{x}) + \varepsilon\mathbf{H}_1(\mathbf{x}) + \varepsilon^2\mathbf{H}_2(\mathbf{x}) + \cdots \qquad \text{for a.e.} \quad \mathbf{x} \in \Omega_-^\varepsilon \cup \Omega_+^\varepsilon\ , \tag{2.3a}$$

$$\mathbf{H}^\varepsilon(\mathbf{x}) \approx \underline{\mathfrak{H}}_0\left(y_\alpha, \frac{h}{\varepsilon}\right) + \varepsilon\underline{\mathfrak{H}}_1\left(y_\alpha, \frac{h}{\varepsilon}\right) + \cdots \qquad \text{for a.e.} \quad \mathbf{x} \in \Omega_o^\varepsilon\ . \tag{2.3b}$$

Here, $\mathbf{x} \in \mathbb{R}^3$ are the cartesian coordinates, (y_α, h) is a local *normal coordinate system* [7] to the surface Γ in the thin layer Ω_o^ε where y_α, $\alpha = 1, 2$ are tangential coordinates on Γ and $h \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ is the normal coordinate to Γ. Moreover, the term $\underline{\mathfrak{H}}_j$ is a "*profile*" defined on $\Gamma \times (-\frac{1}{2}, \frac{1}{2})$. Note, that the intrinsic domain of the "far field terms" $\mathbf{H}_j$ is $\Omega_- \cup \Omega_+$. The first terms $(\underline{\mathfrak{H}}_j, \mathbf{H}_j)$ for $j = 0, 1$ are formally derived step by step in Section 4.

The derivation is based on an expansion of the differential operators inside the thin layer Ω_o^ε in terms of ε, a Taylor expansion of $\mathbf{H}_j|_{\Gamma_\pm^\varepsilon}$ around the mid-surface Γ and a collection of terms of same powers of ε in the governing PDE inside and outside the sheet, the transmission conditions for the traces on $\Gamma_\pm^\varepsilon$ and the boundary conditions. Since, moreover, the terms $\underline{\mathfrak{H}}_j$ of the expansion inside the sheet can be explicitly expressed in terms of $\mathbf{H}_i$, $i = 0, \ldots, j-1$ we obtain formally

$$\operatorname{curl} \mathbf{H}_j^\pm = \mathbf{j}^\pm\, \delta_{0j}\ , \quad \text{in } \Omega_\pm\ , \tag{2.4a}$$

$$\operatorname{div} \mathbf{H}_j^\pm = 0\ , \quad \text{in } \Omega_\pm\ , \tag{2.4b}$$

$$\mathbf{H}_j^+ \cdot \mathbf{n} = 0\ , \qquad \text{on } \partial\Omega\ , \tag{2.4c}$$

for the restrictions $\mathbf{H}_j^\pm$ of $\mathbf{H}_j$ to $\Omega_\pm$, with boundary conditions on Γ:

$$\mu_\pm \mathbf{H}_j^\pm \cdot \mathbf{n} = \sum_{i=0}^{j} \mathcal{G}_i^\pm \begin{pmatrix} \{(\mathbf{H}_{j-i})_\mathbf{T}\}_\Gamma \\ [(\mathbf{H}_{j-i})_\mathbf{T}]_\Gamma \end{pmatrix}\ . \tag{2.4d}$$

Here, $\delta_{0j} = 1$ if $j = 0$ and zero otherwise and $\mathcal{G}_i^\pm$ in the discrete convolution on the right hand side of (2.4d) are differential operators on Γ not depending on ε where $\mathcal{G}_0^\pm = 0$.

Second step : construction of impedance conditions and equivalent models. The second step consists in identifying a simpler problem satisfied by an approximation $\mathbf{H}_\varepsilon^k$ of the truncated expansion $\mathbf{H}_0(\mathbf{x}) + \varepsilon\mathbf{H}_1(\mathbf{x}) + \varepsilon^2\mathbf{H}_2(\mathbf{x}) + \cdots + \varepsilon^k\mathbf{H}_k(\mathbf{x})$ up to a residual term in $\mathcal{O}(\varepsilon^{k+1})$. For this the equations in (2.4) for $i = 0, \ldots, k$ are multiplied with ε^i and added up, and terms in $\mathcal{O}(\varepsilon^{k+1})$ are neglected. In this way we obtain the simpler problem as

$$\operatorname{curl} \mathbf{H}_\varepsilon^k = \mathbf{j}\ , \quad \text{in } \Omega_- \cup \Omega_+\ , \tag{2.5a}$$

$$\operatorname{div} \mathbf{H}_\varepsilon^k = 0\ , \quad \text{in } \Omega_- \cup \Omega_+\ , \tag{2.5b}$$

$$\mathbf{H}_\varepsilon^k \cdot \mathbf{n} = 0\ , \qquad \text{on } \partial\Omega\ , \tag{2.5c}$$

with the following transmission conditions on Γ:

$$\begin{pmatrix} [\mu\mathbf{H}^k_\varepsilon\cdot\mathbf{n}]_\Gamma \\ \{\mu\mathbf{H}^k_\varepsilon\cdot\mathbf{n}\}_\Gamma \end{pmatrix} = \mathcal{G}_{k,\varepsilon}\begin{pmatrix} \{(\mathbf{H}^k_\varepsilon)_\mathbf{T}\}_\Gamma \\ [(\mathbf{H}^k_\varepsilon)_\mathbf{T}]_\Gamma \end{pmatrix}, \tag{2.5d}$$

where $\mathcal{G}_{k,\varepsilon} = \sum_{i=0}^k \varepsilon^i([\mathcal{G}_i]_\Gamma, \{\mathcal{G}_i\}_\Gamma)^\top$ is the truncation of the weigthed sum of operators $\mathcal{G}_i^\pm$ where the jump or mean value is taken respectively. With this derivation it holds at least formally $\mathbf{H}^\varepsilon - \mathbf{H}^k_\varepsilon = \mathcal{O}(\varepsilon^{k+1})$. Hence, we say that $\mathbf{H}^k_\varepsilon$ is an equivalent (or approximate) model of order $k+1$.

In this paper, we give explicitly the equivalent models of order 1 and 2 in Section 3. Their derivations are presented in detail in Section 4.

3. Main results. Equivalent models up to order 2

In this section we present the main results of the paper. We introduce the approximate models of order 1 (Section 3.1) and 2 (Section 3.2) for the magnetic field. Then we deduce equivalent models up to the second order of approximation for the electric field (Section 3.3). Finally we introduce a regularized variational formulation for the second order model (Section 3.4) and we prove well-posedness and stability results for this model.

3.1. **Equivalent model of order 1.**

The equivalent model of order 1 is given by the limit solution $\mathbf{H}_0$ of (2.2) when $\varepsilon \to 0$. The limit solution satisfies the perfectly conducting magnetic (PMC) boundary condition on Γ and can be defined independently in the two subdomains Ω_-, Ω_+. Hence, $\mathbf{H}_0^- = \mathbf{H}_0|_{\Omega_-}$ is a *tangential vector potential* which satisfies

$$\operatorname{curl}\mathbf{H}_0^- = \mathbf{j}_- \,, \quad \text{in } \Omega_-, \tag{3.1a}$$

$$\operatorname{div}\mathbf{H}_0^- = 0 \,, \quad \text{in } \Omega_-, \tag{3.1b}$$

$$\mathbf{H}_0^-\cdot\mathbf{n} = 0 \,, \quad \text{on } \Gamma, \tag{3.1c}$$

whereas $\mathbf{H}_0^+ = \mathbf{H}_0|_{\Omega_+}$ is a tangential vector potential given by

$$\operatorname{curl}\mathbf{H}_0^+ = \mathbf{j}_+ \,, \quad \text{in } \Omega_+, \tag{3.2a}$$

$$\operatorname{div}\mathbf{H}_0^+ = 0 \,, \quad \text{in } \Omega_+, \tag{3.2b}$$

$$\mathbf{H}_0^+\cdot\mathbf{n} = 0 \,, \quad \text{on } \Gamma, \tag{3.2c}$$

$$\mathbf{H}_0^+\cdot\mathbf{n} = 0 \,, \quad \text{on } \partial\Omega. \tag{3.2d}$$

The boundary condition on $\partial\Omega$ is not affected by the limiting process $\varepsilon \to 0$ and transfers simply to the limit solution $\mathbf{H}_0$.

The next proposition ensures that the vector potentials satisfying problems (3.1) and (3.2) are well-defined in the framework above and even in Lipschitz domains. We recall the definition of the classical spaces

$$X_\mathbf{T}(\Omega_-) = \{\mathbf{u}\in\mathbf{H}(\operatorname{curl},\Omega_-) \mid \operatorname{div}\mathbf{u}\in \mathrm{L}^2(\Omega_-)\,,\quad \mathbf{u}\cdot\mathbf{n} = 0\quad \text{on } \Gamma\}\,,$$

and

$$X_\mathbf{T}(\Omega_+) = \{\mathbf{u}\in\mathbf{H}(\operatorname{curl},\Omega_+) \mid \operatorname{div}\mathbf{u}\in \mathrm{L}^2(\Omega_+)\,,\quad \mathbf{u}\cdot\mathbf{n} = 0\quad \text{on } \Gamma\cup\partial\Omega\}\,.$$

Proposition 3.1. *Let* $\mathbf{j} \in \mathbf{L}^2(\Omega)$ *such that* $\operatorname{div} \mathbf{j}^{\pm} = 0$ *in* $\Omega_{\pm}$. *Then there exists a unique vector potential* $\mathbf{H}_0^- \in \mathbf{X_T}(\Omega_-)$ *satisfying* (3.1) *and there exists a unique vector potential* $\mathbf{H}_0^+ \in \mathbf{X_T}(\Omega_+)$ *satisfying* (3.2). *Furthermore the magnetic field* $\mathbf{H}_0^+$ *belongs to* $\mathbf{H}^1(\Omega_+)$ *and* $\mathbf{H}_0^-$ *belongs to* $\mathbf{H}^1(\Omega_-)$.

