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# Automated Constructivization of Proofs

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**Abstract.** No computable function can output a constructive proof from a classical one whenever its associated theorem also holds constructively. We show in this paper that it is however possible, in practice, to turn a large amount of classical proofs into constructive ones. We describe for this purpose a linear-time constructivization algorithm which is provably complete on large fragments of predicate logic.

## 1 Introduction

Classical and constructive provability match on several specific sets of propositions. In propositional logic, as a consequence of Glivenko’s theorem [1], a formula  $\neg A$  is a classical theorem iff it is a constructive one. In arithmetic, a  $\Pi_2^0$  proposition is a theorem in Peano arithmetic iff it is a theorem in Heyting arithmetic [2].

We present in this paper an efficient constructivization algorithm **CONSTRUCT** for predicate logic in general, from cut-free classical sequent calculus **LK** to constructive sequent calculus **LJ**. Unlike the two previous examples, constructivization in predicate logic is as hard as constructive theorem proving. Therefore, as we expect **CONSTRUCT** to terminate, **CONSTRUCT** is incomplete in the sense that it may terminate with a failure output.

**CONSTRUCT** consists of three **linear-time** steps:

1. An algorithm **NORMALIZE**, designed to push occurrences of the right weakening rule towards the root in **LK** proofs. Its purpose is to limit the number of propositions appearing at the right-hand side of sequents in **LK** proofs.
2. A partial translation from cut-free **LK** to a new constructive system **LI**. This algorithm is referred to as **ANNOTATE** as the **LI** system is designed as **LK** equipped with specific annotations – making it a constructive system. **ANNOTATE** is the only step which may fail.
3. A complete translation **INTERPRET** from **LI** to **LJ**.

The **NORMALIZE** step taken alone leads to a simple yet efficient constructivization algorithm **WEAK CONSTRUCT**, which is defined to succeed whenever the result of **NORMALIZE** happens to be directly interpretable in **LJ**, i.e. to have at most one proposition on the right-hand side of sequents in its proof.

The main property of **CONSTRUCT** is to be provably **complete** on large fragments of predicate logic, in the sense that for any proposition  $A$  in one

of these fragments, **CONSTRUCT** is ensured to terminate successfully on any cut-free **LK** proof of  $A$ . Such fragments for which classical and constructive provability match will be referred to as **constructive fragments**. For instance, as a consequence of Glivenko’s theorem [1], the set of negated propositions is a **constructive fragment** of propositional logic. The completeness properties of **CONSTRUCT** lead to the following results:

- The identification of a new constructive fragment  $F$ , the fragment of assertions containing no negative occurrence of the connective  $\vee$  and no positive occurrence of the connective  $\Rightarrow$ . Both **WEAK CONSTRUCT** and **CONSTRUCT** are provably complete on  $F$ .
- The completeness of **CONSTRUCT** on two already known constructive fragments. The first one, referred to as  $F_{Ku}$ , appears as the set of fix points of a polarized version of Kuroda’s double-negation translation [3, 4]. The second one, referred to as  $F_{Ma}$ , appears as a set of assertions for which any cut-free **LK** proof can be directly interpreted as a proof in Maehara’s multi-succedent calculus [5]. Hence, the completeness of **CONSTRUCT** on these two fragments yields a uniform proof of two results coming from very different works.

After the introduction of basic notations and definitions, the two already known constructive fragments  $F_{Ku}$  and  $F_{Ma}$  are presented. Then, the **NORMALIZE** step is presented along with the simple constructivization algorithm **WEAK CONSTRUCT**. In the following section, the new constructive fragment  $F$  is defined, and **WEAK CONSTRUCT** is proved complete on  $F$ . Then, the full constructivization algorithm **CONSTRUCT** is introduced together with the proof of its completeness on  $F$ ,  $F_{Ku}$  and  $F_{Ma}$ . In the last part, experimental results of constructivization using **WEAK CONSTRUCT** and **CONSTRUCT** are presented. These experiments are based the classical theorem prover **Zenon** [10] and the constructive proof checker **Dedukti** [9].

## 2 Notations and definitions

In the following, we only consider as primitive the connectives and quantifiers  $\forall, \exists, \wedge, \vee, \Rightarrow$  and  $\perp$ .  $\neg A$  is defined as  $A \Rightarrow \perp$ .  $\top$ , which doesn’t appear in this paper, could be defined as  $\perp \Rightarrow \perp$ .

We use a definition of sequents based on **multisets**. The size of a multiset  $\Gamma$  will be referred to as  $|\Gamma|$ . We will use the notation  $(A)$  to refer to a multiset containing either zero or one element. Given a multiset  $\Gamma = A_1, \dots, A_n$ , we will use the notations  $\neg\Gamma$  and  $\Gamma \Rightarrow B$  as shorthands for  $\neg A_1, \dots, \neg A_n$ , and  $A_1 \Rightarrow B, \dots, A_n \Rightarrow B$  respectively. Finally, we use the notation  $\bigvee$  to refer to an arbitrary encoding of the  $n$ -ary disjunction from the binary one – using  $\perp$  for the nullary case.

**Definition 1.** We define the cut-free classical sequent calculus **LK** with the following rules:

$$\frac{}{\perp \vdash} \perp_L \frac{}{A \vdash A} \text{ axiom}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta} \text{ weak}_L \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Delta'} \text{ weak}_R$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr}_L \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ contr}_R$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow_L \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \Rightarrow_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall_L \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta} \forall_R$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists_L \quad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \exists_R$$

with the standard freshness constraints for the variables introduced in the rules  $\forall_R$  and  $\exists_L$ .

**Definition 2.** We define the constructive sequent calculus **LJ** from **LK**, applying the following changes:

- All rules except  $\text{contr}_R$ ,  $\vee_R$ ,  $\Rightarrow_L$  are restricted to sequents with at most one proposition on the right-hand side of sequents.

