VARIATIONAL ANALYSIS FOR OPTIONS WITH STOCHASTIC VOLATILITY AND MULTIPLE FACTORS[*]

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Abstract. This paper performs a variational analysis for a class of European or American options with stochastic volatility models, including those of Heston and Achdou-Tchou. Taking into account partial correlations and the presence of multiple factors, we obtain the well-posedness of the related partial differential equations, in some weighted Sobolev spaces. This involves a generalization of the commutator analysis introduced by Achdou and Tchou in [3].

Key words. finance, options, partial differential equations, variational formulation, parabolic variational inequalities

AMS subject classifications. 35K20, 35K85, 91G80

1. Introduction. In this paper we consider variational analysis for the partial differential equations associated with the pricing of European or American options. For an introduction to these models, see Fouque et al., [11]. We will set up a general framework of variable volatility models, which is in particular applicable on the following standard models which are well established in mathematical finance. The well-posedness of PDE formulations of variable volatility poblems was studied in [2, 3, 1, 18], and in the recent work [9, 10].

Let the $W_i(t)$ be Brownian motions on a filtered probability space. The variable s denotes a financial asset, and the components of y are factors that influence the volatility:

(i) The *Achdou-Tchou model* [3], see also Achdou, Franchi, and Tchou [1]:

$$(1.1) \qquad \begin{cases} \mathrm{d}s(t) = rs(t)\mathrm{d}t + \sigma(y(t))s(t)\mathrm{d}W_1(t), \\ \mathrm{d}y(t) = \theta(\mu - y(t))\mathrm{d}t + \nu\mathrm{d}W_2(t), \end{cases}$$

with the interest rate r, the volatility coefficient σ function of the factor y whose dynamics involves a parameter $\nu > 0$, and positive constants θ and μ.

(ii) The *Heston model* [14]

$$(1.2) \qquad \begin{cases} \mathrm{d}s(t) &=& s(t)\left(r\mathrm{d}t + \sqrt{y(t)}\mathrm{d}W_1(t)\right), \\ \mathrm{d}y(t) &=& \theta(\mu - y(t))\mathrm{d}t + \nu\sqrt{y(t)}\mathrm{d}W_2(t). \end{cases}$$

(iii) The *Double Heston model*, see Christoffersen, Heston and Jacobs [17], and also Gauthier and Possamaï [12]:

$$(1.3) \qquad \begin{cases} \mathrm{d}s(t) &=& s(t)\left(r\mathrm{d}t + \sqrt{y_1(t)}\mathrm{d}W_1(t) + \sqrt{y_2(t)}\mathrm{d}W_2(t)\right), \\ \mathrm{d}y_1(t) &=& \theta_1(\mu_1 - y_1(t))\mathrm{d}t + \nu_1\sqrt{y_1(t)}\mathrm{d}W_3(t), \\ \mathrm{d}y_2(t) &=& \theta_2(\mu_2 - y_2(t))\mathrm{d}t + \nu_2\sqrt{y_2(t)}\mathrm{d}W_4(t). \end{cases}$$

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31   In the last two models we have similar interpretations of the coefficients;
32   in the double Heston model, denoting by $\langle \cdot, \cdot \rangle$ the correlation coefficients, we
33   assume that there are correlations only between W_1 and W_3, and W_2 and W_4.
34   Consider now the *general multiple factor model*

35   (1.4)
$$
\begin{aligned}
\mathrm{d}s \quad &= rs(t)\mathrm{d}t + \sum_{k=1}^{N} f_k(y_k(t))s^{\beta_k}(t)\mathrm{d}W_k(t), \\
\mathrm{d}y_k \quad &= \theta_k(\mu_k - y_k(t))\mathrm{d}t + g_k(y_k(t))\mathrm{d}W_{N+k}(t), \quad k = 1, \ldots, N.
\end{aligned}
$$

36   Here the y_k are volatility factors, $f_k(y_k)$ represents the volatility coefficient due to
37   y_k, $g_k(y_k)$ is a volatility coefficient in the dynamics of the kth factor with positive
38   constants θ_k and μ_k. Let us denote the correlation between the ith and jth Brownian
39   motions by κ_{ij}: this is a measurable function of (s, y, t) with value in $[0, 1]$ (here
40    $s \in (0, \infty)$ and y_k belongs to either $(0, \infty)$ or $\mathbb{R}$, see below. We asssume that we have
41   nonzero correlations only between the Brownian motions W_k and W_{N+k}, for $k = 1$
42   to N, i.e.

43   (1.5) $\kappa_{ij} = 0 \quad \text{if } i \neq j \text{ and } |j - i| \neq N.$

44   Note that, in some of the main results, we will assume for the sake of simplicity that
45   the correlations are constant.
46   We apply the developed analysis to a subclass of stochastic volatility models, obtained
47   by assuming that κ is constant and

48   (1.6) $|f_k(y_k)| = |y_k|^{\gamma_k}; \quad |g_k(y_k)| = \nu_k|y_k|^{1-\gamma_k}; \quad \beta_k \in (0, 1]; \quad \nu_k > 0; \quad \gamma_k \in (0, \infty).$

49   This covers in particular a variant of the Achdou and Tchou model with multiple
50   factors (VAT), when $\gamma_k = 1$, as well as a generalized multiple factor Heston model
51   (GMH), when $\gamma_k = 1/2$, i.e., for $k = 1$ to N:

52   (1.7)
$$
\begin{aligned}
\text{VAT:} \quad & f_k(y_k) = y_k, \quad & g_k(y_k) = \nu_k, \\
\text{GMH:} \quad & f_k(y_k) = \sqrt{y_k}, \quad & g_k(y_k) = \nu_k\sqrt{y_k}.
\end{aligned}
$$

53   For a general class of stochastic volatility models with correlation we refer to Lions
54   and Musiela [16].
55   The main contribution of this paper is variational analysis for the pricing equa-
56   tion corresponding to the above general class in the sense of the Feynman-Kac theory.
57   This requires in particular to prove continuity and coercivity properties of the corre-
58   sponding bilinear form in weighted Sobolev spaces H and V, respectively, which have
59   the Gelfand property and allow the application of the Lions and Magenes theory [15]
60   recalled in Appendix A and the regularity theory for parabolic variational inequalities
61   recalled in Appendix B. A special emphasis is given to the continuity analysis of the
62   rate term in the pricing equation. Two approaches are presented, the standard one
63   and an extension of the one based on the commutator of first-order differential oper-
64   ators as in Achdou and Tchou [3], extended to the Heston model setting by Achdou
65   and Pironneau [18]. Our main result is that the commutator analysis gives stronger
66   results for the subclass defined by (1.6), generalizing the particular cases of the VAT
67   and GMH classes, see remarks 6.2 and 6.4. In particular we extend some of the results
68   by [3].
69   This paper is organized as follows. In section 2 we give the expression of the bi-
70   linear form associated with the original PDE, and check the hypotheses of continuity
71   and semi-coercivity of this bilinear form. In section 3 we show how to refine this anal-
72   ysis by taking into account the commutators of the first-order differential operators

73 associated with the variational formulation. In section 4 we show how to compute
74 the weighting function involved in the bilinear form. In section 5 we develop the re-
75 sults for a general class introduced in the next section. In section 6 we specialize the
76 results to stochastic volatility models. The appendix recalls the main results of the
77 variational theory for parabolic equations, with a discussion on the characterization
78 of the V functional spaces in the case of one dimensional problems.

79 **Notation**. We assume that the domain Ω of the PDEs to be considered in the
80 sequel of this paper has the following structure. Let (I, J) be a partition of $\{0, \ldots, N\}$,
81 with $0 \in J$, and

82 (1.8)
$$\Omega := \prod_{k=0}^{N} \Omega_k; \quad \text{with} \quad \Omega_k := \begin{cases} \mathbb{R} & \text{when } k \in I, \\ (0, \infty) & \text{when } k \in J. \end{cases}$$

83 Let $L^0(\Omega)$ denote the space of measurable functions over Ω. For a given weighting
84 function $\rho : \Omega \to \mathbb{R}$ of class C^1, with positive values, we define the *weighted space*

85 (1.9)
$$L^{2,\rho}(\Omega) := \{v \in L^0(\Omega); \int_\Omega v(x)^2 \rho(x) \mathrm{d}x < \infty\},$$

86 which is a Hilbert space endowed with the norm

87 (1.10)
$$\|v\|_\rho := \left(\int_\Omega v(x)^2 \rho(x) \mathrm{d}x \right)^{1/2}.$$

88 By $\mathcal{D}(\Omega)$ we denote the space of C^∞ functions with compact support in Ω. By
89 $H^2_{loc}(\Omega)$ we denote the space of functions over Ω whose product with an element of
90 $\mathcal{D}(\Omega)$ belongs to the Sobolev space $H^2(\Omega)$.

91 Besides, let Φ be a vector field over Ω (i.e., a mapping $\Omega \to \mathbb{R}^n$). The *first-order*
92 *differential operator* associated with Φ is, for $u : \Omega \to \mathbb{R}$ the function over Ω defined
93 by

94 (1.11)
$$\Phi[u](x) := \sum_{i=0}^{n} \Phi_i(x) \frac{\partial u}{\partial x_i}(x), \quad \text{for all } x \in \Omega.$$

95 **2. General setting.** Here we give compute the bilinear form associated with
96 the original PDE, in the setting of the general multiple factor model (1.4). Then we
97 will check the hypotheses of continuity and semi-coercivity of this bilinear form.

98 **2.1. Variational formulation.** We compute the bilinear form of the variational
99 setting, taking into account a general weight function. We wil see how to choose the
100 functional spaces for a given ρ, and then how to choose the weight itself.

101 **2.1.1. The elliptic operator.** In financial models the underlying is solution of
102 stochastic differential equations of the form

103 (2.1)
$$dX(t) = b(t, X(t))\mathrm{d}t + \sum_{i=1}^{n_\sigma} \sigma_i(t, X(t))dW_i.$$

104

105 Here $X(t)$ takes values in Ω, defined in (1.8). That is, X_1 corresponds to the s variable,
106 and X_{k+1}, for $k = 1$ to N, corresponds to y_k. We have that $n_\sigma = 2N$.
107 So, b and σ_i, for $i = 1$ to n_σ, are mappings $(0, T) \times \Omega \to \mathbb{R}^n$, and the W_i, for
108 $i = 1$ to n_σ, are standard Brownian processes with correlation $\kappa_{ij} : (0, T) \times \Omega \to \mathbb{R}$

109 between W_i and W_j for $i, j \in \{1, \ldots, n_\sigma\}$. The $n_\sigma \times n_\sigma$ symmetric *correlation matrix*
110 $\kappa(\cdot, \cdot)$ is nonnegative with unit diagonal:

111 (2.2) $\kappa(t, x) \succeq 0; \quad \kappa_{ii} = 1, \ i = 1, \ldots, n_\sigma, \quad$ for a.a. $(t, x) \in (0, T) \times \Omega.$

112 Here, for symmetric matrices B and C of same size, by "$C \succeq B$" we mean that $C - B$
113 is positive semidefinite. The expression of the second order differential operator A
114 corresponding to the dynamics (2.1) is, skipping the time and space arguments, for
115 $u \colon (0, T) \times \Omega \to \mathbb{R}$:

116 (2.3) $$Au := ru - b \cdot \nabla u - \tfrac{1}{2} \sum_{i,j=1}^{n_\sigma} \kappa_{ij} \sigma_j^\top u_{xx} \sigma_i,$$

117 where

118 (2.4) $$\sigma_j^\top u_{xx} \sigma_i := \sum_{k,\ell=1}^{n_\sigma} \sigma_{kj} \frac{\partial u^2}{\partial x_k \partial x_\ell} \sigma_{\ell i},$$

119 $r(x, t)$ represents an interest rate, and u_{xx} is the matrix of second derivatives in space
120 of u. The associated backward PDE for a European option is of the form

121 (2.5) $$\begin{cases} -\dot{u}(t, x) + A(t, x)u(t, x) = f(t, x), & (t, x) \in (0, T) \times \Omega; \\ u(x, T) = u_T(x), & x \in \Omega, \end{cases}$$

122 with $\dot{u}$ the notation for the time derivative of u, $u_T(x)$ payoff at final time (horizon)
123 T and the r.h.s. $f(t, x)$ represents dividends (often equal to zero).
124   In case of an American option we obtain a variational inequality; for details we
125 refer to Appendix D.

