Output-only subspace and transfer matrix-based damage localization and quantification

Md Delwar Hossain Bhuyan, Eva Viefhues, Michael Döhler, Yann Lecieux, Laurent Mevel, Falk Hille, Franck Schoefs

IOMAC'17

7th International Operational Modal Analysis Conference
2017 May10-12 Ingolstadt - Germany

OUTPUT-ONLY SUBSPACE AND TRANSFER MATRIX- BASED DAMAGE LOCALIZATION AND QUANTIFICATION

Md Delwar H. Bhuyan[1,], Eva Viefhues[2,*], Michael Döhler[1], Yann Lecieux[3], Laurent Mevel[1], Falk Hille[2], Franck Schoefs[3]*

[1] Inria/IFSTTAR, I4S, Rennes, France, {md-delwar-hossain.bhuyan, michael.doehler, laurent.mevel}@inria.fr

[2] BAM Federal Institute of Materials Research and Testing, Berlin, Germany, {eva.viefhues, falk.hille}@bam.de

[3] GeM, Université de Nantes, France, {yann.lecieux, franck.schoefs}@univ-nantes.fr

[*] PhD students

ABSTRACT

This paper deals with vibration-based damage localization and quantification from output-only measurements. We describe an approach which operates on a data-driven residual vector that is statistically evaluated using information from a finite element model, without updating the parameters of the model. First, the damaged elements are detected in statistical tests, and second, the damage is quantified only for the damaged elements. We propose a new residual vector in this context that is based on the transfer matrix difference between reference and damaged states, and compare it with a previously introduced subspace-based residual. We show localization and quantification on both residuals in simulations.

Keywords: damage localization, damage quantification, hypothesis tests

1. INTRODUCTION

The detection, localization and quantification of damages based on measured vibration data are fundamental tasks for structural health monitoring (SHM) to allow an automated damage diagnosis [1]. We consider the case of output-only vibration measurements of a structure subject to ambient excitation. Damages are considered as changes in the structural stiffness.

The problem of vibration-based damage assessment is considered by many methods in the literature. For example, purely data-driven methods for damage localization [2] are designed for particular structural types (beams, plates, rotating machinery, ...), but are in general not easily generalizable to arbitrary structural types, and often do not include damage quantification. Model-based methods update the parameters of a finite element (FE) model from the reference state of the structure such that the dynamic response from the data of the damaged state is reproduced. Comparing the updated stiffness matrices with the original ones provides damage location and extent [3]. While model updating-based approaches are in principle applicable to arbitrary structures, they are often too poorly conditioned to be successful in practice since the parameter size of FE models is usually much larger than the number of identified parameters from measurements, leading to an ill-posed problem [4].

In this paper we consider an approach that operates on a data-driven residual vector that is statistically evaluated using information from a FE model [5], combining elements of both data-driven and model-based methodologies. In this approach, the problems of damage localization and quantification are divided into two separate problems. First, the damaged elements are detected in statistical tests, and second, the damage is quantified only for the damaged elements. We propose a residual vector that is based on the transfer matrix difference between reference and damaged states, as in the SDDLV approach [6–9], and compare to a subspace-based residual vector used in previous works [5, 10].

This paper is organized as follows. In Section 2., the underlying vibration models are recalled. In Section 3., the damage localization and quantification framework is detailed and the new transfer matrix-based residual function is derived. Localization and quantification is then applied in simulations in Sections 4. and 5..

2. MODELS

The behavior of a mechanical structure is assumed to be described by a linear time-invariant (LTI) dynamic system

$$M\ddot{\mathcal{X}}(t) + C\dot{\mathcal{X}}(t) + K\mathcal{X}(t) = f(t) \tag{1}$$

where M, C, $K \in \mathbb{R}^{d\times d}$ are the mass, damping and stiffness matrices, respectively, t indicates continuous time and $\mathcal{X} \in \mathbb{R}^d$ denotes the displacements at the d degrees of freedom (DOF) of the structure. The external force $f(t)$ is not measurable and modeled as white noise. Let the dynamic system (1) be observed at r coordinates. Since $f(t)$ is unmeasured, it can be substituted with a fictive force $e(t) \in \mathbb{R}^r$ acting only in the measured coordinates and that regenerates the measured output. Furthermore, defining $x = [\mathcal{X} \;\; \dot{\mathcal{X}}]^T$, this leads to the corresponding continuous-time state-space model

$$\begin{cases} \dot{x}(t) = A_c x(t) + B_c e(t) \\ y(t) = C_c x(t) + D_c e(t) \end{cases} \tag{2}$$

