Three examples of the stability properties of the invariant extended Kalman filter

Axel Barrau, Silvere Bonnabel

Three examples of the stability properties of the invariant extended Kalman filter ⋆

A. Barrau * **S. Bonnabel** **

* *SAFRAN TECH, Groupe Safran, Rue des Jeunes Bois - Châteaufort, 78772 Magny Les Hameaux CEDEX.*
** *MINES ParisTech, PSL Reasearch University, Centre for Robotics, 60 bd Saint-Michel, 75006 Paris, France.*

Abstract: In the aerospace industry the (multiplicative) extended Kalman filter (EKF) is the most common method for sensor fusion for guidance and navigation. However, from a theoretical point of view, the EKF has been shown to possess local convergence properties only under restrictive assumptions. In a recent paper, we proved a slight variant of the EKF, namely the invariant extended Kalman filter (IEKF), *when used as a nonlinear observer*, possesses local convergence properties under the same assumptions as those of the linear case, for a class of systems defined on Lie groups. This is especially interesting as the IEKF also retains all the desirable features of the standard EKF, especially its relevant tuning in the presence of noises. In the present paper we provide three examples of engineering interest where the theory is shown to apply, yielding three EKF-like algorithms with guaranteed local convergence properties. Beyond those contributions, the present article is sufficiently accessible to help the practitioner understand through concrete examples the general IEKF theory, and to provide him with guidelines for the design of IEKFs.

Keywords: Estimation, Kalman filtering, nonlinear systems, Lie groups, navigation.

1. INTRODUCTION

In the aerospace industry the (multiplicative) extended Kalman filter (EKF) is the most popular method for sensor fusion for guidance and navigation. However, from a theoretical point of view, the EKF has been shown to possess local convergence properties only under restrictive assumptions Song and Grizzle (1995); Krener (2003), and as a matter of fact it can actually diverge, even for small initial errors. The recent paper Barrau and Bonnabel (2017) proves a slight variant of the EKF, namely the invariant extended Kalman filter (IEKF), possesses local convergence properties under highly reasonable assumptions for a well characterized class of systems defined on Lie groups. The IEKF was originally introduced in Bonnabel (2007); Bonnabel et al. (2009b), and builds upon the theory of symmetry-preserving observers Bonnabel et al. (2009a). It can also be related to the generalized multiplicative EKF of Martin and Salaün (2010), the discrete EKF on Lie groups Bourmaud et al. (2013), see also de Ruiter and Forbes (2016).

The principles of the IEKF theory as presented in Barrau and Bonnabel (2017) are not easy to grasp. The main goal of the present paper is thus to provide a user friendly presentation and discussion of the IEKF as described in Barrau and Bonnabel (2017), and to illustrate its stability properties on three examples of engineering interest. Even though the purpose of the present article is essentially tutorial, it also contains novel theoretical results as we derive three non-linear filters for three examples of engineering interest, and guarantee stability of the derived filters.

Those examples could certainly be tackled through non-linear observers, along the lines of e.g., Hua et al. (2014); Wolfe et al. (2011); Batista et al. (2014); Izadi and Sanyal (2014); Sanyal and Nordkvist (2012); Zamani et al. (2014); Lee et al. (2007); Hua et al. (2016); Tayebi et al. (2007), and (almost) global convergence properties could be - or have already been - obtained. The interest (and difference) of our approach with respect to the non-linear observer literature though, is that the three non-linear proposed filters (namely IEKFs) 1- accomodate discrete time measurements with arbitrary and varying sampling times, 2- the gain tuning matches the modeled variance of the noises through (linearized) Kalman's theory 3- this implies the gains easily accomodate time-varying features, such as time-varying covariance matrices, 4- contrarily to non-linear observers, the filter provides an indication (through the covariance matrix P_t) of the extent of uncertainty conveyed by the estimate and 5- the filters viewed as observers, that is, when noise is turned off, converge around *any* trajectory, with an attraction radius which is uniform over time. It is worthy to note that, to that respect, the three filters achieve the same goals as the ones pursued by the very recent XKF Johansen and Fossen (2016), albeit a wholly different method. Note though, it has not yet been shown an XKF may be built on the following examples.

In a nutshell, the IEKFs proposed here should be appealing to the aerospace engineers: they retain all the characteristics of the standard EKF (first-order optimality, realtive ease of tuning, adaptivity to time-varying features

⋆ This work is supported by the company Safran.

and to discrete aperiodic measurements), but with additional stability properties, when studied in a deterministic setting using the tools of dynamical systems theory.