126   **2.1.2. The bilinear form.** In the sequel we assume that

127 (2.6) b, σ, κ are C^1 mappings over $[0, T] \times \Omega.$

128 Multiplying (2.3) by the test function $v \in \mathcal{D}(\Omega)$ and the continuously differentiable
129 weight function $\rho \colon \Omega \to \mathbb{R}$ and integrating over the domain we can integrate by parts;
130 since $v \in \mathcal{D}(\Omega)$ there will be no contribution from the boundary. We obtain

131 (2.7) $$-\tfrac{1}{2} \int_\Omega \sigma_j^\top u_{xx} \sigma_i v \kappa_{ij} \rho = \sum_{p=0}^{3} a_{ij}^p(u, v),$$

132 with

133 (2.8) $a_{ij}^0(u, v) := \tfrac{1}{2} \int_\Omega \sum_{k,\ell=1}^{n} \sigma_{kj} \sigma_{\ell i} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_\ell} \kappa_{ij} \rho = \tfrac{1}{2} \int_\Omega \sigma_j[u] \sigma_i[v] \kappa_{ij} \rho,$

134

135 (2.9) $a_{ij}^1(u, v) := \tfrac{1}{2} \int_\Omega \sum_{k,\ell=1}^{n} \sigma_{kj} \sigma_{\ell i} \frac{\partial u}{\partial x_k} \frac{\partial(\kappa_{ij}\rho)}{\partial x_\ell} v = \tfrac{1}{2} \int_\Omega \sigma_j[u] \sigma_i[\kappa_{ij}\rho] \frac{v}{\rho} \rho,$

136

137 (2.10) $a_{ij}^2(u, v) := \tfrac{1}{2} \int_\Omega \sum_{k,\ell=1}^{n} \sigma_{kj} \frac{\partial(\sigma_{\ell i})}{\partial x_\ell} \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho = \tfrac{1}{2} \int_\Omega \sigma_j[u](\operatorname{div} \sigma_i) v \kappa_{ij} \rho,$

138

$$(2.11) \qquad a_{ij}^3(u,v) := \tfrac{1}{2} \int_\Omega \sum_{k,\ell=1}^n \frac{\partial(\sigma_{kj})}{\partial x_\ell} \sigma_{\ell i} \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho = \tfrac{1}{2} \int_\Omega \sum_{k=1}^n \sigma_i[\sigma_{kj}] \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho.$$

Also, for the contributions of the first and zero order terms resp. we get

$$(2.12) \qquad a^4(u,v) := -\int_\Omega b[u]v\rho; \quad a^5(u,v) := \int_\Omega ruv\rho.$$

Set

$$(2.13) \qquad a^p := \sum_{i,j=1}^{n_\sigma} a_{ij}^p, \quad p = 0, \ldots, 3.$$

The *bilinear* form associated with the above PDE is

$$(2.14) \qquad a(u,v) := \sum_{p=0}^5 a^p(u,v).$$

From the previous discussion we deduce that

LEMMA 2.1. *Let* $u \in H_{\ell oc}^2(\Omega)$ *and* $v \in \mathcal{D}(\Omega)$. *Then we have that*

$$(2.15) \qquad a(u,v) = \int_\Omega A(t,x)u(x)v(x)\rho(x)\mathrm{d}x.$$

2.1.3. The Gelfand triple. We can view a^0 as the principal term of the bilinear form $a(u,v)$. Let σ denote the $n \times n_\sigma$ matrix whose σ_j are the columns. Then

$$(2.16) \qquad a^0(u,v) = \sum_{i,j=1}^{n_\sigma} \int_\Omega \sigma_j[u]\sigma_i[v]\kappa_{ij}\rho = \int_\Omega \nabla u^\top \sigma \kappa \sigma^\top \nabla v \rho.$$

Since $\kappa \succeq 0$, the above integrand is nonnegative when $u = v$; therefore, $a^0(u,u) \geq 0$. When κ is the identity we have that $a^0(u,u)$ is equal to the seminorm $a^{00}(u,u)$, where

$$(2.17) \qquad a^{00}(u,u) := \int_\Omega |\sigma^\top \nabla u|^2 \rho.$$

In the presence of correlations it is natural to assume that we have a coercivity of the same order. That is, we assume that

$$(2.18) \qquad \text{For some } \gamma \in (0,1]: \quad \sigma\kappa\sigma^\top \succeq \gamma\sigma\sigma^\top, \quad \text{for all } (t,x) \in (0,T) \times \Omega.$$

Therefore, we have

$$(2.19) \qquad a^0(u,u) \geq \gamma a^{00}(u,u).$$

REMARK 2.2. *Condition* (2.18) *holds in particular if*

$$(2.20) \qquad \kappa \succeq \gamma I,$$

but may also hold in other situations, e.g., when $n = 1$, $n_\sigma = 2$, $\kappa_{12} = 1$, *and* $\sigma_1 = \sigma_2 = 1$. *Yet when the* σ_i *are linearly independent,* (2.19) *is equivalent to* (2.20).

164   We need to choose a pair (V, H) of Hilbert spaces satisfying the Gelfand condi-
165 tions for the variational setting of Appendix A, namely V densely and continuously
166 embedded in H, $a(\cdot, \cdot)$ continuous and semi-coercive over V. Additionally, the r.h.s.
167 and final condition of (2.5) should belong to $L^2(0, T; V^*)$ and H resp. (and for the
168 second parabolic estimate, to $L^2(0, T; H)$ and V resp.).

169   We do as follows: for some measurable function $h : \Omega \to \mathbb{R}_+$ to be specified later
170 we define

171 (2.21)
$$\begin{cases} H := \{v \in L^0(\Omega); \ hv \in L^{2,\rho}(\Omega)\}, \\ \mathcal{V} := \{v \in H; \ \sigma_i[v] \in L^{2,\rho}(\Omega), \ i = 1, \dots, n_\sigma\}, \\ V := \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}\}, \end{cases}$$

172 endowed with the natural norms,

173 (2.22)
$$\|v\|_H := \|hv\|_\rho; \quad \|u\|_V^2 := a^{00}(u, u) + \|u\|_H^2.$$

174 We do not try to characterize the space V since this is problem dependent.

175   Obviously, $a^0(u, v)$ is a bilinear continuous form over $\mathcal{V}$. We next need to choose
176 h so that $a(u, v)$ is a bilinear and semi-coercive continuous form, and $u_T \in H$.

177   **2.2. Continuity and semi-coercivity of the bilinear form over $\mathcal{V}$.** We will
178 see that the analysis of a^0 to a^2 is relatively easy. It is less obvious to analyze the
179 term

180 (2.23)
$$a^{34}(u, v) := a^3(u, v) + a^4(u, v).$$

181 Let $\bar{q}_{ij}(t, x) \in \mathbb{R}^n$ be the vector with kth component equal to

182 (2.24)
$$\bar{q}_{ijk} := \kappa_{ij}\sigma_i[\sigma_{kj}].$$

183 Set

184 (2.25)
$$\hat{q} := \sum_{i,j=1}^{n_\sigma} \bar{q}_{ij}, \quad q := \hat{q} - b.$$

185 Then by (2.11)-(2.12), we have that

186 (2.26)
$$a^{34}(u, v) = \int_\Omega q[u]v\rho.$$

187 We next need to assume that it is possible to choose η_k in $L^0((0, T) \times \Omega)$, for $k = 1$
188 to n_σ, such that

189 (2.27)
$$q = \sum_{k=1}^{n_\sigma} \eta_k \sigma_k.$$

190 Often the $n \times n_\sigma$ matrix $\sigma(t, x)$ has a.e. rank n. Then the above decomposition is
191 possible. However, the choice for η is not necessarily unique. We will see in examples
192 how to do it. Consider the following hypotheses:

193 (2.28) $h_\sigma \le c_\sigma h$, where $h_\sigma := \displaystyle\sum_{i,j=1}^{n_\sigma} |\sigma_i[\kappa_{ij}\rho]/\rho + \kappa_{ij}\operatorname{div}\sigma_i|$, a.e., for some $c_\sigma > 0$,

194 (2.29) $h_r \le c_r h$, where $h_r := |r|^{1/2}$, a.e., for some $c_r > 0$,

195 (2.30) $h_\eta \le c_\eta h$, where $h_\eta := |\eta|$, a.e., for some $c_\eta > 0$.

197 REMARK 2.3. *Let us set for any differentiable vector field $Z \colon \Omega \to \mathbb{R}^n$*

198 (2.31)
$$G_\rho(Z) := \operatorname{div} Z + \frac{Z[\rho]}{\rho}.$$

199 *Since $\kappa_{ii} = 1$, (2.28) implies that*

200 (2.32)
$$|G_\rho(\sigma_i)| \leq c_\sigma h, \quad i = 1; \ldots, n_\sigma.$$

201 REMARK 2.4. *Since*

202 (2.33)
$$\sigma_i[\kappa_{ij}\rho] = \sigma_i[\kappa_{ij}]\rho + \sigma_i[\rho]\kappa_{ij},$$

203 *and $|\kappa_{ij}| \leq 1$ a.e., a sufficient condition for (2.28) is that there exist a positive con-*
204 *stants c'_σ such that*

205 (2.34)
$$h'_\sigma \leq c'_\sigma h; \quad h'_\sigma := \sum_{i,j=1}^{n_\sigma} |\sigma_i[\kappa_{ij}]| + \sum_{i=1}^{n_\sigma} \left(|\operatorname{div} \sigma_i| + |\sigma_i[\rho]/\rho| \right).$$

206 *We will see in section 4 how to choose the weight ρ so that $|\sigma_i[\rho]/\rho|$ can be easily*
207 *estimated as a function of σ.*

208 LEMMA 2.5. *Let (2.28)-(2.30) hold. Then the bilinear form $a(u,v)$ is both* (i)
209 *continuous over V, and* (ii) *semi-coercive, in the sense of (A.5).*

210 *Proof.* (i) We have that $a^1 + a^2$ is continuous, since by (2.9)-(2.10), (2.28) and
211 the Cauchy-Schwarz inequality:

212 (2.35)
$$\begin{aligned}
|a^1(u,v) + a^2(u,v)| &\leq \sum_{i,j=1}^{n_\sigma} |a_{ij}^1(u,v) + a_{ij}^2(u,v)| \\
&\leq \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho \sum_{i=1}^{n_\sigma} \|(\sigma_i[\kappa_{ij}\rho]/\rho + \kappa_{ij} \operatorname{div} \sigma_i) v\|_\rho \\
&\leq c_\sigma n_\sigma \|v\|_H \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho.
\end{aligned}$$

213
214 (ii) Also, a^{34} is continuous, since by (2.27) and (2.30):

215 (2.36)
$$|a^{34}(u,v)| \leq \sum_{k=1}^{n_\sigma} \|\sigma_k[u]\|_\rho \|\eta_k v\|_\rho \leq c_\eta \|v\|_H \sum_{k=1}^{n_\sigma} \|\sigma_k[u]\|_\rho.$$

216 Set $c := c_\sigma n_\sigma + c_\eta^2$. By (2.35)–(2.36), we have that

217 (2.37)
$$\begin{cases}
|a^5(u,v)| \leq \||r|^{1/2}u\|_{2,\rho} \||r|^{1/2}v\|_{2,\rho} \leq c_r^2 \|u\|_H \|v\|_H, \\
|a^1(u,v) + a^2(u,v) + a^{34}(u,v)| \leq c a^{00}(u)^{1/2} \|v\|_H.
\end{cases}$$

218 Since a^0 is obviously continuous, the continuity of $a(u,v)$ follows.
219 (iii) Semi-coercivity. Using (2.37) and Young's inequality, we get that

220 (2.38)
$$\begin{aligned}
a(u,u) &\geq a_0(u,u) - |a^1(u,u) + a^2(u,u) + a^{34}(u,u)| - |a^5(u,u)| \\
&\geq \gamma a^{00}(u) - c a^{00}(u)^{1/2} \|u\|_H - c_r \|u\|_H^2 \\
&\geq \tfrac{1}{2}\gamma a^{00}(u) - \left(\tfrac{1}{2}\frac{c^2}{\gamma} + c_r \right) \|u\|_H^2,
\end{aligned}$$

221 which means that a is semi-coercive. $\qquad\square$

222 The above consideration allow to derive well-posedness results for parabolic equations
223 and parabolic variational inequalities.