For instance,  $\wedge_R$  becomes  $\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_R$

- There is no  $\text{contr}_R$  rule
- The  $\vee_R$  rule is split into two rules  $\frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \vee A_1} \vee_R$
- The  $\Rightarrow_L$  rule becomes  $\frac{\Gamma \vdash A \quad \Gamma, B \vdash (C)}{\Gamma, A \Rightarrow B \vdash (C)} \Rightarrow_L$

- We add a cut rule  $\frac{\Gamma \vdash A \quad \Gamma, A \vdash (B)}{\Gamma \vdash (B)} \text{ cut}$

*Remark 1.* In these presentations of **LK** and **LJ**,

- weakenings are applied to multisets instead of propositions
- $\perp_L$  and *axiom* are not relaxed to  $\frac{}{\Gamma, \perp \vdash \Delta} \perp_L$  and  $\frac{}{\Gamma, A \vdash A, \Delta} \text{ axiom}$

These specific conventions are chosen to ease the definition of the algorithm NORMALIZE in Section 5, which requires pushing weakenings towards the root of the proof.

**Definition 3.** We introduce the following notations in **LK**, along with their constructive analogs in **LJ**:

- $\frac{}{\Gamma, A \vdash A, \Delta} \text{ axiom}^*$  for  $\frac{\frac{}{A \vdash A} \text{ axiom}}{\Gamma, A \vdash A} \text{ weak}_L$   
 $\frac{}{\Gamma, A \vdash A, \Delta} \text{ weak}_R$
- $\frac{}{\Gamma, \perp \vdash \Delta} \perp_L^*$  for  $\frac{\frac{}{\perp \vdash} \perp_L}{\Gamma, \perp \vdash} \text{ weak}_L$   
 $\frac{}{\Gamma, \perp \vdash \Delta} \text{ weak}_R$
- $\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg_L$  for  $\frac{\Gamma \vdash A, \Delta \quad \frac{}{\Gamma, \perp \vdash \Delta} \perp_L^*}{\Gamma, \neg A \vdash \Delta} \Rightarrow_L$
- $\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg_R$  for  $\frac{\frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \perp, \Delta} \text{ weak}_R}{\Gamma \vdash \neg A, \Delta} \Rightarrow_R$

### 3 State of the art: two constructive fragments of predicate logic

Constructive sequent calculus – as well as constructive natural deduction – extends the notion of constructive provability from propositions to sequents of the shape  $\Gamma \vdash (G)$ , which will be referred to as **mono-succedent sequents**. As a consequence, we will define constructive fragments of predicate logic as sets of mono-succedent sequents instead of sets of simple propositions.

The definitions of these fragments will be based on the usual notion of polarity of occurrences of connectives, quantifiers and atoms in a sequent: given a sequent  $\Gamma \vdash \Delta$ ,

- the root of a proposition in  $\Gamma$  is negative, the root of a proposition in  $\Delta$  is positive
- polarity only changes between an occurrence of  $A \Rightarrow B$  and the occurrence of its direct subformula  $A$  (in particular, as  $\neg A$  is defined as  $A \Rightarrow \perp$ , it changes between  $\neg A$  and its direct subformula  $A$ )

**Definition 4.** *We define the following fragments of predicate logic:*

- $F_{Ku}$ , the fragment of sequents of the shape  $\Gamma \vdash$  containing no positive occurrence of  $\forall$ .
- $F_{Ma}$ , the fragment of mono-succedent sequents containing no positive occurrence of  $\forall$  and no positive occurrence of  $\Rightarrow$ .

**Theorem 1.**  *$F_{Ku}$  is a constructive fragment of predicate logic: for any sequent  $\Gamma \vdash$  in  $F_{Ku}$ ,  $\Gamma \vdash$  is classically provable iff it is constructively provable.*

The key arguments to prove this theorem as an adaptation of Kuroda’s double negation translation [3] are the following:

1. Kuroda’s double negation translation [3] is based on a double negation translation  $|\cdot|_{Ku}$  inserting double-negations after any occurrence of  $\forall$ . The original theorem is that a proposition  $A$  is classically provable iff  $\neg\neg|A|_{Ku}$  is constructively provable.
2. It can be adapted in two ways. First,  $|\cdot|_{Ku}$  can be lightened to insert double negations only after positive occurrences of  $\forall$  as shown in [4], and extended from propositions to contexts. Second, the main statement can be turned to the following one: a classical sequent  $\Gamma \vdash \Delta$  is classically provable iff  $|\Gamma, \neg\Delta|_{Ku} \vdash$  is constructively provable
3. By definition of  $F_{Ku}$ , a sequent  $\Gamma \vdash$  in  $F_{Ku}$  admits the property  $\Gamma = |\Gamma|_{Ku}$ , hence  $\Gamma \vdash$  is classically provable iff it is constructively provable.

We don’t give more details on this proof as the completeness of CONSTRUCT on  $F_{Ku}$  shown in Section 6 will yield a new proof of this result.

*Remark 2.* One could expect similar constructive fragments to be found using other double negation translations, such as Gödel-Gentzen’s [7, 6] or Kolmogorov’s [8]. Unfortunately, these two translations always insert double-negations in front of atoms, hence they cannot be easily modified to leave a large fragment of propositions unchanged.