The paper is organized as follows. Section II is a concise tutorial summary and discussion on Barrau and Bonnabel (2017). Each following section deals with an example. A conclusion seemed not necessary, so it was omitted.

2. REVIEW OF THE IEKF METHODOLOGY AND CONVERGENCE PROPERTIES

In this section, we review the IEKF methodology as presented in Barrau and Bonnabel (2017), that is, for continuous time dynamics with discrete time observations for systems defined on Lie groups. The exposure is meant to be concise and tutorial, and is enhanced by discussions. See the Appendix for more details on matrix Lie groups.

2.1 Considered class of systems and IEKF equations

Consider in this section a dynamics on a matrix Lie group $G \subset \mathbb{R}^{N \times N}$ with state $\chi_t \in G$ satisfying:

$$\frac{d}{dt}\chi_t = f_{u_t}(\chi_t) + \chi_t w_t, \tag{1}$$

where w_t is a continuous white noise belonging to the Lie algebra $\mathfrak{g}$ (see the Appendix). Let $q = \dim G$ denote the dimension of the Lie group G (or alternatively defined by $q = \dim\, \mathfrak{g}$). Assume moreover the following relation holds

$$f_u(ab) = af_u(b) + f_u(a)b - af_u(Id)b \tag{2}$$

for all $(u, a, b) \in U \times G \times G$. This system can be associated with two different kinds of discrete observations at arbitrary times $t_0 < t_1 < t_2, \cdots$.

Left-invariant observations The first family of outputs we are interested in write:

$$Y^1_{t_n} = \chi_{t_n}\left(d^1 + B^1_n\right) + V^1_n, ..., Y^k_{t_n} = \chi_{t_n}\left(d^k + B^k_n\right) + V^k_n, \tag{3}$$

where $(d^i)_{i\leq k}$ are known vectors of $\mathbb{R}^N$, and where the $(V^i_n)_{i\leq k}$, $(B^i_n)_{i\leq k}$ are centered Gaussian varibables noises with known covariance matrices.

The outputs are said to be "left-invariant" as, in the absence of noise, the outputs are of the form $h(\chi) = \chi d$ so that, $\chi_2 h(\chi_1) = h(\chi_2\chi_1)$. This property is also referred to as left equivariance in the mathematics literature and in the theory of symmetry-preserving observers. For left-invariant observations, a Left-Invariant EKF (LIEKF) should always be used.

The Left-Invariant Extended Kalman Filter (LIEKF) is defined through the usual following propagation and update steps:

$$\frac{d}{dt}\hat{\chi}_t = f_{u_t}(\hat{\chi}_t), \quad t_{n-1} \leq t < t_n, \qquad \text{Propagation} \tag{4}$$

$$\hat{\chi}^+_{t_n} = \hat{\chi}_{t_n} \exp\left[L_n\begin{pmatrix}\hat{\chi}^{-1}_{t_n}Y^1_{t_n} - d^1\\ ...\\ \hat{\chi}^{-1}_{t_n}Y^k_{t_n} - d^k\end{pmatrix}\right], \quad \text{Update} \tag{5}$$

where the function $L_n : \mathbb{R}^{kN} \to \mathbb{R}^q$ is defined through linearizations as in the conventional EKF theory. But here, instead of considering the usual linear state error $\hat{\chi}_t - \chi_t$, one must consider the following left-invariant error between true state χ_t and the estimated state $\hat{\chi}_t$:

$$\eta^L_t = \chi^{-1}_t\hat{\chi}_t. \tag{6}$$

which is the counterpart of the linear error $\hat{\chi}_t - \chi_t$ (which has no proper meaning in the present context), when dealing with a state space that is a Lie group. Note that, this error is nominally equal to identity matrix and not zero. The rationale of the IEKF theory, and more generally the theory of symmetry-preserving observers, is to linearize the error system at the propagation and update state. It turns out the error system, with an error defined this way, has remarkable properties, that are key to prove the IEKF stability properties of Barrau and Bonnabel (2017).

