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A Discontinuous Adaptive Sliding-Mode Observer for a Class of Uncertain Nonlinear Systems ⋆

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Abstract: In this paper the problem of simultaneous state and parameter estimation is studied for a class of uncertain nonlinear systems. A discontinuous adaptive sliding-mode observer is proposed based on a discontinuous nonlinear parameter estimation algorithm. It is shown that such an algorithm provides a rate of convergence faster than exponential. Then, the proposed discontinuous parameter estimation algorithm is included in the structure of a sliding-mode state observer providing an ultimate bound for the full estimation error. Some simulation results illustrate the feasibility of the proposed adaptive sliding-mode observer.

Keywords: Adaptive observer, Sliding-modes, Nonlinear systems.

1. INTRODUCTION

The adaptive control design has received a great deal of attention in control theory during the last decades. Due to this attention, this area has grown to turn into one of the widest in terms of algorithms, techniques for design, analytical tools, and so on (see, for instance Ioannou and Sun (1996) and Astolfi et al. (2008)). One important problem in the adaptive control area is the design of adaptive observers, *i.e.* the design of observers estimating simultaneously the whole state and the parameters of the system by some on-line adaptation law Besançon (2007).

In this context, there exist a lot of literature related to the adaptive observers design for linear systems (see, for instance Sastry and Bodson (1989), Lüders and Narendra (1973), Carroll and Lindorff (1973), and Narendra and Annaswamy (2005)). Most of these results are based on appropriated change of state coordinates to some canonical form in order to provide a state estimation together with persistence of excitation conditions to ensure the parameter estimation.

For nonlinear systems, one of the first results was proposed by Bastin and Gevers (1988) extending some linear results. In the same vein, some results based on output injection transformations are given for nonlinear systems that are equivalent to linear observable systems in the Brunovsky observer form (see, for example Marino and Tomei (1992) and Marino and Tomei (1995)). More recently, in Besançon (2000) a unifying adaptive observer form is proposed for nonlinear systems providing asymptotic state estimation as well as parameter estimation under some passivity-like conditions. For multiple-input multiple-output (MIMO) linear time-varying systems, an adaptive observer is proposed by Zhang (2002) and it is also valid for affine state nonlinear systems. In Xu and Zhang (2004), a more general design of adaptive observers is proposed for a class of single-output uniformly observable nonlinear systems. For the case of uniformly observable multiple-input multiple-output nonlinear systems, in Farza et al. (2009) an adaptive observer is proposed to exponentially estimate the state and the unknown parameters under a persistent excitation condition. The structure of this observer gives some flexibility to obtain high-gain-like observers and adaptive sliding-mode-like observers. In Stamnes et al. (2011), a redesign of adaptive observers is proposed for nonlinear systems based on adaptive laws that use delayed measurements. These delayed observers improve the performance of the parameter estimation but increase the computa-

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tional load. However, all the works previously mentioned do not consider external disturbances.

Some robust adaptive observers have been proposed in the literature. For instance, in Liu (2009) a robust adaptive observer is provided for nonlinear systems with disturbances and unmodeled dynamics based on adaptive nonlinear damping. Nevertheless, such an observer is just able to estimate the state. Some other solutions have been proposed in the sliding-mode area due to the insensitivity that these algorithms present for certain class of external disturbances Shtessel et al. (2014). In the context of fault detection, in Yan and Edwards (2008) an adaptive sliding-mode observer is provided for a class of nonlinear systems with unknown parameters and faults. Using the inherent features of the sliding-mode observers, a fault reconstruction is given under relative degree of the output with respect to the fault equal to one. Finally, it is worth saying that most of the mentioned adaptive observers propose linear parameter estimation algorithms.

This paper contributes with an adaptive sliding-mode observer based on a nonlinear parameter estimation algorithm for uncertain nonlinear systems. The proposed adaptive sliding-mode observer is a modified version of that one proposed in Efimov et al. (2016). Such a modification lies in the inclusion of a discontinuous nonlinear parameter estimation algorithm that provides a rate of convergence faster than exponential (Ríos et al., 2017). Then, the proposed parameter estimation algorithm is included in the structure of a sliding-mode state observer providing an ultimate bound for the state and parameter estimation error. Some simulation results illustrate the feasibility of the proposed adaptive sliding-mode observer.