**Theorem 2.**  *$F_{Ma}$  is a constructive fragment of predicate logic: for any sequent  $\Gamma \vdash (G)$  in  $F_{Ma}$ ,  $\Gamma \vdash (G)$  is classically provable iff it is constructively provable.*

It lays on a key idea: polarity restrictions have a direct influence on the shape of cut-free proofs. It can be presented in the following way:

**Lemma 1.** *For any connective or quantifier  $X$  and any cut-free **LK** proof  $\Pi$  of a sequent  $\Gamma \vdash \Delta$ :*

- *If  $\Gamma \vdash \Delta$  contains no positive occurrence of  $X$ , then  $\Pi$  doesn't contain the rule  $X_R$ .*
- *If  $\Gamma \vdash \Delta$  contains no negative occurrence of  $X$ , then  $\Pi$  doesn't contain the rule  $X_L$ .*

This lemma can be proved directly by induction on cut-free **LK** proofs. Using this lemma, the key arguments to prove Theorem 2 are the following:

1. All **LK** rules except  $\Rightarrow_R$  and  $\forall_R$  rules belong in Maehara's multi-succedent calculus [5], a constructive multi-succedent sequent calculus.
2. By lemma 1,  $F_{Ma}$  sequents are proved by cut-free **LK** proofs without the  $\Rightarrow_R$  and  $\forall_R$  rules.
3. Hence, a sequent  $\Gamma \vdash (G)$  in  $F_{Ma}$  is classically provable iff it is constructively provable.

Again, we don't give more details on this proof as the completeness of CONSTRUCT on  $F_{Ma}$  shown in Section 6 will yield a new proof of this result.

*Remark 3.* The same fragment  $F_{Ma}$  can be found using similar multi-succedent constructive systems, such as Dragalin's calculus GHPC [11].

## 4 The weakening normalization

A naive constructivization algorithm can be defined by selecting **LK** proofs which can be directly interpreted in **LJ**.

In this direct interpretation, premises of the classical rules  $\forall_R$  and  $\Rightarrow_L$  may be multi-succedent only when they are introduced by a  $weak_R$  whose premise is a mono-succedent sequent. For instance, the classical derivation

$$\frac{\frac{\Gamma \vdash A}{\Gamma \vdash A, B} weak_R}{\Gamma \vdash A \vee B} \forall_R \text{ can be interpreted as } \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \forall_R .$$

However, in practice, the  $weak_R$  rule doesn't appear as low as possible – in presentations using multi-succedents *axiom* rules, they may not appear at all. Such situations are problematic for constructive interpretations: for instance, a classical proof such as

$$\frac{\frac{\frac{\overline{A \vdash A} axiom}{A \vdash A, B} weak_R}{\vdash A \Rightarrow A, B} \Rightarrow_R}{\vdash (A \Rightarrow A) \vee B} \forall_R$$

cannot be interpreted in **LJ** directly because the  $weak_R$  rule doesn't occur immediately above the  $\vee_R$  rule.

The **NORMALIZE** algorithm is designed to address this issue, pushing the application of  $weak_R$  as low as possible in proofs. In its definition, we need to consider all possible configuration of  $weak_R$  appearing above a **LK** rule. In order to factor this definition, we partition all such configurations into three classes **A**, **B**, and **C**.

These definitions will be based on the following notation of **LK** proofs:

**Definition 5.** We write any cut-free **LK** rule  $X$  as

$$\frac{\Gamma, L_1 \vdash R_1, \Delta \quad \cdots \quad \Gamma, L_n \vdash R_n, \Delta}{\Gamma, L \vdash R, \Delta} X$$

where  $L_1, \dots, L_n, R_1, \dots, R_n, L$  and  $R$  are the (possibly empty) **multisets** of propositions containing the active propositions of the rule  $X$ .

For instance, in the rule  $\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \Rightarrow_R$ ,

$L_1 = \{A\}$ ,  $R_1 = \{B\}$ ,  $L = \emptyset$ , and  $R = \{A \Rightarrow B\}$ .

The classes **A**, **B**, and **C** are defined as follows:

**Definition 6.** We consider all configurations where  $weak_R$  appears above a **LK** rule  $X$ , in its  $i$ -th premise:

$$\frac{\cdots \quad \frac{\Gamma, L_i \vdash \Delta_i}{\Gamma, L_i \vdash R_i, \Delta} weak_R \quad \cdots}{\Gamma, L \vdash R, \Delta} X$$

This weakening can be done on propositions in  $R_i$ , in  $\Delta$  or both: in the general case, we only know  $\Delta_i \subseteq (R_i, \Delta)$ . We define the following partition of all cases:

- **A**:  $R_i \subseteq \Delta_i$
- **B**:  $R_i \not\subseteq \Delta_i$  and  $\Delta_i \subseteq \Delta$
- **C**:  $R_i \not\subseteq \Delta_i$  and  $\Delta_i \not\subseteq \Delta$ . This only happens when  $|R_i| = 2$ , when exactly one proposition of  $R_i$  is in  $\Delta_i$ .

**Definition 7.** **NORMALIZE** is a linear-time algorithm associating any cut-free **LK** proof of a sequent  $\Gamma \vdash \Delta$  to a proof of a sequent  $\Gamma \vdash \Delta'$ , where  $\Delta' \subseteq \Delta$ . It is defined recursively. Using the conventions of Definition 5, we describe the original proof  $\Pi$  as

$$\frac{\frac{\Pi_1}{\Gamma, L_1 \vdash R_1, \Delta} \quad \cdots \quad \frac{\Pi_n}{\Gamma, L_n \vdash R_n, \Delta}}{\Gamma, L \vdash R, \Delta} X$$

The definition of  $\text{NORMALIZE}(\Pi)$  is based on the analysis of the proof

$$\frac{\frac{\text{NORMALIZE}(\Pi_1)}{\Gamma, L_1 \vdash \Delta_1} \text{weak}_R \quad \dots \quad \frac{\text{NORMALIZE}(\Pi_n)}{\Gamma, L_n \vdash \Delta_n} \text{weak}_R}{\Gamma, L \vdash R, \Delta} X$$

The different cases are the following:

- Case 1: for all index  $i$ ,  $\mathbf{A}$  holds, i.e.  $R_i \subseteq \Delta_i$ .  
If  $X$  is  $\text{weak}_R$ , we define  $\text{NORMALIZE}(\Pi)$  as  $\text{NORMALIZE}(\Pi_1)$ .