224   THEOREM 2.6. (i) *Let (V, H) be given by* (2.21), *with h satisfying* (2.28)-(2.30),
225 $(f, u_T) \in L^2(0, T; V^*) \times H$. *Then equation* (2.5) *has a unique solution u in $L^2(0, T; V)$*
226 *with $\dot{u} \in L^2(0, T; V^*)$, and the mapping $(f, u_T) \mapsto u$ is nondecreasing.* (ii) *If in*
227 *addition the semi-symmetry condition* (A.8) *holds, then u in $L^\infty(0, T; V)$ and $\dot{u} \in$*
228 $L^2(0, T; H)$.

229   *Proof.* This is a direct consequence of Propositions A.1, A.2 and C.1.   □

230 We next consider the case of parabolic variational inequalities associated with the set

231 (2.39) $K := \{\psi \in V : \psi(x) \geq \Psi(x) \quad \text{a.e. in } \Omega\}$,

232 where $\Psi \in V$. The strong and weak formulations of the parabolic variational in-
233 equality are defined in (B.2) and (B.5) resp. The abstract notion of monotonicity is
234 discussed in appendix B. We denote by $\mathcal{K}$ the closure of K in V.

235   THEOREM 2.7. (i) *Let the assumptions of theorem* 2.6 *hold, with $u_T \in \mathcal{K}$. Then*
236 *the weak formulation* (B.5) *has a unique solution u in $L^2(0, T; K) \cap C(0, T; H)$, and*
237 *the mapping $(f, u_T) \mapsto u$ is nondecreasing.*
238 (ii) *Let in addition the semi-symmetry condition* (A.8) *be satisfied. Then u is the*
239 *unique solution of the strong formulation* (B.2), *belongs to $L^\infty(0, T; V)$, and $\dot{u}$ belongs*
240 *to $L^2(0, T; H)$.*

241   *Proof.* This follows from Propositions B.1 and C.2.   □

242   **3. Variational analysis using the commutator analysis.** In the following a
243 commutator for first order differential operators is introduced, and calculus rules are
244 derived.

245   **3.1. Commutators.** Let $u : \Omega \to \mathbb{R}$ be of class C^2. Let Φ and Ψ be two vector
246 fields over Ω, both of class C^1. Recalling (1.11), we may define the *commutator* of
247 the first-order differential operators associated with Φ and Ψ as

248 (3.1) $[\Phi, \Psi][u] := \Phi[\Psi[u]] - \Psi[\Phi[u]]$.

249 Note that

250 (3.2) $\Phi[\Psi[u]] = \sum_{i=1}^{n} \Phi_i \frac{\partial(\Psi u)}{\partial x_i} = \sum_{i=1}^{n} \Phi_i \left(\sum_{k=1}^{n} \frac{\partial \Psi_k}{\partial x_i} \frac{\partial u}{\partial x_k} + \Psi_k \frac{\partial^2 u}{\partial x_k \partial x_i} \right)$.

251 So, the expression of the commutator is

$$
\begin{aligned}
[\Phi, \Psi][u] &= \sum_{i=1}^{n} \left(\Phi_i \sum_{k=1}^{n} \frac{\partial \Psi_k}{\partial x_i} \frac{\partial u}{\partial x_k} - \Psi_i \sum_{k=1}^{n} \frac{\partial \Phi_k}{\partial x_i} \frac{\partial u}{\partial x_k} \right) \\
&= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \Phi_i \frac{\partial \Psi_k}{\partial x_i} - \Psi_i \frac{\partial \Phi_k}{\partial x_i} \right) \frac{\partial u}{\partial x_k}.
\end{aligned}
$$
252 (3.3)

253 It is another first-order differential operator associated with a vector field (which
254 happens to be the Lie bracket of Φ and Ψ, see e.g.[4]).

3.2. Adjoint. Remembering that H was defined in (2.21), given two vector fields Φ and Ψ over Ω, we define the spaces

$$(3.4) \qquad \mathcal{V}(\Phi, \Psi) := \{v \in H; \ \Phi[v], \ \Psi[v] \in H\},$$

$$(3.5) \qquad V(\Phi, \Psi) := \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}(\Phi, \Psi)\}.$$

We define the adjoint $\Phi^\top$ of Φ (view as an operator over say $C^\infty(\Omega, \mathbb{R})$, the latter being endowed with the scalar product of $L^{2,\rho}(\Omega)$), by

$$(3.6) \qquad \langle \Phi^\top[u], v \rangle_\rho = \langle u, \Phi[v] \rangle_\rho \quad \text{for all } u, v \in \mathcal{D}(\Omega),$$

where $\langle \cdot, \cdot \rangle_\rho$ denotes the scalar product in $L^{2,\rho}(\Omega)$. Thus, there holds the identity

$$(3.7) \qquad \int_\Omega \Phi^\top[u](x)v(x)\rho(x)\mathrm{d}x = \int_\Omega u(x)\Phi[v](x)\rho(x)\mathrm{d}x \quad \text{for all } u, v \in \mathcal{D}(\Omega).$$

Furthermore,

$$(3.8) \qquad \begin{aligned} \int_\Omega u \sum_{i=1}^n \Phi_i \frac{\partial v}{\partial x_i} \rho \mathrm{d}x &= -\sum_{i=1}^n \int_\Omega v \frac{\partial}{\partial x_i}(u\rho\Phi_i)\mathrm{d}x \\ &= -\sum_{i=1}^n \int_\Omega v \left(\frac{\partial}{\partial x_i}(u\Phi_i) + \frac{u}{\rho}\Phi_i\frac{\partial\rho}{\partial x_i} \right) \rho\mathrm{d}x. \end{aligned}$$

Hence,

$$(3.9) \qquad \Phi^\top[u] = -\sum_{i=1}^n \frac{\partial}{\partial x_i}(u\Phi_i) - u\Phi_i\frac{\partial\rho}{\partial x_i}/\rho = -u\operatorname{div}\Phi - \Phi[u] - u\Phi[\rho]/\rho.$$

Remembering the definition of $G_\rho(\Phi)$ in (2.31), we obtain that

$$(3.10) \qquad \Phi[u] + \Phi^\top[u] + G_\rho(\Phi)u = 0.$$

3.3. Continuity of the bilinear form associated with the commutator. Setting, for v and w in $V(\Phi, \Psi)$:

$$(3.11) \qquad \Delta(u, v) := \int_\Omega [\Phi, \Psi][u](x)v(x)\rho(x)\mathrm{d}x,$$

we have
(3.12)
$$\begin{aligned} \Delta(u, v) &= \int_\Omega (\Phi[\Psi[u]]v - \Psi[\Phi[u]]v)\rho\mathrm{d}x = \int_\Omega \Psi[u]\Phi^\top[v] - \Phi[u]\Psi^\top[v])\rho\mathrm{d}x \\ &= \int_\Omega (\Phi[u]\Psi[v] - \Psi[u]\Phi[v])\rho\mathrm{d}x + \int_\Omega (\Phi[u]G_\rho(\Psi)v - \Psi[u]G_\rho(\Phi)v)\rho\mathrm{d}x. \end{aligned}$$

LEMMA 3.1. *For $\Delta(\cdot, \cdot)$ to be a continuous bilinear form on $V(\Phi, \Psi)$, it suffices that, for some $c_\Delta > 0$:*

$$(3.13) \qquad |G_\rho(\Phi)| + |G_\rho(\Psi)| \le c_\Delta h \quad a.e.,$$

and we have then:

$$(3.14) \quad |\Delta(u, v)| \le \|\Psi[u]\|_\rho \left(\|\Phi[v]\|_\rho + c_\Delta \|v\|_H \right) + \|\Phi[u]\|_\rho \left(\|\Psi[v]\|_\rho + c_\Delta \|v\|_H \right).$$

281   *Proof.* Apply the Cauchy Schwarz inequality to (3.12), and use (3.13) combined
282   with the definition of the space H.   □

283   We apply the previous results with $\Phi := \sigma_i$, $\Psi := \sigma_j$. Set for v, w in V:

284   (3.15) $$\Delta_{ij}(u, v) := \int_\Omega [\sigma_i, \sigma_j][u](x)v(x)\rho(x)\mathrm{d}x, \quad i, j = 1, \ldots, n_\sigma.$$

285   We recall that V was defined in (2.21).

286   COROLLARY 3.2. *Let* (2.28) *hold. Then the* $\Delta_{ij}(u, v)$, $i, j = 1, \ldots, n_\sigma$, *are con-*
287   *tinuous bilinear forms over* V.

288   *Proof.* Use remark 2.3 and conclude with lemma 3.1.   □

289   **3.4. Redefining the space H.** In section 2.2 we have obtained the continuity
290   and semi-coercivity of a by decomposing q, defined in (2.26), as a linear combination
291   (2.27) of the σ_i. We now take advantage of the previous computation of commutators
292   and assume that, more generally, instead of (2.27), we can decompose q in the form

293   (3.16) $$q = \sum_{k=1}^{n_\sigma} \eta_k'' \sigma_k + \sum_{1 \le i < j \le n_\sigma} \eta_{ij}'[\sigma_i, \sigma_j] \quad \text{a.e.}$$

294   We assume that η' and η'' are measurable functions over $[0, T] \times \Omega$, that η' is weakly
295   differentiable, and that for some $c_\eta' > 0$:

296   (3.17) $$h_\eta' \le c_\eta' h, \text{ where } h_\eta' := |\eta''| + \sum_{i,j=1}^N \left|\sigma_i[\eta_{ij}']\right| \quad \text{a.e.,} \ \ \eta' \in L^\infty(\Omega).$$

297   LEMMA 3.3. *Let* (2.28), (2.29), *and* (3.17) *hold. Then the bilinear form* $a(u, v)$
298   *defined in* (2.14) *is both* (i) *continuous and* (ii) *semi-coercive over* V.