The outline of this work is as follows. The problem statement is presented in the Section 2. The proposed adaptive sliding-mode observer is given in Section 3. The simulation results are illustrated by Section 4. Finally, some concluding remarks are discussed in Section 5.

Notation: Let $\|q\|$ denote the Euclidean norm of a vector $q \in \mathbb{R}^n$; $\overline{1,n}$ a sequence of integers $1,...,n$; and I_n an identity matrix of dimension $n \times n$. The induced norm for a matrix $Q \in \mathbb{R}^{m\times n}$ is given as $\|Q\| := \sqrt{\lambda_{\max}(Q^TQ)} = \sigma_{\max}(Q)$, where $\lambda_{\max}$ (respectively, $\lambda_{\min}$) is the maximum (respectively, the minimum) eigenvalue, and $\sigma_{\max}$ is the largest singular value. For a Lebesgue measurable function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ define the norm $\|u\|_{(t_0,t_1)} := \operatorname{ess\,sup}_{t\in(t_0,t_1)} \|u(t)\|$, then $\|u\|_\infty := \|u\|_{(0,+\infty)}$ and the set of functions u with the property $\|u\|_\infty < +\infty$ is denoted as $\mathcal{L}_\infty$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if for each fixed s, $\beta(r,s) \in \mathcal{K}$ with respect to r, and for each fixed r, $\beta(r,s)$ is decreasing to zero with respect to s. The notation $\nabla V(x)f(x)$ denote the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at any point x.

2. PROBLEM STATEMENT

Consider a class of uncertain nonlinear systems that can be written, essentially after a change of coordinates, as follows

$$\dot{x} = Ax + \phi(y,u) + G(t,y,u)\theta + Dw, \tag{1}$$
$$y = Cx, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the measurable output vector, $u \in \mathbb{R}^m$ is the control input vector, $\theta \in \mathbb{R}^q$ is a vector of unknown constant parameters, and $w \in \mathbb{R}^l$ is a vector of external disturbances. The matrices A, C and D are known, they have corresponding dimensions, and the pair (A,C) is detectable. The functions $\phi : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ and $G : \mathbb{R}_{\geq 0} \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^{n\times q}$ are also known and they ensure uniqueness and existence of solutions for system (1) for all admissible disturbances.

The aim of this paper is to provide estimations of the state and parameter vectors, *i.e.* x and θ, respectively; only using the information of the output y and attenuating as much as possible the effects of the external disturbances w.

The following assumptions are introduced for the system (1)-(2).

Assumption 1. *The trajectories of the system, the control input, and the external disturbances belong to* $\mathcal{L}_\infty$, i.e. $\|x\|_\infty < +\infty$, $\|u\|_\infty < +\infty$, *and* $\|w\|_\infty < +\infty$, *respectively; and* $\|G(t,y(t),u(t))\| < +\infty$ *for all* $t \geq 0$.

2.1 Preliminaries

Consider the following nonlinear system

$$\dot{x} = f(x,w), \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^l$ is the external disturbances, and $f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ is a locally Lipschitz function. For an initial condition $x_0 \in \mathbb{R}^n$ and an external disturbance $w \in \mathcal{L}_\infty$, denote the solution by $x(t,x_0,w)$ for any $t \geq 0$ for which the solution exists.

The following stability properties for system (3) are introduced (for more details see Jiang et al. (1996), Dashkovskiy et al. (2011) and Bernuau et al. (2013)).

Definition 1. *The system* (3) *is said to be* Input-to-State practically Stable (ISpS) *if for any* $w \in \mathcal{L}_\infty$ *and any* $x_0 \in \mathbb{R}^n$ *there exist some functions* $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ *and a constant* $\kappa \in \mathbb{R}_{\geq 0}$ *such that*

$$\left\|x(t,x_0,w)\right\| \leq \beta(\left\|x_0\right\|,t) + \gamma(\|w\|_\infty) + \kappa, \ \forall t \geq 0.$$

The system (3) *is said to be* Input-to-State Stable (ISS) *if* $\kappa = 0$.

