Switched gain differentiator with fixed-time convergence

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Abstract Acceleration of estimation for a class of nonlinear systems in the output canonical form is considered in this work. The acceleration is achieved by a supervisory algorithm design that switches among different values of observer gain. The presence of bounded matched disturbances, Lipschitz uncertainties and measurement noises is taken into account. The proposed switched-gain observer guarantees global uniform time of convergence of the estimation error to the origin in the noise-free case. In the presence of noise our commutation strategy pursuits the goals of overshoot reducing for the initial phase, acceleration of convergence and improvement of asymptotic precision of estimation. Efficacy of the proposed switching-gain observer is illustrated by numerical comparison with a sliding mode and linear high-gain observers.

Keywords: High-gain observers, Differentiator, Switched system

1. INTRODUCTION

State estimation for linear and nonlinear systems is one of the central problems in the control systems theory. There are many methods proposed for linear systems and plenty nonlinear solutions Crassidis and Junkins (2012); Khalil (1996); Luenberger (1979); Sontag (1998); Utkin (1992), which are differing by the requirements imposed on the plant model and by the guaranteed performance (asymptotic precision, initial overshooting, rate and domain of convergence, and robustness with respect to external disturbances, measurement noises and small delays, *etc.*) of the estimation error dynamics.

There are different kinds of convergence rates. For example, if a system is homogeneous with a negative/zero/positive degree and asymptotically stable, then actually it has a finite-time/exponential/asymptotic rate of convergence (in the case of positive degree the time of convergence to a sphere is uniformly bounded by a constant for any initial conditions, if the system is also locally finite-time converging, then it is called fixed-time stable) Moulay and Perruquetti (2008); Cruz-Zavala et al. (2011); Polyakov (2012); Lopez-Ramirez et al. (2016); Ríos and Teel (2016).

Convergence rate and asymptotic estimation errors are the main optimization criteria for state observers in applications. The present work studies the problem of adjusting the convergence rate for a class of observers designed for nonlinear systems in the output canonical form. It is a well-known fact that augmenting the observer gains it is possible to accelerate the speed of convergence of the estimates to the evaluated values, however, this also leads to the robustness degradation with respect to the measurement noises and peaking phenomenon (initial huge overshooting) Luenberger (1979). In the present work a switching algorithm between different values of the observer gain is designed, which is aimed on resolving these issues. Noise dependence optimization of the asymptotic precision via a single switch of the observer gains has been proposed in Elbeheiry and Elmaraghy (2003); Ahrens and Khalil (2009). Continuous-time gain adaptation has been investigated in many works, see Andrieu et al. (2009); Boizot et al. (2010); Sanfelice and Praly (2011) for interesting examples. For the problem of state feedback stabilization there are also supervisory algorithms aiming on acceleration of convergence: in Ananyevskii (2003) for a scalar linear system with bounded perturbation a switching rule, which increases the scalar gain of linear feedback, has been proposed making the closed-loop system finite-time stable, an extension to planar mechanical systems is given in Ananyevskii (2001). In Dvir and Levant (2015a,b) for sliding mode control systems Utkin (1992); Fridman (2011); Moreno and Osorio (2012); Poznyak et al. (2004) it has been proposed an algorithm of on-line switching between parameters, which ensures a desired accelerated rate of convergence for the closed-loop system. An algorithm of parameter switching for finite-time and fixed-time convergence to the origin (or

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a ball) is developed in Efimov et al. (2016a) for homogeneous systems with different degrees. The present work is based on ideas of gain commutation presented in Ahrens and Khalil (2009) and Efimov et al. (2016a).

The outline of this paper is as follows. Notation and preliminary results are introduced in sections 2 and 3. The precise problem statement and some auxiliary results are given in Section 4. The proposed supervisory algorithm of switching among different sets of values of the observer gain ensuring a required acceleration is presented in Section 5. The proposed supervisory algorithm is illustrated and compared by computer experiments with a sliding-mode differentiator in Section 6. Concluding remarks and discussion appear in Section 7.