Proof. The proof of existence and uniqueness of tangential vector potentials $\mathbf{H}_0^{\pm} \in \mathbf{X_T}(\Omega_{\pm})$ can be found in [2, Th 3.12] for Lipschitz domains in $\mathbb{R}^3$. The proof of the regularity result can be found in [2, Th 2.9] when the domains $\Omega_{\pm}$ are of class $\mathcal{C}^{1,1}$ (see also [14, chapter I, section 3.5]). We refer the reader to the reference [12] for the proof of this regularity result when the domains $\Omega_{\pm}$ are smooths. $\square$

3.2. **Equivalent model of order** 2. We define a second order approximate solution $\mathbf{H}_\varepsilon^1$, which shall be much more accurate approximation of $\mathbf{H}^\varepsilon$ than the limit solution $\mathbf{H}_0$ when $\varepsilon \to 0$. The equations defining $\mathbf{H}_\varepsilon^1$ outside the mid-surface Γ remain the same, i.e., it solves

$$\operatorname{curl} \mathbf{H}_\varepsilon^1 = \mathbf{j}\,, \qquad \text{in } \Omega_- \cup \Omega_+\,, \tag{3.3a}$$

$$\operatorname{div} \mathbf{H}_\varepsilon^1 = 0\,, \qquad \text{in } \Omega_- \cup \Omega_+\,, \tag{3.3b}$$

$$\mathbf{H}_\varepsilon^1 \cdot \mathbf{n} = 0\,, \qquad \text{on } \partial\Omega, \tag{3.3c}$$

and at the mid-surface Γ the transmission conditions

$$\begin{pmatrix} \left[\mu\, \mathbf{H}_\varepsilon^1 \cdot \mathbf{n}\right]_\Gamma \\ \left\{\mu\, \mathbf{H}_\varepsilon^1 \cdot \mathbf{n}\right\}_\Gamma \end{pmatrix} = \varepsilon \begin{pmatrix} \mathrm{L}_1 & \mathrm{L}_3 \\ \mathrm{L}_3 & \mathrm{L}_2 \end{pmatrix} \begin{pmatrix} \left\{(\mathbf{H}_\varepsilon^1)_\mathbf{T}\right\}_\Gamma \\ \left[(\mathbf{H}_\varepsilon^1)_\mathbf{T}\right]_\Gamma \end{pmatrix} \tag{3.3d}$$

are posed. Here, $[\cdot]_\Gamma$ and $\{\cdot\}_\Gamma$ denote the jump and averages introduced in Sec. 2.1 and L_i are first order differential operators given by

$$\mathrm{L}_i = C_i \operatorname{div}_\Gamma\,, \qquad i = 1, 2, 3,$$

in which C_i are constants defined by

$$\begin{aligned} C_1 &= \{\mu\}_\Gamma - 2\frac{\mu_o}{\gamma}\tanh\left(\frac{\gamma}{2}\right), \\ C_2 &= \frac{\{\mu\}_\Gamma}{4} - \frac{\mu_o}{2\gamma}\coth\left(\frac{\gamma}{2}\right), \\ C_3 &= \frac{1}{4}\,[\mu]_\Gamma\,, \end{aligned} \tag{3.4}$$

and

$$\gamma = \exp\left(\frac{3i\pi}{4}\right)\sqrt{\omega\mu_o\tilde{\sigma}}\,. \tag{3.5}$$

Equivalent model of order 2 *in a "symmetric" configuration.* If the electromagnetic properties on both sides of the sheet are the same, i.e., $\mu_+ = \mu_- =: \mu$ in $\Omega_+ \cup \Omega_-$, then the transmission conditions (3.3d) of the second order model simplify to

$$\left[\mu\, \mathbf{H}_\varepsilon^1 \cdot \mathbf{n}\right]_\Gamma = \varepsilon C \operatorname{div}_\Gamma \left\{(\mathbf{H}_\varepsilon^1)_\mathbf{T}\right\}_\Gamma\,, \tag{3.6a}$$

$$\left\{\mu\, \mathbf{H}_\varepsilon^1 \cdot \mathbf{n}\right\}_\Gamma = \varepsilon D \operatorname{div}_\Gamma \left[(\mathbf{H}_\varepsilon^1)_\mathbf{T}\right]_\Gamma\,, \tag{3.6b}$$

where

$$C = \mu - \frac{2\mu_o}{\gamma}\tanh\left(\frac{\gamma}{2}\right), \qquad D = \frac{\mu}{4} - \frac{\mu_o}{2\gamma}\coth\left(\frac{\gamma}{2}\right). \tag{3.7}$$

3.3. **Impedance transmission conditions for the electric field.** Hereafter we deduce a second order model for the electric field $\mathbf{E}_\varepsilon^1$ from $\mathbf{H}_\varepsilon^1$. Using the Faraday's law and the Stokes formula, there holds

$$(\mathbf{H}_\varepsilon^1)_{\mathbf{T}} = \frac{1}{\mathrm{i}\omega\mu}(\operatorname{curl}\mathbf{E}_\varepsilon^1)_{\mathbf{T}} \quad \text{and} \quad \mu\,\mathbf{H}_\varepsilon^1\cdot\mathbf{n} = \frac{1}{\mathrm{i}\omega}\operatorname{div}_\Gamma(\mathbf{E}_\varepsilon^1\times\mathbf{n}) \quad \text{on} \quad \Gamma\,.$$

Then using the transmission conditions (3.3d) and applying the inverse of the operator $\operatorname{div}_\Gamma$, we can identify the following transmission conditions for the electric field

$$\left(\begin{array}{c} \big[(\mathbf{E}_\varepsilon^1\times\mathbf{n})\big]_\Gamma \\ \big\{(\mathbf{E}_\varepsilon^1\times\mathbf{n})\big\}_\Gamma \end{array}\right) = \varepsilon\left(\begin{array}{cc} C_1 & C_3 \\ C_3 & C_2 \end{array}\right)\left(\begin{array}{c} \Big\{\frac{1}{\mu}(\operatorname{curl}\mathbf{E}_\varepsilon^1)_{\mathbf{T}}\Big\}_\Gamma \\ \Big[\frac{1}{\mu}(\operatorname{curl}\mathbf{E}_\varepsilon^1)_{\mathbf{T}}\Big]_\Gamma \end{array}\right), \tag{3.8}$$

where the constants C_i are defined in (3.4). The transmission conditions (3.8) can be regarded as a Poincaré-Steklov map $\mathbf{H}_{\mathbf{T}}$–to–$\mathbf{E}\times\mathbf{n}$ which tends to the PEC boundary condition for $\varepsilon\to 0$. As a consequence the equivalent model of order 1 for the electric field $\mathbf{E}_0$ (A.2)-(A.3) (in Appendix A) is defined independently in the two subdomains $\Omega_\pm$.

Finally the second order model $\mathbf{E}_\varepsilon^1$ solves

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\varepsilon^{1,\pm} = \mathrm{i}\omega\mathbf{j}^\pm \quad \text{in} \quad \Omega_\pm\,, \tag{3.9a}$$

$$\operatorname{div}\mathbf{E}_\varepsilon^{1,\pm} = 0 \quad \text{in} \quad \Omega_\pm\,, \tag{3.9b}$$

$$\int_\Gamma \mathbf{E}_\varepsilon^{1,\pm}\cdot\mathbf{n}\,\mathrm{d}S = 0\,, \tag{3.9c}$$

$$\mathbf{E}_\varepsilon^1\times\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{3.9d}$$

and the transmission conditions (3.8) at the mid-surface Γ.

Remark 3.2. The conditions (3.8) are simpler than the second order conditions for the time-harmonic Maxwell equations [26, Eq. (7c)] since the new Poincaré-Steklov map (3.8) is the scalar part of the Poincaré-Steklov map of Wentzell type [26, Eq. (7c)].

3.4. **Regularized variational formulation for the electric field $\mathbf{E}_\varepsilon^1$.** Following Section 3.3, we can deduce a regularized variational formulation for the electric field $\mathbf{E}_\varepsilon^1$: we search for $\mathbf{E}_\varepsilon^1$ in the Hilbert space

$$\begin{aligned}\mathbf{Y} =\Big\{&\mathbf{E}\in\mathbf{L}^2(\Omega),\ \operatorname{curl}\mathbf{E}^\pm\in\mathbf{L}^2(\Omega_\pm),\ \operatorname{div}\mathbf{E}^\pm\in\mathrm{L}^2(\Omega_\pm),\ \mathbf{E}^\pm\times\mathbf{n}\in\mathbf{L}_t^2(\Gamma),\\ &\int_\Gamma\mathbf{E}^\pm\cdot\mathbf{n}\,\mathrm{d}S=0,\ \mathbf{E}^+\times\mathbf{n}=0\text{ on }\partial\Omega\Big\}\,.\end{aligned} \tag{3.10}$$

This space is equipped with the norm

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{Y}}^2 = \|\mathbf{u}\|_{0,\Omega}^2 + \|\operatorname{curl}\mathbf{u}^+\|_{0,\Omega_+}^2 + \|\operatorname{curl}\mathbf{u}^-\|_{0,\Omega_-}^2 \\ + \|\operatorname{div}\mathbf{u}^+\|_{0,\Omega_+}^2 + \|\operatorname{div}\mathbf{u}^-\|_{0,\Omega_-}^2 + \|\mathbf{u}^+\times\mathbf{n}\|_{0,\Gamma}^2 + \|\mathbf{u}^-\times\mathbf{n}\|_{0,\Gamma}^2\,.\end{aligned}$$

Variational problem for the electric field $\mathbf{E}_\varepsilon^1$. For all $\varepsilon>0$ we consider the variational problem :

Find $\mathbf{E}\in\mathbf{Y}$ such that for all $\mathbf{v}\in\mathbf{Y}$,

$$a_1^\varepsilon(\mathbf{E},\mathbf{v}) = \mathrm{i}\omega\int_\Omega \mathbf{j}\cdot\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x}\,. \tag{3.11}$$

Here the sesquilinear form (in its regularized version) a_1^ε is defined as

(3.12)

$$a_1^\varepsilon(\mathbf{u},\mathbf{v}) = \int_{\Omega_-} \frac{1}{\mu_-}\operatorname{curl}\mathbf{u}^- \cdot \operatorname{curl}\overline{\mathbf{v}^-}\,\mathrm{d}\mathbf{x} + \int_{\Omega_+} \frac{1}{\mu_+}\operatorname{curl}\mathbf{u}^+ \cdot \operatorname{curl}\overline{\mathbf{v}^+}\,\mathrm{d}\mathbf{x} + \int_{\Omega_-} \operatorname{div}\mathbf{u}^- \operatorname{div}\overline{\mathbf{v}^-}\,\mathrm{d}\mathbf{x}$$

$$+ \int_{\Omega_+} \operatorname{div}\mathbf{u}^+ \operatorname{div}\overline{\mathbf{v}^+}\,\mathrm{d}\mathbf{x} - \varepsilon^{-1} \int_\Gamma \mathsf{A}^{-1} \begin{pmatrix} [\mathbf{u}\times\mathbf{n}] \\ \{\mathbf{u}\times\mathbf{n}\} \end{pmatrix} \cdot \begin{pmatrix} [\,\overline{\mathbf{v}}\times\mathbf{n}\,] \\ \{\,\overline{\mathbf{v}}\times\mathbf{n}\,\} \end{pmatrix} \mathrm{d}S\,,$$

and A is the nonsingular matrix given by

$$\mathsf{A} = \begin{pmatrix} C_1 & C_3 \\ C_3 & C_2 \end{pmatrix},$$

where the constants C_i are defined in (3.4).