These properties also have a Lyapunov function characterization.

Definition 2. *A smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is said to be an* ISpS Lyapunov function *for system* (3) *if for all* $x \in \mathbb{R}^n$ *and any* $w \in \mathcal{L}_\infty$ *there exist some functions* $\psi_1, \psi_2, \psi_3 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, *and a constant* $\kappa \in \mathbb{R}_{\geq 0}$ *such that*

$$\psi_1(\|x\|) \leq V \leq \psi_2(\|x\|),$$
$$\|x\| \geq \chi(\|w\|_\infty) + \kappa \Rightarrow \nabla V(x) f(x,w) \leq -\psi_3(\|x\|).$$

The function V *is said to be an* ISS Lyapunov function *for system* (3) *if* $\kappa = 0$.

Theorem 1. (Dashkovskiy et al., 2011). *The system* (3) *is* ISpS (ISS) *if and only if it admits an* ISpS (ISS) Lyapunov function.

Let us consider the following interconnected nonlinear system

$$\dot{x}_1 = f_1(x_1, x_2, w), \tag{4}$$
$$\dot{x}_2 = f_2(x_1, x_2, w), \tag{5}$$

where $x_i \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^l$, and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^l \to \mathbb{R}^{n_i}$ ensures existence of the system solutions at least locally, for $i = 1, 2$. Suppose that there exist ISpS Lyapunov functions V_1 and V_2, for (4) and (5), respectively; such that, for all $x_i \in \mathbb{R}^{n_i}$ and any $w \in \mathcal{L}_\infty$ there exist some functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in \mathcal{K}_\infty$, $\gamma_i, \chi_i \in \mathcal{K}$ and some constants $\kappa_i \in \mathbb{R}_{\geq 0}$ with $i = 1, 2$, the following holds

$$\psi_{i1}(\|x_i\|) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|),\ i = 1, 2, \tag{6}$$
$$V_1(x_1) \geq \max[\chi_1(V_2(x_2)), \gamma_1(\|w\|) + \kappa_1]$$
$$\Rightarrow \nabla V_1(x_1) f_1(x_1, x_2, w) \leq -\psi_{13}(V_1), \tag{7}$$
$$V_2(x_2) \geq \max[\chi_2(V_1(x_1)), \gamma_2(\|w\|) + \kappa_2]$$
$$s \Rightarrow \nabla V_2(x_2) f_2(x_1, x_2, w) \leq -\psi_{23}(V_2). \tag{8}$$

Then, the following nonlinear small-gain result is introduced for the interconnected system (4)-(5) in terms of ISpS Lyapunov functions.

Theorem 2. (Jiang et al., 1996). *Suppose that the interconnected system* (4)-(5) *has ISpS Lyapunov functions* V_1 *and* V_2 *satisfying the condition* (6)-(8). *If there exists some constant* $\kappa_0 \in \mathbb{R}_{\geq 0}$ *such that*

$$\chi_1 \circ \chi_2(\mathrm{r}) < \mathrm{r},\ \forall \mathrm{r} > \kappa_0, \tag{9}$$

then the interconnected system (4)-(5) *is* ISpS. *The system* (4)-(5) *is* ISS *if* $\kappa_0 = \kappa_1 = \kappa_2 = 0$. *Moreover, with no external disturbances,* i.e. $w = 0$, *the system* (4)-(5) *is* globally asymptotically stable.