2. NOTATION

Through the paper the following notation is used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real number.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$, $\|x\|_{\mathcal{A}} = \inf_{\xi \in \mathcal{A}} \|x - \xi\|$ is the distance from a point $x \in \mathbb{R}^n$ to a set $\mathcal{A} \subset \mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $||d||_{[t_0,t_1)} = \text{ess}\sup_{t\in[t_0,t_1)}\|d(t)\|$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of $d(t)$ with the property $||d||_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions from $\mathbb{R}_+$ to $\mathbb{R}^m$); $\mathcal{L}_D^m = \{d \in \mathcal{L}_\infty^m : ||d||_\infty \leq D\}$ for any $D > 0$ ($\mathcal{L}_D^1 = \mathcal{L}_D$).
- For a symmetric matrix $A \in \mathbb{R}^{n\times n}$ denote $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimum and the maximum eigenvalues of A, respectively.

3. PRELIMINARIES

Consider the following nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input, $d \in \mathcal{L}_\infty^m$; $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ ensures forward existence of the system solutions (understood in the Filippov sense Filippov (1988)) at least locally, $f(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty^m$ define the corresponding solution by $X(t, x_0, d)$ for any $t \geq 0$ for which the solution exists. A set $\mathcal{A} \subset \mathbb{R}^n$ is called uniformly forward invariant for (1) if $x_0 \in \mathcal{A}$ implies that $X(t, x_0, d) \in \mathcal{A}$ for all $t \geq 0$ and all $d \in \mathcal{L}_D^m$ for given $D > 0$.

Following Roxin (1966); Khalil (1996); Lin et al. (1996); Polyakov (2012), let Ω be an open neighborhood of non-empty, compact and uniformly forward invariant set $\mathcal{A} \subset \mathbb{R}^n$ of (1) with some $D > 0$.

Definition 1. At the set $\mathcal{A}$ the system (1) for $d \in \mathcal{L}_D^m$ is said to be

(a) *uniformly Lyapunov stable* if for any $x_0 \in \Omega$ and $d \in \mathcal{L}_D^m$ the solution $X(t, x_0, d)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x_0 \in \Omega$, if $\|x_0\|_{\mathcal{A}} \leq \delta$ then $\|X(t, x_0, d)\|_{\mathcal{A}} \leq \epsilon$ for all $t \geq 0$;

(b) *uniformly asymptotically stable* if it is uniformly Lyapunov stable and for any $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa, \epsilon) \geq 0$ such that for any $x_0 \in \Omega$ and $d \in \mathcal{L}_D^m$, if $\|x_0\|_{\mathcal{A}} \leq \kappa$ then $\|X(t, x_0, d)\|_{\mathcal{A}} \leq \epsilon$ for all $t \geq T(\kappa, \epsilon)$;

(c) *uniformly finite-time stable* if it is uniformly Lyapunov stable and *uniformly finite-time converging from* Ω, *i.e.* for any $x_0 \in \Omega$ and all $d \in \mathcal{L}_D^m$ there exists $0 \leq T < +\infty$ such that $X(t, x_0, d) \in \mathcal{A}$ for all $t \geq T$. The function $T_{\mathcal{A}}(x_0) = \inf\{T \geq 0 : X(t, x_0, d) \in \mathcal{A}\ \forall t \geq T,\ \forall d \in \mathcal{L}_D^m\}$ is called the *uniform settling time* of the system (1);

(d) *uniformly fixed-time stable* if it is uniformly finite-time stable and $\sup_{x_0\in\Omega} T_{\mathcal{A}}(x_0) < +\infty$.

The set Ω is called a *domain of stability/attraction.*

If $\Omega = \mathbb{R}^n$, then the corresponding properties are called *global* uniform Lyapunov/asymptotic/finite-/fixed-time stability of (1) for $d \in \mathcal{L}_D^m$ at $\mathcal{A}$.

4. PROBLEM INTRODUCTION

In this section the system of interest is introduced with a basic observer, next some their properties used in the sequel are discussed, and finally the problem statement is given.

4.1 Basic system and its observer

Consider a nonlinear system in a canonical form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \varphi(u(t), y(t)) \\ &\quad + b[g(x(t)) + d(t)],\ t \geq 0, \\ y(t) &= c^T x(t) + v(t), \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector with $n > 1$, $y(t) \in \mathbb{R}$ is the output available for measurements with the noise $v(t) \in \mathbb{R}$, $v \in \mathcal{L}_V$ for some $V > 0$; $u(t) \in \mathbb{R}^m$ is control, $u \in \mathcal{L}_\infty^m$; $d(t) \in \mathbb{R}$ is the exogenous disturbance, $d \in \mathcal{L}_D$ for some $D > 0$; the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