Else, writing  $\Delta_i = R_i, \Delta'_i$ , we define  $\text{NORMALIZE}(\Pi)$  as

$$\frac{\frac{\text{NORMALIZE}(\Pi_1)}{\Gamma, L_1 \vdash R_1, \Delta'_1} \text{weak}_R \quad \dots \quad \frac{\text{NORMALIZE}(\Pi_n)}{\Gamma, L_n \vdash R_n, \Delta'_n} \text{weak}_R}{\Gamma, L \vdash R, \Delta'} X$$

where  $\Delta'$  is the smallest multiset containing all multisets  $\Delta'_i$

- Case 2: there exists a smallest premise  $i$  for which  $\mathbf{B}$  holds, i.e.  $R_i \not\subseteq \Delta_i$  and  $\Delta_i \subseteq \Delta$ . As  $R_i \neq \emptyset$ , either  $X$  is  $\Rightarrow_R$  or  $L_i = \emptyset$ .

If  $X$  is  $\Rightarrow_R$ , we define  $\text{NORMALIZE}(\Pi)$  as

$$\frac{\frac{\text{NORMALIZE}(\Pi_1)}{\Gamma, A \vdash \Delta_1} \text{weak}_R}{\Gamma \vdash A \Rightarrow B, \Delta_1} \Rightarrow_R$$

Else,  $L_i = \emptyset$  and we define  $\text{NORMALIZE}(\Pi)$  as

$$\frac{\text{NORMALIZE}(\Pi_i)}{\Gamma, L \vdash \Delta_i} \text{weak}_L$$

- Case 3: there exists a smallest premise  $i$  for which the case  $\mathbf{C}$  applies, i.e.  $R_i \not\subseteq \Delta_i$  and  $\Delta_i \not\subseteq \Delta$ . This only happens when  $|R_i| = 2$ , when exactly one proposition of  $R_i$  is in  $\Delta_i$ . In this case,  $X$  is either  $\text{contr}_R$  or  $\vee_R$ .

If  $X$  is  $\text{contr}_R$ , we can write  $R_1 = A, A$ , and  $\Delta_1 = (A, \Delta'_1)$  with  $\Delta'_1 \subseteq \Delta$ . We define  $\text{NORMALIZE}(\Pi)$  as  $\text{NORMALIZE}(\Pi_1)$ .

If  $X$  is  $\vee_R$ , we can write  $R_1 = A_0, A_1$ , and  $\Delta_1 = (A_k, \Delta'_1)$  with  $\Delta'_1 \subseteq \Delta$ .

We define  $\text{NORMALIZE}(\Pi)$  as

$$\frac{\frac{\text{NORMALIZE}(\Pi_1)}{\Gamma \vdash A_k, \Delta'_1} \text{weak}_R}{\Gamma \vdash A_0 \vee A_1, \Delta'_1} \vee_R$$

*Remark 4.* The nullary rules *axiom* and  $\perp_L$  having no premise, they match the first case.

**Definition 8.** We define a first constructivization algorithm `WEAK CONSTRUCT`, which

- takes as input a cut-free **LK** proof  $\frac{\Pi}{\Gamma \vdash (G)}$ ,
- computes the proof  $\frac{\text{NORMALIZE}(\Pi)}{\Gamma \vdash (G)} \text{weak}_R$ ,
- outputs its **LJ** interpretation if it exists and fails otherwise

## 5 A new constructive fragment

**Definition 9.** We define  $F$  as the fragment of mono-succedent sequents containing no negative occurrence of  $\vee$  and no positive occurrence of  $\Rightarrow$ .

**Theorem 3.** `WEAK CONSTRUCT` is complete on  $F$ : if  $\Pi$  is a cut-free **LK** proof of a sequent  $\Gamma \vdash (G) \in F$ , then `WEAK CONSTRUCT`( $\Pi$ ) succeeds.

*Proof.* By Lemma 1,  $F$  sequents are proved by cut-free **LK** proofs containing no  $\vee_L$  or  $\Rightarrow_R$  rule. We prove that for any such proof  $\Pi$ , `NORMALIZE`( $\Pi$ ) proves a mono-succedent sequent interpretable in **LJ**. This proof is done by induction on cut-free **LK** proofs containing no  $\vee_L$  or  $\Rightarrow_R$  rule, following the partition of cases and the notations introduced in the definition of `NORMALIZE`:

- Case 1: we split this case according to the rule  $X$ .
  - nullary rules: *axiom* and  $\perp_L$  are interpretable in **LJ**.
  - $\text{weak}_R$ : The result follows directly by induction hypothesis.
  - other unary rules: In these cases  $\Delta' = \Delta'_1$ , hence `NORMALIZE`( $\Pi$ ) is

$$\frac{\frac{\text{NORMALIZE}(\Pi_1)}{\Gamma, L_1 \vdash R_1, \Delta'_1} \text{weak}_R}{\Gamma, L \vdash R, \Delta'_1} X$$

By induction hypothesis, `NORMALIZE`( $\Pi_1$ ) is interpretable in **LJ**. Hence,  $|R_1| \leq 1$ , which ensures that  $X$  is neither  $\text{contr}_R$  nor  $\vee_R$ . All other unary rules lead to a proof interpretable in **LJ**, therefore the result is interpretable in **LJ**.