Right-invariant observations The second family of observations we are interested in have the form:

$$Y^1_{t_n} = \chi^{-1}_{t_n}\left(d^1 + V^1_n\right) + B^1_n, ..., Y^k_{t_n} = \chi^{-1}_{t_n}\left(d^k + V^k_n\right) + B^k_n. \tag{7}$$

with the same notation as in the previous paragraph. The Right-Invariant EKF (RIEKF), always to be used for right-invariant observations of the form (7) is defined here in the same way, alternating between continous time propagation and discrete time update steps:

$$\frac{d}{dt}\hat{\chi}_t = f_{u_t}(\hat{\chi}_t), \quad t_{n-1} \leq t < t_n, \tag{8}$$

$$\hat{\chi}^+_{t_n} = \exp\left[L_n\begin{pmatrix}\hat{\chi}_{t_n}Y^1_{t_n} - d^1\\ ...\\ \hat{\chi}_{t_n}Y^k_{t_n} - d^k\end{pmatrix}\right]\hat{\chi}_{t_n}. \tag{9}$$

To tune the gain L_n the state error must be linearized, but in this case we rather consider the right-invariant error

$$\eta^R_t = \hat{\chi}_t\chi^{-1}_t. \tag{10}$$

Gain tuning To tune the gain matrix L_n, one must linearize the error equation associated to (6), or respectively (10). To do so, the user can refer to the general theory of Barrau and Bonnabel (2017), or rather proceed to a case by case derivation as done in the examples below, which is recommended. In any case, one can associate to the nonlinear error (6), or (10), a vector $\xi_t \in \mathbb{R}^q$ that captures the error up to the first order. It can be used to obtain a linear approximation to the true error system, of the form:

$$\frac{d}{dt}\xi_t = A_t\xi_t + D(\hat{\chi}_t)\tilde{w}_t \tag{11}$$

where $\tilde{w}_t$ is a continuous noise in $\mathbb{R}^q$ and to a linearized error update equation of the form

$$\xi^+_{t_n} = \xi_{t_n} - L_n(H\xi_{t_n} + E(\hat{\chi}_{t_n})V_n) \tag{12}$$

with V_n a centered Gaussian. To account for the fact the stochastic terms entering the system depend on the estimated trajectory, we define as in the standard EKF theory with non-additive noises (see e.g. Stengel (1986)) the covariance matrices

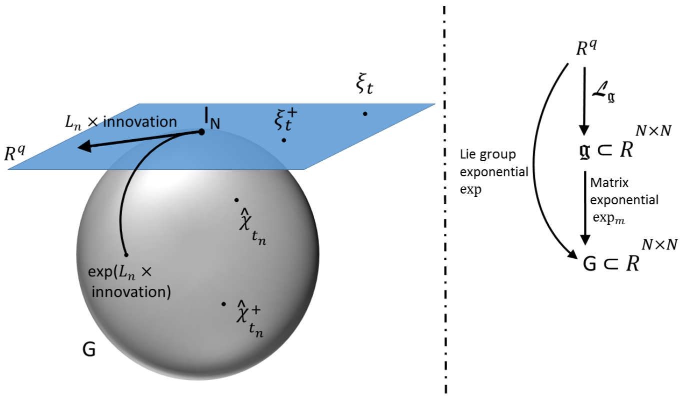
$$Q(\hat{\chi}_t) = D(\hat{\chi}_t)Cov(\tilde{w}_t)D(\hat{\chi}_t)^T$$
$$N(\hat{\chi}_{t_n}) = E(\hat{\chi}_{t_n})Cov(V_n)E(\hat{\chi}_{t_n})^T$$

As in the standard EKF methodology, the "optimal" gain L_n is then obtained through the Kalman equations:

$$
\begin{aligned}
\frac{d}{dt}P_t &= A_t P_t + P_t A_t^T + Q(\hat{\chi}_t), \\
S_n &= H P_{t_n} H^T + N(\hat{\chi}_{t_n}), \\
L_n = P_{t_n} H^T S_n^{-1}, &\quad P_{t_n}^+ = (I - L_n H) P_{t_n}.
\end{aligned} \tag{13}
$$