Remark 3.3. Here we assume that $\det\mathsf{A} \neq 0$. Straightforward calculi lead to

$$\det\mathsf{A} = \frac{\mu_+\mu_-}{4} - \{\mu\}_\Gamma \coth(\gamma)\frac{\mu_o}{\gamma} + \frac{\mu_o^2}{\gamma^2}\,,$$

and we have

$$\mathsf{A}^{-1} = (\det\mathsf{A})^{-1} \begin{pmatrix} C_2 & -C_3 \\ -C_3 & C_1 \end{pmatrix},$$

where the constants C_i are defined in (3.4).

Lemma 3.4. *Assume that the domains $\Omega_\pm$ are of class $\mathcal{C}^{1,1}$. Let the positive constants $\mu_\pm$, μ_o, $\tilde\sigma$ and ω be fixed such that $\det\mathsf{A}\neq 0$ and assume that there exists $\beta \in \{z\in\mathbb{C} \mid \operatorname{Re} z > 0\}$ such that*

(3.13)

$$\forall\mathbf{u}\in\mathbf{Y}\,,\quad -\operatorname{Re}\beta\int_\Gamma \mathsf{A}^{-1} \begin{pmatrix} [\mathbf{u}\times\mathbf{n}] \\ \{\mathbf{u}\times\mathbf{n}\} \end{pmatrix} \cdot \begin{pmatrix} [\,\overline{\mathbf{u}}\times\mathbf{n}\,] \\ \{\,\overline{\mathbf{u}}\times\mathbf{n}\,\} \end{pmatrix} \mathrm{d}S \gtrsim \|\mathbf{u}^+\times\mathbf{n}\|^2_{0,\Gamma} + \|\mathbf{u}^-\times\mathbf{n}\|^2_{0,\Gamma}\,.$$

Then there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, a_1^ε is strongly coercive on $\mathbf{Y}$: there exists $c_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and for all $\mathbf{u}\in\mathbf{Y}$

$$\operatorname{Re}\left(\beta a_1^\varepsilon(\mathbf{u},\mathbf{u})\right) \geqslant c_0\|\mathbf{u}\|^2_{\mathbf{Y}}\,. \tag{3.14}$$

Remark 3.5. Assumption (3.13) is satisfied in the "symmetric" configuration (i.e., $\mu_+ = \mu_- =: \mu$ in $\Omega_+\cup\Omega_-$) when $|\gamma|$ is small enough and $\mu_o > \mu$ since in this framework we can check the following inequality

$$-\operatorname{Re}\beta\int_\Gamma \mathsf{A}^{-1}\begin{pmatrix}\mathsf{u}\\ \mathsf{v}\end{pmatrix}\cdot\begin{pmatrix}\bar{\mathsf{u}}\\ \bar{\mathsf{v}}\end{pmatrix}\mathrm{d}S \gtrsim \|\mathsf{u}\|^2_{0,\Gamma} + \|\mathsf{v}\|^2_{0,\Gamma}\,, \tag{3.15}$$

for all $\beta\in\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}$ such that $\frac{\mu}{4}\operatorname{Re}\beta + \frac{1}{\omega\tilde\sigma}\operatorname{Im}\beta \leqslant 0$.

We will prove Lemma 3.4 in section 3.5. As a consequence of this lemma we infer the following result as an application of the Lax-Milgram lemma.

Theorem 3.6. *Assume that the domains $\Omega_\pm$ are of class $\mathcal{C}^{1,1}$. Let $\mathbf{j}\in\mathbf{L}^2(\Omega)$ satisfy $\operatorname{div}\mathbf{j}=0$ in Ω and $\mathbf{j}\cdot\mathbf{n}=0$ on $\Gamma_\pm$. Let the positive constants $\mu_\pm$, μ_o, $\tilde\sigma$ and ω be fixed such that $\det\mathsf{A}\neq 0$ and assume that (3.13) holds.*

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a unique solution $\mathbf{E}^1_\varepsilon \in \mathbf{Y}$ to the variational problem (3.11) *and we have uniform estimates*

$$\|\mathbf{E}^1_\varepsilon\|_{\mathbf{Y}} \leqslant C\|\mathbf{j}\|_{0,\Omega} \,.$$

3.5. **Proof of Lemma 3.4.** Let $\mathbf{u}$ be any function in the space $\mathbf{Y}$. The proof of this Lemma involved Helmholtz decompositions for $\mathbf{u}^-$ (Section 3.5.1) and $\mathbf{u}^+$ (Section 3.5.2).

Proof of Lemma 3.4. Using Assumption (3.13), one notes that we have the following inequality

$$\operatorname{Re}\left(\beta a_1^\varepsilon(\mathbf{u},\mathbf{u})\right) \gtrsim \|\operatorname{curl}\mathbf{u}^+\|^2_{0,\Omega_+} + \|\operatorname{curl}\mathbf{u}^-\|^2_{0,\Omega_-} + \|\operatorname{div}\mathbf{u}^+\|^2_{0,\Omega_+} + \|\operatorname{div}\mathbf{u}^-\|^2_{0,\Omega_-} \\ + \varepsilon^{-1}(\|\mathbf{u}^+\times\mathbf{n}\|^2_{0,\Gamma} + \|\mathbf{u}^-\times\mathbf{n}\|^2_{0,\Gamma}) \,.$$

Thus there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and for all $\mathbf{u} \in \mathbf{Y}$

$$\operatorname{Re}\left(\beta a_1^\varepsilon(\mathbf{u},\mathbf{u})\right) \gtrsim \|\operatorname{curl}\mathbf{u}^+\|^2_{0,\Omega_+} + \|\operatorname{curl}\mathbf{u}^-\|^2_{0,\Omega_-} + \|\operatorname{div}\mathbf{u}^+\|^2_{0,\Omega_+} + \|\operatorname{div}\mathbf{u}^-\|^2_{0,\Omega_-} \\ + \|\mathbf{u}^+\times\mathbf{n}\|^2_{0,\Gamma} + \|\mathbf{u}^-\times\mathbf{n}\|^2_{0,\Gamma} \,.$$

It remains to prove that the right hand-side above is an upper bound for $\|\mathbf{u}\|^2_{0,\Omega} = \|\mathbf{u}^-\|^2_{0,\Omega_-} + \|\mathbf{u}^+\|^2_{0,\Omega_+}$. We are going to prove this result in two independent steps :
(i) The first step consists in proving the following estimate for $\mathbf{u}^-$

$$\|\operatorname{curl}\mathbf{u}^-\|_{0,\Omega_-} + \|\operatorname{div}\mathbf{u}^-\|_{0,\Omega_-} + \|\mathbf{u}^-\times\mathbf{n}\|_{0,\Gamma} \gtrsim \|\mathbf{u}^-\|_{0,\Omega_-} \,. \tag{3.16}$$

The proof of this estimate is based on a Helmholtz decomposition for $\mathbf{u}^-$ (Section 3.5.1).
(ii) The second step consists in proving the following estimate for $\mathbf{u}^+$

$$\|\operatorname{curl}\mathbf{u}^+\|_{0,\Omega_+} + \|\operatorname{div}\mathbf{u}^+\|_{0,\Omega_+} + \|\mathbf{u}^+\times\mathbf{n}\|_{0,\Gamma} \gtrsim \|\mathbf{u}^+\|_{0,\Omega_+} \,. \tag{3.17}$$

The proof of this estimate is based on a Helmholtz decomposition for $\mathbf{u}^+$ (Section 3.5.2).

Finally combining estimates (3.16) and (3.17) we obtain the coerciveness result (3.14) which ends the proof of Lemma 3.4. □

3.5.1. *Proof of Step* (i).

Proof of (3.16). We recall that we assume that the domain Ω_- is simply connected of class $\mathcal{C}^{1,1}$ and it has a connected boundary. One notes that $\mathbf{v}^- := \operatorname{curl}\mathbf{u}^-$ satisfies $\operatorname{div}\mathbf{v}^- = 0$ in Ω_- and $\langle \mathbf{v}^-\cdot\mathbf{n}, 1\rangle_\Gamma = 0$. Then relying to Theorem 3.12 and Theorem 2.9 in [2], we deduce that there exists a unique tangential vector potential $\mathbf{w}^- \in \mathbf{H}^1(\Omega_-)$ such that

$$\operatorname{curl}\mathbf{w}^- = \operatorname{curl}\mathbf{u}^- \,, \qquad \operatorname{div}\mathbf{w}^- = 0 \quad \text{in } \Omega_-, \quad \text{and} \quad \mathbf{w}^-\cdot\mathbf{n} = 0 \quad \text{on } \Gamma. \tag{3.18}$$

Moreover we have the estimate

$$\|\mathbf{w}^-\|_{1,\Omega_-} \leqslant C\|\operatorname{curl}\mathbf{u}^-\|_{0,\Omega_-} \,, \tag{3.19}$$

where C is independent of ε.

Since Ω_- is simply connected and $\operatorname{curl}\mathbf{w}^- = \operatorname{curl}\mathbf{u}^-$ in Ω_- , we obtain that there exists $\varphi^- \in \mathrm{H}^1(\Omega_-)$ (which is unique up to an additive constant) such that

$$\mathbf{u}^- = \mathbf{w}^- + \nabla\varphi^- \quad \text{in} \quad \Omega_- \,. \tag{3.20}$$

Thus φ^- is solution of the Neumann problem

$$\Delta\varphi^- = \operatorname{div}\mathbf{u}^- \,, \quad \text{and} \quad \partial_{\mathbf{n}}\varphi^- = \mathbf{u}^-\cdot\mathbf{n} \quad \text{on } \Gamma. \tag{3.21}$$

The Neumann problem is compatible since $\int_{\Omega_-} \operatorname{div} \mathbf{u}^- \mathrm{d}\mathbf{x} = \langle \mathbf{u}^- \cdot \mathbf{n}, 1\rangle_\Gamma$ (note that the Gauge condition $\langle \mathbf{u}^- \cdot \mathbf{n}, 1\rangle_\Gamma = 0$ has not been used here). As a consequence, there exists a unique solution $\varphi^- \in \mathrm{V}_N = \{\varphi \in \mathrm{H}^1(\Omega_-) \mid \int_{\Omega_-} \varphi \,\mathrm{d}\mathbf{x} = 0\}$ to the problem (3.21) and φ^- satisfies the following uniform estimate

$$\|\varphi^-\|_{1,\Omega_-} \leqslant C_N(\|\operatorname{div} \mathbf{u}^-\|_{0,\Omega_-} + \|\mathbf{u}^- \cdot \mathbf{n}\|_{0,\Gamma}) \,.$$

One notes that we have also the uniform estimate

$$\|\nabla\varphi^-\|_{0,\Omega_-} \leqslant C(\|\operatorname{div} \mathbf{u}^-\|_{0,\Omega_-} + \|\varphi^-|_\Gamma\|_{\mathrm{H}^{\frac12}(\Gamma)/\mathbb{C}} \,, \tag{3.22}$$

and it remains to bound $\|\varphi^-|_\Gamma\|_{\mathrm{H}^{\frac12}(\Gamma)/\mathbb{C}}$ since (3.19) and (3.20) imply that

$$\|\mathbf{u}^-\|_{0,\Omega_-} \leqslant \|\mathbf{w}^-\|_{0,\Omega_-} + \|\nabla\varphi^-\|_{0,\Omega_-} \lesssim \|\operatorname{curl} \mathbf{u}^-\|_{0,\Omega_-} + \|\nabla\varphi^-\|_{0,\Omega_-} \,.$$

Since we have the trace estimates (See e.g. [14] or the proof of Lemma 2.2 in [2])

$$\|\varphi^-|_\Gamma\|_{\mathrm{H}^{\frac12}(\Gamma)/\mathbb{C}} \leqslant \|\nabla\varphi^- \times \mathbf{n}\|_{-\frac12,\Gamma} \leqslant \|\mathbf{u}^- \times \mathbf{n}\|_{-\frac12,\Gamma} + \|\mathbf{w}^- \times \mathbf{n}\|_{-\frac12,\Gamma} \,, \tag{3.23}$$

and since we have also the estimates

$$\|\mathbf{u}^- \times \mathbf{n}\|_{-\frac12,\Gamma} \leqslant \|\mathbf{u}^- \times \mathbf{n}\|_{0,\Gamma} \quad \text{and} \quad \|\mathbf{w}^- \times \mathbf{n}\|_{-\frac12,\Gamma} \leqslant \|\mathbf{w}^-\|_{1,\Omega_-} \,,$$

we can combine the two above estimates (3.22)-(3.23) with (3.19) to infer the uniform estimate (3.16). ☐

3.5.2. *Proof of Step* (ii).