299   *Proof.* (i) We only have to analyze the contribution of a^{34} (defined in (2.23)),
300   since the other contributions to $a(\cdot, \cdot)$ do not change. For the terms in the first sum
301   in (3.16) we have, as was done in (2.36):

302   (3.18) $$\left|\int_\Omega \sigma_k[u]\eta_k''v\rho\right| \le \|\sigma_k[u]\|_\rho \|\sigma_k[u]\eta_k''v\|_\rho \le \|\sigma_k[u]\|_\rho \|v\|_H.$$

303   (ii) Setting $w := \eta_{ij}'v$ and taking here $(\Phi, \Psi) = (\sigma_i, \sigma_j)$, we get that

304   (3.19) $$\int_\Omega \eta_{ij}'[\sigma_i, \sigma_j][u]v\rho = \Delta(u, w),$$

305   where $\Delta(\cdot, \cdot)$ was defined in (3.11). Combining with lemma 3.1, we obtain

306   (3.20) $$\begin{aligned} |\Delta_{ij}(u, v)| \le \ & \|\sigma_j[u]\|_\rho \left(\|\sigma_i[w]\|_\rho + c_\sigma \|\eta_{ij}'\|_\infty \|v\|_H\right) \\ & + \|\sigma_i[u]\|_\rho \left(\|\sigma_j[w]\|_\rho + c_\sigma \|\eta_{ij}'\|_\infty \|v\|_H\right). \end{aligned}$$

307   Since

308   (3.21) $$\sigma_i[\eta_{ij}'v] = \eta_{ij}'\sigma_i[v] + \sigma_i[\eta_{ij}']v,$$

309   by (3.17):

310   (3.22) $\|\sigma_i[w]\|_\rho \leq \|\eta'_{ij}\|_\infty \|\sigma_i[v]\|_\rho + \left\|\sigma_i[\eta'_{ij}]v\right\|_\rho \leq \|\eta'_{ij}\|_\infty \|\sigma_i[v]\|_\rho + c_\eta \|v\|_H.$

311   Combining these inequalities, point (i) follows.

312   (ii) Use $u = v$ in (3.21) and (3.12). We find after cancellation in (3.12) that

313   (3.23)
$$\Delta_{ij}(u, \eta'_{ij}u) = \int_\Omega u(\sigma_i[u]\sigma_j[\eta'_{ij}] - \sigma_j[u]\sigma_i(\eta'_{ij}))\rho$$
$$+ \int_\Omega (\sigma_i[u]G_\rho(\sigma_j) - \sigma_j[u]G_\rho(\sigma_i))\,\eta'_{ij}u\rho.$$

314   By (3.17), an upper bound for the absolute value of the first integral is

315   (3.24) $\left(\|\sigma_i[u]\|_\rho + \|\sigma_j[u]\|_\rho\right) \|hu\|_\rho \leq 2\,\|u\|_\mathcal{V}\,\|u\|_H\,.$

316   With (2.28), we get an upper bound for the absolute value of the second integral in
317   the same way, so, for any $\varepsilon > 0$:

318   (3.25) $|\Delta_{ij}(u, \eta'_{ij}u)| \leq 4\,\|u\|_\mathcal{V}\,\|u\|_H\,.$

319   We finally have that for some $c > 0$

320   (3.26)
$$\begin{aligned} a(u, u) \quad &\geq a_0(u, u) - c\,\|u\|_\mathcal{V}\,\|u\|_H\,, \\ &\geq a_0(u, u) - \tfrac{1}{2}\,\|u\|_\mathcal{V}^2 - \tfrac{1}{2}c^2\,\|u\|_H^2\,, \\ &= \tfrac{1}{2}\,\|u\|_\mathcal{V}^2 - \tfrac{1}{2}(c^2 + 1)\,\|u\|_H^2\,. \end{aligned}$$

321   The conclusion follows.   □

322   REMARK 3.4. *The statements analogous to theorems 2.6 and 2.7 hold, assuming*
323   *now that h satisfies (2.28), (2.29), and (3.17) (instead of (2.28)-(2.30)).*

324   **4. The weight ρ.** Classes of weighting functions characterized by their growth
325   are introduced. A major result is the independence of the growth order of the function
326   h on the choice of the weighting function ρ in the class under consideration.

327   **4.1. Classes of functions with given growth.** In financial models we usually
328   have nonnegative variables and the related functions have polynomial growth. Yet,
329   after a logarithmic transformation, we get real variables whose related functions have
330   exponential growth. This motivates the following definitions.
331   We remind that (I, J) is a partition of $\{0, \ldots, N\}$, with $0 \in J$ and that Ω was
332   defined in (1.8).

333   DEFINITION 4.1. *Let γ' and γ'' belong to $\mathbb{R}_+^{N+1}$, with index from 0 to N. Let*
334   *$\mathcal{G}(\gamma', \gamma'')$ be the class of functions $\varphi : \Omega \to \mathbb{R}$ such that for some $c > 0$:*

335   (4.1) $|\varphi(x)| \leq c \left(\prod_{k \in I} (e^{\gamma'_k x_k} + e^{-\gamma''_k x_k}) \right) \left(\prod_{k \in J} (x_k^{\gamma'_k} + x_k^{-\gamma''_k}) \right).$

336   *We define $\mathcal{G}$ as the union of $\mathcal{G}(\gamma', \gamma'')$ for all nonnegative (γ', γ''). We call γ'_k and γ''_k*
337   *the growth order of φ, w.r.t. x_k, at $-\infty$ and $+\infty$ (resp. at zero and $+\infty$).*

338   Observe that the class $\mathcal{G}$ is stable by the operations of sum and product, and that
339   if f, g belong to that class, so does $h = fg$, h having growth orders equal to the sum
340   of the growth orders of f and g. For $a \in \mathbb{R}$, we define

341   (4.2) $a^+ := \max(0, a); \quad a^- := \max(0, -a); \quad N(a) := (a^2 + 1)^{1/2},$

as well as

(4.3) $$\rho := \rho_I \rho_J,$$

where

(4.4) $$\rho_I(x) := \prod_{k \in I} e^{-\alpha'_k N(x_k^+) - \alpha''_k N(x_k^-)},$$

(4.5) $$\rho_J(x) := \prod_{k \in J} \frac{x_k^{\alpha'_k}}{1 + x_k^{\alpha'_k + \alpha''_k}},$$

for some *nonnegative* constants α'_k, α''_k, to be specified later.

LEMMA 4.2. *Let* $\varphi \in \mathcal{G}(\gamma', \gamma'')$. *Then* $\varphi \in L^{1,\rho}(\Omega)$ *whenever* ρ *is as above, with* α *satisfying, for some positive* ε' *and* ε'', *for all* $k = 0$ *to* N:

(4.6) $$\begin{cases} \alpha'_k = \varepsilon' + \gamma'_k, & \alpha''_k = \varepsilon'' + \gamma''_k, & k \in I, \\ \alpha'_k = (\varepsilon' + \gamma''_k - 1)_+, & \alpha''_k = 1 + \varepsilon'' + \gamma'_k, & k \in J. \end{cases}$$

In addition we can choose for $k = 0$ *(if element of J):*

(4.7) $$\begin{cases} \alpha'_0 := (\varepsilon' + \gamma''_0 - 1)_+; \ \alpha''_0 := 0 & \text{if } \varphi(s, y) = 0 \text{ when } s \text{ is far from 0,} \\ \alpha'_0 := 0, \ \alpha''_0 := 1 + \varepsilon'' + \gamma'_0, & \text{if } \varphi(s, y) = 0 \text{ when } s \text{ is close to 0.} \end{cases}$$

Proof. It is enough to prove (4.6), the proof of (4.7) is similar. We know that φ satisfy (4.1) for some $c > 0$ and γ. We need to check the finiteness of

(4.8) $$\int_\Omega \left(\prod_{k \in I} (e^{\gamma'_k y_k} + e^{-\gamma''_k y_k}) \right) \left(\prod_{k \in J} (y_k^{\gamma'_k} + y_k^{-\gamma''_k}) \right) \rho(s, y) \mathrm{d}(s, y).$$

But the above integral is equal to the product $p_I p_J$ with

(4.9) $$p_I := \prod_{k \in I} \int_\mathbb{R} (e^{\gamma'_k x_k} + e^{-\gamma''_k x_k}) e^{-\alpha'_k N(x_k^+) - \alpha''_k N(x_k^-)} \mathrm{d}x_k,$$

(4.10) $$p_J := \prod_{k \in J} \int_{\mathbb{R}_+} \frac{x_k^{\alpha'_k + \gamma'_k} + x_k^{\alpha'_k - \gamma''_k}}{1 + x_k^{\alpha'_k + \alpha''_k}} \mathrm{d}x_k.$$

Using (4.6) we deduce that p_I is finite since for instance

(4.11) $$\int_{\mathbb{R}_+} (e^{\gamma'_k x_k} + e^{-\gamma''_k x_k}) e^{-\alpha'_k N(x_k^+) - \alpha''_k N(x_k^-)} \mathrm{d}x_k \\ \leq 2 \int_{\mathbb{R}_+} e^{\gamma'_k x_k} e^{-(1+\gamma'_k) x_k} \mathrm{d}x_k = 2 \int_{\mathbb{R}_+} e^{-x_k} \mathrm{d}x_k = 2,$$

and p_J is finite since

(4.12) $$p_J = \prod_{k \in J} \int_{\mathbb{R}_+} \frac{x_k^{\varepsilon' + \gamma'_k + \gamma''_k} + x_k^{\varepsilon' - 1}}{1 + x_k^{\varepsilon' + \varepsilon'' + \gamma'_k + \gamma''_k}} \mathrm{d}x_k < \infty.$$

The conclusion follows.   □

4.2. On the growth order of h. Set for all k

(4.13) $$\alpha_k := \alpha'_k + \alpha''_k.$$

Remember that we take ρ in the form (4.3)-(4.4).

LEMMA 4.3. *We have that:*
(i) *We have that*

(4.14) $$\left\| \frac{\rho_{x_k}}{\rho} \right\|_\infty \le \alpha_k, \quad k \in I; \quad \left\| \frac{x}{\rho} \rho_{x_k} \right\|_\infty \le \alpha_k, \quad k \in J.$$

(ii) *Let h satisfying either (2.28)-(2.30) or (2.28)-(2.29), and (3.17). Then the growth order of h does not depend on the choice of the weighting function ρ.*

Proof. (i) For $k \in I$ this is an easy consequence of the fact that $N(\cdot)$ is non expansive. For $k \in J$, we have that

(4.15) $$\frac{x}{\rho} \rho_{x_k} = \frac{x}{\rho} \frac{\alpha'_k x^{\alpha'_k - 1}(1 + x^{\alpha_k}) - x^{\alpha'_k} \alpha_k x^{\alpha_k - 1}}{(1 + x^{\alpha_k})^2} = \frac{\alpha'_k - \alpha''_k x^{\alpha_k}}{1 + x^{\alpha_k}}.$$

We easily conclude, discussing the sign of the numerator.
(ii) The dependence of h w.r.t. ρ is only through the last term in (2.28), namely, $\sum_i |\sigma_i[\rho]/\rho$. By (i) we have that

(4.16) $$\left| \frac{\sigma_i^k[\rho]}{\rho} \right| \le \left\| \frac{\rho_{x_k}}{\rho} \right\|_\infty |\sigma_i^k| \le \alpha_k |\sigma_i^k|, \quad k \in I,$$

(4.17) $$\left| \frac{\sigma_i^k[\rho]}{\rho} \right| \le \left\| \frac{x_k \rho_{x_k}}{\rho} \right\|_\infty \left| \frac{\sigma_i^k}{x_k} \right| \le \alpha_k \left| \frac{\sigma_i^k}{x_k} \right|, \quad k \in J.$$

In both cases, the choice of α has no influence on the growth order of h. $\qquad\square$

4.3. European option. In the case of a European option with payoff $u_T(x)$, we need to check that $u_T \in H$, that is, ρ must satisfy

(4.18) $$\int_\Omega |u_T(x)|^2 h(x)^2 \rho(x) \mathrm{d}x < \infty.$$

In the framework of the semi-symmetry hypothesis (A.8), we need to check that $u_T \in V$, which gives the additional condition

(4.19) $$\sum_{i=1}^{n_\sigma} \int_\Omega |\sigma_i[u_T](x)|^2 \rho(x) \mathrm{d}x < \infty.$$

In practice the payoff depends only on s and this allows to simplify the analysis.

5. Applications using the commutator analysis. The commutator analysis is applied to the general multiple factor model and estimates for the function h characterizing the space H (defined in (2.21)) are derived. The estimates are compared to the case when the commutator analysis is not applied. The resulting improvement wil be established in the next section.