- $\vee_L$ : This case doesn't occur by hypothesis

- $\Rightarrow_L$ : By induction hypothesis,  $\text{NORMALIZE}(II_1)$  and  $\text{NORMALIZE}(II_2)$  are interpretable in  $\mathbf{LJ}$ , hence  $|R_1, \Delta'_1| \leq 1$ . As  $|R_1| = 1$ ,  $\Delta'_1 = \emptyset$ , and  $\Delta' = \Delta'_2$ .

$$\text{As } \frac{\frac{\Gamma \vdash A}{\Gamma \vdash A, \Delta'_2} \text{weak}_R \quad \frac{\Gamma, B \vdash \Delta'_2}{\Gamma, B \vdash \Delta'_2} \text{weak}_R}{\Gamma, A \Rightarrow B \vdash \Delta'_2} \Rightarrow_L$$

is interpretable as  $\frac{\Gamma \vdash A \quad \Gamma, B \vdash \Delta'_2}{\Gamma, A \Rightarrow B \vdash \Delta'_2} \Rightarrow_L$  in  $\mathbf{LJ}$ , the result follows.

- $\wedge_R$ : By induction hypothesis,  $\text{NORMALIZE}(II_1)$  and  $\text{NORMALIZE}(II_2)$  are interpretable in  $\mathbf{LJ}$ , hence  $|R_1, \Delta'_1| \leq 1$  and  $|R_2, \Delta'_2| \leq 1$ . As  $|R_1| = |R_2| = 1$ ,  $\Delta'_1 = \Delta'_2 = \emptyset$ . Therefore  $\Delta' = \emptyset$ , from which the result follows.

- Case **2**: By hypothesis,  $X$  is not  $\Rightarrow_R$ , hence  $\text{NORMALIZE}(II)$  is defined as

$$\frac{\text{NORMALIZE}(II_i)}{\frac{\Gamma \vdash \Delta_i}{\Gamma, L \vdash \Delta_i} \text{weak}_L}$$

The result follows by induction hypothesis.

- Case **3**: If  $X$  is  $\text{contr}_R$ , the result follows directly by induction hypothesis. Else,  $X$  is  $\vee_R$ . By induction hypothesis,  $\text{NORMALIZE}(II_1)$  is interpretable in  $\mathbf{LJ}$ , thus  $|A_k, \Delta'_1| \leq 1$ , and  $\Delta'_1 = \emptyset$ .

$$\text{As } \frac{\frac{\Gamma \vdash A_k}{\Gamma \vdash A_0, A_1} \text{weak}_R}{\Gamma \vdash A_0 \vee A_1} \vee_R \text{ is interpretable as } \frac{\Gamma \vdash A_k}{\Gamma \vdash A_0 \vee A_1} \vee_R \text{ in } \mathbf{LJ},$$

the result follows.

**Corollary 1.** *The fragment  $F$  is a constructive fragment of predicate logic: a sequent  $\Gamma \vdash (G)$  is classically provable iff it is constructively provable.*

## 6 The full constructivization algorithm

The previous algorithm `WEAK CONSTRUCT` was based on the reject of multi-succedent sequents. The idea leading to our main algorithm `CONSTRUCT` is to try to interpret multi-succedent sequents constructively as well. This interpretation is based on a new multi-succedent constructive system, which will be referred to as  $\mathbf{LI}$  in the following. As mentioned in the introduction, the constructivization algorithm `CONSTRUCT` comprises three steps: first the algorithm `NORMALIZE`, then a partial translation `ANNOTATE` from  $\mathbf{LK}$  proofs to  $\mathbf{LI}$  proofs, and finally a complete translation `INTERPRET` from  $\mathbf{LI}$  proof to  $\mathbf{LJ}$  proofs.

There are several ways to interpret multi-succedent sequents constructively. For instance,  $\Gamma \vdash \bigvee \Delta$  and  $\Gamma, \neg \Delta \vdash$  are two possible interpretations of a multi-succedent sequent  $\Gamma \vdash \Delta$ . These interpretation are equivalent classically but not constructively: for instance, the classical sequent  $\vdash A, \neg A$  is not provable constructively under the first interpretation, but it is provable constructively under the second one. As a consequence, some classical rules may be constructively valid or not according to the chosen interpretation of classical sequents.

The new system **LI** is built to benefit from the freedom left in the constructive interpretation of classical sequents. **LI** is designed as a sequent calculus based on **annotated sequents**, where the annotation will refer to the choice of constructive interpretation of the underlying classical sequent. We formalize first the notion of **annotated sequents**.

**Definition 10.** We define the set of **annotated sequents** as sequents of the shape  $\Gamma \vdash \Delta_1; \Delta_2$ .

We define the following interpretation INTERPRET on annotated sequents:  
 $\text{INTERPRET}(\Gamma \vdash \Delta_1; \Delta_2) = \Gamma, \neg \Delta_2 \vdash \bigvee \Delta_1$ .

In the following, this function will be extended from **LI** proofs to **LJ** proofs.

We define the following erasure function ERASE on annotated sequents:  
 $\text{ERASE}(\Gamma \vdash \Delta_1; \Delta_2) = \Gamma \vdash \Delta_1, \Delta_2$ .

In the following, this function will be extended from **LI** proofs to **LK** proofs.

Then, we define the system **LI** in the following way:

**Definition 11.** **LI** is based on the following rules:

$$\begin{array}{c}
\frac{}{\perp \vdash;} \quad \perp_L \frac{}{A \vdash A;} \text{ axiom}^1 \quad \frac{}{A \vdash; A} \text{ axiom}^2 \\
\\
\frac{\Gamma \vdash \Delta_1; \Delta_2}{\Gamma, \Gamma' \vdash \Delta_1; \Delta_2} \text{ weak}_L \quad \frac{\Gamma \vdash \Delta_1; \Delta_2}{\Gamma \vdash \Delta_1, \Delta'_1; \Delta_2, \Delta'_2} \text{ weak}_R \\
\\
\frac{\Gamma, A, A \vdash \Delta_1; \Delta_2}{\Gamma, A \vdash \Delta_1; \Delta_2} \text{ contr}_L \quad \frac{\Gamma \vdash A, A, \Delta_1; \Delta_2}{\Gamma \vdash A, \Delta_1; \Delta_2} \text{ contr}_R^1 \quad \frac{\Gamma \vdash \Delta_1; A, A, \Delta_2}{\Gamma \vdash \Delta_1; A, \Delta_2} \text{ contr}_R^2 \\
\\
\frac{\Gamma, A, B \vdash \Delta_1; \Delta_2}{\Gamma, A \wedge B \vdash \Delta_1; \Delta_2} \wedge_L \quad \frac{\Gamma \vdash A, \Delta_1; \Delta_2 \quad \Gamma \vdash B, \Delta_1; \Delta_2}{\Gamma \vdash A \wedge B, \Delta_1; \Delta_2} \wedge_R^1 \\
\\
\frac{\Gamma \vdash; A, \Delta_2 \quad \Gamma \vdash; B, \Delta_2}{\Gamma \vdash; A \wedge B, \Delta_2} \wedge_R^2 \quad \frac{\Gamma \vdash A, \Delta_1; \Delta_2 \quad \Gamma \vdash B, \Delta_1; \Delta_2}{\Gamma \vdash \Delta_1; A \wedge B, \Delta_2} \wedge_R^3, |\Delta_1| \geq 1 \\
\\
\frac{\Gamma, A \vdash \Delta_1; \Delta_2 \quad \Gamma, B \vdash \Delta_1; \Delta_2}{\Gamma, A \vee B \vdash \Delta_1; \Delta_2} \vee_L \\
\\
\frac{\Gamma \vdash A, B, \Delta_1; \Delta_2}{\Gamma \vdash A \vee B, \Delta_1; \Delta_2} \vee_R^1 \quad \frac{\Gamma \vdash \Delta_1; A, B, \Delta_2}{\Gamma \vdash \Delta_1; A \vee B, \Delta_2} \vee_R^2 \\
\\
\frac{\Gamma \vdash; A, \Delta_2 \quad \Gamma, B \vdash; \Delta_2}{\Gamma, A \Rightarrow B \vdash; \Delta_2} \Rightarrow_L^1 \quad \frac{\Gamma \vdash A, \Delta_1; \Delta_2 \quad \Gamma, B \vdash \Delta_1; \Delta_2}{\Gamma, A \Rightarrow B \vdash \Delta_1; \Delta_2} \Rightarrow_L^2, |\Delta_1| \geq 1
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma, A \vdash B; \Delta_2}{\Gamma \vdash A \Rightarrow B; \Delta_2} \Rightarrow_R^1 \quad \frac{\Gamma, A \vdash; B, \Delta_2}{\Gamma \vdash; A \Rightarrow B, \Delta_2} \Rightarrow_R^2 \\
\\
\frac{\Gamma, A[t/x] \vdash \Delta_1; \Delta_2}{\Gamma, \forall x A \vdash \Delta_1; \Delta_2} \forall_L \quad \frac{\Gamma \vdash A; \Delta_2}{\Gamma \vdash \forall x A; \Delta_2} \forall_R^1 \quad \frac{\Gamma \vdash A; \Delta_2}{\Gamma \vdash; \forall x A, \Delta_2} \forall_R^2 \\
\\
\frac{\Gamma, A \vdash \Delta_1; \Delta_2}{\Gamma, \exists x A \vdash \Delta_1; \Delta_2} \exists_L \quad \frac{\Gamma \vdash A[t/x], \Delta_1; \Delta_2}{\Gamma \vdash \exists x A, \Delta_1; \Delta_2} \exists_R^1 \quad \frac{\Gamma \vdash \Delta_1; A[t/x], \Delta_2}{\Gamma \vdash \Delta_1; \exists x A, \Delta_2} \exists_R^2
\end{array}$$

with the standard freshness constraints for the variables introduced in the rules  $\forall_R^i$  and  $\exists_L$ .

All **LI** rules correspond to a **LK** rule through the erasure of the premises and the conclusions. Hence, we can extend the ERASE function from **LI** rules to **LK** rules, and consequently from **LI** proofs to **LK** proofs.

In the same way, we would like to extend the INTERPRET function from **LI** proofs to **LJ** proofs. This can be done associating each **LI** rule to a partial **LJ** proof deriving the interpretation of its conclusion from the interpretation of its premises. However, such an approach would be heavy: as the disjunction in **LJ** is binary,  $\bigvee$  is based on a nesting of binary disjunctions, and a proposition in  $\Gamma \vdash \Delta_1; \Delta_2$  can occur deep in  $\Gamma, \neg \Delta_2 \vdash \bigvee \Delta_1$ . As INTERPRET will be part of the constructivization algorithm CONSTRUCT, we need to find another method to define it as a linear-time algorithm.

For this reason, we will define the interpretation of rules using the property that  $\Gamma \vdash \bigvee \Delta$  is constructively provable iff  $\Gamma, \Delta \Rightarrow G \vdash G$  is provable for any proposition  $G$ .

**Definition 12.** We define the function  $\text{INTERPRET}'(\cdot|G)$  on annotated sequents as  $\text{INTERPRET}'(\Gamma \vdash \Delta_1; \Delta_2|G) = (\Gamma, \Delta_1 \Rightarrow G, \neg \Delta_2 \vdash G)$ .