are in the canonical form; the function $\varphi : \mathbb{R}^{m+1} \to \mathbb{R}^n$ ensures existence and uniqueness of solutions of the system (2), the function $g : \mathbb{R}^n \to \mathbb{R}$ is globally Lipschitz, then there exists $\gamma > 0$ such that for all $x', x \in \mathbb{R}^n$:

$$\|g(x) - g(x')\| \leq \gamma\|x - x'\|.$$

Following Khalil (1996); Ahrens and Khalil (2009); Sanfelice and Praly (2011) the simplest observer for (2) takes the form:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \varphi(u(t), y(t)) + bg(\hat{x}(t)) \\ &\quad + l(y(t) - c^T\hat{x}(t)), \end{aligned} \tag{3}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and $l \in \mathbb{R}^n$ is the observer gain to be designed.

Assumption 1. For given $\gamma > 0$, $\kappa > 0$, $\rho_d > 0$ and $\rho_v > 0$ there exist $P = P^T \in \mathbb{R}^{n\times n}$ and $w \in \mathbb{R}^n$ such that

$$P > 0, \begin{bmatrix} \tilde{S} & Pb & -w & -Pb \\ b^T P & -\rho_d^2 & 0 & 0 \\ -w^T & 0 & -\rho_v^2 & 0 \\ -b^T P & 0 & 0 & -\gamma^{-2} \end{bmatrix} \leq 0,$$
$$\tilde{S} = A^T P + PA - cw^T - wc^T + I_n + \kappa P.$$

In this assumption κ, ρ_d and ρ_v are design parameters, which meaning will be explained later.

Remark 2. For $\gamma = 0$, *i.e.* if there is no uncertainty $g(\cdot)$, then the above linear matrix inequalities (LMIs) take the form:

$$P > 0, \begin{bmatrix} \tilde{S} & Pb & -w \\ b^T P & -\rho_d^2 & 0 \\ -w^T & 0 & -\rho_v^2 \end{bmatrix} \leq 0,$$
$$\tilde{S} = A^T P + PA - cw^T - wc^T + \kappa P.$$

Lemma 3. Let Assumption 1 be satisfied, then in (2), (3) with $l = P^{-1}w$ for all $t \geq 0$:

$$\|x(t) - \hat{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-0.5\kappa t} \|x(0) - \hat{x}(0)\| + \sqrt{\frac{1}{\kappa \lambda_{\min}(P)}} (\rho_d D + \rho_v V).$$

All proofs are excluded due to space limitations.

4.2 The problem statement

As it has been established in Lemma 3, the observer (3) ensures a robust estimation of the state of (2) with an exponential rate of convergence (if LMIs from Assumption 1 are satisfied). The problem further studied in this work: is it possible to ensure the uniform fixed-time estimation of (2), with convergence of estimation error to a ball proportional to D and V, by switching among different sets of coefficients in the gain l, for $d \in \mathcal{L}_D$ and $v \in \mathcal{L}_V$ with given $D > 0$ and $V > 0$. Such a problem in stabilization context has been already considered in Ananyevskii (2001, 2003) (finite-time case), in Dvir and Levant (2015a,b) for the sliding mode feedback, and in Efimov et al. (2016a) the general case of homogeneous systems has been analyzed. A high-gain observer similar to (3) with a switched gain has been proposed in Ahrens and Khalil (2009), but there the switching has been performed just once in order to minimize influence of noise on asymptotic behavior. The problem of multiple commutation of gains, with accelerating the convergence and making it uniform other $\mathbb{R}^n$, is considered in the present work.

Following Ahrens and Khalil (2009); Efimov et al. (2016a), instead of (3) we will consider for all integer $i \geq 0$ the observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), y(t)) + bg(\hat{x}(t)) + \mu_i M_i^{-1} l(y(t) - c^T M_i \hat{x}(t)),\ t \in [t_i, t_{i+1}), \tag{4}$$

where $\hat{x}(t) \in \mathbb{R}^n$ has meaning as previously, l is as before comes from Assumption 1; $M_i = \mathrm{diag}\{\mu_i^{1-k}\}_{k=1}^n$ and scalars $\mu_i \geq 1$ form a sequence of parameters, which stay constant on the interval $[t_i, t_{i+1})$ and change their values at instants t_i, $i \geq 0$ ($t_0 = 0$). It is required to determine the instants t_i, $i \geq 0$ and the discrete-time update law for μ_i such that for (4) the estimation error variable $e(t) = x(t) - \hat{x}(t)$ becomes uniformly fixed-time stable with respect to the origin.