Proof of (3.17). We recall that the domain Ω_+ is simply connected of class $\mathcal{C}^{1,1}$ and its boundary has two connected components $\partial\Omega_+ = \Gamma \cup \partial\Omega$. One notes that $\mathbf{v}^+ := \operatorname{curl} \mathbf{u}^+$ satisfies

$$\operatorname{div} \mathbf{v}^+ = 0 \quad \text{in} \quad \Omega_+ \,, \mathbf{v}^+ \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \big\langle \mathbf{v}^+ \cdot \mathbf{n}, 1\big\rangle_\Gamma = 0 \,.$$

Then relying to [2, Eq. (3.39)] and Theorem 2.12 in [2], we deduce that there exists a unique vector potential (with mixed boundary conditions) $\mathbf{w}^+ \in \mathbf{H}^1(\Omega_+)$ such that

$$\begin{aligned} \operatorname{curl} \mathbf{w}^+ = \operatorname{curl} \mathbf{u}^+ \,, \qquad \operatorname{div} \mathbf{w}^+ = 0 \quad \text{in } \Omega_-, \quad \mathbf{w}^+ \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \,, \\ \mathbf{w}^+ \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \,, \quad \text{and} \quad \big\langle \mathbf{w}^+ \cdot \mathbf{n}, 1\big\rangle_{\partial\Omega} = 0 \,. \end{aligned} \tag{3.24}$$

Moreover we have the estimate

$$\|\mathbf{w}^+\|_{1,\Omega_+} \leqslant C\|\operatorname{curl} \mathbf{u}^+\|_{0,\Omega_+} \,, \tag{3.25}$$

where C is independent of ε.

Since Ω_+ is simply connected and $\operatorname{curl} \mathbf{w}^+ = \operatorname{curl} \mathbf{u}^+$ in Ω_+, we obtain that there exists $\varphi^+ \in \mathrm{H}^1(\Omega_+)$ (which is unique up to an additive constant) such that

$$\mathbf{u}^+ = \mathbf{w}^+ + \nabla\varphi^+ \quad \text{in} \quad \Omega_+ \,. \tag{3.26}$$

Thus φ^+ is solution to the problem

$$\Delta\varphi^+ = \operatorname{div} \mathbf{u}^+ \,, \quad \varphi^+ = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_\Gamma \partial_{\mathbf{n}}\varphi^+ \,\mathrm{d}S = 0. \tag{3.27}$$

Note that the gauge condition $\int_\Gamma \mathbf{u}^+ \cdot \mathbf{n}\, \mathrm{d}S = 0$ has been used here. According to Lemma 2.1 in [2], there exists a unique solution $\varphi^+ \in \mathrm{H}^1_{0,\partial\Omega}(\Omega_+) = \{\varphi \in \mathrm{H}^1(\Omega_+) \,|\, \varphi = 0 \quad \text{on} \quad \partial\Omega\}$ to the problem (3.27) and we have a uniform estimate

$$\|\varphi^+\|_{1,\Omega_+} \leqslant C \left(\|\operatorname{div} \mathbf{u}^+\|_{0,\Omega_+} + \|\varphi^+|_\Gamma\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma)/\mathbb{C}} \right) .$$

It remains to bound $\|\varphi^+|_\Gamma\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma)/\mathbb{C}}$ since (3.25) and (3.26) implies that

$$\|\mathbf{u}^+\|_{0,\Omega_+} \leqslant \|\mathbf{w}^+\|_{0,\Omega_+} + \|\nabla\varphi^+\|_{0,\Omega_+} \lesssim \|\operatorname{curl} \mathbf{u}^+\|_{0,\Omega_+} + \|\varphi^+\|_{1,\Omega_+} .$$

Finally since we have the trace estimates

$$\|\varphi^+|_\Gamma\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma)/\mathbb{C}} \leqslant \|\nabla\varphi^+ \times \mathbf{n}\|_{-\frac{1}{2},\Gamma} \leqslant \|\mathbf{u}^+ \times \mathbf{n}\|_{-\frac{1}{2},\Gamma} + \|\mathbf{w}^+ \times \mathbf{n}\|_{-\frac{1}{2},\Gamma}$$

and

$$\|\mathbf{u}^+ \times \mathbf{n}\|_{-\frac{1}{2},\Gamma} \leqslant \|\mathbf{u}^+ \times \mathbf{n}\|_{0,\Gamma} \quad \text{and} \quad \|\mathbf{w}^+ \times \mathbf{n}\|_{-\frac{1}{2},\Gamma} \leqslant \|\mathbf{w}^+\|_{1,\Omega_+} ,$$

we can combine the above estimates with (3.25) to infer the uniform estimate (3.17). □

4. A MULTISCALE EXPANSION FOR THE MAGNETIC FIELD

In the guidelines on the derivation of the equivalent transmission conditions in Sect. 2.4 we have already argued that this derivation is based on an asymptotic expansion for the magnetic field $\mathbf{H}^\varepsilon$ (2.2) inside and outside the sheet. More precisely, we search $\mathbf{H}^\varepsilon$ as the asymptotic expansion (2.3), which is

$$\begin{aligned}
&\mathbf{H}^\varepsilon(\mathbf{x}) \approx \mathbf{H}_0(\mathbf{x}) + \varepsilon\mathbf{H}_1(\mathbf{x}) + \varepsilon^2\mathbf{H}_2(\mathbf{x}) + \cdots && \text{for a.e.} \quad \mathbf{x} \in \Omega^\varepsilon_- \cup \Omega^\varepsilon_+ ,\\
&\mathbf{H}^\varepsilon(\mathbf{x}) \approx \mathfrak{H}_0\left(y_\alpha, \frac{h}{\varepsilon}\right) + \varepsilon\mathfrak{H}_1\left(y_\alpha, \frac{h}{\varepsilon}\right) + \cdots && \text{for a.e.} \quad \mathbf{x} \in \Omega^\varepsilon_o .
\end{aligned}$$

In this section, we will derive the terms of this asymptotic expansion step by step up to order 2 as well as their governing equations, having in mind that the magnetic field $\mathbf{H}^\varepsilon$ satisfies the following transmission problem

$$\begin{aligned}
&\text{(4.1a)} && \operatorname{curl} \mathbf{H}^\varepsilon_\pm = \mathbf{j}_\pm && \text{in} \quad \Omega^\varepsilon_\pm ,\\
&\text{(4.1b)} && \operatorname{div} \mathbf{H}^\varepsilon_\pm = 0 && \text{in} \quad \Omega^\varepsilon_\pm ,\\
&\text{(4.1c)} && \operatorname{curl}\operatorname{curl} \mathbf{H}^\varepsilon_o - (\kappa^\varepsilon_o)^2\mathbf{H}^\varepsilon_o = 0 && \text{in} \quad \Omega^\varepsilon_o ,\\
&\text{(4.1d)} && \mathbf{H}^\varepsilon_\pm \times \mathbf{n} = \mathbf{H}^\varepsilon_o \times \mathbf{n}, && \text{on} \quad \Gamma^\varepsilon_\pm ,\\
&\text{(4.1e)} && \mu_\pm\mathbf{H}^\varepsilon_\pm \cdot \mathbf{n} = \mu_o\mathbf{H}^\varepsilon_o \cdot \mathbf{n}, && \text{on} \quad \Gamma^\varepsilon_\pm ,\\
&\text{(4.1f)} && \mathbf{H}^\varepsilon \cdot \mathbf{n} = 0 && \text{on} \quad \partial\Omega .
\end{aligned}$$

In (4.1c) we denote by κ^ε_o the complex wave number given by

$$\kappa^\varepsilon_o = \varepsilon^{-1}\sqrt{\omega\mu_o\tilde{\sigma}}\ \mathrm{e}^{\mathrm{i}\pi/4} .$$

This derivation is order by order and for each order n it is in four steps:

(i) Writing the operator $\operatorname{curl}\operatorname{curl}$ in the domain Ω_o^ε in *local coordinates* (y_α, h) [7] and performing the change of variable $Y_3 = \varepsilon^{-1}h$, i.e., rewriting it in in (y_α, Y_3)–coordinates, leads to an algebraic equation fixing the normal component $\mathfrak{h}_n$ of $\underline{\mathfrak{H}}_n$ and a differential equation for the two tangential components $\mathfrak{H}_n$, which are not completely defined yet.

(ii) We expand the far field term $\mathbf{H}_n$ at the two surfaces $\Gamma_\pm^\varepsilon$ of the conductor around its mid-surface Γ using the Taylor expansion.

(iii) Using the the transmission conditions (4.1e) on $\Gamma_\pm^\varepsilon$ and the Taylor expansion of $\mathbf{H}_n$ we obtain boundary conditions for $\mathbf{H}_n^\pm$ on Γ, which depend on the first terms $\mathbf{H}_k$, $k = 0, \ldots, n-1$ on Γ.

(iv) Using the transmission conditions (4.1d) on $\Gamma_\pm^\varepsilon$ together with the Taylor expansion of $\mathbf{H}_n$ we obtain boundary conditions for the tangential components $\mathfrak{H}_n$ inside the sheet. They can now be explicitly defined as a function of the terms $\mathbf{H}_k$, $k = 0, \ldots, n$.

We expand the "magnetic" operator inside the thin layer Ω_o^ε in powers of ε, in Section 4.1. We deduce in Section 4.2 the equations satisfied by the magnetic profiles $\underline{\mathfrak{H}}_n$ and the far field terms $\mathbf{H}_n^\pm$. We derive explicitly the first terms in Section 4.3.