We extend  $\text{INTERPRET}'$  from **LI** rules to partial **LJ** derivations in the following way:

$$\text{From a } \mathbf{LI} \text{ rule } \frac{\Gamma^1 \vdash \Delta_1^1; \Delta_2^1 \quad \dots \quad \Gamma^n \vdash \Delta_1^n; \Delta_2^n}{\Gamma \vdash \Delta_1; \Delta_2} R$$

and a proposition  $G$ , we define a partial **LJ** derivation  $\text{INTERPRET}'(R|G)$  as a partial derivation of the form

$$\frac{\text{INTERPRET}'(\Gamma^1 \vdash \Delta_1^1; \Delta_2^1|G^1) \quad \dots \quad \text{INTERPRET}'(\Gamma^n \vdash \Delta_1^n; \Delta_2^n|G^n)}{\text{INTERPRET}'(\Gamma \vdash \Delta_1; \Delta_2|G)}$$

The **LI** system is designed to ensure that such definitions rely on simple constructive tautologies. As an illustration, we present here the case of the rule

$$\frac{\Gamma \vdash A, \Delta_1; \Delta_2 \quad \Gamma, B \vdash \Delta_1; \Delta_2}{\Gamma, A \Rightarrow B \vdash \Delta_1; \Delta_2} \Rightarrow_L^3$$

From a proposition  $G$ , defining  $\Sigma = \Gamma, \Delta_1 \Rightarrow G, \neg \Delta_2$ , we derive

$$\frac{\frac{\frac{}{\Sigma, A \vdash A} \text{ axiom}^* \quad \frac{\Sigma, B \vdash G}{\Sigma, B, A \vdash G} \text{ weak}_L}{\Sigma, A \Rightarrow B, A \vdash G} \Rightarrow_L}{\Sigma, A \Rightarrow B \vdash A \Rightarrow G} \Rightarrow_R \quad \frac{\frac{\Sigma, A \Rightarrow G \vdash G}{\Sigma, A \Rightarrow B, A \Rightarrow G \vdash G} \text{ weak}_L}{\Sigma, A \Rightarrow B \vdash G} \text{ cut}$$

where the two open premises correspond to  $\text{INTERPRET}'(\Gamma, B \vdash \Delta_1; \Delta_2 | G)$  and  $\text{INTERPRET}'(\Gamma \vdash A, \Delta_1; \Delta_2 | G)$  respectively.

*Remark 5.* In this case, we chose  $G_1 = G_2 = G$ . Other choices for  $G_i$  appear in the cases  $\wedge_R^2$ ,  $\Rightarrow_L^1$ ,  $\Rightarrow_R^2$ , and  $\forall_R^2$ .

In a second step, we extend  $\text{INTERPRET}'(\cdot | G)$  from **LI** proofs to **LJ** proofs recursively. Finally, we extend  $\text{INTERPRET}(\cdot)$  from **LI** proofs of sequents of the shape  $\Gamma \vdash (G)$ ; to **LJ** proofs:

- for  $\Pi$  a **LI** proof of a sequent  $\Gamma \vdash \perp$ ; , we define  $\text{INTERPRET}(\Pi)$  as

$$\frac{\frac{}{\Gamma, \perp \vdash} \perp_L^* \quad \frac{\text{INTERPRET}'(\Pi | \perp)}{\Gamma \vdash \perp} \text{ cut}}{\Gamma \vdash} \text{ cut}$$

- for  $\Pi$  a **LI** proof of a sequent  $\Gamma \vdash G$ ; , we define  $\text{INTERPRET}(\Pi)$  as

$$\frac{\frac{\text{INTERPRET}'(\Pi | G)}{\Gamma, G \Rightarrow G \vdash G} \quad \frac{\frac{}{\Gamma, G \vdash G} \text{ axiom}^*}{\Gamma \vdash G \Rightarrow G} \Rightarrow_R}{\Gamma \vdash G} \text{ cut}$$

**Definition 13.** We define the linear-time partial algorithm  $\text{ANNOTATE}(\cdot | \cdot)$  with, as inputs, a **LI** sequent  $S$  and a cut-free **LK** proof  $\Pi$  of  $\text{ERASE}(S)$  and, as output, either a **LI** proof of  $S$  or a failure. This annotation is done from the root to the leaves: at each step, the first argument  $S$  prescribes how the current conclusion must be annotated. The definition is recursive on the second argument.

$$\text{Describing } S \text{ as } \Gamma \vdash \Delta_1; \Delta_2 \text{ and } \Pi \text{ as } \frac{\frac{\Pi^1}{\Gamma^1 \vdash \Delta_1^1} \quad \dots \quad \frac{\Pi^n}{\Gamma^n \vdash \Delta_2^n}}{\Gamma \vdash \Delta_1, \Delta_2} R,$$

- If there exists a **LI** rule

$$\frac{\Gamma^1 \vdash \Delta_1^1; \Delta_2^1 \quad \dots \quad \Gamma^n \vdash \Delta_1^n; \Delta_2^n}{\Gamma \vdash \Delta_1; \Delta_2} R'$$

such that for all  $i$ ,  $\Delta_1^i, \Delta_2^i = \Delta^i$ , then the output is

$$\frac{\frac{\text{ANNOTATE}(\Gamma^1 \vdash \Delta_1^1; \Delta_2^1 | \Pi^1)}{\Gamma^1 \vdash \Delta_1^1; \Delta_2^1} \quad \dots \quad \frac{\text{ANNOTATE}(\Gamma^n \vdash \Delta_1^n; \Delta_2^n | \Pi^n)}{\Gamma^n \vdash \Delta_1^n; \Delta_2^n}}{\Gamma \vdash \Delta_1; \Delta_2} R'$$

– Else,  $\text{ANNOTATE}(\cdot, \cdot)$  fails.

*Remark 6.* The only failing cases appear when the rule  $R$  is either  $\Rightarrow_R$  or  $\vee_R$ , and exclusively for sequents  $\Gamma \vdash \Delta_1; \Delta_2$  such that  $|\Delta_1, \Delta_2| > 1$ .

**Definition 14.** We define the linear-time constructivization algorithm  $\text{CONSTRUCT}$ , which

- takes as input a cut-free  $\mathbf{LK}$  proof  $\Pi$  of a sequent  $\Gamma \vdash (G)$ ,
- computes the proof  $\Pi' = \frac{\text{NORMALIZE}(\Pi)}{\Gamma \vdash (G)} \text{weak}_R$ ,
- outputs  $\text{INTERPRET}(\text{ANNOTATE}(\Gamma \vdash (G); |\Pi'|))$  if it exists and fails otherwise.