5. DESIGN OF SWITCHED-GAIN OBSERVER

Let us introduce an auxiliary dynamical system for $t \in [t_i, t_{i+1})$:

$$\dot{z}(t) = \mu_i\{(A - lc^T)z(t) - lv(t) + \mu_i^{-n} b[g(x(t)) - g(x(t) - M_i^{-1} z(t)) + d(t)]\}, \tag{5}$$

where μ_i is the same as in (4), and after update of μ_i to μ_{i+1} at the instant of time t_{i+1} we have a state resetting for $z(t)$:

$$z(t_{i+1}) = M_{i+1} M_i^{-1} z(t_{i+1}^-), \tag{6}$$

where $z(t_{i+1}^-)$ denotes the left limit of $z(t)$ as t is approaching t_{i+1} from the left. As we can conclude, (5), (6) is a hybrid system, which has to be augmented by rules for assignment of switching instants t_i and for update of μ_i, for all $i \geq 0$.

Obviously, $e(t) = M_i^{-1} z(t)$ for $t \in [t_i, t_{i+1})$ is the corresponding solution of the estimation error dynamics of (4):

$$\dot{e}(t) = (A - \mu_i M_i^{-1} l c^T M_i) e(t) + b[g(x(t)) - g(\hat{x}(t)) + d(t)] - \mu_i M_i^{-1} l v(t).$$

Therefore, in order to design the supervisory algorithms for selection of t_i and μ_i we will consider below the hybrid system (5), (6).

Lemma 4. Let Assumption 1 be satisfied, then in (5) with $l = P^{-1}w$ for all $t \in [t_i, t_{i+1})$:

$$\|z(t)\| \leq e^{-0.5\kappa\mu_i(t-t_i)} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|z(t_i)\| + \sqrt{\frac{1 - e^{-\kappa\mu_i(t-t_i)}}{\kappa\lambda_{\min}(P)}} (\rho_d \mu_i^{-n} D + \rho_v V).$$

Thus, for the system (5) augmenting value of $\mu_i \geq 1$ leads to increase of the convergence speed and to decrease of the gain with respect to the disturbance d (the system becomes uniform in d for $\lim_{i\to+\infty} \mu_i = +\infty$).

Corollary 5. Let Assumption 1 be satisfied, then for all $t \in [t_i, t_{i+1})$:

$$\|x(t) - \hat{x}(t)\| \leq \mu_i^{n-1} e^{-0.5\kappa\mu_i(t-t_i)} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|x(t_i) - \hat{x}(t_i)\| + \sqrt{\frac{1 - e^{-\kappa\mu_i(t-t_i)}}{\kappa\lambda_{\min}(P)}} (\rho_d \mu_i^{-1} D + \mu_i^{n-1} \rho_v V),$$

$$|c^T[x(t) - \hat{x}(t)]| \leq e^{-0.5\kappa\mu_i(t-t_i)} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|x(t_i) - \hat{x}(t_i)\| + \sqrt{\frac{1 - e^{-\kappa\mu_i(t-t_i)}}{\kappa\lambda_{\min}(P)}} (\rho_d \mu_i^{-n} D + \rho_v V).$$

Thus, for the system (2), (4) augmenting value of $\mu_i \geq 1$ leads to increase of the convergence speed and overshooting, and to decrease of the gain with respect to the disturbance d, however, at the price that the gain with respect to the measurement noise v grows drastically. An important observation is that the measured part of the estimation error $c^T[x(t)-\hat{x}(t)]$ is free from these shortages (noise gain and overshooting growth), while inheriting acceleration of the convergence rate and decrease of the disturbance gain.

5.1 Supervisory algorithm for the noise-free case

First, assume that $V = 0$, *i.e.* there is no measurement noise $v(t)$ in (2). Then, according to Corollary 5, augmentation of μ_i will lead to uniform in d estimation of the state of the system (2) by the observer (4). Let us design an algorithm for commutation of μ_i guaranteeing the global fixed-time convergence of the estimation error to the origin in (2), (4).