with state vector $x \in \mathbb{R}^n$, output vector $y \in \mathbb{R}^r$, the state transition matrix $A_c \in \mathbb{R}^{n\times n}$ and output matrix $C_c \in \mathbb{R}^{r\times n}$, where $n = 2d$ is the system order and r is the number of outputs. Since the input of the system is replaced by the fictive force $e \in \mathbb{R}^r$, the input influence matrix and direct transmission matrix are of size $B_c \in \mathbb{R}^{n\times r}$ and $D_c \in \mathbb{R}^{r\times r}$ respectively. However, only the system matrices A_c and C_c are relevant from output-only system identification, and the non-identified matrices B_c and D_c are only relevant in the derivation of estimates related to the transfer matrix. From Stochastic Subspace Identification (SSI) [11, 12], estimates $\widehat{A}_c$ and $\widehat{C}_c$ can be obtained from output only measurements, based on the respective discrete-time state-space model

$$\begin{cases} x_{k+1} = A_d x_k + B_d e_k \\ \quad y_k = C_d x_k + D_d e_k \end{cases}, \tag{3}$$

which results from sampling system (2) at time steps $t = k\tau$ where τ is the time step.

3. DAMAGE LOCALIZATION AND QUANTIFICATION

3.1. Framework

Let $\theta \in \mathbb{R}^l$ be a parameter vector that describes the monitored system in the current state, and $\theta_0 \in \mathbb{R}^l$ its value in the healthy reference system. For damage localization and quantification we assume that damage is linked to stiffness changes. In this case, assume that θ is the collection of stiffness parameters of the elements of the structure, where θ_0 is obtained from a finite element model. For example, the components of θ can be the stiffnesses of a mass-spring chain system, Young modulus of beam elements or it can be basically any quantity linked to damage-sensitive properties of the system.

In [5], a statistical framework has been set up for Gaussian residual vectors parametrized by θ with the purpose to decide which parts of θ have changed (for damage localization) and then to estimate this change (for damage quantification). The Gaussian residual vector $\zeta \in \mathbb{R}^h$ is computed from the measurements of the system and needs to satisfy

$$\zeta \sim \begin{cases} \mathcal{N}(0,\Sigma) & \text{in reference state} \\ \mathcal{N}(\mathcal{J}\delta,\Sigma) & \text{in damaged state,} \end{cases} \tag{4}$$

with $\delta = \sqrt{N}(\theta - \theta_0) \in \mathbb{R}^l$ is the unknown change in parameter vector, N is the data length used for the computation of ζ, the sensitivity matrix $\mathcal{J} \in \mathbb{R}^{h\times l}$ has full column rank and the residual covariance matrix $\Sigma \in \mathbb{R}^{h\times h}$ is positive definite.

Before introducing two explicit residual functions, the statistical tests and estimators for damage localization and quantification are recalled.

3.2. Damage localization tests

For damage localization it has to be decided which parts of vector δ are non-zero, i.e. which parts of the parameter vector are changed. The structural elements corresponding to the changed parameters are thus damaged. To this end, each component of δ will be tested one after another. Denote the component to be tested by δ_a, and the remaining components by δ_b, such that

$$\delta = \begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix}. \tag{5}$$

Then, $\delta_a = 0$ is tested against $\delta_a \neq 0$. Following (5), the sensitivity matrix $\mathcal{J}$ and the Fisher information matrix $F = \mathcal{J}^T\Sigma^{-1}\mathcal{J}$ are analogously arranged as

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_a \; \mathcal{J}_b \end{bmatrix}, F = \begin{bmatrix} F_{aa} & F_{ab} \\ F_{ba} & F_{bb} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_a^T\Sigma^{-1}\mathcal{J}_a & \mathcal{J}_a^T\Sigma^{-1}\mathcal{J}_b \\ \mathcal{J}_b^T\Sigma^{-1}\mathcal{J}_a & \mathcal{J}_b^T\Sigma^{-1}\mathcal{J}_b \end{bmatrix}. \tag{6}$$

Sensitivity tests Assuming that $\delta_b = 0$ for testing $\delta_a = 0$ against $\delta_a \neq 0$, the generalized likelihood ratio (GLR) test follows as

$$t_{sens} = \zeta^T\Sigma^{-1}\mathcal{J}_a^T(\mathcal{J}_a^T\Sigma^{-1}\mathcal{J}_a)^{-1}\mathcal{J}_a^T\Sigma^{-1}\zeta, \tag{7}$$

which is called sensitivity test. The test statistic t_{sens} is χ^2 distributed with non-centrality parameter $\delta_a^T F_{aa}\delta_a$. For making decision about the damage location, the test variable is compared to a threshold.