2.2 *Geometrical insight for the practitioner*

In this section we provide the user with some (novel) geometrical insight about the IEKF linearization procedure, which may prove helpful, albeit not strictly necessary to apply the theory. Even if it is not apparent when looking at the original problem, the fact that the state space may be identified to a matrix Lie group G, which is a *subset* of $\mathbb{R}^{N\times N}$ means the state space is a *curved* space. Albeit not rigorous, one can think of the state space as a sphere. The error η_t of (6), resp. (10), is an element of G, and is equal to I_N (only) when $\hat{\chi}_t = \chi_t$. Thus, the error system is always linearized around the same point, that is, the identity matrix. To linearize it, the rationale (see Bonnabel et al. (2009a)) is to identify a small error η_t with an element of the *tangent space* at I_N (called the Lie algebra of G, and denoted by $\mathfrak{g}$). In turn this element of the tangent space can be identified (through the operator $\mathcal{L}_\mathfrak{g}$, see the appendix) to an element ξ_t of the vector space $\mathbb{R}^q$. And the Kalman gain matrix L_n always acts on this vector space, which is identified with the *tangent space* at I_N. Thus "$L_n \times$ the innovation" is always an element of the tangent space to G at I_N. The Lie exponential map appearing in (5), resp. (9), is in turn a way to map this tangent vector at I_N (the correction term) *back* to the group G, and premultiplication in (5), resp. post-multiplication in (9), by $\hat{\chi}_{t_n}$, is a way to apply this correction to the latest estimate $\hat{\chi}_{t_n} \in G$. This is illustrated on the following cartoon.



2.3 *IEKF general stability properties*

Consider *deterministic* non-linear continuous time systems with discrete time observations, that is, general systems of the form $\frac{d}{dt}x_t = f(x_t, u_t)$, $y_{t_n} = h(x_{t_n})$. Even under observability conditions, designing non-linear observers is always a challenge. Notably, local convergence around *any* trajectory is a very rare property to obtain (see e.g. Aghannan and Rouchon (2003) which deals with the apparently simple problem of estimating the velocity from position and acceleration measurements of a mechanical system). A simple way to design an observer that might yield good results, is to apply a conventional EKF to the system $\frac{d}{dt}x_t = f(x_t, u_t)$, $\quad y_{t_n} = h(x_{t_n})$, considering the process noise and measurement noise covariance matrices as design parameters. But it turns out the obtained filter is by no means guaranteed to converge around any trajectory of the system, unless some rather strong assumptions hold, see e.g. Song and Grizzle (1995); Bonnabel and Slotine (2015). By contrast, for systems defined on matrix Lie groups of the form (1)-(3) or (1)-(7) with noise turned off, it turns out the IEKF used as a deterministic non-linear observer converges around *any* trajectory, as follows.

Theorem 1. Barrau and Bonnabel (2017) Consider the system (1) with noise turned off, i.e.,

$$\frac{d}{dt}\chi_t = f_{u_t}(\chi_t)$$

Either one can assume left-invariant discrete output measurements (3) with noise turned off, i.e.

$$Y_{t_n}^1 = \chi_{t_n} d^1 \quad , \quad \dots \quad , \quad Y_{t_n}^k = \chi_{t_n} d^k,$$

and use the Left-invariant EKF (4)-(5).

Or, one can assume right-invariant output measurements (7) with noise turned off, i.e.

$$Y_{t_n}^1 = \chi_{t_n}^{-1} d^1 \quad , \quad \dots \quad , \quad Y_{t_n}^k = \chi_{t_n}^{-1} d^k,$$

and use the Right-invariant EKF (8)-(9).

Let A_t, Q, N be defined as in the IEKF algorithm, and let $\Phi_{t_0}^t$ denote the square matrix defined by $\Phi_{t_0}^{t_0} = I_q$, $\frac{d}{dt}\Phi_{t_0}^t = A_t \Phi_{t_0}^t$. For simplicity of notation let $Q(\chi_t) := Q_t$ and $N(\chi_{t_n}) := N_n$. Assume there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \delta_3, M$ such that:

(1) $(\Phi_{t_n}^{t_{n+1}})^T \Phi_{t_n}^{t_{n+1}} \succeq \delta_1 I_p \succeq 0$,
(2) $\exists q \in \mathbb{N}^*, \forall s > 0, \exists G_s \in \mathbb{R}^{p\times q}, Q_s = G_s Q' G_s^T$ where $Q' \succeq \delta_2 I_q \succeq 0$,
(3) $N_n \succeq \delta_3 I_N \succeq 0$,
(4) $\alpha_1 I_p \leq \int_{s=t_{n-M}}^{t_n} (\Phi_s^{t_n}) Q_s (\Phi_s^{t_n})^T \leq \alpha_2 I_p$,
(5) $\beta_1 I_p \leq \sum_{i=n-M}^{n-1} \left(\Phi_{t_{i+1}}^{t_n}\right)^T H^T N_n^{-1} H \left(\Phi_{t_{i+1}}^{t_n}\right) \leq \beta_2 I_p$.