To this end, the following algorithm is proposed in this work:

$$t_{i+1} = t_i + T_i,\ t_0 = 0, \tag{7}$$

$$T_i = -\frac{2}{\kappa\mu_i}\ln\left(\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\frac{\mu_i^{1-n}}{q}\right)$$

for all $i \geq 0$, where $q > 1$ is a tuning parameter, and

$$\mu_i = q^{\alpha i}, \tag{8}$$

where $\alpha > 0$ is another tuning parameters. Note that for (8),

$$\begin{aligned}\|M_{i+1}M_i^{-1}\| &= \left\|\mathrm{diag}\left\{\frac{\mu_{i+1}^{1-k}}{\mu_i^{1-k}}\right\}_{k=1}^{n}\right\| \\ &= \left\|\mathrm{diag}\{q^{\alpha(1-k)}\}_{k=1}^{n}\right\| \leq 1\end{aligned}$$

and in variable $z(t)$ the state jumps at instants t_i are not stretching, thus all properties are predefined by the continuous-time dynamics, which, as we already recognized above, is just accelerated by μ_i.

Let us prove that for the supervisory algorithm (7), (8) the estimation error of (2), (4) converges in a fixed time to the origin.

Theorem 6. Let $D = 0$ and $V = 0$ for the system (2), and for the observer (4) the supervisory algorithm be selected as in (7), (8) with $q > 1$ and $\alpha \in (0, \frac{1}{n-1}]$. Then the estimation error dynamics is globally fixed-time stable at the origin. If $d \in \mathcal{L}_D$ for some $D > 0$, then the estimation error is globally fixed-time convergent. In addition, the time of convergence is less than

$$\overline{T}_0 = -\frac{2}{\kappa}\frac{q^{\alpha}}{q^{\alpha}-1}\ln\left(\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}q^{\frac{\alpha(1-n)}{q^{\alpha}-1}-1}\right).$$

Thus, in the ideal case ($D = V = 0$) observer (4) provides for global fixed-time stability of the estimation error for the system (2) with Lipschitz uncertainty. If $D \neq 0$, then the rate of convergence is preserved in the system, but the overshoots during transients may have a complex behavior.

Remark 7. As we can conclude from Theorem 6, the lengths of intervals between switching T_i are monotonously decreasing with $i \to +\infty$ and approaching zero in a finite time, then the dynamics exhibits a Zeno behavior, like in sliding mode control systems Filippov (1988); Edwards and Spurgeon (1998); Boiko and Fridman (2005); Levant (2010); Fridman (2011). In practice the switching with a frequency higher than the sampling frequency of the system (computer) is not possible, and the number of switches is always finite (the switching stops when T_i becomes too small), thus the Zeno behavior in applications is never presented for (4).

Remark 8. The convergence speed can be increased by considering

$$\mu_{i+1} = q^{\alpha}\mu_i \tag{9}$$

with $\mu_0 > 1$ instead of (8), where $\mu_0 = 1$ always. The proofs stay almost the same in this case.

5.2 Supervision algorithm for the noisy case

Now let us consider the general case with $V \neq 0$. From Corollary 5 we conclude that for any fixed $\mu_i \geq 1$ the estimation error in (2), (4) converges asymptotically to the ball of radius $r(\mu_i) = \sqrt{\frac{1}{\kappa\lambda_{\min}(P)}}(\rho_d\mu_i^{-1}D + \mu_i^{n-1}\rho_v V)$. Since $D > 0$ and $V > 0$ are external restrictions, and P, κ, ρ_d, ρ_v are fixed under Assumption 1, then imposing a reasonable in practice assumption that

$$\rho_d D > (n-1)\rho_v V \tag{10}$$

we can calculate the optimal value $\mu_{\min}$ of μ_i, which minimize $r(\mu_i)$ (it is the solution of the equation $r'(\mu_{\min}) = 0$, and since direct computation shows that $r''(\mu_{\min}) > 0$, then $\mu_i = \mu_{\min}$ is the minimum of $r(\mu_i)$):

$$\mu_{\min} = \sqrt[n]{\frac{1}{n-1}\frac{\rho_d D}{\rho_v V}}.$$

Under (10) $\mu_{\min} > 1$, and (10) can be guaranteed while solving LMIs for Assumption 1 by a proper selection of ρ_d and ρ_v. The maximal value $\mu_{\max}$ of μ_i, which does not lead to deterioration of the asymptotic estimation precision in (2), (4), can be found as the solution higher than $\mu_{\min}$ of the following equation:

$$r(\mu_{\max}) = r(\mu_0),$$

such a solution always exists under (10). For example, for $n = 2$ and $\mu_0 = 1$ we obtain:

$$\mu_{\min} = \sqrt{\frac{\rho_d D}{\rho_v V}},\ \mu_{\max} = \frac{\rho_d D}{\rho_v V}.$$