Minmax tests Instead of assuming the components of $\delta_b = 0$, the variable δ_b is substituted by its least favorable value for making a decision about δ_a, which leads to the minmax test as follows. Define the partial residuals

$$\zeta_a = \mathcal{J}_a^T\Sigma^{-1}\zeta \tag{8a}$$
$$\zeta_b = \mathcal{J}_b^T\Sigma^{-1}\zeta, \tag{8b}$$

and the robust residual

$$\zeta_a^* = \zeta_a - F_{aa}F_{bb}^{-1}\zeta,$$

whose mean is sensitive to changes δ_a but not to δ_b. Testing $\delta_a = 0$ against $\delta_a \neq 0$ with the GLR test yields

$$t_{mm} = \zeta_a^* F_a^{*-1}\zeta_a^*, \tag{9}$$

where $F_a^* = F_{aa} - F_{ab}F_{bb}^{-1}F_{ba}$. The test statistic t_{mm} is χ^2 distributed with non-centrality parameter $\delta_a^T F_a^*\delta_a$.

3.3. Estimators for damage quantification

When damage is localized in the first step, the task for damage quantification is to estimate δ_a for the damaged components in the second step. Then, the parameter change follows as $\theta - \theta_0 = \delta/\sqrt{N}$.

Sensitivity approach An estimate of δ_a can be derived from the residual vector ζ as

$$\hat{\delta}_a^{sens} = (\mathcal{J}_a^T \Sigma^{-1} \mathcal{J}_a)^{-1} \mathcal{J}_a^T \Sigma^{-1} \zeta, \tag{10}$$

where $\hat{\delta}_a^{sens} \sim \mathcal{N}(\delta_a, F_{aa}^{-1})$ for the assumption $\delta_b = 0$.

Minmax approach Similarly, an estimate of the δ_a can be derived based on minmax approach as

$$\hat{\delta}_a^{mm} = F_a^{*-1} \zeta_a^*, \tag{11}$$

with $\hat{\delta}_a^{mm} \sim \mathcal{N}(\delta_a, F_a^{*-1})$.

3.4. Transfer matrix-based residual

Inspired by the SDDLV damage localization approach [6–9], where the null space of the transfer matrix difference between reference and damaged states is evaluated based on mechanical properties, we define a similar residual and evaluate it in the statistical framework above.

The transfer matrix of system (2) is defined as

$$G(s) = C_c(sI - A_c)^{-1} B_c + D_c \in \mathbb{C}^{r \times r},$$

but it cannot be estimated from output-only measurements since matrices B_c and D_c cannot be estimated. However, under the condition that the system order satisfies $n \leq 2r$, i.e. the number m of identified modes satisfies $m \leq r$, it holds [6]

$$G(s) = R(s) D_c + D_c,$$

where

$$R(s) = C_c(sI - A_c)^{-1} \begin{bmatrix} C_c A_c \\ C_c \end{bmatrix}^{\dagger} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{12}$$

In (12), I is the identity matrix of size $r \times r$, 0 is the zero matrix of size $r \times r$, and † denotes the Moore-Penrose pseudoinverse. Matrix $R(s)$ can be estimated from output-only measurements. Denote matrices in the damaged state with tilde, and matrices in the reference state without tilde. Assume that damage is due to changes in stiffness and mass is constant. Then $D_c = \tilde{D}_c$, and the matrix differences $\tilde{G}(s) - G(s)$ and $\tilde{R}(s)^T - R(s)^T$ are identical up to the multiplication by an invertible matrix.

Define the real-valued vector for any vector q as

$$q_{\text{re}} \overset{\text{def}}{=} \begin{bmatrix} \Re(q) \\ \Im(q) \end{bmatrix},$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary parts, respectively. Denote $\text{vec}(\cdot)$ as the column stacking vectorization operator. Now, consider the Taylor expansion of $\text{vec}(\tilde{R}(s)^T)_{\text{re}}$, corresponding to θ, in $\text{vec}(R(s)^T)_{\text{re}}$, corresponding to θ_0:

$$\text{vec}(\tilde{R}(s)^T)_{\text{re}} \approx \text{vec}(R(s)^T)_{\text{re}} + \mathcal{J}_{R,\theta}(\theta - \theta_0), \tag{13}$$

where $\mathcal{J}_{R,\theta} = \frac{\partial \text{vec}(R(s)^T)_{\text{re}}}{\partial \text{vec}(\theta)}\Big|_{\theta=\theta_0}$ is the sensitivity matrix, which is detailed in the following. It follows

$$\sqrt{N} \text{vec}(\tilde{R}(s)^T - R(s)^T)_{\text{re}} \approx \mathcal{J}_{R,\theta} \sqrt{N}(\theta - \theta_0), \tag{14}$$

and thus

$$\sqrt{N}\text{vec}(\tilde{R}(s)^T - R(s)^T)_{\text{re}} \approx \mathcal{J}_{R,\theta}\, \delta. \tag{15}$$