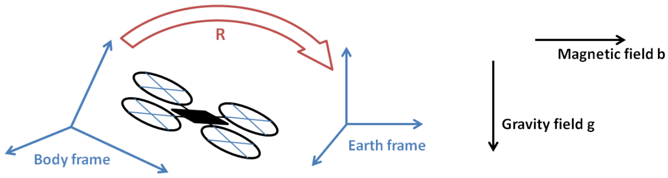
Then, there exists $\epsilon > 0$ such that, for *any* $t_0 \geq 0$, if the distance between $\hat{\chi}_{t_0}$ and χ_{t_0} is less than ϵ, then the distance between the estimate $\hat{\chi}_t$ and the true state χ_t tends to zero as $t \to \infty$.

The theorem is proved in Barrau and Bonnabel (2017). The proof is technical, non trivial, and long. It brings to bear the very rich structure of the class of systems considered. Contrarily to previous literature on EKF convergence, the main feature here is that no assumption is whatsoever made on the estimate $\hat{\chi}_t$'s behavior.

2.4 *Discussion on the role of noises*

The theorem holds for the noise-free system, whereas the matrices $Q(\hat{\chi}_t)$ and $N(\hat{\chi}_{t_n})$ involved in the gain tuning (13) are based on the noises's characteristics. This may seem a contradictory approach. But, proving the stochastic counterpart of Theorem 1 would definitely be out of reach, whereas the theory of deterministic dynamical systems allows using systematic and relatively basic tools for stability analysis. And proving that a non-linear filter, once tuned for an actually noisy system, enjoys deterministic convergence properties in the absence of noise, is certainly desirable and reassuring: a guarantee of local stability in the absence of noise is a great indication that the filter should not diverge for sufficiently small errors and noises.

3. FIRST EXAMPLE: ATTITUDE ESTIMATION



In this section we consider the problem of attitude estimation with gyrometers and observation of two vectors with known directions in the fixed frame. This problem has received a lot of attention over the past decades, and the literature is too broad to be covered here (see e.g. Mahony et al. (2008); Izadi and Sanyal (2014); Sanyal and Nordkvist (2012); Zamani et al. (2014); Lee et al. (2007)). We insist that this example is very simple, and that we included it essentially for tutorial reasons. However, note that, contrarily to the non-linear attitude estimation literature just mentioned, the following filter accomodates discrete-time observations, which can be relevant in practice (for instance a star tracking system in a sattelite could use infrequent snapshots to save energy), and time varying model uncertainties (for instance the quasi-static assumption below for a UAV does not hold in dynamic flight phases, so that the noise variance on the gravity measurement should be temporarily increased when manoeuvers are made). Note also that we already published the equations of the IEKF for this problem, but we never proved its local stability properties.

Model Consider the attitude of a vehicle, represented by the rotation matrix $R_t \in SO(3)$ mapping the coordinates of a vector expressed in the vehicle frame to its coordinates in the static frame. The vehicle is endowed with gyrometers measuring an angular velocity ω_t. The dynamics read:

$$\frac{d}{dt}R_t = R_t(\omega_t + w_t)_\times \tag{14}$$

where $(b)_\times$ is the skew symmetric matrix associated to vector b and w_t is the gyroscopes' noise vector. Note that, (14) bears a resemblance to a linear system. The resemblance is yet wholly artificial as $SO(3)$ is not a vector space. Consider as observations for $t_0 < t_1 < t_2 < \cdots$

$$Y_n = (R_{t_n}^T g + V_n^g; R_{t_n}^T b + V_n^b), \tag{15}$$

where g and b are two non-collinear vectors of $\mathbb{R}^3$ and V_n^g, V_n^b two centered noises in $\mathbb{R}^3$. Generally those two measurements are respectively associated to an accelerometer that is supposed to measure the earth gravity under the quasi-static hypothesis, and to a magnetometer that measures the earth magnetic field. Relation (2) holds.