3. ADAPTIVE SLIDING-MODE OBSERVER

Let us introduce the following adaptive observer

$$\dot{\Omega} = A_L \Omega + G(t, y, u), \tag{10}$$
$$\dot{\hat{\theta}} = \Gamma \Omega^T C^T \lceil y - C\hat{x} \rfloor^\alpha, \tag{11}$$
$$\dot{\hat{x}} = A\hat{x} + \phi(y, u) + G(t, y, u)\hat{\theta} + L(y - C\hat{x}) + kD\mathrm{sign}[F(y - C\hat{x})] + \Omega\dot{\hat{\theta}}, \tag{12}$$

where $\Omega \in \mathbb{R}^{n \times q}$ represents an auxiliary variable, $\hat{\theta} \in \mathbb{R}^q$ is the estimation of θ while $\hat{x} \in \mathbb{R}^n$ is the estimation of x. The function $\lceil \cdot \rfloor^\alpha := |\cdot|^\alpha \mathrm{sign}(\cdot)$, with $|\cdot|$ and $\mathrm{sign}(\cdot)$ understood in the component-wise sense, the function $\mathrm{sign}[q] := q/\|q\|$ for any vector $q \in \mathbb{R}^m$, the design Hurwitz matrix $A_L := A - LC$, with $L \in \mathbb{R}^{n \times p}$ and $0 < \Gamma^T = \Gamma \in \mathbb{R}^{q \times q}$, while $F \in \mathbb{R}^{l \times p}$, $k,\ \alpha, \in \mathbb{R}_{\geq 0}$ are designed later.

The adaptive sliding-mode observer (10)-(12) represents a modified version of the one proposed in Efimov et al. (2016). Such a modification lies in the nonlinear parameter estimation algorithm (11). It is worth mentioning that in Efimov et al. (2016) just the case when $\alpha = 1$ was studied, *i.e.* the linear case. In this paper, it will be shown that the nonlinear algorithm (11) may improve the rate of convergence and the accuracy of the given estimation. However, from another side, the nonlinearity in (11) also complicates the proof drastically with respect to Efimov et al. (2016).

In the following, some properties of the nonlinear parameter estimation algorithm (11) are presented but before let us introduce the following assumption.

Assumption 2. *The term* $\Omega^T C^T$ *is such that* $\sigma_{\Omega_{\min}} := \min_{t \geq 0}(\sigma_{\min}(\Omega^T(t) C^T)) > 0$, *for all* $t \geq 0$.

The previous assumption implies that $p \geq q$ and it is equivalent to the classic identifiability condition corresponding to the injectivity of the term $\Omega^T C^T$, *i.e.* $\mathrm{rank}(\Omega^T C^T) = q$, for each instant of time t Narendra and Annaswamy (2005). Note also that under Assumption 2 and for a Hurwitz matrix $A - LC$, the variable Ω stays bounded and $\sigma_{\Omega_{\max}} < \infty$.

3.1 Nonlinear Parameter Estimation Algorithm

Let us define the errors $\tilde{\theta} := \hat{\theta} - \theta$ and $\delta := x - \hat{x} + \Omega\tilde{\theta}$. Hence, taking into account (10)-(12), the error dynamics are given by

$$\dot{\tilde{\theta}} = -\Gamma \Omega^T C^T \lceil C\Omega\tilde{\theta} - C\delta \rfloor^\alpha, \tag{13}$$
$$\dot{\delta} = A_L \delta + D(w - k\mathrm{sign}[F(y - C\hat{x})]). \tag{14}$$

The following lemma shows that the system (13) is *ISS* with respect to the input δ for $\alpha = 0$.

Lemma 3. *Let Assumption 2 be satisfied. Then, the system* (13), *with* $\alpha = 0$ *and* $\Gamma = \Gamma^T > 0$, *is* ISS *with respect to the input* δ. *Moreover, its trajectories satisfy the following bounds:*

$$\left\|\tilde{\theta}(t)\right\| \leq \sqrt{2\lambda_{\max}(\Gamma)} \left(2\lambda_{\min}(\Gamma))^{-\frac{1}{2}} \left\|\tilde{\theta}(0)\right\| - \frac{\sigma_{\Omega_{\min}} \sqrt{2\lambda_{\min}(\Gamma)}}{2} t \right),\ \forall t \leq T_{\tilde{\theta}}(\tilde{\theta}(0)), \tag{15}$$