Hence, the residual defined by

$$\zeta^t \stackrel{\text{def}}{=} \sqrt{N}\text{vec}(\tilde{R}(s)^T - R(s)^T)_{\text{re}} \tag{16}$$

satisfies the distribution property (4). Note that asymptotic normality follows from the estimation of the modal parameters from subspace identification that are used in the computation of $R(s)$ and $\tilde{R}(s)$, as detailed in [8]. The covariance of the quantity $\text{vec}(\tilde{R}(s)^T - R(s)^T)_{\text{re}}$ is derived in detail in [8], as well as the sensitivity of $\text{vec}(R(s))$ with respect to the modal parameters. To obtain the required sensitivity matrix $\mathcal{J}_{R,\theta}$, the derivative of the modal parameters with respect to the structural parameters is needed in addition, which is described in detail in [13].

Note that the transfer matrix can be evaluated for several Laplace variables s_i, $i = 1, \ldots, n_s$, as well as for different mode sets $\mathcal{M}_j$, $j = 1, \ldots, n_m$ to aggregate information [8, 9]. The evaluation for different mode sets is particularly useful when less sensors than identified modes are present, since then the constraint $m \leq r$ becomes active and not all modes could be used in the computation. Instead, the transfer matrix estimates $R^j(s)$ can be computed on different mode sets $\mathcal{M}_j$ in this case, where the number of modes in each set satisfies this constraint. Then, the residual can be defined as

$$\zeta^t \stackrel{\text{def}}{=} \begin{bmatrix} \text{vec}(\tilde{R}^1(s_1)^T - R^1(s_1)^T)_{\text{re}} \\ \vdots \\ \text{vec}(\tilde{R}^1(s_{n_s})^T - R^1(s_{n_s})^T)_{\text{re}} \\ \vdots \\ \text{vec}(\tilde{R}^{n_m}(s_{n_s})^T - R^{n_m}(s_{n_s})^T)_{\text{re}} \end{bmatrix}. \tag{17}$$

The respective sensitivity matrices $\mathcal{J}_{R,\theta}$ are stacked analogously, and the joint covariance is detailed in [9].

3.5. Subspace-based residual

In previous works on damage detection and localization [5, 10], the subspace-based residual function

$$\zeta^s \stackrel{\text{def}}{=} \sqrt{N}\, \text{vec}(S^T \widehat{\mathcal{H}}) \tag{18}$$

has been introduced, where $\widehat{\mathcal{H}}$ is an estimate of the block Hankel matrix of the output covariances of the current system (3)

$$\mathcal{H} \stackrel{\text{def}}{=} \begin{bmatrix} R_1 & R_2 & \ldots & R_q \\ R_2 & R_3 & \ldots & R_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p+1} & R_{p+2} & \ldots & R_{p+q} \end{bmatrix}, \quad R_i = \mathbf{E}(y_k y_{k-i}^T), \quad \hat{R}_i = \frac{1}{N}\sum_{k=1}^{N} y_k y_{k-i}^T$$

and S is the left null space of $\mathcal{H}$ from the reference system. It can be obtained from a Hankel matrix $\mathcal{H}_0$ in the reference state through a singular value decomposition

$$\mathcal{H}_0 = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \approx 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

as $S = U_2$. When the system is in the reference state, the expected value of the product $S^T \widehat{\mathcal{H}}$ is zero, and when the system is damaged the product deviates from zero. The residual in (18) satisfies relation (4) asymptotically, and its sensitivity and covariance are given in detail in [10].

4. APPLICATION: MASS SPRING CHAIN

In the first numerical application, a damped mass-spring chain system is considered with 6 degrees of freedom (Figure 1). The stiffness parameters are defined as $k_1 = k_3 = k_5 = 4000$, $k_2 = k_4 = k_6 = 2000$, and the mass of all elements is 1. Damping is defined such that the damping ratio of each mode is 2%. Damage is introduced in element 4 by decreasing the stiffness in different steps. For damaged and undamaged states, acceleration data of length $N = 50{,}000$ has been generated from collocated white noise excitation using three sensors at elements 2, 4 and 6 with sampling frequency of 50 Hz. White measurement noise with 5% magnitude of the standard deviation of each output was added.