Error equation The outputs (15) are right-invariant. Thus one should devise a right-invariant EKF. The RIEKF (8)-(9) for the system (14)-(15) is defined by:

$$\frac{d}{dt}\hat{R}_t = \hat{R}_t(\omega_t)_\times \quad \hat{R}_{t_n}^+ = \exp(L_n[\hat{R}_{t_n}Y_n - (g;b)])\hat{R}_{t_n},$$

with the notation $R(x_1;x_2) = (Rx_1; Rx_2)$ for $R \in SO(3)$ anx $x_1, x_2 \in \mathbb{R}^3$. To compute the gains L_n we write the evolution of the right invariant error (10): $\eta_t = \hat{R}_t R_t^T$. As concerns the propagation step, using that $(\omega_t + w_t)_\times^T = -(\omega_t + w_t)_\times = -(\omega_t)_\times - (w_t)_\times$, we have

$$\begin{aligned}\frac{d}{dt}\eta_t &= (\frac{d}{dt}\hat{R}_t)R_t^T + \hat{R}_t(\frac{d}{dt}R_t^T) \\ &= \hat{R}_t(\omega_t)_\times R_t^T + \hat{R}_t(\omega_t + w_t)_\times^T R_t^T \\ &= 0 - \hat{R}_t(w_t)_\times R_t^T \\ &= -(\hat{R}_t w_t)_\times \eta_t\end{aligned}$$

since $\hat{R}_t(x \times y) = (\hat{R}_t x \times \hat{R}_t y)$. The update step writes

$$\eta_{t_n}^+ = \exp(L_n[\eta_t g - g + \hat{R}_t V_n^g; \eta_t b - b + \hat{R}_t V_n^b])\eta_{t_n}$$

Linearized error equation and gain tuning To linearize the error equation we introduce the linearized state error *vector* $\xi_t \in \mathbb{R}^3$ in the Lie algebra $\mathfrak{so}(3)$, defined by $\eta_t := \exp(\xi_t) = I_3 + \mathcal{L}_{\mathfrak{so}(3)}(\xi_t) + O(\|\xi_t\|^2) = I_3 + (\xi_t)_\times + O(\|\xi_t\|^2)$. See the appendix for more details. Replacing η_t with $I_3 + (\xi_t)_\times$, the error propagation equation above reads

$$\frac{d}{dt}(\xi_t)_\times = -(\hat{R}_t w_t)_\times (I_3 + (\xi_t)_\times) = -(\hat{R}_t w_t)_\times$$

where we neglected terms of magnitude $\|w_t\|\|\xi_t\|$, as in the standard EKF methodology in the presence of non-additive noises Stengel (1986) p 386. This yields the linearized equation

$$\frac{d}{dt}\xi_t = 0 - \hat{R}_t w_t \tag{16}$$

and by identification with (11) the matrices

$$A_t = 0_{3,3}, \qquad Q(\hat{R}_t) = \hat{R}_t Cov(w_t)\hat{R}_t^T. \tag{17}$$

In the same way, discarding terms of magnitude $\|\xi_t\|^2$, and $\|\xi_t\|\|V_n\|$, and using the subsequent approximations $\eta_t^+ = I_3 + (\xi_t^+)_\times$, $\exp(u) = I_3 + (u)_\times$, $\eta_t^{-1} = I_3 - (\xi_t)_\times$ and $\exp^{-1}[I_3 + (u)_\times] = u$, we get

$$\xi_{t_n}^+ = \xi_{t_n} - L_n\left[\begin{pmatrix}(g)_\times \\ (b_\times)\end{pmatrix}\xi_{t_n} - \begin{pmatrix}\hat{R}_{t_n}V_n^g \\ \hat{R}_{t_n}V_n^b\end{pmatrix}\right]$$

yielding the matrices for the linearized update error equation (12)

$$H = \begin{pmatrix}(g)_\times \\ (b_\times)\end{pmatrix}, \; N(\hat{R}_{t_n}) = \hat{R}_{t_n}Cov(V_n)\hat{R}_{t_n}^T \tag{18}$$

The gains L_n are thus computed using the Riccati equation (13), where the matrices $A_t, H, Q(\hat{R}_t), N(\hat{R}_{t_n})$ are defined by (17) and (18).