$$\left\|\tilde{\theta}(t)\right\| \leq \sqrt{\frac{\lambda_{\max}(\Gamma)}{\lambda_{\min}(\Gamma)}} \mu_{\tilde{\theta}},\ \forall t > T_{\tilde{\theta}}(\tilde{\theta}(0)), \tag{16}$$

with

$$\mu_{\tilde{\theta}} := \frac{\|C\|}{\sigma_{\Omega_{\max}}} \|\delta\|_\infty,$$

$$T_{\tilde{\theta}}(\tilde{\theta}(0)) \leq \max\left[0, \frac{2\left((2\lambda_{\min}(\Gamma))^{-\frac{1}{2}} \left\|\tilde{\theta}(0)\right\| - (2\lambda_{\max}(\Gamma))^{-\frac{1}{2}} \mu_{\tilde{\theta}}\right)}{\sigma_{\Omega_{\min}} \sqrt{2\lambda_{\min}(\Gamma)}}\right],$$

and any $\tilde{\theta}(0) \in \mathbb{R}^q$.

Hence, it is concluded that the solutions of system (13) are *ultimately bounded* with its trajectories satisfying the bounds given by (15) and (16) for $\alpha = 0$. Moreover, some important *ISS* properties with respect to the input δ are provided for the system (13).

Remark 1. *Lemma 3 shows that the solutions of the system* (13) *enter into the bound* (16) *at most in a finite time* $T_{\tilde{\theta}}(\tilde{\theta}(0))$.

Now, the following lemma shows that system (14) is *ISS* with respect to the inputs $\tilde{\theta}$ and w.

Lemma 4. *Let Assumption 1 be satisfied. If the following matrix inequalities*

$$A_L^T P + P A_L + \beta^{-1} P + (\beta r + 2\varpi) C^T C \leq 0, \tag{17}$$
$$PD = C^T F^T, \tag{18}$$

are feasible for a matrix $0 < P^T = P \in \mathbb{R}^{n\times n}$, *matrices* $F \in \mathbb{R}^{l\times p}$, $L \in \mathbb{R}^{n\times p}$, *and constants* $\beta, r, \varpi > 0$, *then the system* (14), *with* $k = \|w\|_\infty$, *is* ISS *with respect to the inputs* $\tilde{\theta}$ *and* w. *Moreover, its trajectories satisfy the following bounds:*

$$\left\|\delta(t)\right\| \leq e^{-\frac{\zeta_1}{2}t}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left\|\delta(0)\right\|, \ \forall t \leq T_\delta(\delta(0)), \quad (19)$$

$$\left\|\delta(t)\right\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\mu_\delta, \ \forall t > T_\delta(\tilde{\theta}(0)), \quad (20)$$

with

$$\zeta_1 := \frac{(1-\rho)(\beta^{-1}\lambda_{\min}(P) + \varpi\left\|C\right\|^2)}{\lambda_{\max}(P)},$$

$$\mu_\delta := \sqrt{\frac{\varpi\sigma^2_{\Omega_{\max}}}{\rho\alpha_\delta}}\left\|\tilde{\theta}\right\|_\infty + \frac{2\left\|F\right\|\left\|w\right\|_\infty}{\sqrt{\rho\alpha_\delta}},$$

$$T_\delta(\delta(0)) \leq \max\left(0, \frac{2\left[\ln\left(\left\|\delta(0)\right\|\right) - \ln\left(\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\mu_\delta\right)\right]}{\zeta_1}\right),$$

$\rho \in (0,1)$, *and any* $\delta(0) \in \mathbb{R}^n$.

3.2 Convergence of the Adaptive Observer

In the following, based on the statements given by Lemmas 3 and 4, it will be shown that the interconnected error system (13)-(14) is *ISS* with respect to the external disturbance w, for $\alpha = 0$.