In the subspace-based approach, no modal identification on the test datasets is necessary. For the transfer matrix-based approach, all 6 modes were identified from the generated data of the structure using covariance-driven subspace identification, and the identified modes are split into two mode sets namely $\mathcal{M}_1$ and $\mathcal{M}_2$ of 3 modes each.

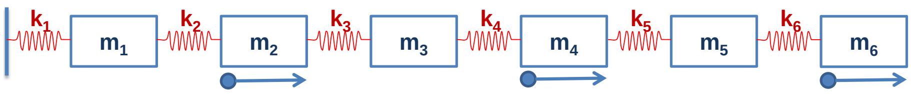
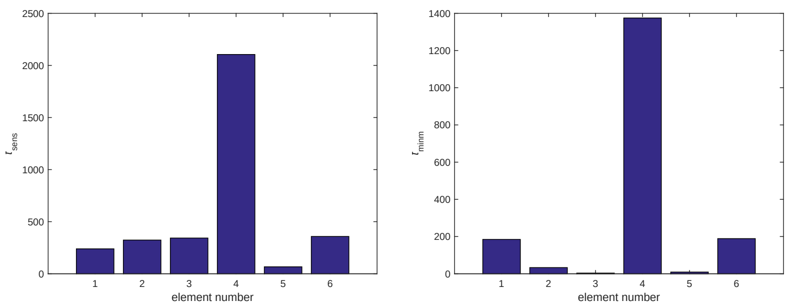


Figure 1: Mass-spring chain system (6 DOFs), three sensors.

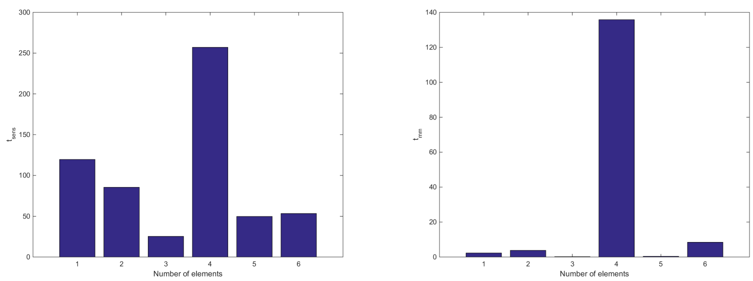
4.1. Localization and quantification from one dataset

First, the localization test statistics at all elements are computed using one dataset in both damaged and healthy states, where the damaged element is simulated by decreasing stiffness by 10% of its original value. Note that the highest value of the test statistic indicates damage localization. In Figures 2 and 3, the damage indicators are shown for each structural element for the subspace-based and transfer-matrix



(a) Sensitivity tests **(b)** Minmax tests

Figure 2: Subspace-based localization for mass-spring chain.



(a) Sensitivity tests **(b)** Minmax tests

Figure 3: Transfer matrix-based localization for mass-spring chain.

based residuals. In all cases, the test statistic at the damaged element has the highest value, correctly localizing the damage at element 4. In particular in Figures 2a and 3a it can be seen that the sensitivity test also reacts slightly at the undamaged elements due to the violation of $\delta_b = 0$. Though there is also some light reaction in the undamaged elements in the minmax tests in Figures 2b and 3b, its performance is much better than the sensitivity test. For the quantification of the damage extent, the values of $\hat{\delta}_{sens}$ and $\hat{\delta}_{mm}$ were estimated, leading to the parameter change of 15.0% and 11.7% for the respective sensitivity and minmax approaches with the subspace-based method, and of 10.3% and 10.1% with the transfer matrix-based method.

4.2. Damage quantification for several damage extents and datasets

Based on 100 simulated datasets, respectively for different damage extents between 5% and 30% damage in element 4, the mean and standard deviation of the estimated damage extent have been calculated. The results are shown in Figures 4 and 5 for several damage cases in both the sensitivity and the minmax approaches for the subspace-based and transfer matrix-based approaches, respectively. From the results, it can be seen that the damage is sometimes underestimated and sometimes overestimated, but always – at least roughly – in the order of the expected values. The subspace-based approach shows smaller uncertainties than the transfer matrix-based approach, while the mean values seem to be more accurate in the transfer matrix-based approach. In both approaches, results with the sensitivity approach are slightly more accurate than with the minmax approach.