4.1. **Expansion of differential operators inside the conductor.** Due to the small thickness of the conductor the derivatives in normal and the tangential directions scale differently in ε. Hence, it is convenient to use a local *normal coordinate system* in Ω_o^ε, see e.g., [7, App. A.1]. For this coordinate system we call D_α the covariant derivative on the mean surface Γ and ∂_3^h is the partial derivative with respect to the normal coordinate $y_3 = h$. Let furthermore $a_{\alpha\beta}(h)$ be the metric tensor of the manifold Γ_h, which is the surface contained in Ω_o^ε at a distance h of Γ. The metric tensor in such a coordinate system writes [7, App. A.1, Eq. (A.7)]

$$a_{\alpha\beta}(h) = a_{\alpha\beta} - 2b_{\alpha\beta}h + b_\alpha^\gamma b_{\gamma\beta}h^2 \,, \tag{4.2}$$

and its inverse expands in power series of h

$$a^{\alpha\beta}(h) = a^{\alpha\beta} + 2b^{\alpha\beta}h + \mathcal{O}(h^2)\,.$$

Subsequently, we use a property of the covariant derivative, that it acts on scalar functions $\mathfrak{h}$ like the partial derivative: $D_\alpha\mathfrak{h} = \partial_\alpha\mathfrak{h}$.

We denote by $\mathbf{L}(y_\alpha, h; D_\alpha, \partial_3^h)$ the second order Maxwell operator

$$\operatorname{curl}\operatorname{curl} - (\kappa_o^\varepsilon)^2\,\mathbb{I}$$

in Ω_o^ε in the normal coordinate system. The operator $\mathbf{L}$ expand in power series of h with intrinsic coefficients with respect to Γ, see [7, App. A, §A.4].

Now, we scale the normal coordinate $Y_3 = \varepsilon^{-1}h$ to obtain a coordinate, this is Y_3, which does not change with ε. We use from now on the same symbol $\underline{\mathfrak{H}}$ for three-dimensional one-form field in these scaled coordinates and call $\mathbf{L}[\varepsilon]$ the three-dimensional harmonic Maxwell operators in Ω_o^ε. This operator expands in powers of ε

$$\mathbf{L}[\varepsilon] = \varepsilon^{-2}\sum_{n=0}^{\infty}\varepsilon^n\mathbf{L}^n \,,$$

whose coefficients are intrinsic operators on Γ, which are completely determined by the shape of Γ and the material parameters of the conducting sheet. We denote by L_α^n the surface components of $\mathbf{L}^n$. With the summation convention of repeated two dimensional indices (represented by greek letters), there holds [7, App. A.1, Eq. (A.10)]

$$L^0_\alpha(\underline{\mathfrak{H}}) = -\partial_3^2\mathfrak{H}_\alpha + \gamma^2\mathfrak{H}_\alpha \quad \text{and} \quad L^1_\alpha(\underline{\mathfrak{H}}) = -2b^\beta_\alpha\partial_3\mathfrak{H}_\beta + \partial_3 D_\alpha\mathfrak{h} + b^\beta_\beta\partial_3\mathfrak{H}_\alpha \ , \tag{4.3}$$

(we recall that γ is defined in (3.5), so that $(\kappa^\varepsilon_o)^2 = -\varepsilon^{-2}\gamma^2$) and [7, App. A.1, Eq. (A.28)]

$$B^0_\alpha(\underline{\mathfrak{H}}) = \partial_3\mathfrak{H}_\alpha \quad \text{and} \quad B^1_\alpha(\underline{\mathfrak{H}}) = -D_\alpha\mathfrak{h} \ . \tag{4.4}$$

Here, ∂_3 is the partial derivative with respect to Y_3. We denote by L^n_3 the transverse components of $\mathbf{L}^n$. There holds [7, App. A.1, Eq. (A.12)]

$$L^0_3(\underline{\mathfrak{H}}) = \gamma^2\mathfrak{h} \quad \text{and} \quad L^1_3(\underline{\mathfrak{H}}) = \gamma^\alpha_\alpha(\partial_3\underline{\mathfrak{H}}) + b^\beta_\beta\partial_3\mathfrak{h} \ , \tag{4.5}$$

where $\gamma_{\alpha\beta}(\underline{\mathfrak{H}}) = \frac{1}{2}(D_\alpha\mathfrak{H}_\beta + D_\beta\mathfrak{H}_\alpha) - b_{\alpha\beta}\mathfrak{h}$ is the change of metric tensor and $\gamma^\alpha_\alpha = a^{\alpha\beta}\gamma_{\alpha\beta}$.

4.2. **Equations for the coefficients of the magnetic field.** Writing the partial differential equation (4.1c) in the thin conductor Ω^ε_o in the scaled local coordinate system we find that the profiles $\underline{\mathfrak{H}}_j$ satisfy the following system (with $I = (-\frac{1}{2}, \frac{1}{2})$)

$$\mathbf{L}[\varepsilon]\sum_{j\geqslant 0}\varepsilon^j\underline{\mathfrak{H}}_j(y_\alpha, Y_3) = 0 \ , \qquad \text{in} \quad \Gamma\times I \ . \tag{4.6}$$

It is not very convenient that the terms $\mathbf{H}^\pm_j$ of the magnetic field (see (2)) (which are involved on the left hand side of (4.1d)-(4.1e)) are evaluated on $\Gamma^\varepsilon_\pm$ which moves with ε. However, as the expansion (2) of $\mathbf{H}^\varepsilon$ is assumed to be valid for any small $\varepsilon > 0$, the terms $\mathbf{H}^\pm_j$ are defined in $\Omega^\varepsilon_\pm$ for any $\varepsilon > 0$, and, hence, in $\Omega_\pm$. As we have assumed that the thin conductor, and so its mid-surface Γ, are smooth, that $\mu_\pm$ are constants, and that the current $\mathbf{j}$ is zero close to Γ it makes sense to accept that the vector fields $\mathbf{H}^\pm_j$ are regular in a neighbourhood of Γ. This can be justified using the regularity theory, see e.g. [23, Chap. 4]. Hence, we can use the Taylor expansion and infer for $n\in\mathbb{N}$ that

$$\mathbf{H}^\pm_n\times\mathbf{n}|_{h=\pm\frac{\varepsilon}{2}} = \quad \mathbf{H}_n\times\mathbf{n}|_{0^\pm} \pm \frac{\varepsilon}{2}\partial_h\mathbf{H}_n\times\mathbf{n}|_{0^\pm} + \cdots \ , \tag{4.7a}$$

and

$$\mathbf{H}^\pm_n\cdot\mathbf{n}|_{h=\pm\frac{\varepsilon}{2}} = \quad \mathbf{H}_n\cdot\mathbf{n}|_{0^\pm} \pm \frac{\varepsilon}{2}\partial_h\mathbf{H}_n\cdot\mathbf{n}|_{0^\pm} + \cdots \ , \tag{4.7b}$$

where $\cdot|_{0^\pm}$ means the limit for positive or negative $h\to 0$, respectively. Furthermore, it is convenient to define $\mathbf{H}_n$ for $n\in\mathbb{N}$ by $\mathbf{H}_n = \mathbf{H}^+_n$ in Ω_+, and $\mathbf{H}_n = \mathbf{H}^-_n$ in Ω_-.

Using the expression of the operator $\mathbf{L}^0$, and expanding $\mathbf{H}^\varepsilon$ in $\Omega^\varepsilon_\pm$, we deduce that, according to the system (4.1) and using (4.6) and (4.7), the profiles $\underline{\mathfrak{H}}_n = (\mathfrak{H}_n, \mathfrak{h}_n)$ and the terms $\mathbf{H}_n$ have to satisfy, for all $n \geq 0$

$$\begin{aligned}
L_3^0(\underline{\mathfrak{H}}_n) = \gamma^2 \mathfrak{h}_n &= -\sum_{j=1}^{n} L_3^j(\underline{\mathfrak{H}}_{n-j}) && \text{in} \quad \Gamma \times I,\\
\operatorname{curl} \mathbf{H}_n^{\pm} &= \delta_n^0 \mathbf{j}_{\pm} && \text{in} \quad \Omega_{\pm},\\
\operatorname{div} \mathbf{H}_n^{\pm} &= 0 && \text{in} \quad \Omega_{\pm},\\
\mathbf{H}_n^{\pm} \cdot \mathbf{n}|_{0^\pm} &= \frac{\mu_o}{\mu_\pm} \mathfrak{h}_n|_{\pm\frac{1}{2}} - \sum_{j=1}^{n} \frac{1}{(\pm 2)^j} \partial_h^j \mathbf{H}_{n-j}^{\pm} \cdot \mathbf{n}|_{0^\pm} && \text{on} \quad \Gamma,\\
\mathbf{H}_n^{+} \cdot \mathbf{n} &= 0 && \text{on} \quad \partial\Omega,\\
L_\alpha^0(\underline{\mathfrak{H}}_n) = -\partial_3^2 \mathfrak{H}_{n,\alpha} + \gamma^2 \mathfrak{H}_{n,\alpha} &= -\sum_{j=1}^{n} L_\alpha^j(\underline{\mathfrak{H}}_{n-j}) && \text{in} \quad \Gamma \times I,\\
\mathfrak{H}_n|_{\pm\frac{1}{2}} &= \mathbf{n} \times \mathbf{h}_n^{\pm} \times \mathbf{n}|_{0^\pm} + \sum_{j=1}^{n} \frac{1}{(\pm 2)^j} \partial_h^j \mathbf{n} \times \mathbf{h}_{n-j}^{\pm} \times \mathbf{n}|_{0^\pm} && \text{on} \quad \Gamma\,,
\end{aligned}$$

where $\cdot|_{\pm\frac{1}{2}}$ abbreviates the trace on $Y_3 = \pm\frac{1}{2}$, and $\mathbf{h}_n^{\pm}$ denotes the trace of $\mathbf{H}_n^{\pm}$ on $\Gamma_{\pm}$.