Theorem 5. *Let Assumptions 1 and 2 be satisfied. If the matrix inequalities* (17)-(18) *and*

$$\left(\frac{\left\|C\right\|^2\sqrt{\beta\varpi}\lambda_{\min}(\Gamma)}{\sigma_{\Omega_{\max}}\lambda_{\max}(P)\lambda_{\max}(\Gamma)\sqrt{\rho\lambda_{\min}(P)}}\right) < 1, \quad (21)$$

are feasible for matrices $0 < P^T = P \in \mathbb{R}^{n\times n}$, $0 < \Gamma^T = \Gamma \in \mathbb{R}^{q\times q}$, $F \in \mathbb{R}^{l\times p}$, $L \in \mathbb{R}^{n\times p}$, *constants* $\beta, r, \varpi > 0$, $\rho \in (0,1)$ *and* $p \in \mathbb{N}$ *the dimension of* y, *then the interconnected error system* (13)-(14) *is* ISS *with respect to the input* w. *Moreover, with no external disturbances,* i.e. $w = 0$, *the system* (13)-(14) *is globally asymptotically stable.*

The proofs of all these results are omitted due to lack of space. Note that Theorem 5 implies that the estimation error $e := x - \hat{x} = \delta + \Omega\tilde{\theta}$ is also *ISS* since

$$\|e(t)\| \leq (1 + ||\Omega||)\left\|\begin{pmatrix}\tilde{\theta}(t)\\ \delta(t)\end{pmatrix}\right\|, \ \forall t \geq 0, \quad (22)$$

for $\alpha = 0$.

After the statements given by Lemmas 3 and 4; and Theorem 5, one can highlight the following points:

(1) For the ideal case, *i.e.* $w = 0$ with $k = 0$, the estimations $\hat{\theta}$ and $\hat{x}$ converge to the real values θ and x, respectively; and the rate of convergence for $\tilde{\theta}$ is faster than exponential for $\alpha = 0$, which is our motivation for design of such a nonlinear estimation scheme.

(2) For the perturbed case, *i.e.* $w \neq 0$, one can show by taking $V_e = e^T P e$ that

$$\dot{V}_e \leq e^T\left[A_L^T P + P A_L\right]e + 2e^T P(\Omega\Gamma\Omega^T C^T\lceil Ce\rfloor^\alpha + G(t,y,y)\tilde{\theta}) + 2e^T C^T F^T(w - k\text{sign}[FCe]),$$

and therefore, if one fixes $k = \|w\|_\infty$, the effect of the external disturbance w is completely attenuated.

(3) The condition (18) introduces structural restrictions over the triple (A, D, C); specifically, it must not have invariant zeros, and the relative degree of the output y with respect to the input w must be equal to one. In order to avoid these restrictions some approaches are proposed in Efimov and Fradkov (2006) and Edwards et al. (2007).

(4) To find a solution of the matrix inequality (17), one can rewrite it as follows

$$A_L^T P + P A_L + \tau_1 P + \tau_2 C^T C \leq 0, \quad (23)$$

where $\tau_1 = \beta^{-1}$ and $\tau_2 = \beta r + 2\varpi$ are new variables. Then, using Λ-inequality (see, for instance Poznyak (2008)) and Schur's complement, it follows that

$$\begin{pmatrix} A_L^T P + P A_L + \tau_2 C^T C & \tau_1 I_n & P \\ \tau_1 I_n & -\Lambda^{-1} & 0 \\ P & 0 & -\Lambda \end{pmatrix} \leq 0, \quad (24)$$

is equivalent to (23) for any $0 < \Lambda^T = \Lambda \in \mathbb{R}^{n\times n}$. Note that, for a fixed Λ, (24) is now a linear matrix inequality with respect to matrix P, and parameters τ_1 and τ_2. Then, the matrix inequality (21) can be numerically verified with the corresponding values of the solution of (24) and fixing α and ρ.

(5) The feasibility of (24) is ensured for sufficiently small τ_1 and τ_2, due to the fact that the pair (A, C) is observable.