*Example 1.* We consider the law of excluded middle  $A \vee \neg A$  given with the

following  $\mathbf{LK}$  proof:  $\frac{\frac{\overline{A \vdash A} \text{ axiom}}{\vdash A, \neg A} \Rightarrow_R}{\vdash A \vee \neg A} \vee_R$ . This proof is unchanged by  $\text{NORMALIZE}$ .

The  $\text{ANNOTATE}$  step fails as follows:  $\frac{\text{FAILURE}}{\vdash A, \neg A; \vdash A \vee \neg A} \vee_R^1$

*Example 2.* We consider a variant of the non contradiction of law of excluded

middle,  $(\neg(A \vee \neg A)) \Rightarrow B$ , given with the proof:  $\frac{\frac{\frac{\overline{A \vdash A, B} \text{ axiom}^*}{\vdash A, \neg A, B} \Rightarrow_R}{\vdash A \vee \neg A, B} \vee_R}{\frac{\overline{\perp \vdash B}}{\perp \vdash B} \perp_L^*}{\frac{\neg(A \vee \neg A) \vdash B}{\vdash (\neg(A \vee \neg A)) \Rightarrow B} \Rightarrow_R} \Rightarrow_L^*$

The result of  $\text{NORMALIZE}$  is  $\frac{\frac{\frac{\overline{A \vdash A} \text{ axiom}}{\vdash A, \neg A} \Rightarrow_R}{\vdash A \vee \neg A} \vee_R}{\frac{\overline{\perp \vdash}}{\perp \vdash} \perp_L}{\frac{\neg(A \vee \neg A) \vdash}{\neg(A \vee \neg A) \vdash B} \text{weak}_R} \Rightarrow_L$

Then, the result of  $\text{ANNOTATE}$  is  $\frac{\frac{\frac{\overline{A \vdash; A} \text{ axiom}^2}{\vdash; A, \neg A} \Rightarrow_R^2}{\vdash; A \vee \neg A} \vee_R^2}{\frac{\overline{\perp \vdash;}}{\perp \vdash;} \perp_L}{\frac{\neg(A \vee \neg A) \vdash;}{\neg(A \vee \neg A) \vdash B; \text{weak}_R} \text{weak}_R} \Rightarrow_L^1$

As  $\text{ANNOTATE}$  is the only step which may fail,  $\text{CONSTRUCT}$  succeeds on this example. We see on the example that the application of  $\text{NORMALIZE}$  was crucial for  $\text{ANNOTATE}$  to succeed.

**Theorem 4.** *CONSTRUCT is complete on  $F$ ,  $F_{Ku}$ , and  $F_{Ma}$ : for any proof  $\Pi$  of a sequent  $S$  in one of these fragments,  $\text{CONSTRUCT}(\Pi)$  succeeds.*

*Proof.* We consider  $F$ ,  $F_{Ku}$ , and  $F_{Ma}$  separately:

- For  $F$ : we consider a cut-free **LK** proof  $\Pi$  of a sequent  $\Gamma \vdash (G) \in F$ .

By Theorem 3,  $\Pi' = \frac{\text{NORMALIZE}(\Pi)}{\Gamma \vdash (G)} \text{weak}_R$  is interpretable in **LJ**.

As a consequence, the only multi-succedent sequents in  $\Pi'$  are conclusions of weakenings. As all failing cases (c.f. Remark 6) involve sequents  $\Gamma \vdash \Delta_1; \Delta_2$  such that  $|\Delta_1, \Delta_2| > 1$  which are conclusions of  $\Rightarrow_R$  or  $\forall_R$  rules, **ANNOTATE** succeeds. Hence, **CONSTRUCT** succeeds.

- For  $F_{Ku}$ : the result follows directly from a stronger assertion: for any cut-free **LK** proof  $\Pi$  of a sequent  $\Gamma \vdash \Delta$  containing no  $\forall_R$  rule,  $\text{ANNOTATE}(\Gamma \vdash; \Delta | \Pi)$  succeeds. This assertion is proved by induction on such sequents and proofs, noticing that all induction hypotheses refer to sequents of the shape  $\Gamma' \vdash; \Delta'$ .
- For  $F_{Ma}$ : we consider a cut-free **LK** proof  $\Pi$  of a sequent in  $F_{Ma}$ . As mentioned in Remark 6 the only failing cases involve the  $\Rightarrow_R$  or  $\forall_R$  rules, which don't occur in a proof of a sequent in  $F_{Ma}$ . Hence, **CONSTRUCT** succeeds.

## 7 Experimental results

In order to measure the success of **CONSTRUCT** in practice, experiments were made on the basis of **TPTP** [13] first-order problems. The classical theorem prover **Zenon** [10] was used to prove such problems. **Zenon** builds cut-free **LK** proofs internally. It was instrumented to use these internal proofs as inputs for an implementation of **WEAK CONSTRUCT** and **CONSTRUCT**. The **LJ** proofs obtained as outputs were expressed and checked in the constructive logical framework **Dedukti** [9].

A set of 724 **TPTP** problems was selected for the experimentations, corresponding to all **TPTP** problems in the category **FOF** which could be proved in less than 1 second using the uninstrumented version of **Zenon**. The results are the following:

- **WEAK CONSTRUCT** led to constructive proofs in 51% of tested cases.
- **CONSTRUCT** led to constructive proofs in 85% of tested cases (including all **WEAK CONSTRUCT** successes).

All constructive proofs generated were successfully checked using **Dedukti**. Among all cases where **CONSTRUCT** failed, 35% are proved to be invalid constructively using the constructive theorem prover **ileanCoP** [12].

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