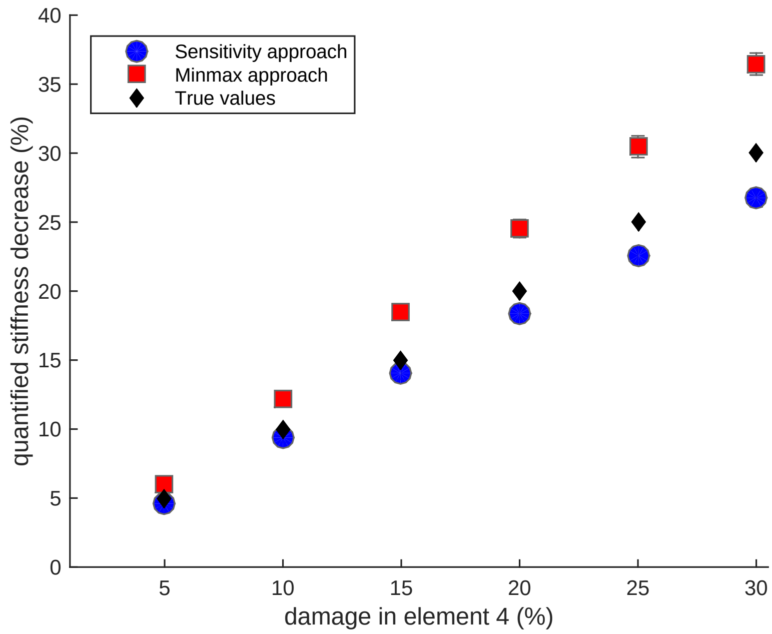


Figure 4: Quantification of different damage extents with subspace-based approach.

5. APPLICATION: CANTILEVER BEAM

In a second study, a 2D Beam model has been considered (see Figure 6) for damage localization and quantification. The structure is modeled with 5 beam elements of total length 1 m. The beam elements are circular with external diameter of 0.02 m. The mass density, Young modulus (E) and Poisson ratio are 7800 kg.m^{-3}, 207 GPa and 0.3, respectively. The total number of degrees of freedom of the structure is 15. Damping is defined such that the damping ratio of all modes is 1%. Damaged is modeled in element 3 by decreasing Young and Shear modulus in different extents. For the damaged and undamaged states,

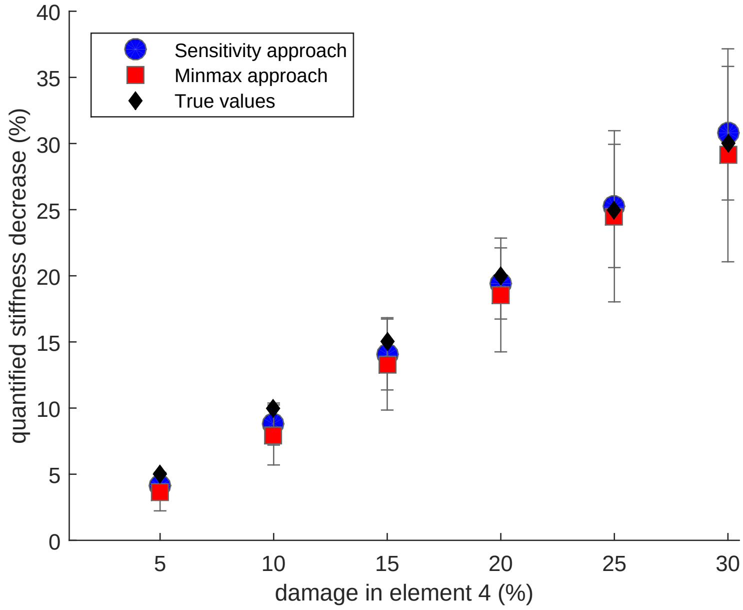


Figure 5: Quantification of different damage extents with transfer matrix-based approach.

the acceleration data length for each simulated set is N = 25,000, generated from collocated white noise excitation using five sensors in the Y-direction at each node with sampling frequency of 3125 Hz, and 5% white noise was added to the output data. For the transfer matrix-based approach, the first five modes of the structure were identified with subspace identification.

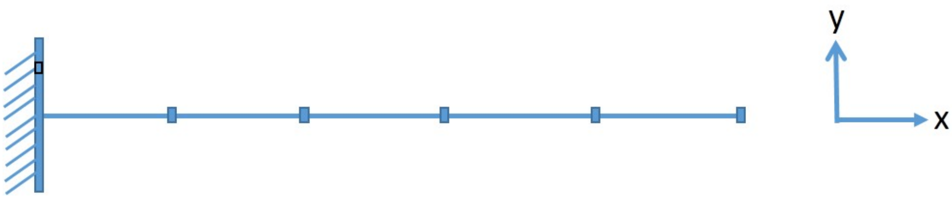
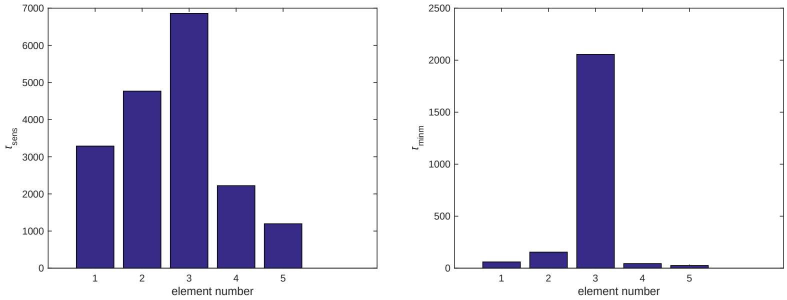