396   **5.1. Commutator and continuity analysis.** We analyze the general multiple
397 factor model (1.4), which belongs to the class of models (2.1) with $\Omega \subset \mathbb{R}^{1+N}$, $n_\sigma =$
398 $2N$, and for $i = 1$ to N:

399 (5.1) $$\sigma_i[v] = f_i(y_i)s^{\beta_i}v_s; \quad \sigma_{N+i}[v] = g_i(y_i)v_i,$$

400 with f_i and g_i of class C^1 over Ω. We need to compute the commutators of the first-
401 order differential operators associated with the σ_i. The correlations will be denoted
402 by

403 (5.2) $$\hat{\kappa}_k := \kappa_{k,N+k}, \quad k = 1, \ldots, N.$$

404   REMARK 5.1. *We use many times the following rule. For $\Omega \subset \mathbb{R}^n$, where $n =$*
405 *$1 + N$, $u \in H^1(\Omega)$, $a, b \in L^0$, and vector fields $Z[u] := au_{x_1}$ and $Z'[u] := bu_{x_2}$, we*
406 *have $Z[Z'[u]] = a(bu_{x_2})_{x_1} = ab_{x_1}u_{x_2} + abu_{x_1x_2}$, so that*

407 (5.3) $$[Z, Z'][u] = ab_{x_1}u_{x_2} - ba_{x_2}u_{x_1}.$$

408   We obtain that

409 (5.4) $$[\sigma_i, \sigma_\ell][u] = (\beta_\ell - \beta_i)f_i(y_i)f_\ell(y_\ell)s^{\beta_i+\beta_\ell-1}u_s, \quad 1 \leq i < \ell \leq N,$$

410

411 (5.5) $$[\sigma_i, \sigma_{N+i}][u] = -s^{\beta_i}f_i'(y_i)g_i(y_i)u_s, \quad i = 1, \ldots, N,$$

412 and

413 (5.6) $$[\sigma_i, \sigma_{N+\ell}][u] = [\sigma_{N+i}, \sigma_{N+\ell}][u] = 0, \quad i \neq \ell.$$

414 Also,

415 (5.7)
$$\begin{aligned}
\operatorname{div}\sigma_i + \frac{\sigma_i[\rho]}{\rho} &= f_i(y_i)s^{\beta_i-1}(\beta_i + s\frac{\rho_s}{\rho}), \\
\operatorname{div}\sigma_{N+i} + \frac{\sigma_{N+i}[\rho]}{\rho} &= g_i'(y_i) + g_i(y_i)\frac{\rho_i}{\rho}.
\end{aligned}$$

416   **5.1.1. Computation of q.** Remember the definitions of $\bar{q}$, $\hat{q}$ and q in (2.24) and
417 (2.25), where δ_{ij} denote the Kronecker operator. We obtain that, for $1 \leq i, j, k \leq N$:

418 (5.8)
$$\begin{cases}
\bar{q}_{ij0} &= \delta_{ij}\beta_j f_i^2(y_i)s^{2\beta_i-1}; \quad \bar{q}_{iik} = 0; \\
\bar{q}_{i,N+j} &= 0; \\
\bar{q}_{N+i,j,0} &= \delta_{ij}\hat{\kappa}_i f'(y_i)g_i(y_i)s^{\beta_i}; \quad \bar{q}_{N+i,j,k} = 0; \\
\bar{q}_{N+i,N+j,k} &= \delta_{ijk}g_i(y_i)g_i'(y_i).
\end{cases}$$

419 That means, we have for $\hat{q} = \sum_{i,j=1}^{2N}\bar{q}_{ij}$ and $q = \hat{q} - b$ that

420 (5.9)
$$\begin{aligned}
\hat{q}_0 &= \sum_{i=1}^N \left(\beta_i f_i^2(y_i)s^{2\beta_i-1} + \hat{\kappa}_i f'(y_i)g_i(y_i)s^{\beta_i}\right); \quad &q_0 &= \hat{q}_0 - rs, \\
\hat{q}_k &= g_k(y_k)g_k'(y_k); \quad &q_k &= \hat{q}_k - \theta_k(\mu_k - y_k), \\
& & &k = 1, \ldots, N.
\end{aligned}$$

5.2. Computation of η' and η''. The coefficients η', η'' are solution of (3.16). We can write $\eta = \hat{\eta} + \tilde{\eta}$, where

(5.10)
$$\hat{q} = \sum_{i=1}^{n_\sigma} \hat{\eta}_i'' \sigma_i + \sum_{1 \leq i,j \leq n_\sigma} \hat{\eta}_{ij}'[\sigma_i, \sigma_j], \quad \hat{\eta}_{ij}' = 0 \text{ if } i = j.$$
$$-b = \sum_{i=1}^{n_\sigma} \tilde{\eta}_i'' \sigma_i + \sum_{1 \leq i,j \leq n_\sigma} \tilde{\eta}_{ij}'[\sigma_i, \sigma_j], \quad \tilde{\eta}_{ij}' = 0 \text{ if } i = j.$$

For $k = 1$ to N, this reduces to

(5.11)
$$\begin{cases} \hat{\eta}_{N+k}'' g_k(y_k) = g_k'(y_k) g_k(y_k); \\ \tilde{\eta}_{N+k}'' g_k(y_k) = -\theta_k(\mu_k - y_k). \end{cases}$$

So, we have that

(5.12)
$$\begin{cases} \hat{\eta}_{N+k}'' = g_k'(y_k); \\ \tilde{\eta}_{N+k}'' = \dfrac{-\theta_k(\mu_k - y_k)}{g_k(y_k)}. \end{cases}$$

For the 0th component, (5.10) can be expressed as

(5.13)
$$\begin{cases} \displaystyle\sum_{k=1}^{N} \left(-\hat{\eta}_{k,N+k}' f_k'(y_k) g_k(y_k) s^{\beta_k} - \hat{\kappa}_k f_k'(y_k) g_k(y_k) s^{\beta_k} \right) \\ \displaystyle+ \sum_{k=1}^{N} \left(\hat{\eta}_k'' f_k(y_k) s^{\beta_k} - \beta_k f_k^2(y_k) s^{2\beta_k - 1} \right) \\ \displaystyle+ \sum_{k=1}^{N} \left(-\tilde{\eta}_{k,N+k}' f_k'(y_k) g_k(y_k) s^{\beta_k} + \tilde{\eta}_k'' f_k(y_k) s^{\beta_k} \right) - rs = 0. \end{cases}$$

We choose to set each term in parenthesis in the first two lines above to zero. It follows that

(5.14)
$$\hat{\eta}_{k,N+k}' = -\hat{\kappa}_k \in L^\infty(\Omega), \quad \hat{\eta}_k'' = \beta_k f_k(y_k) s^{\beta_k - 1}.$$

If $N > 1$ we (arbitrarily) choose then to set the last line to zero with

(5.15)
$$\tilde{\eta}_k'' = \tilde{\eta}_k' = 0, \quad k = 2, \ldots, N.$$

It remains that

(5.16)
$$\tilde{\eta}_1'' f_1(y_1) s^{\beta_1} - \tilde{\eta}_{1,N+1}' f_1'(y_1) g_1(y_1) s^{\beta_1} = rs.$$

Here, we can choose to take either $\tilde{\eta}_1'' = 0$ or $\tilde{\eta}_{1,N+1}' = 0$. We obtain then two possibilities:

(5.17)
$$\begin{cases} \text{(i)} \quad \tilde{\eta}_1'' = 0 \text{ and } \tilde{\eta}_{1,N+1}' = \dfrac{-rs^{1-\beta_1}}{f_1'(y_1)g_1(y_1)}, \\ \text{(ii)} \quad \tilde{\eta}_1'' = \dfrac{rs^{1-\beta_1}}{f_1(y_1)} \text{ and } \tilde{\eta}_{1,N+1}' = 0. \end{cases}$$

5.2.1. Estimate of the h function. We decide to choose case (i) in (5.17). The function h needs to satisfy (2.28), (2.29), and (3.17) (instead of (2.30)). Instead of (2.28), we will rather check the stronger condition (2.34). We compute

$$(5.18) \qquad h'_\sigma := \sum_{k=1}^N |f_k(y_k)| s^{\beta_k} \left(|(\hat\kappa_k)_s| + |\frac{\rho_s}{\rho}| \right) + |g_k(y_k)| \left(|(\hat\kappa_k)_k| + |\frac{\rho_k}{\rho}| \right)$$

$$+ \sum_{k=1}^N \left(\beta_k |f_k(y_k) s^{\beta_k - 1}| + |g'_k(y_k)| \right),$$

$$(5.19) \qquad h_r := |r|^{\frac{1}{2}},$$

$$(5.20) \qquad h'_\eta := \hat h'_\eta + \tilde h'_\eta,$$

where we have

$$(5.21)$$

$$\hat h'_\eta := \sum_{k=1}^N \left(\beta_k |f_k(y_k)| s^{\beta_k - 1} + |g'_k(y_k)| + \left| f_k(y_k)| s^{\beta_k} \frac{\partial \hat\kappa_k}{\partial s} \right| + \left| g_k(y_k) \frac{\partial \hat\kappa_k}{\partial y_k} \right| \right),$$

$$(5.22)$$

$$\tilde h'_\eta := \sum_{k=1}^N \left| \frac{\theta_k(\mu_k - y_k)}{g_k(y_k)} \right| + \left| r \frac{f_1(y_1)}{f'_1(y_1) g_1(y_1)} \right| + \left| r g_1(y_1) s^{1-\beta_1} \frac{\partial}{\partial y_1} \left[\frac{1}{f'_1(y_1) g_1(y_1)} \right] \right|.$$

REMARK 5.2. *Had we chosen (ii) instead of (i) in (5.17), this would only change the expression of $\tilde h'_\eta$ that would then be*

$$(5.23) \qquad \tilde h'_\eta = \sum_{k=1}^N \left| \frac{\theta_k(\mu_k - y_k)}{g_k(y_k)} \right| + \left| \frac{r s^{1-\beta_1}}{f_1(y_1)} \right|.$$

5.2.2. Estimate of the h function without the commutator analysis. The only change in the estimate of h will be the contribution of h'_η and h''_η. We have to satisfy (2.28)-(2.30). In addition, ignoring the commutator analysis, we would solve (5.13) with $\hat\eta' = 0$, meaning that we choose

$$(5.24) \qquad \hat\eta''_k := \beta_k f_k(y_k) s^{\beta_k - 1} + \hat\kappa_k \frac{f'_k(y_k) g_k(y_k)}{f_k(y_k)}, \ k = 1, \dots, N,$$

and take $\tilde\eta''_1$ out of (5.16). Then condition (3.17), with here $\hat\eta' = 0$, would give

$$(5.25) \qquad h \ge c_\eta h_\eta, \ \text{where} \ h_\eta := h_{\hat\eta} + h_{\tilde\eta},$$

with

$$(5.26) \qquad h_{\hat\eta} := \sum_{k=1}^N \left(\beta_k |f_k(y_k)| s^{\beta_k - 1} + |\hat\kappa_k| \left| \frac{f'_k(y_k) g_k(y_k)}{f_k(y_k)} \right| + |g'_k(y_k)| \right),$$

$$(5.27) \qquad h_{\tilde\eta} := \sum_{k=1}^N \left| \frac{\theta_k(\mu_k - y_k)}{g_k(y_k)} \right| + \left| \frac{r s^{1-\beta_1}}{f_1(y_1)} \right|.$$

We will see in applications that this is in general worse.

469   **6. Application to stochastic volatility models.** The results of Section 5 are
470 specified for a subclass of the multiple factor model, in particular for the VAT and
471 GMH models. We show that the commutator analysis allows to take smaller values
472 for the function h (and consequently to include a larger class of payoff functions).