4.3. **First terms of the asymptotics.** In the previous section we have derived the coupled systems for the terms of the asymptotic expansions to any order n. Hence we can determine now the first terms $\underline{\mathfrak{H}}_n = (\mathfrak{H}_n, \mathfrak{h}_n)$ and $\mathbf{H}_n$ by induction.
The coupled system of order 1. For $n = 0$ in the previous system, we find that $\underline{\mathfrak{H}}_0 = (\mathfrak{H}_0, \mathfrak{h}_0)$ and $\mathbf{H}_0$ satisfy

$$\gamma^2 \mathfrak{h}_0 = 0 \quad \text{in} \quad \Gamma \times I\,, \tag{4.9a}$$

$$\operatorname{curl} \mathbf{H}_0^{\pm} = \mathbf{j}_{\pm} \quad \text{in} \quad \Omega_{\pm}\,, \tag{4.9b}$$

$$\operatorname{div} \mathbf{H}_0^{\pm} = 0 \quad \text{in} \quad \Omega_{\pm}\,, \tag{4.9c}$$

$$\mathbf{H}_0^{\pm} \cdot \mathbf{n}|_{0^\pm} = \frac{\mu_o}{\mu_\pm} \mathfrak{h}_0|_{\pm\frac{1}{2}} \quad \text{on} \quad \Gamma\,, \tag{4.9d}$$

$$\mathbf{H}_0^{+} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{4.9e}$$

$$-\partial_3^2 \mathfrak{H}_{0,\alpha} + \gamma^2 \mathfrak{H}_{0,\alpha} = 0 \quad \text{in} \quad \Gamma \times I, \tag{4.9f}$$

$$\mathfrak{H}_0|_{\pm\frac{1}{2}} = \mathbf{n} \times \mathbf{h}_0^{\pm} \times \mathbf{n}|_{0^\pm} \quad \text{on} \quad \Gamma\,. \tag{4.9g}$$

Obviously, (4.9a) implies with $\gamma \neq 0$ that $\mathfrak{h}_0 = 0$ and in view of (4.9b), (4.9c), (4.9d) and (4.9e) we can assert that the magnetic field satisfies the PMC boundary conditions, and we obtain the limit system (3.1)–(3.2) for $\mathbf{H}_0^{\pm}$. Then the unique solution of the ODE (4.9f)-(4.9g) is, with the choice (3.5) for γ, the tangential field

$$\underline{\mathfrak{H}}_{0,0}(y_\beta)\ \cosh(\gamma Y_3) + \underline{\mathfrak{H}}_{0,1}(y_\beta)\ \sinh(\gamma Y_3) \tag{4.10}$$

with

$$\underline{\mathfrak{H}}_{0,0} = \frac{1}{\cosh\left(\frac{\gamma}{2}\right)} \{\mathbf{h}_0\}_\Gamma \qquad \text{and} \qquad \underline{\mathfrak{H}}_{0,1} = \frac{1}{2\sinh\left(\frac{\gamma}{2}\right)} [\mathbf{h}_0]_\Gamma\ .$$

The coupled system of order 2. Then in the same way as above we find that $\underline{\mathfrak{H}}_1 = (\mathfrak{H}_1, \mathfrak{h}_1)$ and $\mathbf{H}_1$ satisfy

$$\gamma^2 \mathfrak{h}_1 = -L_3^1(\underline{\mathfrak{H}}_0) \qquad \text{in} \quad \Gamma \times I\,, \tag{4.11a}$$

$$\operatorname{curl} \mathbf{H}_1^{\pm} = 0\,, \qquad \text{in} \quad \Omega_{\pm}\,, \tag{4.11b}$$

$$\operatorname{div} \mathbf{H}_1^{\pm} = 0 \qquad \text{in} \quad \Omega_{\pm}\,, \tag{4.11c}$$

$$\mathbf{H}_1^{\pm} \cdot \mathbf{n}|_{0^{\pm}} = \frac{\mu_o}{\mu_{\pm}} \mathfrak{h}_1|_{\pm\frac{1}{2}} \mp \frac{1}{2} \partial_h \mathbf{H}_0^{\pm} \cdot \mathbf{n}|_{0^{\pm}} \qquad \text{on} \quad \Gamma\,, \tag{4.11d}$$

$$\mathbf{H}_1^{+} \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{4.11e}$$

$$-\partial_3^2 \mathfrak{H}_{1,\alpha} + \gamma^2 \mathfrak{H}_{1,\alpha} = -L_\alpha^1(\underline{\mathfrak{H}}_0) \qquad \text{in} \quad \Gamma \times I\,, \tag{4.11f}$$

$$\mathfrak{H}_1|_{\pm\frac{1}{2}} = \mathbf{n} \times \mathbf{h}_1^{\pm} \times \mathbf{n}|_{0^{\pm}} \pm \frac{1}{2} \partial_h \mathbf{n} \times \mathbf{h}_0^{\pm} \times \mathbf{n}|_{0^{\pm}} \qquad \text{on} \quad \Gamma\,. \tag{4.11g}$$

The normal component $\mathfrak{h}_1$ of the profile $\underline{\mathfrak{H}}_1$ (of order 1) is given by equation (4.11a). According to (4.10) and using (4.5) we obtain

$$\mathfrak{h}_1(y_\beta, Y_3) = -\gamma^{-1} \left(D_\alpha \left\{ \mathsf{h}_0^\alpha \right\}(y_\beta) \, \frac{\sinh(\gamma Y_3)}{\cosh(\frac{\gamma}{2})} + D_\alpha \left[\mathsf{h}_0^\alpha \right](y_\beta) \, \frac{\cosh(\gamma Y_3)}{2\sinh(\frac{\gamma}{2})} \right), \tag{4.12}$$

where h_0^α denote the tangential components of $\mathbf{h}_0$.

Now, inserting this explicit representation into the conditions (4.11d), we find that the term $\mathbf{H}_1^-$ solves the following boundary value problem :

$$\left\{ \begin{array}{lll} \operatorname{curl} \mathbf{H}_1^- & = 0 & \text{in} \quad \Omega_-\,, \\ \operatorname{div} \mathbf{H}_1^- & = 0 & \text{in} \quad \Omega_-\,, \\ \mu_- \mathbf{H}_1^- \cdot \mathbf{n} & = \mathsf{g}_1^- & \text{on} \quad \Gamma\,, \end{array} \right. \tag{4.13a}$$

and the term $\mathbf{H}_1^+$ satisfies the problem

$$\left\{ \begin{array}{lll} \operatorname{curl} \mathbf{H}_1^+ & = 0 & \text{in} \quad \Omega_+\,, \\ \operatorname{div} \mathbf{H}_1^+ & = 0 & \text{in} \quad \Omega_+\,, \\ \mu_+ \mathbf{H}_1^+ \cdot \mathbf{n} & = \mathsf{g}_1^+ & \text{on} \quad \Gamma\,, \\ \mathbf{H}_1^+ \cdot \mathbf{n} & = 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{4.13b}$$

with

$$\mathsf{g}_1^{\pm} := \mu_o \mathfrak{h}_1|_{\pm\frac{1}{2}} \mp \frac{\mu_{\pm}}{2} \partial_h \mathbf{H}_0^{\pm} \cdot \mathbf{n}|_{0^{\pm}}\,, \tag{4.13c}$$

i.e.

(4.14)
$$\mathsf{g}_1^{\pm} = -\gamma^{-1} \mu_o \left(\pm D_\alpha \left\{ \mathsf{h}_0^\alpha \right\}(y_\beta) \, \tanh(\frac{\gamma}{2}) + D_\alpha \left[\mathsf{h}_0^\alpha \right](y_\beta) \, \frac{1}{2\tanh(\frac{\gamma}{2})} \right) \mp \frac{\mu_{\pm}}{2} \partial_h \mathbf{H}_0^{\pm} \cdot \mathbf{n}|_{0^{\pm}}$$

The next term which can be determined is the tangential field $\mathfrak{H}_1$: $\mathfrak{H}_1$ is the unique solution of the ODE (4.11f)-(4.11g).

4.4. **Further notes for deriving the second order condition.** We have explained in Sec. 2.4 guidelines to derive impedance conditions from the boundary conditions (2.4d) on Γ for the terms of the asymptotic expansions. It is convenient to write the impedance conditions with mean and jump traces.

One may think that the term $\partial_h \mathbf{H}_0^\pm \cdot \mathbf{n}$ in (4.14), as it is neither a Dirichlet nor a Neumann trace of $\mathbf{H}_0^\pm \in \mathbf{H}(\mathrm{curl}, \Omega_\pm)$, is not suitable to use in variational formulations or the finite element method. However, using the relation (recalling that $\mathbf{u}_\mathbf{T} = \mathbf{n} \times (\mathbf{u}_{|\Gamma} \times \mathbf{n})$)

$$\mathrm{div}\, \mathbf{u} = \mathrm{div}_\Gamma\, \mathbf{u}_\mathbf{T} + \partial_h(\mathbf{u}\cdot\mathbf{n}) + \mathbf{u}\cdot\mathbf{n}\,\mathrm{div}\,\mathbf{n}\,,$$

since we have the equalities $\mathbf{H}_0^\pm \cdot \mathbf{n} = 0$ and $\mathrm{div}\,\mathbf{H}_0^\pm = 0$ on Γ, we can write

$$\partial_h \mathbf{H}_0^\pm \cdot \mathbf{n} = -\,\mathrm{div}_\Gamma\, \mathbf{H}_{0,\mathbf{T}}^\pm \quad \text{on} \quad \Gamma\,.$$

Since we can write the last term $\mathrm{div}_\Gamma\, \mathbf{H}_{0,\mathbf{T}}^\pm$ with the covariant derivative

$$\mathrm{div}_\Gamma\, \mathbf{H}_\mathbf{T} = D_\alpha \mathsf{h}^\alpha \quad \text{on} \quad \Gamma\,,$$

we find that the right hand side in (4.14) can be written as

$$\mathsf{g}_1^\pm = -\gamma^{-1}\mu_o \left(\pm\,\mathrm{div}_\Gamma \{\mathbf{H}_{0,\mathbf{T}}\}\ \tanh(\frac{\gamma}{2}) + \mathrm{div}_\Gamma\,[\mathbf{H}_{0,\mathbf{T}}]\ \frac{1}{2\tanh(\frac{\gamma}{2})}\right) \pm \frac{\mu_\pm}{2}\,\mathrm{div}_\Gamma\,\mathbf{H}_{0,\mathbf{T}}^\pm \tag{4.15}$$

Finally the boundary conditions for $\mathbf{H}_1$ on Γ in (4.13) can be written as

$$[\mu \mathbf{H}_1 \cdot \mathbf{n}]_\Gamma = \left(\{\mu\}_\Gamma - 2\frac{\mu_o}{\gamma}\tanh(\frac{\gamma}{2})\right) \mathrm{div}_\Gamma\, \{\mathbf{H}_{0,\mathbf{T}}\}_\Gamma + \frac{1}{4}[\mu]_\Gamma\, \mathrm{div}_\Gamma\,[\mathbf{H}_{0,\mathbf{T}}]_\Gamma\ ,$$

and

$$\{\mu \mathbf{H}_1 \cdot \mathbf{n}\}_\Gamma = \left(\frac{\{\mu\}}{4} - \frac{\mu_o}{2\gamma}\coth(\frac{\gamma}{2})\right) \mathrm{div}_\Gamma\,[\mathbf{H}_{0,\mathbf{T}}]_\Gamma + \frac{1}{4}[\mu]_\Gamma\, \mathrm{div}_\Gamma\,\{\mathbf{H}_{0,\mathbf{T}}\}_\Gamma$$

where we have used the equalities $\{\mathbf{A}\}_\Gamma = \{\mu\}_\Gamma\{\frac{1}{\mu}\mathbf{A}\}_\Gamma + \frac{1}{4}[\mu]_\Gamma[\frac{1}{\mu}\mathbf{A}]_\Gamma$ and $[\mathbf{A}]_\Gamma = \{\mu\}_\Gamma[\frac{1}{\mu}\mathbf{A}]_\Gamma + [\mu]_\Gamma\{\frac{1}{\mu}\mathbf{A}\}_\Gamma$ for any vector field $\mathbf{A}$.