Figure 6: 2D Model with beam elements (15 DOFs)

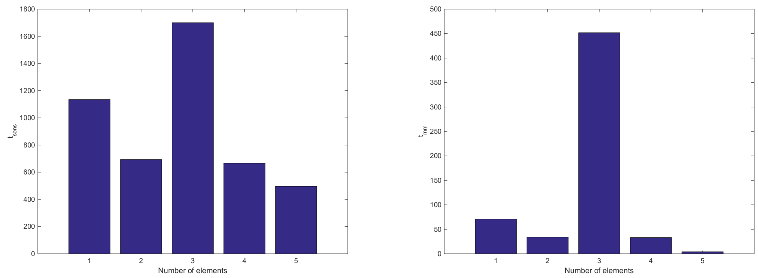
5.1. Damage localization and quantification for one dataset

Analogously as in the previous application, damage localization results are shown for one test case at all elements. Damage is simulated by decreasing stiffness of element 3 by 20%. From the sensitivity and minmax tests in Figures 7 and 8, it is seen that both the subspace-based and the transfer matrix-based approaches show a similar performance. The damaged element is correctly located at element 3. In comparison to the mass-spring chain, the reaction of the sensitivity test is much stronger now at the undamaged elements, while the minmax test performs very well. For the damage quantification, the values $\hat{\delta}_{sens}$ and $\hat{\delta}_{mm}$ were estimated, leading to an estimated parameter change of 20.5% and 20.7% in the damaged element for the subspace-based approach, and of 22.8% and 18.6% for the transfer matrix-based approach.



(a) Sensitivity tests **(b)** Minmax tests

Figure 7: Subspace-based localization for beam.



(a) Sensitivity tests **(b)** Minmax tests

Figure 8: Transfer matrix-based localization for beam.

5.2. Damage quantification for several damage extents

Based again on 100 simulated datasets, respectively for different damage extents between 5% and 30% damage in element 3, damage quantification results are shown in Figures 9 and 10 for the subspace-

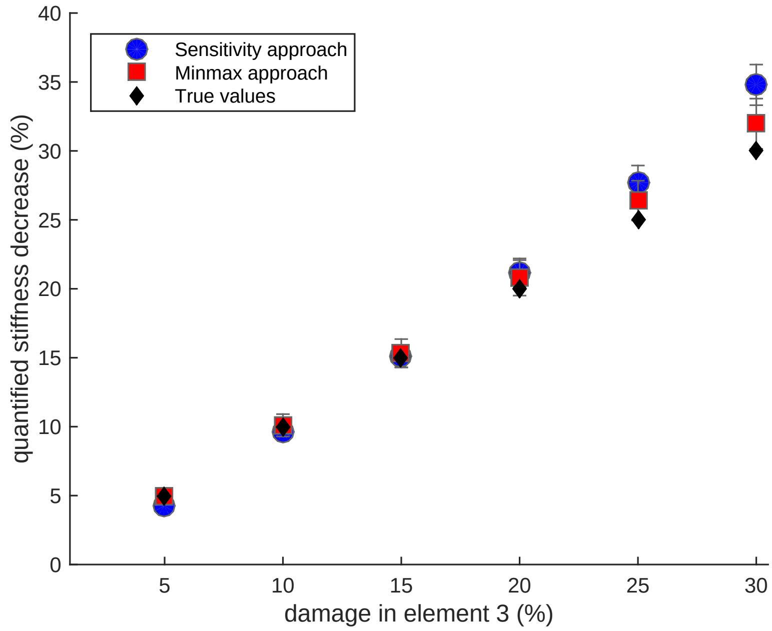


Figure 9: Quantification of different damage extents with subspace-based approach.