473   **6.1. A useful subclass.** Here we assume that

474   (6.1) $|f_k(y_k)| = |y_k|^{\gamma_k}; \quad |g_k(y_k)| = \nu_k |y_k|^{1-\gamma_k}; \quad \beta_k \in (0,1]; \quad \nu_k > 0; \quad \gamma_k \in (0,\infty).$

475 Furthermore, we assume κ to be constant and

476   (6.2) $|f_k'(y_k)g_k(y_k)| = \text{const} \quad \text{for all } y_k, \quad k = 1,\ldots,N.$

477 Set

478   (6.3)
$$c_s := \|s\rho_s/\rho\|_\infty;$$
$$c_k' = \begin{cases} \|\rho_k/\rho\|_\infty & \text{if } \Omega_k = \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$
$$c_k'' = \begin{cases} 0 & \text{if } \Omega_k = \mathbb{R}, \\ \|y_k\rho_k/\rho\|_\infty & \text{otherwise.} \end{cases}$$

479 We get, assuming that $\gamma_1 \neq 0$:

480   (6.4)
$$h_\sigma' := \sum_{k=1}^N \left(c_s|y_k|^{\gamma_k} s^{\beta_k-1} + \nu_k c_k'|y_k|^{1-\gamma_k} \right.$$
$$\left. + \nu_k c_k''|y_k|^{-\gamma_k} + \beta_k|y_k|^{\gamma_k} s^{\beta_k-1} + (1-\gamma_k)\nu_k|y_k|^{-\gamma_k} \right),$$

481

482   (6.5) $\hat{h}_\eta' := \sum_{k=1}^N \left(\beta_k|y_k|^{\gamma_k} s^{\beta_k-1} + (1-\gamma_k)\nu_k|y_k|^{-\gamma_k} \right),$

483   (6.6) $\tilde{h}_\eta' := \sum_{k=1}^N \left(\frac{\theta_k|\mu_k - y_k|}{\nu_k|y_k|^{1-\gamma_k}} + \frac{r|y_1|^{\gamma_1}}{\gamma_1\nu_1} \right).$
484

485 Therefore when all $y_k \in \mathbb{R}$, we can choose h' as

486   (6.7)
$$h' := \quad 1 + \sum_{k=1}^N \left(|y_k|^{\gamma_k}(1 + s^{\beta_k-1}) + (1-\gamma_k)|y_k|^{-\gamma_k} + |y_k|^{\gamma_k-1} \right)$$
$$+ \sum_{k \in I} |y_k|^{1-\gamma_k} + \sum_{k \in J} |y_k|^{-\gamma_k}.$$

487 Without the commutator analysis we would get

488   (6.8) $\hat{h}_\eta := \sum_{k=1}^N (\beta_k|y_k|^{\gamma_k} s^{\beta_k-1} + \nu_k|\hat{\kappa}_k||y_k|^{-\gamma_k} + (1-\gamma_k)\nu_k|y_k|^{-\gamma_k}),$

489   (6.9) $\tilde{h}_\eta := \sum_{k=1}^N \left(\theta_k \frac{|\mu_k - y_k|}{\nu_k|y_k|^{1-\gamma_k}} + rs^{1-\beta_1}|y_1|^{-\gamma_1} \right).$
490

491 Therefore we can choose

492 (6.10) $h := h''; \quad h'' := h' + rs^{1-\beta_1}/|y_1|^{\gamma_1} + \sum_k \nu_k |\hat{\kappa}_k| |y_k|^{-\gamma_k}.$

493 So, we always have that $h' \leq h''$, meaning that it is advantageous to use the commu-
494 tator analysis, due to the term $rs^{1-\beta_1}/|y_1|^{\gamma_1}$ above in particular. The last term in
495 the above r.h.s. has as contribution only when $\gamma_k \neq 1$ (since otherwise h' includes a
496 term of the same order).

497   **6.2. Application to the VAT model.** For the variant of the Achdou and
498 Tchou model with multiple factors (VAT), i.e. when $\gamma_k = 1$, for $k = 1$ to N, we can
499 take h equal to

500 (6.11) $h'_{TA} := 1 + \sum_{k=1}^{N} |y_k|(1 + s^{\beta_k - 1}),$

501 when the commutator analysis is used, and when it is not, take h equal to

502 (6.12) $h_{TA} := h_{TA} + rs^{1-\beta_1}|y_1|^{-1} + \sum_{k=1}^{N} \nu_k |\hat{\kappa}_k| |y_k|^{-1}.$

503   Remember that $u_T(s) = (s - K)_+$ for a call option, and $u_T(s) = (K - s)_+$ for a
504 put option, both with strike $K > 0$.

505   LEMMA 6.1. *For the VAT model, using the commutator analysis, in case of a call*
506 *(resp. put) option with strike $K > 0$, we can take $\rho = \rho_{call}$, (resp. $\rho = \rho_{put}$), with*

507 (6.13)
$$\rho_{call}(s, y) := (1 + s^{3+\varepsilon''})^{-1}\Pi_{k=1}^{N}e^{-\varepsilon N(y_k)},$$
$$\rho_{put}(s, y) := \frac{s^{\alpha_P}}{1 + s^{\alpha_P}}\Pi_{k=1}^{N}e^{-\varepsilon N(y_k)},$$

508 *where $\alpha_P := \left(\varepsilon' + 2\sum_{k=1}^{N}(1 - \beta_k) - 1\right)_+.$*

509   *Proof.* (i) In the case of a call option, we have that

510 (6.14) $1 \geq c_0 s^{\beta_k - 1}$ for $c_0 > 0$ small enough over the domain of integration,

511 so that we can as well take

512 (6.15) $h(s, y) = 1 + \sum_{k=1}^{N} |y_k| \leq \Pi_{k=1}^{N}(1 + |y_k|).$

513 So, we need that $\varphi(s, y) \in L^{1,\rho}(\Omega)$, with

514 (6.16) $\varphi(s, y) = h^2(s, y)u_T^2(s) = (s - K)_+^2 \Pi_{k=1}^{N}(1 + |y_k|)^2.$

515 By lemma 4.2, where here $J = \{0\}$ and $I = \{1, \ldots, N\}$, we may take resp.

516 (6.17) $\gamma_0' = 2, \ \gamma_0'' = 0, \ \gamma_k' > 0, \ \gamma_k'' > 0, \quad k = 1, \ldots, N,$

517 and so we may choose for $\varepsilon' > 0$ and $\varepsilon'' > 0$:

518 (6.18) $\alpha_0' = 0, \ \alpha_0'' = 3 + \varepsilon'', \ \alpha_k' = \varepsilon', \ \alpha_k'' = \varepsilon'', \quad k = 1, \ldots, N,$

519 so that setting $\varepsilon := \varepsilon' + \varepsilon''$, we can take $\rho = \rho_{call}$.

520 (ii) For a put option with strike $K > 0$, $1 \le c_0 s^{\beta_k - 1}$ for big enough $c_0 > 0$, over the

521 domain of integration, so that we can as well take

522 (6.19) $\quad h(s, y) = 1 + \sum_{k=1}^{N} |y_k| s^{\beta_k - 1} \le \Pi_{k=1}^{N}(1 + |y_k| s^{\beta_k - 1})^2 \le \Pi_{k=1}^{N} s^{2\beta_k - 2}(1 + |y_k|)^2$

523 and

524 (6.20) $\qquad \varphi(s, y) = h^2(s, y) u_T^2(s) \le (K - s)_+^2 \Pi_{k=1}^{N} s^{2\beta_k - 2}(1 + |y_k|)^2.$

525 By lemma 4.2, in the case of a put option and since $(K - s)_+^2$ is bounded, we can take

526 γ_k', γ_k'', α_k', α_k'' as before, for $k = 1$ to N, and

527 (6.21) $\qquad \gamma_0' = 0, \; \gamma_0'' = 2\sum_{k=1}^{N}(1 - \beta_k), \; \alpha_0' = \left(\varepsilon' + 2\sum_{k=1}^{N}(1 - \beta_k) - 1 \right)_+, \; \alpha_0'' = 0$

528 the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

529   REMARK 6.2. *If we do not use the commutator analysis, then we have a greater*

530 *"h" function; we can check that our previous choice of ρ does not apply any more (so*

531 *we should consider a smaller weight function, but we do not need to make it explicit).*

532 *And indeed, we have then a singularity when say y_1 is close to zero so that the previous*

533 *choice of ρ makes the p integral undefined.*

534   **6.3. Application to the GMH model.** For the generalized multiple factor

535 Heston model (GMH), i.e. when $\gamma_k = 1/2$, $k = 1$ to N, we can take h equal to

536 (6.22) $\qquad\qquad h_H' := 1 + \sum_{k=1}^{N} \left(|y_k|^{\frac{1}{2}}(1 + s^{\beta_k - 1}) + |y_k|^{-\frac{1}{2}} \right),$

537 when the commutator analysis is used, and when it is not, take h equal to

538 (6.23) $\qquad\qquad\qquad h_H := h_H + rs^{1-\beta_1}|y_1|^{-\frac{1}{2}}.$

539   LEMMA 6.3. (i) *For the GMH model, using the commutator analysis, in case of*

540 *a* call option *with strike K, meaning that $u_T(s) = (s - K)_+$, we can take $\rho = \rho_{call}$,*

541 *with*

542 (6.24) $\qquad\qquad \rho_{call}(s, y) := (1 + s^{\varepsilon'' + 3})^{-1} \Pi_{k=1}^{N} y_k^{\varepsilon'}(1 + y_k^{\varepsilon+2})^{-1}.$

543 (ii) *For a* put option *with strike $K > 0$, we can take $\rho = \rho_{put}$, with*

544 (6.25) $\qquad\qquad \rho_{put}(s, y) := \Pi_{k=1}^{N} y_k^{\varepsilon'}(1 + y_k^{\varepsilon+2})^{-1}.$

545   *Proof.* (i) For the call option, using (6.14) we see that we can as well take

546 (6.26) $\quad h(s, y) \le 1 + \sum_{k=1}^{N} \left(y_k^{1/2} + y_k^{-1/2} \right) \le (s - K)_+^2 \Pi_{k=1}^{N}(1 + y_k^{1/2} + y_k^{-1/2}).$

547 So, we need that $\varphi(s, y) \in L^{1, \rho}(\Omega)$, with

548 (6.27) $\qquad \varphi(s, y) = h^2(s, y) u_T^2(s) = (s - K)_+^2 \Pi_{k=1}^{N}(1 + y_k^{1/2} + y_k^{-1/2}).$

549   By lemma 4.2, where here $J = \{0, \ldots, N\}$, we may take resp.

550   (6.28) $\gamma_0' = 2, \; \gamma_0'' = 0, \; \gamma_k' = 1, \; \gamma_k'' = 1, \quad k = 1, \ldots, N,$

551   and so we may choose for $\varepsilon' > 0$ and $\varepsilon'' > 0$:

552   (6.29) $\alpha_0' = 0, \; \alpha_0'' = 3 + \varepsilon'', \; \alpha_k' = \varepsilon', \; \alpha_k'' = \varepsilon'' + 2, \quad k = 1, \ldots, N,$

553   so that setting $\varepsilon := \varepsilon' + \varepsilon''$, we can take $\rho = \rho_{call}$.
554   (ii) For a put option with strike $K > 0, 1 \leq c_0 s^{\beta_k - 1}$ for big enough $c_0 > 0$, over the
555   domain of integration, so that we can as well take

556   (6.30) $h(s,y) = 1 + \sum_{k=1}^{N} |y_k| s^{\beta_k - 1} \leq \Pi_{k=1}^{N}(1 + |y_k| s^{\beta_k - 1})^2 \leq \Pi_{k=1}^{N} s^{2\beta_k - 2}(1 + |y_k|)^2$

557   and

558   (6.31) $\varphi(s,y) = h^2(s,y) u_T^2(s) \leq (K - s)_+^2 \Pi_{k=1}^{N} s^{2\beta_k - 2}(1 + |y_k|)^2.$

559   By lemma 4.2, in the case of a put option and since $(K - s)_+^2$ is bounded, we can take
560   $\gamma_k', \gamma_k'', \alpha_k', \alpha_k''$ as before, for $k = 1$ to N, and

561   (6.32) $\gamma_0' = 0, \; \gamma_0'' = 0, \; \alpha_0' = 0, \; \alpha_0'' = 0.$

562   the result follows.   □

563   REMARK 6.4. *If we do not use the commutator analysis, then, again, we have a*
564   *greater "h" function; we can check that our previous choice of ρ does not apply any*
565   *more (so we should consider a smaller weight function, but we do not need to make it*
566   *explicit). And indeed, by the behaviour of the integral for large s the previous choice*
567   *of ρ makes the p integral undefined.*

568   **Appendix A. Regularity results by Lions and Magenes [15, Ch. 1].**
569   Let H be a Hilbert space identified with its dual and scalar product denoted by
570   $(\cdot, \cdot)$. Let V be a Hilbert space, densely and continuously embedded in H, with duality
571   product denoted by $\langle \cdot, \cdot \rangle_V$. Set

572   (A.1) $W(0, T) := \{u \in L^2(0, T; V); \; \dot{u} \in L^2(0, T; V^*)\}.$

573   It is known [15, Ch. 1] that

574   (A.2) $W(0, T) \subset C(0, T; H)$ with continuous inclusion,

575   and that for any u, v in $W(0, T)$, and $0 \leq t < t' \leq T$, the following integration by
576   parts formula holds:

577   (A.3) $\int_t^{t'} (\langle \dot{u}(s), v(s) \rangle_V + \langle \dot{v}(s), u(s) \rangle_V) \, ds = (u(t'), v(t'))_H - (u(t), v(t))_H.$

578   Equivalently,

579   (A.4) $2 \int_t^{t'} \langle \dot{u}(s), u(s) \rangle_V ds = \|u(t')\|_H^2 - \|u(t)\|_H^2, \quad$ for all $u \in W(0, T)$.