4. SIMULATION RESULTS

Let us consider the following excited Duffing system

$$\dot{x} = \begin{pmatrix}0 & 1\\ 1 & -\mu\end{pmatrix}x + \begin{pmatrix}0\\ u\end{pmatrix} + \begin{pmatrix}0\\ -y^3\end{pmatrix}\theta + \begin{pmatrix}1\\ 0\end{pmatrix}w,$$
$$y = x_1,$$

where $u = 0.3\cos(t)$, $w = 0.5\sin(2t)$, $\mu = 0.2$, $\theta = 3$ and $x(0) = (2,1)^T$. For these parameters, the system develops a chaotic behavior and Assumption 1 is satisfied. Let us apply the statements given by Theorem 5 for both cases, *i.e.* the ideal and the perturbed case.

Let us fix the matrices $L = (2.80, 2.44)^T$ and $\Gamma = 100I_2$; and the gain $k = 1$. For this matrix L and the given $G(t,y,u) = (0, -y^3)^T$, it is possible to show that Assumption 2 is satisfied.

Then, SeDuMi solver among YALMIP in Matlab is used to find a solution for the LMIs (18), (24), and (21), respectively. The following feasible solution, with $\Lambda = I_2$, is found

$$P = \begin{pmatrix}0.6652 & 0\\ 0 & 0.4455\end{pmatrix}, \ \tau_1 = 0.0187, \ \tau_2 = 3.3096,$$
$$F = 0.6652, \ \beta = 53.4569, \ r = 0.0393, \ \varpi = 0.6048.$$

The simulations have been done in Matlab with the Euler discretization method, sample time equal to 0.001, and initial conditions $\Omega(0) = (0,0)^T$, $\hat{\theta}(0) = 0$ and $\hat{x}(0) = (0,0)^T$. The results for the ideal case, *i.e.* $w = 0$, with $\alpha = 0.05$ (discontinuous nonlinear algorithm) and $\alpha = 1.0$

(linear algorithm) for the parameter estimation algorithm, are depicted by Figure 1. One can see that the estimations $\hat{\theta}$ and $\hat{x}$, given by the nonlinear algorithm, converge to the real values θ and x, respectively; faster than the linear algorithm.

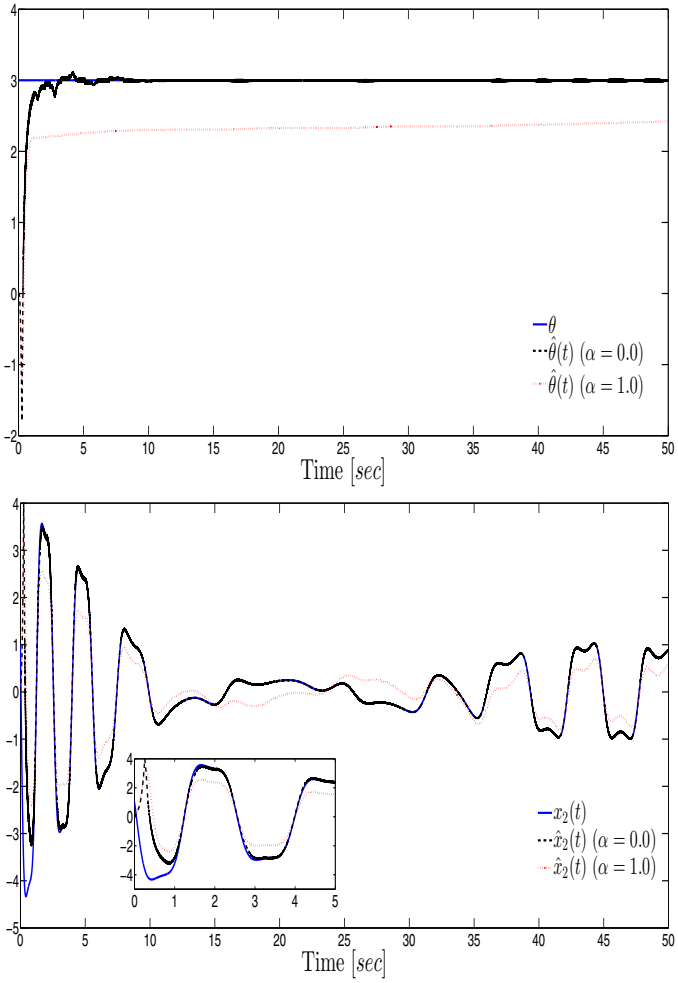


Fig. 1. The estimations of $\hat{\theta}$ and $\hat{x}$ - Ideal Case ($\alpha = 0.0$ and $\alpha = 1.0$)