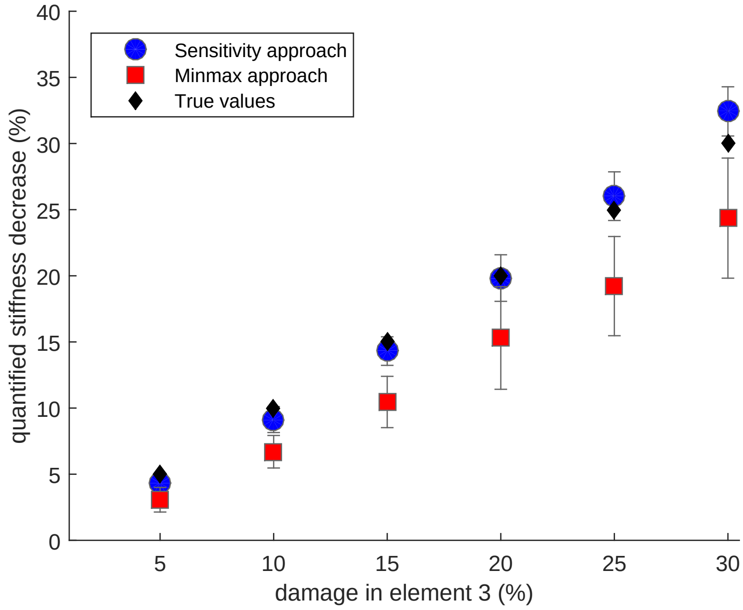


Figure 10: Quantification of different damage extents with transfer matrix-based approach.

based and transfer matrix-based approaches. In this case, the subspace-based approach seems to be more accurate than the transfer matrix-based approach and deviates only for large damages from the expected values. In the transfer matrix-based approach, the sensitivity approach overestimates and the minmax approach underestimates the damage extents. Figures 9 and 10 show that the error increases for large damage extents which can be expected since the sensitivity matrix is computed in the reference state and is thus not accurate anymore for large changes. Further error sources are modal truncation since only five out of 15 modes are taken into account in the sensitivity computation.

6. CONCLUSION

In this paper, two residuals were presented for a statistical output-only damage localization and quantification approach in a Gaussian framework. While the transfer matrix-based residual has interesting properties and relations to the SDDLV damage localization approach, its performance in this framework was not as good as the previously introduced subspace-based residual. Further investigation of the reasons behind this phenomenon are part of future research.

REFERENCES

[1] C.R. Farrar and K. Worden. An introduction to structural health monitoring. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 365(1851):303–315, 2007.

[2] W. Fan and P. Qiao. Vibration-based damage identification methods: a review and comparative study. *Structural Health Monitoring*, 10(1):83–111, 2011.

[3] J. M. W. Brownjohn, P.-Q. Xia, H. Hao, and Y. Xia. Civil structure condition assessment by FE model updating: methodology and case studies. *Finite Elements in Analysis and Design*, 37(10):761–775, 2001.

[4] M.I. Friswell. Damage identification using inverse methods. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 365(1851):393–410, 2007.

[5] M. Döhler, L. Mevel, and Q. Zhang. Fault detection, isolation and quantification from Gaussian residuals with application to structural damage diagnosis. *Annual Reviews in Control*, 42:244–256, 2016.

[6] D. Bernal. Load vectors for damage location in systems identified from operational loads. *Journal of Engineering Mechanics*, 136(1):31–39, 2010.

[7] M. Döhler, L. Marin, D. Bernal, and L. Mevel. Statistical decision making for damage localization with stochastic load vectors. *Mechanical Systems and Signal Processing*, 39(1-2):426–440, 2013.

[8] L. Marin, M. Döhler, D. Bernal, and L. Mevel. Robust statistical damage localization with stochastic load vectors. *Structural Control and Health Monitoring*, 22(3):557–573, 2015.

[9] M.D.H. Bhuyan, M. Döhler, and L. Mevel. Statistical damage localization with stochastic load vectors using multiple mode sets. In *Proc. 8th European Workshop on Structural Health Monitoring*, Bilbao, Spain, 2016.

[10] É. Balmès, M. Basseville, L. Mevel, H. Nasser, and W. Zhou. Statistical model-based damage localization: a combined subspace-based and substructuring approach. *Structural Control and Health Monitoring*, 15(6):857–875, 2008.

[11] B. Peeters and G. De Roeck. Reference-based stochastic subspace identification for output-only modal analysis. *Mechanical Systems and Signal Processing*, 13(6):855–878, 1999.

[12] M. Döhler and L. Mevel. Fast multi-order computation of system matrices in subspace-based system identification. *Control Engineering Practice*, 20(9):882–894, 2012.

[13] W. Heylen, S. Lammens, and P. Sas. *Modal Analysis Theory and Testing*. Katholieke Universiteit Leuven, Belgium, 1998.