580   Let $A(t) \in L^\infty(0,T; L(V,V^*))$ satisfy the hypotheses of uniform continuity and semi-
581   coercivity, i.e., for some $\alpha > 0$, $\lambda \geq 0$, and $c > 0$:

582   (A.5)
$$\begin{cases} \langle A(t)u, u \rangle_V \geq \alpha\|u\|_V^2 - \lambda\|u\|_H, & \text{for all } u \in V, \text{ and a.a. } t \in [0,T], \\ \|A(t)u\|_{V^*} \leq c\|u\|_V, & \text{for all } u \in V, \text{ and a.a. } t \in [0,T]. \end{cases}$$

583   Given $(f, u_T) \in L^2(0,T; V^*) \times H$, we consider the following (backward) parabolic
584   equation: find u in $W(0,T)$ such

585   (A.6)
$$\begin{cases} -\dot{u}(t) + A(t)u(t) = f & \text{in } L^2(0,T; V^*), \\ u(T) = u_T & \text{in } H, \end{cases}$$

586   and recall classical results from [15, Ch. 1].

587   PROPOSITION A.1 (first parabolic estimate).   *The parabolic equation* (A.6) *has*
588   *a unique solution* $u \in W(0,T)$, *and for some* $c > 0$ *not depending on* (f, u_T):

589   (A.7) $\|u\|_{L^2(0,T;V)} + \|u\|_{L^\infty(0,T;H)} \leq c(\|u_T\|_H + \|f\|_{L^2(0,T;V^*)})$.

590   We next derive a stronger result with the hypothesis of *semi-symmetry* below:
      (A.8)
591
$$\begin{cases} A(t) = A_0(t) + A_1(t), \ A_0(t) \text{ and } A_1(t) \text{ continuous linear mappings } V \to V^*, \\ A_0(t) \text{ symmetric and continuously differentiable } V \to V^* \text{ w.r.t. } t, \\ A_1(t) \text{ is measurable with range in } H, \text{ and for positive numbers } \alpha_0, c_{A,1}: \\ \text{(i)} \quad \langle A_0(t)u, u \rangle_V \geq \alpha_0\|u\|_V^2, \quad \text{for all } u \in V, \text{ and a.a. } t \in [0,T], \\ \text{(ii)} \quad \|A_1(t)u\|_H \leq c_{A,1}\|u\|_V, \quad \text{for all } u \in V, \text{ and a.a. } t \in [0,T], \\ f \in L^2(0,T; H) \text{ and } u_T \in V. \end{cases}$$

592   PROPOSITION A.2 (second parabolic estimate).   *Let* (A.8) *hold. Then the solu-*
593   *tion* $u \in W(0,T)$ *of* (A.6) *belongs to* $L^\infty(0,T; V)$, $\dot{u}$ *belongs to* $L^2(0,T; H)$, *and for*
594   *some* $c > 0$ *not depending on* (f, u_T):

595   (A.9) $\|u\|_{L^\infty(0,T;V)} + \|\dot{u}\|_{L^2(0,T;H)} \leq c(\|u_T\|_V + \|f\|_{L^2(0,T;H)})$.

596   **Appendix B. Parabolic variational inequalities.**
597   Let $K \subset V$ be a non-empty, closed and convex set, $\mathcal{K}$ be the closure of K in H,
598   and $u_T \in K$. Let

599   (B.1)
$$\begin{cases} L^2(0,T; K) := \{u \in L^2(0,T; V); \ u(t) \in K \text{ a.e.}\}, \\ W(0,T; K) := W(0,T) \cap L^2(0,T; K). \end{cases}$$

600   We consider parabolic variational inequalities as follows: find $u \in W(0,T; K)$ such
601   that

602   (B.2)
$$\begin{cases} \langle -\dot{u}(t) + A(t)u(t) - f(t), v - u(t) \rangle_V \geq 0 & \text{for all } v \in K, \quad \text{a.a. } t, \\ u(T) = u_T & \text{in } H. \end{cases}$$

603   Take $v \in W(0,T; K)$.   Adding to the previous inequality the integration by parts
604   formula

605   (B.3) $-\displaystyle\int_0^T \langle \dot{v}(s) - \dot{u}(s), v(s) - u(s) \rangle_V \mathrm{d}s = \frac{1}{2}\|u(0) - v(0)\|_H^2 - \frac{1}{2}\|u(T) - v(T)\|_H^2$

606   and since $u(T) = u_T$ we find that

(B.4)

607
$$
\begin{cases}
\displaystyle\int_0^T \langle -\dot{v}(t) + A(t)u(t) - f(t), v - u(t)\rangle_V \geq \tfrac{1}{2}\|u(0) - v(0)\|_H^2 - \tfrac{1}{2}\|u(T) - v(T)\|_H^2 \\[2mm]
\quad \text{for all } v \in W(0, T; K), \quad u(T) = u_T.
\end{cases}
$$

608   It can be proved that the two formulation (B.2) and (B.4) are equivalent (they have the
609   same set of solutions), and that they have at most one solution. The weak formulation
610   is as follows: find $u \in L^2(0, T; K) \cap C(0, T; H)$ such that

611   (B.5)
$$
\begin{cases}
\displaystyle\int_0^T \langle -\dot{v}(t) + A(t)u(t) - f(t), v - u(t)\rangle_V \geq -\tfrac{1}{2}\|u(T) - v(T)\|_H^2 \\[2mm]
\quad \text{for all } v \in L^2(0, T; K), \quad u(T) = u_T.
\end{cases}
$$

612   Clearly a solution of the strong formulation (B.2) is solution of the weak one.

613   PROPOSITION B.1 (Brézis [6]).   *The following holds:*
614   *(i) Let $u_T \in \mathcal{K}$ and $f \in L^2(0, T; V^*)$. Then the weak formulation (B.5) has a*
615   *unique solution u and, for some $c > 0$, given $v_0 \in K$:*

616   (B.6) $\|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)} \leq c(\|u_T\|_H + \|f\|_{L^2(0,T;V^*)} + \|v_0\|_V).$

617   *(ii) Let in addition the semi-symmetry hypothesis (A.8) hold, and let u_T belong*
618   *to K. Then $u \in L^\infty(0, T; V)$, $\dot{u} \in L^2(0, T; H)$, and u is the unique solution*
619   *of the original formulation (B.2). Furthermore, for some $c > 0$:*

620   (B.7) $\|u\|_{L^\infty(0,T;V)} + \|\dot{u}\|_{L^2(0,T;H)} \leq c(\|u_T\|_V + \|f\|_{L^2(0,T;H)}).$

621   **Appendix C. Monotonicity.**   Assume that H is an Hilbert lattice, i.e., is
622   endowed with an order relation $\succeq$ compatible with the vector space structure:

623   (C.1) $x_1 \succeq x_2$ implies that $\gamma x_1 + x \succeq \gamma x_2 + x$, for all $\gamma \geq 0$ and $x \in H$,

624   such that the maxima and minima denoted by $\max(x_1, x_2)$ and $\min(x_1, x_2)$ are well
625   defined, the operator max, min be continuous, with $\min(x_1, x_2) = -\max(-x_1, -x_2)$.
626   Setting $x_+ := \max(x, 0)$ and $x_- := -\min(x, 0)$ we have that $x = x_+ - x_-$. Assuming
627   that the maximum of two elements of V belong to V we see that we have an induced
628   lattice structure on V. The induced dual order over V^* is as follows: for v_1^* and v_2^* in
629   V^*, we say that $v_1^* \geq v_2^*$ if $\langle v_1^* - v_2^*, v\rangle_V \geq 0$ whenever $v \geq 0$.
630   Assume that we have the following extension of the integration by parts formula
631   (B.3): for all u, v in $W(0, T)$ and $0 \leq t < t' \leq T$,

632   (C.2) $2 \displaystyle\int_t^{t'} \langle \dot{u}(s), u_+(s)\rangle_V \mathrm{d}s = \|u_+(t')\|_H^2 - \|u_+(t)\|_H^2.$

633   and that

634   (C.3) $\langle A(t)u, u_+\rangle_V = \langle A(t)u_+, u_+\rangle_V.$

635   PROPOSITION C.1. *Let u_i be solution of the parabolic equation (A.6) for $(f, u_T) =$*
636   *(f^i, u_T^i), $i = 1, 2$. If $f^1 \geq f2$ and $u_T^1 \geq u_T^2$, then $u_1 \geq u_2$.*

637   This type of result may be extended to the case of variational inequalities. If K
638 and K' are two subsets of V, we say that K *dominates* K' if for any $u \in K$ and
639 $u' \in K'$, $\max(u, u') \in K$ and $\min(u, u') \in K'$.

640   PROPOSITION C.2. *Let u_i be solution of the weak formulation* (B.5) *of the parabolic*
641 *variational inequality for* $(f, u_T, K) = (f^i, u_T^i, K^i)$, $i = 1, 2$. *If* $f_1 \geq f_2$, $u_T^1 \geq u_T^2$, *and*
642 K^1 *dominates* K^2, *then* $u_1 \geq u_2$.

643   The monotonicity w.r.t. the convex K is due to Haugazeau [13] (in an elliptic
644 setting, but the result is easily extended to the parabolic one). See also Brézis [7].