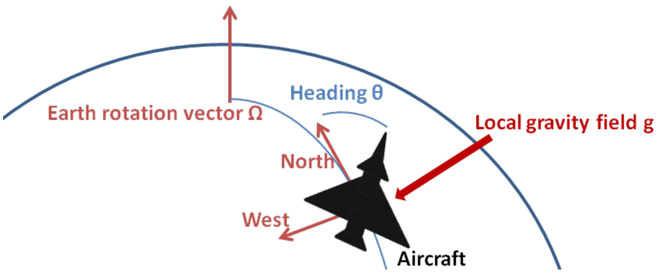
Stability results The IEKF, when used as an observer, enjoys the following desirable convergence property:

Proposition 2. Assume that the eigenvalues of $Cov(V_n)$, $Cov(w_t)$ are (uniformly) lower bounded by some $\alpha > 0$. Then the IEKF, used as an attitude observer for the continuous time system $\frac{d}{dt}R_t = R_t(\omega_t)_\times$ with an infinite number of discrete observations $Y_n = (R_{t_n}^T g; R_{t_n}^T b)$ where g, b are known non-collinear vectors of $\mathbb{R}_3$, is such that $\hat{R}_t R_t^T \to I_3$ when $t \to \infty$ for $\hat{R}_0, R_0$ sufficiently close.

Proof. It suffices to verify the assumptions of the theorem, which is particularly simple as $A_t \equiv 0$. Using the latter equality and the assumptions on the noise covariance matrices readily implies assumptions (1), (2), (3), (4) hold, and using moreover that g, b are not collinear, the matrix H is of rank 3 so that assumption (5) holds. The infinite number of observations is implicit in the theorem, it is explicitly mentioned here to underline to the practitioner the existence of this assumption.

4. SECOND EXAMPLE: GYROCOMPASS

This example is also well-known. Note that, in the recent paper Batista et al. (2014), the authors have obtained a globally converging observer for it in continuous time. The filter proposed in the present paper is only locally convergent, but it has the merit to strictly be an EKF variant, with all the associated advantages we have already largely discussed.



Model Consider the attitude of a gyrocompass, represented by the rotation matrix $R_t \in SO(3)$ mapping the coordinates of a vector expressed in the gyrocompass frame to its coordinates in the static frame. The vehicle is endowed with gyrometers giving an angular velocity ω_t, and supposed precise enough to measure the earth rotation. The equation of the dynamics then reads:

$$\frac{d}{dt}R_t = (\Omega)_\times R_t + R_t(\omega_t + w_t)_\times \qquad (19)$$

where Ω is the earth rotation axis known in a inertial frame linked to remote stars, and w_t is the gyro noise. The gyrocompass being at rest, an accelerometer measures moreover the gravity field in the reference frame of the gyrocompass (quasi-static hypothesis):

$$Y_n = R_t^T g + V_n^g \qquad (20)$$

The reader can verify the relation (2).

Error equation Due to similarities with the latter example, we skip quite a number of details in the sequel. The RIEKF (8)-(9) for the system (19)-(20) is defined by:

$$\frac{d}{dt}\hat{R}_t = (\Omega)_\times \hat{R}_t + \hat{R}_t(\omega_t)_\times, \ \hat{R}_{t_n}^+ = \exp(L_n[\hat{R}_{t_n}Y_n - g])\hat{R}_{t_n}$$

The invariant error is $\eta_t = \hat{R}_t R_t^T$ and its evolution reads:

$$\frac{d}{dt}\eta_t = (\Omega)_\times \eta_t - \eta_t(\Omega)_\times - (\hat{R}_t w_t)_\times \eta_t$$
$$\eta_{t_n}^+ = \exp(L_n[\eta_t g - g + \hat{R}_t V_n^g])\eta_{t_n}$$

Linearized error equation and gain tuning To linearize this equation we introduce the linearized error ξ_t defined as $\eta_t = I_3 + (\xi_t)_\times$. Introducing $\eta_t = I_3 + (\xi_t)_\times$, $\eta_t^+ = I_3 + (\xi_t^+)_\times$, $\exp(u) = I_3 + (u)_\times$, $\eta_t^{-1} = I_3 - (\xi_t)_\times$ and $\exp^{-1}[I_3 + (u)_\times] = u$ and removing the second-order terms in ξ_t, V_n^g, w_t and products of those terms, we obtain the following linearized error equation:

$$\frac{d}{dt}\xi_t = (\Omega)_\times \xi_t - \hat{R}_t w_t, \ \xi_{t_n}^+ = \xi_{t_n} - L_n[(g)_\times \xi_{t_n} - \hat{R}_{t_n} V_n^g]$$

The gains L_n are thus computed using the Riccati equation (13) with $A_t = (\Omega)_\times$, $H = -(g)_\times$, $Q(\hat{R}_t) = \hat{R}_t Cov(w_t)\hat{R}_t^T$, $N(\hat{R}_{t_n}) = \hat{R}_t Cov(V_n)\hat{R}_t^T$.

Stability results We start with a simple lemma.