Therefore, in the noisy case the maximum number of switching $i^* > 0$ can be calculated, and the switching stops either when $\mu_{i^*+1} \geq \mu_{\max}$ or when $T_{i^*+1} \leq T_{\min}$, where $T_{\min} > 0$ is the time constant related with the maximal admissible frequency of commutation in the system. Hence,

$$i^* = \mathrm{floor}\left(\min\{\alpha^{-1}\log_q \mu_{\max}, i'\}\right),$$

where the function floor$(\cdot)$ returns the biggest integer not higher than the argument, and i' is the solution of the following equation:

$$T_{\min} = -\frac{2}{\kappa q^{i'\alpha}}\ln\left(\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\frac{q^{i'\alpha(1-n)}}{q}\right).$$

Remark 9. Note that if the condition $\mu_{i^*+1} \geq \mu_{\max}$ is realized (or $i \geq i^*$), then after some dwell time the value of μ_i can be reset back to μ_0 or $\mu_{\min}$, since a similar asymptotic accuracy is guaranteed in this case. This idea has been proposed in Ahrens and Khalil (2009).

Another possibility to orchestrate the switching law is by analyzing the value of $c^T[x(t) - \hat{x}(t)]$, which is available for measurements, and which, according to Corollary 5, is monotonously decreasing for $i \geq 0$. It can be used to stop (7), (8) when acceleration phase is finished, next reset the value of μ_i to optimize the asymptotic precision (Remark 9), and finally to activate acceleration again if the signal has been changed and acceleration phase is needed again.

5.3 Numerical implementation of (4)

The main issue with application of the proposed observer with the switched gains is that μ_i is monotonously increasing in accordance with (8), and if the explicit Euler discretization method is used for implementation of (4), then such a realization may become unstable for some sufficiently high values of μ_i. In order to avoid such a drawback, for the case $g(\cdot) = 0$, the implicit Euler method is proposed to use in implementation of (4) (see Efimov et al. (2016b) for a discussion on advantages of the implicit Euler method other the explicit one for calculation of solutions of fixed-time stable systems). Let $h > 0$ be the discretization step and $\hat{x}_\ell = \hat{x}(h\ell)$ be the value of estimate of the state at discrete time instant $h\ell$, then in accordance with the implicit Euler method Butcher (2008) (the substitutions $y(h(\ell+1)) \simeq y(h\ell)$ and $u(h(\ell+1)) \simeq u(h\ell)$ have been used to ensure the algorithm causality):

$$\hat{x}_{\ell+1} = O_{\mu_i}^{-1}\{\hat{x}_\ell + h[\varphi(y(h\ell), u(h\ell)) + \mu_i M_i^{-1} l y(h\ell)]\}, \quad (11)$$

where $O_{\mu_i} = I_n - hA + \mu_i h M_i^{-1} l c^T$, and while $1 + h\mu_i c^T S M_i^{-1} l \neq 0$, where $S = (I_n - hA)^{-1}$, the inverse of O_{μ_i} can be derived using Sherman–Morrison formula:

$$O_{\mu_i}^{-1} = S - \frac{S M_i^{-1} l c^T S}{h^{-1}\mu_i^{-1} + c^T S M_i^{-1} l}.$$

Thus, the matrix S can be calculated in advance, and the inverse of diagonal matrix M_i is not costly. It is well known fact that the implicit Euler method is converging for any h Butcher (2008); Efimov et al. (2016b).