The impedance conditions (3.3d) of order 2 are then obtained by adding the previous equations multiplied by ε to the PMC conditions $\{\mathbf{H}_0\cdot\mathbf{n}\}_\Gamma = [\mathbf{H}_0\cdot\mathbf{n}]_\Gamma = 0$ (see (3.1c) and (3.2c)) for $\mathbf{H}_0$ and by replacing $\mathbf{H}_0 + \varepsilon\mathbf{H}_1$ on the left hand side by the new unknown $\mathbf{H}_\varepsilon^1$ and by replacing $\varepsilon\mathbf{H}_0$ on the right hand side by $\varepsilon\mathbf{H}_\varepsilon^1$.

Appendix A. Variational formulations for the electric field

In this section we introduce variational formulations for the electric fields $\mathbf{E}^\varepsilon$ and $\mathbf{E}_0$, and we present elements of proofs for stability and convergence results.

A.1. **Strong form of equations.**

Equations for the electric field $\mathbf{E}^\varepsilon$. According to (2.1a)-(2.1b), the electric field $\mathbf{E}^\varepsilon$ solves the following problem for any $\varepsilon > 0$ [8]

$$\operatorname{curl} \frac{1}{\mu_\pm} \operatorname{curl} \mathbf{E}^\varepsilon_\pm = \mathrm{i}\omega \mathbf{j}^\pm \quad \text{in} \quad \Omega^\varepsilon_\pm , \tag{A.1a}$$

$$\operatorname{div} \mathbf{E}^\varepsilon_\pm = 0 \quad \text{in} \quad \Omega^\varepsilon_\pm , \tag{A.1b}$$

$$\operatorname{curl} \frac{1}{\mu_o} \operatorname{curl} \mathbf{E}^\varepsilon_o - \mathrm{i}\omega\sigma^\varepsilon_o \mathbf{E}^\varepsilon_o = 0 \quad \text{in} \quad \Omega^\varepsilon_o , \tag{A.1c}$$

$$\operatorname{div} \mathbf{E}^\varepsilon_o = 0 \quad \text{in} \quad \Omega^\varepsilon_o , \tag{A.1d}$$

$$\mathbf{E}^\varepsilon_\pm \times \mathbf{n} = \mathbf{E}^\varepsilon_o \times \mathbf{n}, \quad \text{on} \quad \Gamma^\varepsilon_\pm , \tag{A.1e}$$

$$\mathbf{E}^\varepsilon_o \cdot \mathbf{n} = 0, \quad \text{on} \quad \Gamma^\varepsilon_\pm , \tag{A.1f}$$

$$\int_{\Gamma^\varepsilon_\pm} \mathbf{E}^\varepsilon_\pm \cdot \mathbf{n} \, \mathrm{d}S = 0 , \tag{A.1g}$$

$$\mathbf{E}^\varepsilon_+ \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega. \tag{A.1h}$$

Equations for the electric field $\mathbf{E}_0$. The electric field $\mathbf{E}^+_0$ solves the following problem

$$\operatorname{curl} \frac{1}{\mu_-} \operatorname{curl} \mathbf{E}^-_0 = \mathrm{i}\omega \mathbf{j}^- \quad \text{in} \quad \Omega_- , \tag{A.2a}$$

$$\operatorname{div} \mathbf{E}^-_0 = 0 \quad \text{in} \quad \Omega_- , \tag{A.2b}$$

$$\int_{\Gamma} \mathbf{E}^-_0 \cdot \mathbf{n} \, \mathrm{d}S = 0 , \tag{A.2c}$$

$$\mathbf{E}^-_0 \times \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{A.2d}$$

and $\mathbf{E}^+_0$ solves the following problem

$$\operatorname{curl} \frac{1}{\mu_+} \operatorname{curl} \mathbf{E}^+_0 = \mathrm{i}\omega \mathbf{j}^+ \quad \text{in} \quad \Omega_+ , \tag{A.3a}$$

$$\operatorname{div} \mathbf{E}^+_0 = 0 \quad \text{in} \quad \Omega_+ , \tag{A.3b}$$

$$\int_{\Gamma} \mathbf{E}^+_0 \cdot \mathbf{n} \, \mathrm{d}S = 0 , \tag{A.3c}$$

$$\mathbf{E}^+_0 \times \mathbf{n} = 0 \qquad \text{on } \Gamma \cup \partial\Omega . \tag{A.3d}$$

A.2. **Variational framework.** The variational space for $\mathbf{E}^\varepsilon$ is the Hilbert space [8, Rem. 3]

$$\mathbf{Y}_\varepsilon = \{\mathbf{u} \in \mathbf{H}_0(\operatorname{curl}, \Omega) \ : \ \operatorname{div} \mathbf{u}_\pm \in \mathrm{L}^2(\Omega^\varepsilon_\pm) \, , \ \operatorname{div} \mathbf{u}_o \in \mathrm{L}^2(\Omega^\varepsilon_o) \, , \ \int_{\Gamma^\varepsilon_\pm} \mathbf{u}_\pm \cdot \mathbf{n} \, \mathrm{d}S = 0\} \, ,$$

equipped with the norm

$$\|\mathbf{u}\|^2_{\mathbf{Y}_\varepsilon} = \|\mathbf{u}\|^2_{0,\Omega} + \|\operatorname{curl} \mathbf{u}\|^2_{0,\Omega} + \|\operatorname{div} \mathbf{u}_+\|^2_{0,\Omega^\varepsilon_+} + \|\operatorname{div} \mathbf{u}_-\|^2_{0,\Omega^\varepsilon_-} + \|\operatorname{div} \mathbf{u}_o\|^2_{0,\Omega^\varepsilon_o} \, .$$

The variational spaces for $\mathbf{E}^+_0$ and $\mathbf{E}^-_0$ are

$$\mathbf{Y}_0(\Omega_+) = \{\mathbf{u} \in \mathbf{H}_0(\operatorname{curl}, \Omega_+) \ : \ \operatorname{div} \mathbf{u} \in \mathrm{L}^2(\Omega_+) \, , \ \int_\Gamma \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = 0\} \, ,$$

and

$$\mathbf{Y}_0(\Omega_-) = \{\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega_-) \ : \ \mathrm{div}\, \mathbf{u} \in \mathrm{L}^2(\Omega_-) \ , \ \int_\Gamma \mathbf{u} \cdot \mathbf{n} \,\mathrm{d}S = 0\} \ ,$$

respectively. The spaces $\mathbf{Y}_0(\Omega_+)$ and $\mathbf{Y}_0(\Omega_-)$ are equipped with the norms $\|.\|_{\mathbf{Y}_0(\Omega_+)}$ and $\|.\|_{\mathbf{Y}_0(\Omega_-)}$, respectively

$$\|\mathbf{u}\|^2_{\mathbf{Y}_0(\Omega_\pm)} = \|\mathbf{u}\|^2_{0,\Omega_\pm} + \|\,\mathrm{curl}\,\mathbf{u}\|^2_{0,\Omega_\pm} + \|\,\mathrm{div}\,\mathbf{u}\|^2_{0,\Omega_\pm} \ .$$

A.3. **Variational problems.**

Variational problem for the electric field $\mathbf{E}^\varepsilon$. For all $\varepsilon > 0$ we consider the variational problem :
Find $\mathbf{E} \in \mathbf{Y}_\varepsilon$ such that for all $\mathbf{v} \in \mathbf{Y}_\varepsilon$,

$$a^\varepsilon_R(\mathbf{E}, \mathbf{v}) = \mathrm{i}\omega \int_{\Omega^\varepsilon_- \cup \Omega^\varepsilon_+} \mathbf{j} \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} \ . \tag{A.4}$$

Here the sesquilinear form (in its regularized version) a^ε_R is defined as

$$a^\varepsilon_R(\mathbf{u}, \mathbf{v}) = \int_\Omega \frac{1}{\underline{\mu}^\varepsilon} \mathrm{curl}\,\mathbf{u} \cdot \mathrm{curl}\,\overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \int_{\Omega^\varepsilon_- \cup \Omega^\varepsilon_o \cup \Omega^\varepsilon_+} \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} - \mathrm{i}\omega \int_{\Omega^\varepsilon_o} \sigma^\varepsilon_o \mathbf{u} \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} \ .$$

Lemma A.1. *Let the positive constants $\mu_\pm$, μ_o, $\tilde\sigma$ and ω be fixed. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, a^ε_R is strongly coercive on $\mathbf{Y}_\varepsilon$: there exist $\alpha \in \mathbb{C}$ and $c_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\mathbf{E} \in \mathbf{Y}_\varepsilon$*

$$\mathrm{Re}\,(\alpha a^\varepsilon_R(\mathbf{E}, \mathbf{E})) \geqslant c_0 \|\mathbf{E}\|^2_{\mathbf{Y}_\varepsilon} \ . \tag{A.5}$$

Proof. Let us fix $\alpha = \mathrm{e}^{\mathrm{i}\pi/4}$. Since

$$\mathrm{Re}\,(\alpha a^\varepsilon_R(\mathbf{E}, \mathbf{E})) \gtrsim \|\,\mathrm{curl}\,\mathbf{E}\|^2_{0,\Omega} + \|\,\mathrm{div}\,\mathbf{E}_+\|^2_{0,\Omega^\varepsilon_+} + \|\,\mathrm{div}\,\mathbf{E}_-\|^2_{0,\Omega^\varepsilon_-} + \|\,\mathrm{div}\,\mathbf{E}_o\|^2_{0,\Omega^\varepsilon_o} + \varepsilon^{-2}\|\mathbf{E}\|^2_{0,\Omega^\varepsilon_o} \ ,$$

then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\mathbf{E} \in \mathbf{Y}_\varepsilon$

$$\mathrm{Re}\,(\alpha a^\varepsilon_R(\mathbf{E}, \mathbf{E})) \gtrsim \|\,\mathrm{curl}\,\mathbf{E}\|^2_{0,\Omega} + \|\,\mathrm{div}\,\mathbf{E}_+\|^2_{0,\Omega^\varepsilon_+} + \|\,\mathrm{div}\,\mathbf{E}_-\|^2_{0,\Omega^\varepsilon_-} + \|\,\mathrm{div}\,\mathbf{E}_o\|^2_{0,\Omega^\varepsilon_o} + \|\mathbf{E}\|^2_{0,\Omega^\varepsilon_o} \ .$$

Then it is possible to prove that the right hand side is an upper bound for $\|\mathbf{E}\|^2_{0,\Omega^\varepsilon_-} + \|\mathbf{E}\|^2_{0,\Omega^\varepsilon_+}$, we refer the reader to [8, Rem. 3]. The proof is worked out in details in [8, Lemma 2.2] (with $\delta_0 = 0$) in a slightly different configuration. ☐

Variational problems for $\mathbf{E}^\pm_0$. We consider the variational problems :
(i) Find $\mathbf{E}^-_0 \in \mathbf{Y}_0(\Omega_-)$ such that for all $\mathbf{v} \in \mathbf{Y}_0(\Omega_-)$,

$$a^0_-(\mathbf{E}^-_0, \mathbf{v}) = \mathrm{i}\omega \int_{\Omega_-} \mathbf{j}^- \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} \ , \tag{A.6}$$

(ii) Find $\mathbf{E}^+_0 \in \mathbf{Y}_0(\Omega_+)$ such that for all $\mathbf{v} \in \mathbf{Y}_0(\Omega_+)$,

$$a^0_+(\mathbf{E}^+_0, \mathbf{v}) = \mathrm{i}\omega \int_{\Omega_+} \mathbf{j}^+ \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} \ . \tag{A.7}$$

Here the sesquilinear forms (regularized versions) $a^0_\pm$ are defined as

$$a^0_\pm(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\pm} \frac{1}{\mu_\pm} \mathrm{curl}\,\mathbf{u} \cdot \mathrm{curl}\,\overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \int_{\Omega_\pm} \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} \ .$$

We have the following well-posedness and elliptic regularity results.