645   **Appendix D. Link with American options.** An American option is the
646 right to get a payoff $\Psi(t, x)$ at any time $t < T$ and u_T at time T. We can motivate
647 as follows the derivation of the associated variational inequalities. If the option can
648 be exercized only at times $t_k = hk$, with $h = T/M$ and $k = 0$ to M (Bermudean
649 option), then the same PDE as for the European option holds over (t_k, t_{k+1}), $k = 0$
650 to $M - 1$. Denoting by $\tilde{u}_k$ the solution of this PDE, we have that $u(t_k) = \max(\Psi, \tilde{u}_k)$.
651 Assuming that A does not depend on time and that there is a flux $f(t, x)$ of dividents,
652 we compute the approximation u_k of $u(t_k)$ as follows. Discretizing the PDE with the
653 implicit Euler scheme we obtain the continuation value $\hat{u}_k$ solution of

654 (D.1) $\dfrac{\hat{u}_k - u_{k+1}}{h} + A\hat{u}_k = f(t_k, \cdot), \ k = 0, \ldots, M - 1; \quad u_M = \max(\Psi, 0),$

655 so that $u_k = u_{k+1} - hA\hat{u}_k + hf(t_k, \cdot)$, we find that

656 (D.2) $u_k = \max(\hat{u}_k, \Psi) = \max(u_{k+1} - hA\hat{u}_k + hf(t_k, \cdot), \Psi),$

657 which is equivalent to

658 (D.3) $\min(u_k - \Psi, \dfrac{u_k - u_{k+1}}{h} + A\hat{u}_k - f(t_k, \cdot)) = 0.$

659 This suggest for the continuous time model and general operators A and r.h.s. f the
660 following formulation

661 (D.4) $\min(u(t, x) - \Psi(x), -\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)) = 0, \quad (t, x) \in (0, T) \times \Omega.$

662 The above equation has a rigorous mathematical sense in the context of viscosity
663 solution, see Barles [5]. However we rather need the variational formulation which
664 can be derived as follows. Let $v(x)$ satisfy $v(x) \geq \Psi(x)$ a.e., be smooth enough. Then

665 (D.5)
$$
\begin{aligned}
\int_\Omega &\left(-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)\right)\left(v(x) - u(t, x)\right)\mathrm{d}x = \\
&\int_{\{u(t,x)=\Psi(x)\}} \left(-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)\right)\left(v(x) - u(t, x)\right)\mathrm{d}x \\
&+ \int_{\{u(t,x)>\Psi(x)\}} \left(-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)\right)\left(v(x) - u(t, x)\right)\mathrm{d}x.
\end{aligned}
$$

666 The first integrand is nonnegative, being a product of nonnegative terms, and the
667 second integrand is equal to 0 since by (D.3), $-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x) = 0$
668 a.e. when $u(t, x) > \Psi(x)$. So we have that, for all $v \geq \Psi$ smooth enough:

669 (D.6) $\int_\Omega \left(-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)\right)\left(v(x) - u(t, x)\right)\mathrm{d}x \geq 0.$

670 We see that this is of the same nature as a parabolic variational inequality, where K
671 is the set of functions greater or equal to Ψ (in an appropriate Sobolev space).

672 **Appendix E. Some one dimensional problems.** It is not always easy to
673 characterize the space $\mathcal{V}$. Let us give a detailed analysis in a simple case.

674 **E.1. The Black-Scholes setting.** For the Black-Scholes model with zero inter-
675 est rate (the extension to a constant nonzero interest rate is easy) and unit volatility
676 coefficient, we have that $Au = -\frac{1}{2}x^2 u''(x)$, with $x \in (0, \infty)$. In the case of a put
677 option: $u_T(x) = (K - x)_+$ we may take $H := L^2(\mathbb{R}_+)$. For $v \in \mathcal{D}(0, \infty)$ and u
678 sufficiently smooth we have that $-\frac{1}{2}\int_0^\infty x^2 u''(x)\mathrm{d}x = a(u, v)$ with

679 (E.1) $$a(u, v) := \tfrac{1}{2}\int_0^\infty x^2 u'(x)v'(x)\mathrm{d}x + \int_0^\infty x u'(x)v(x)\mathrm{d}x.$$

680 This bilinear form a is continuous and semi coercive over the set

681 (E.2) $$V := \{u \in H;\ x u'(x) \in H\}.$$

682 It is easily checked that $\bar{u}(x) := x^{-1/3}/(1 + x)$ belongs to V. So, some elements of V
683 are unbounded near zero.
684 We now claim that $\mathcal{D}(0, \infty)$ is a dense subset of V. First, it follows from a
685 standard truncation argument and the dominated convergence theorem that $V_\infty :=$
686 $V \cap L^\infty(0, \infty)$ is a dense subset of V. Note that elements of V are continuous over
687 $(0, \infty)$. Given $\varepsilon > 0$ and $u \in V_\infty$, define

688 (E.3) $$u_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in (0, \varepsilon), \\ u(2\varepsilon)(x/\varepsilon - 1) & \text{if } x \in [\varepsilon, 2\varepsilon], \\ u(2\varepsilon) & \text{if } x > 2\varepsilon. \end{cases}$$

689 Obviously $u_\varepsilon \in V_\infty$. By the dominated convergence theorem, $u_\varepsilon \to u$ in H. Set for
690 $w \in V$

691 (E.4) $$\Phi_\varepsilon(w) := \int_0^{2\varepsilon} x^2 w'(x)^2 \mathrm{d}x.$$

692 Since Φ_ε is quadratic and $v_\varepsilon \to u$ in H, we have that:

693 (E.5) $$\tfrac{1}{2}\int_0^\infty x^2(u'_\varepsilon - u')^2 \mathrm{d}x = \tfrac{1}{2}\Phi_\varepsilon(u_\varepsilon - u) \le \Phi_\varepsilon(u_\varepsilon) + \Phi_\varepsilon(u).$$

694 Since $u \in V$, $\Phi_\varepsilon(u) \to 0$ and

695 (E.6) $$\Phi_\varepsilon(u_\varepsilon) \le \|u\|_\infty^2 \int_0^{2\varepsilon} \varepsilon^{-2} x^2 \mathrm{d}x = O(\|u\|_\infty^2 \varepsilon).$$

696 So, the l.h.s. of (E.5) has limit 0 when $\varepsilon \downarrow 0$. We have proved that the set V^0 of
697 functions in V^∞ equal to zero near zero, is a dense subset of V. Now define for $N > 0$

698 (E.7) $$\varphi_N(x) = \begin{cases} 1 & \text{if } x \in (0, N), \\ 1 - \log(x/N) & \text{if } x \in [N, eN], \\ 0 & \text{if } x > eN. \end{cases}$$

699 Given $u \in V_0$, set $u_N := u\varphi_N$. Then $u_N \in H$ and, by a dominated convergence
700 argument, $u_N \to u$ in H. The weak derivative of u_N is $u'_N = u'\varphi_N + u\varphi'_N$. By a

dominated convergence argument, $xu'\varphi_N \to xu'$ in $L^2(\mathbb{R}_+)$. It remains to prove that $xu\varphi'_N \to 0$ in $L^2(\mathbb{R}_+)$. But φ'_N is equal to $1/x$ over its support, so that

(E.8) $$\|xu\varphi'_N\|^2_{L^2(\mathbb{R}_+)} = \int_N^{eN} u^2(x)\mathrm{d}x \leq \int_N^\infty u^2(x)\mathrm{d}x \to 0$$

when $N \uparrow \infty$. The claim is proved.

E.2. The CIR setting. In the Cox-Ingersoll-Ross model [8] the stochastic process satisfies

(E.9) $$\mathrm{d}s(t) = \theta(\mu - s(t))\mathrm{d}t + \sigma\sqrt{s}\,\mathrm{d}W(t), \quad t \geq 0$$

We assume the coefficients θ, μ and σ to be constant and positive. The associated PDE is given by

(E.10) $$\begin{cases} Au := -\theta(\mu - x)u' - \frac{1}{2}x\hat{\sigma}^2 u'' = 0 & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \qquad\qquad u(x,T) = u_T(x) & x \in \mathbb{R}^+. \end{cases}$$

Again for the sake of simplicity we will take $\rho(x) = 1$, which is well-adapted in the case of a payoff with compact support in $(0,\infty)$. For $v \in \mathcal{D}(0,\infty)$ and u sufficiently smooth we have that $\int_0^\infty Au(x)v(x)\mathrm{d}x = a(u,v)$ with

(E.11)
$$a(u,v) := \theta \int_0^\infty (\mu - x)u'(x)v(x)\mathrm{d}x + \frac{1}{2}\hat{\sigma}^2 \int_0^\infty xu'(x)v'(x)\mathrm{d}x + \frac{1}{2}\hat{\sigma}^2 \int_0^\infty u'(x)v(x)\mathrm{d}x.$$

So one should take $\mathcal{V}$ of the form

(E.12) $$\mathcal{V} := \{u \in H; \ \sqrt{x}u'(x) \in L^2(\mathbb{R}_+)\}.$$

We next determine H by requiring that the bilinear form is continuous; by the Cauchy-Schwarz inequality

(E.13)
$$\left|\int_0^\infty u'(x)v(x)\mathrm{d}x\right| \leq \|x^{1/2}u'\|_2\|x^{-1/2}v\|_2; \ \left|\int_0^\infty xu'(x)v(x)\mathrm{d}x\right| \leq \|x^{1/2}u'\|_2\|x^{1/2}v\|_2.$$

We easily deduce that the bilinear form a is continuous and semi coercive over $\mathcal{V}$, when choosing

(E.14) $$H := \{v \in L^2(\mathbb{R}_+); \ (x^{1/2} + x^{-1/2})v \in L^2(\mathbb{R}_+)\},$$

Note that then the integrals below are well defined and finite for any $v \in \mathcal{V}$:

(E.15) $$\int_0^\infty (x^{1/2}v')(x^{-1/2}v) = \int_0^\infty vv' = \frac{1}{2}\int_0^\infty (v^2)'.$$

So $w := v^2$ is the primitive of an integrable function and therefore has a limit at zero. Since v is continuous over $(0,\infty)$ it follows that v has a limit at zero.

However if this limit is nonzero we get a contradiction with the condition that $x^{-1/2}v \in L^2(\mathbb{R}_+)$. So, every element of $\mathcal{V}$ has zero value at zero.

We now claim that $\mathcal{D}(0,\infty)$ is a dense subset of $\mathcal{V}$. First, $\mathcal{V}_\infty := \mathcal{V} \cap L^\infty(0,\infty)$ is a dense subset of $\mathcal{V}$. Note that elements of $\mathcal{V}$ are continuous over $(0,\infty)$. Given $\varepsilon > 0$

and $u \in \mathcal{V}_\infty$, define $u_\varepsilon(x)$ as in (E.3). Then $u_\varepsilon \in \mathcal{V}_\infty$. By the dominated convergence theorem, $u_\varepsilon \to u$ in H. Set for $w \in \mathcal{V}$

(E.16)
$$\Phi_\varepsilon(w) := \int_0^{2\varepsilon} x w'(x)^2 \mathrm{d}x.$$

Since Φ_ε is quadratic and $u_\varepsilon \to u$ in H, we have that:

(E.17)
$$\tfrac{1}{2} \int_0^\infty x^2 (u_\varepsilon' - u')^2 \mathrm{d}x = \tfrac{1}{2}\Phi_\varepsilon(u_\varepsilon - u) \le \Phi_\varepsilon(u_\varepsilon) + \Phi_\varepsilon(u).$$

Since $u \in \mathcal{V}$, $\Phi_\varepsilon(u) \to 0$ and

(E.18)
$$\Phi_\varepsilon(u_\varepsilon) \le \varepsilon^{-2} u(2\varepsilon)^2 \int_0^{2\varepsilon} x \mathrm{d}x = 2u(2\varepsilon)^2 \to 0.$$

So, the l.h.s. of (E.17) has limit 0 when $\varepsilon \downarrow 0$. We have proved that the set $\mathcal{V}^0$ of functions in $\mathcal{V}^\infty$ equal to zero near zero, is a dense subset of $\mathcal{V}$. Define φ_N as in (E.7)

Given $u \in \mathcal{V}_0$, set $u_N := u\varphi_N$. As before, $u_N \to u$ in H, is $u_N' = u'\varphi_N + u\varphi_N'$, $xu'\varphi_N \to xu$ in $L^2(\mathbb{R}_+)$, and it remains to prove that $xu\varphi_N' \to 0$ in $L^2(\mathbb{R}_+)$. But φ_N' is equal to $1/x$ over its support, so that when $N \uparrow \infty$:

(E.19)
$$\|x^{1/2} u\varphi_N'\|_{L^2(\mathbb{R}_+)}^2 = \int_N^{eN} x^{-1} u^2(x) \mathrm{d}x \le \int_N^\infty u^2(x) \mathrm{d}x \to 0.$$

The claim is proved.

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