The results for the perturbed case, *i.e.* $w = 0.5\sin(2t)$, with $\alpha = 0.0$ (discontinuous nonlinear algorithm) and $\alpha = 1.0$ (linear algorithm) for the parameter estimation algorithm, are depicted by Figure 2. In this case, the estimations $\hat{\theta}$ and $\hat{x}$ converge to a neighborhood of the real values θ and x, respectively. One may see that the nonlinear algorithm still converges, to a neighborhood of the real value, faster than the linear algorithm.

5. CONCLUSIONS

In this paper an adaptive sliding-mode observer based on a nonlinear parameter estimation algorithm is proposed for uncertain nonlinear systems. The given adaptive sliding-mode observer is a modified version of that one proposed by Efimov et al. (2016). Such a modification lies in the inclusion of a discontinuous nonlinear parameter estimation algorithm that provides a rate of convergence faster than exponential. Then, the proposed parameter estimation algorithm is included in the structure of a sliding-mode

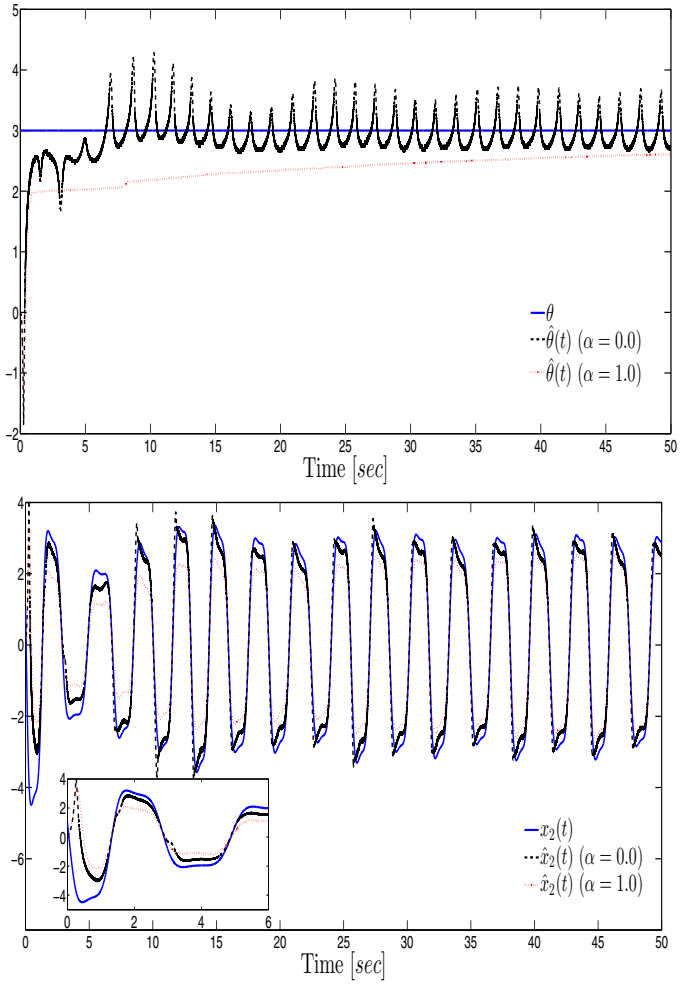


Fig. 2. The estimations of $\hat{\theta}$ and $\hat{x}$ - Perturbed Case ($\alpha = 0.0$ and $\alpha = 1$)

state observer providing an ultimate bound for the state and parameter estimation error. Some simulation results illustrate the feasibility of the proposed adaptive sliding-mode observer.

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