Proposition A.2. *Let* $\mathbf{j} \in \mathbf{L}^2(\Omega)$ *such that* $\operatorname{div} \mathbf{j}^{\pm} = 0$ *in* $\Omega_{\pm}$. *Then there exists a unique solution* $\mathbf{E}_0^- \in \mathbf{Y}_0(\Omega_-)$ *to problem* (A.6) *and there exists a unique solution* $\mathbf{E}_0^+ \in \mathbf{Y}_0(\Omega_+)$ *to problem* (A.7). *The solution* $\mathbf{E}_0^{\pm}$ *satisfies all equations in* (A.2)-(A.3).

Moreover, if $\mathbf{j}^{\pm} \in \mathbf{H}^s(\Omega_{\pm})$, $s \geqslant 0$, *then we have*

$$\mathbf{E}_0^- \in \mathbf{H}^{s+2}(\Omega_-) \quad \textit{and} \quad \mathbf{E}_0^+ \in \mathbf{H}^{s+2}(\Omega_+) \ .$$

Remark A.3. We recall that on the space $\mathbf{X}_{\mathbf{N}}(\Omega_-) = \mathbf{H}(\operatorname{div}, \Omega_-) \cap \mathbf{H}_0(\operatorname{curl}, \Omega_-)$ the seminorm

$$\mathbf{u} \longmapsto \| \operatorname{curl} \mathbf{u} \|_{0,\Omega_-} + \| \operatorname{div} \mathbf{u} \|_{0,\Omega_-} \ ,$$

is equivalent to the norm $\| \cdot \|_{\mathbf{X}(\Omega_-)}$ where $\mathbf{X}(\Omega_-) = \mathbf{H}(\operatorname{div}, \Omega_-) \cap \mathbf{H}(\operatorname{curl}, \Omega_-)$. Then the well-posedness result for $\mathbf{E}_0^-$ in $\mathbf{Y}_0(\Omega_-)$ is obtained as an application of the Lax-Milgram Lemma.

As a consequence of Lemma A.1, we infer the following theorem

Theorem A.4. *Let* $\mathbf{j} \in \mathbf{L}^2(\Omega)$ *such that* $\operatorname{div} \mathbf{j} = 0$ *in* $\Omega_{\pm}^{\varepsilon}$, $\mathbf{j} = 0$ *in* Ω_o^{ε} *and* $\mathbf{j} \cdot \mathbf{n} = 0$ *on* $\Gamma_{\pm}^{\varepsilon}$. *Let the positive constants* $\mu_{\pm}$, μ_o, $\tilde{\sigma}$ *and* ω *be fixed. Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$
(i) *There exists a unique solution* $\mathbf{E}^{\varepsilon} \in \mathbf{Y}_{\varepsilon}$ *to problem* (A.4).
(ii) *The solution* $\mathbf{E}^{\varepsilon}$ *satisfies all equations in* (A.1).
(iii) *Uniform estimates in* $\mathbf{Y}_{\varepsilon}$: *there exists* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$

$$\|\mathbf{E}^{\varepsilon}\|_{\mathbf{Y}_{\varepsilon}} \leqslant C \|\mathbf{j}\|_{0,\Omega} \ . \tag{A.8}$$

(iv) *As* $\varepsilon \to 0$, $\mathbf{E}^{\varepsilon} \to \mathbf{E}_0$ *(at least in* $\mathbf{L}^2(\Omega_o^{\varepsilon})$*) and we have the following estimates : there exists* $C > 0$ *such that for any small parameter* $\varepsilon \in (0, \varepsilon_0)$

$$\|\mathbf{E}^{\varepsilon} - \mathbf{E}_0\|_{0,\Omega_o^{\varepsilon}} + \varepsilon \| \operatorname{curl}(\mathbf{E}^{\varepsilon} - \mathbf{E}_0)\|_{0,\Omega} \leqslant C\sqrt{\varepsilon} \ . \tag{A.9}$$

Proof. *(i)* Accordingly estimate (A.5), a straightforward application of the Lax-Milgram lemma leads to existence and uniqueness to the solution $\mathbf{E}^{\varepsilon}$ to the variational problem (A.4).
(ii) The proof is worked out in details in [8, Theorem 2.3] in a slightly different configuration.
(iii) Using estimates (A.5), estimates (A.8) are obvious.
(iv) According to Prop. A.2, $\mathbf{E}_0^{\pm}$ and $\operatorname{curl} \mathbf{E}_0^{\pm}$ belong to $\mathbf{H}^2(\Omega_{\pm})$. Hence $\mathbf{E}_0^{\pm}$ and $\operatorname{curl} \mathbf{E}_0^{\pm}$ belong to $\mathbf{L}^{\infty}(\Omega_{\pm})$. We denote by $\mathbf{U} = \mathbf{E}^{\varepsilon} - \mathbf{E}_0$.

According to (A.1)-(A.2), and since $\operatorname{curl} \frac{1}{\mu_o} \operatorname{curl} \mathbf{E}_0^{\pm} = \mathrm{i}\omega \mathbf{j}^{\pm} = 0$ in Ω_o^ε, we deduce that $\mathbf{U}$ solves the following equations for any $\varepsilon > 0$

$$\operatorname{curl} \frac{1}{\mu_\pm} \operatorname{curl} \mathbf{U}_\pm = 0 \quad \text{in} \quad \Omega_\pm^\varepsilon \,, \tag{A.10a}$$

$$\operatorname{div} \mathbf{U}_\pm = 0 \quad \text{in} \quad \Omega_\pm^\varepsilon \,, \tag{A.10b}$$

$$\operatorname{curl} \frac{1}{\mu_o} \operatorname{curl} \mathbf{U}_o - \mathrm{i}\omega\sigma_o^\varepsilon \mathbf{U}_o = \mathrm{i}\omega\sigma_o^\varepsilon \mathbf{E}_0^- \quad \text{in} \quad \Omega_o^\varepsilon \cap \Omega_- \,, \tag{A.10c}$$

$$\operatorname{curl} \frac{1}{\mu_o} \operatorname{curl} \mathbf{U}_o - \mathrm{i}\omega\sigma_o^\varepsilon \mathbf{U}_o = \mathrm{i}\omega\sigma_o^\varepsilon \mathbf{E}_0^+ \quad \text{in} \quad \Omega_o^\varepsilon \cap \Omega_+ \,, \tag{A.10d}$$

$$\operatorname{div} \mathbf{U}_o = 0 \quad \text{in} \quad \Omega_o^\varepsilon \cap (\Omega_- \cup \Omega_+) \,, \tag{A.10e}$$

$$[\mathbf{U} \times \mathbf{n}] = 0 \,, \quad \text{on} \quad \Gamma_\pm^\varepsilon \cup \Gamma \,, \tag{A.10f}$$

$$\left[\frac{1}{\mu} \operatorname{curl} \mathbf{U} \times \mathbf{n}\right] = 0 \,, \quad \text{on} \quad \Gamma_\pm^\varepsilon \,, \tag{A.10g}$$

$$\left[\frac{1}{\mu} \operatorname{curl} \mathbf{U} \times \mathbf{n}\right] = -\left[\frac{1}{\mu} \operatorname{curl} \mathbf{E}_0 \times \mathbf{n}\right] \,, \quad \text{on} \quad \Gamma \,, \tag{A.10h}$$

$$\mathbf{U}_+ \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega. \tag{A.10i}$$

One notes that $\operatorname{curl} \mathbf{U}$ belongs to $\mathbf{L}^2(\Omega)$. Multiplying the equations (A.10a)-(A.10c)-(A.10d) by $\overline{\mathbf{U}}$ and integrating by parts we obtain the following identity

$$\int_\Omega \frac{1}{\underline{\mu}^\varepsilon} |\operatorname{curl} \mathbf{U}|^2 \,\mathrm{d}\mathbf{x} - \mathrm{i}\omega\varepsilon^{-2}\tilde{\sigma} \int_{\Omega_o^\varepsilon} |\mathbf{U}_o|^2 \,\mathrm{d}\mathbf{x} = \mathrm{i}\omega\varepsilon^{-2}\tilde{\sigma} \int_{\Omega_o^\varepsilon} \mathbf{E}_0 \cdot \overline{\mathbf{U}_o} \,\mathrm{d}\mathbf{x} + \mathrm{i}\omega \left\langle [\mathbf{H}_0 \times \mathbf{n}]_\Gamma \,, \overline{(\mathbf{E}_o^\varepsilon)_\mathbf{T}} \right\rangle_\Gamma \,, \tag{A.11}$$

since $\left[\frac{1}{\mu} \operatorname{curl} \mathbf{E}_0 \times \mathbf{n}\right] = -\mathrm{i}\omega \left[\mathbf{H}_0 \times \mathbf{n}\right]$ and $\mathbf{U}_\mathbf{T} = (\mathbf{E}_o^\varepsilon)_\mathbf{T}$ on Γ.

One notes that $[\mathbf{H}_0 \times \mathbf{n}]_\Gamma$ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ and according to (A.8) $(\mathbf{E}_o^\varepsilon)_\mathbf{T}$ is bounded in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$. Then taking the imaginary part and the real part of the identity (A.11) and absorbing the right-hand sides, we infer the following estimates

$$\|\mathbf{U}\|_{0,\Omega_o^\varepsilon} + \varepsilon \|\operatorname{curl} \mathbf{U}\|_{0,\Omega} \leqslant C\sqrt{\varepsilon} \,,$$

since

$$\|\mathbf{E}_0\|_{0,\Omega_o^\varepsilon} \lesssim \sqrt{\varepsilon} \,.$$

That ends the proof of estimates (A.9).

□

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