6. NUMERICAL COMPARISON

In order to illustrate the advantages of the proposed switched-gain observer (4), (7), (8) let us consider the problem of differentiation of a harmonic signal, and compare the obtained solution with (4) for a fixed μ_i and the well-known super-twisting differentiator Levant (2005):

$$\dot{\zeta}_1(t) = \zeta_2(t) - 1.5D^{1.5}|\zeta_1(t) - y(t)|^{0.5}\mathrm{sign}(\zeta_1(t) - y(t)),$$
$$\dot{\zeta}_2(t) = -1.1D\mathrm{sign}(\zeta_1(t) - y(t)),$$
$$y(t) = f(t) + v(t),$$

where $\zeta = [\zeta_1 \ \zeta_2]^T \in \mathbb{R}^2$ is the state of differentiator, $f \in \mathbb{R}$ is the useful signal to be differentiated ($f = c^T x$ for a suitably defined state x in (2)), v is the measurement noise as previously, ζ_2 is the estimate of $\dot{f}$, and D is the Lipschitz constant of $\dot{f}$. For simulation purposes we will use:

$$f(t) = 1 + \sin(t) + \sin(\pi t), \ v(t) = V\sin(\omega t),$$

then $D = 1 + \pi^2$, and take $V = 0.1$. For (4) in this case

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \varphi(\cdot) = g(\cdot) = 0,$$

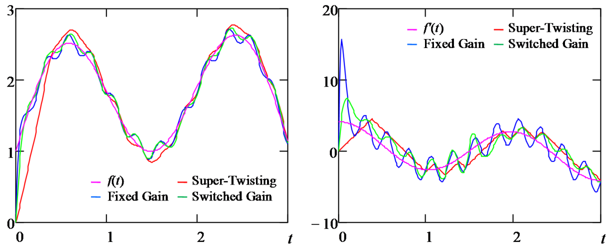

Figure 1. Results of simulation for $h = 0.01$ and $\omega = 25$

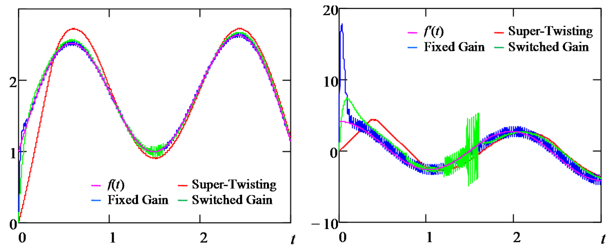

Figure 2. Results of simulation for $h = 0.001$ and $\omega = 250$

and in order to calculate l the values $\rho_d = \rho_v = 1$ and $\kappa = 1$ were selected ($\gamma = 0$), then the LMIs from Assumption 1 (Remark 2) are satisfied for

$$l = [2.6 \ 2.1]^T, \ P = \begin{bmatrix} 0.76 & -0.48 \\ -0.48 & 0.6 \end{bmatrix}.$$

Take $q = 2$ and $\alpha = 1$ in (7), (9) with $\mu_0 = 10$, and let us continue switching while $T_i \leq T_{\min} = 3h$ and $\mu_i \leq \mu_{\max} = 108.69$. Following Remark 9, for $i \geq i^*$ the value of μ_i is reset to be $\mu_{\min} = 10.42$. The super-twisting algorithm is implemented using the explicit Euler method Levant (2005), while for (4) the implicit Euler method based computation (11) is used. We also compare the results with the observer (4) with fixed value of $\mu_i = 30$ (without the supervision algorithm (7), (8)).

For $h = 0.01$ the state trajectories, as well as $f(t)$ and $f'(t)$, are shown in Fig. 1 for $\omega = 25$, the same results for $h = 0.001$ and $\omega = 250$ are presented in Fig. 2. The initial conditions for all observers have been selected to be zero. As we can conclude from these results, gain switching reduces peaking phenomenon comparing with the fixed-gain case, and the obtained differentiator is less sensitive to the discretization step value h than the super-twisting algorithm due to application of the implicit Euler scheme. The switched-gain observer also demonstrates by construction the fastest rate of convergence for big initial errors of estimation (it is not shown in simulation due to its clarity, for the selected values of parameters the uniform global convergence time estimate $\overline{T}_0 = 9.11$).

7. CONCLUSION

The problem of estimation rate acceleration for a class of nonlinear systems in the output canonical form by switching among different values of observer gain is addressed in this work. The presence of bounded matched disturbances, Lipschitz uncertainties and measurement noises is taken into account. The proposed switched-gain observer ensures global fixed-time stability of the estimation error at the origin in the noise-free case. In the presence of noise a modified commutation strategy for the observer gain is proposed, which ensures peaking avoiding for the initial phase, convergence acceleration and optimization of asymptotic precision of estimation. The results are illustrated by computer simulation and comparison in planar case with super-twisting differentiator and observer with fixed gains.

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