Lemma 3. The system (19), (20) defining the gyrocompass is observable if g and Ω are non-collinear.

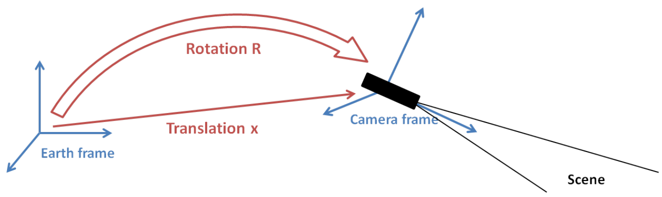
Proof. The observability matrix associated to the deterministic part of (19), (20) for two updates reads: $\begin{pmatrix}(g)_\times \\ (g)_\times R_\Omega\end{pmatrix}$, where $R_\Omega = \exp\left(\Omega_\times(t_{n+1} - t_n)\right)$ denotes the rotation matrix corresponding to the axis-angle representation Ω. The rank is not increased if this matrix is multiplied on the left by $\begin{pmatrix} I_3 & 0_{3\times3} \\ 0_{3\times3} & R_\Omega^T\end{pmatrix}$. We obtain $\begin{pmatrix}(g)_\times \\ R_\Omega^T(g)_\times R_\Omega\end{pmatrix} = \begin{pmatrix}(g)_\times \\ (R_\Omega^T g)_\times\end{pmatrix}$ This is the same observation matrix as in the case where vectors g and $R_\Omega^T g$ are observed. This matrix is of rank 3 if g and $R_\Omega^T g$ are non-collinear, i.e. if g and Ω are non-collinear.

Proposition 4. The IEKF for the gyrocompass problem (19)-(20) is an asymptotically stable observer if g and Ω are non-collinear, and the eigenvalues of $Cov(w_t)$, $Cov(V_n)$ are greater than some $\alpha > 0$.

This is an immediate consequence of Lemma 3 and Theorem 1. Note that, if g and Ω are collinear (in other word the gyrocompass is on the north or south pole), the heading is anyway not observable.

5. THIRD EXAMPLE: 3D CAMERA LOCALIZATION

This example has been partly considered in Hervier et al. (2012), except that here we do not assume to have the gyroscopes. No proof of convergence has been derived whatsoever in prior work.



Model We consider here the problem of a hand-held depth-camera (such as the Microsoft Kinect sensor) in a fixed environment of which we already possess a 3D model. The camera is supposed not to be equipped with motion sensors (such as gyrometers and accelerometers). A classical way to account for the latter, is to assume the camera to be fixed (low dynamics motion prior) and to let the filter know the motion model is uncertain. Mathematically this boils down to assume the camera follows a Brownian motion. The attitude is denoted by the rotation matrix R_t and the position by the 3-dimensionnal vector x_t. The equations then read:

$$\frac{d}{dt}R_t = (w_t^R)_\times R_t \qquad , \qquad \frac{d}{dt}x_t = (w_t^R)_\times x + w_t^x \qquad (21)$$

where w_t^R and w_t^x are noises. The process noise is assumed known in the static frame, as no matter what the camera orientation, the horizontal velocity is generally higher than the vertical one, resulting in a covariance linked to the static frame axes. We assume a scan matching algorithm that compares the captured depth image with the 3D model, returns the whole state (see Hervier et al. (2012); Barczyk et al. (2014)), that is,

$$Y_n^R = R_{t_n}^T V_n^R \quad , \quad Y_n^x = R_{t_n}^T (x + V_n^x) \tag{22}$$

where V_n^R, V_n^x are centered Gaussian noises whose covariance can be evaluated, see e.g. Censi (2007).

Matrix form The system must be embedded into a matrix Lie group in order to apply the theory above. The chosen matrix Lie group is $SE(3)$ (see Appendix A.2), and the embedding is done through the following alternative state, output, and noise (matrix) variables:

$$\chi_t = \begin{pmatrix} R_t & x_t \\ 0_{1\times 3} & 1 \end{pmatrix} \quad , \quad w_t = \begin{pmatrix} (w_t^R)_\times & w_t^x \\ 0_{1\times 3} & 0 \end{pmatrix}$$
$$Y_n = \begin{pmatrix} Y_n^R & -Y_n^x \\ 0_{1,3} & 1 \end{pmatrix} \quad , \quad V_n = \begin{pmatrix} V_n^R & -V_n^x \\ 0_{1,3} & 1 \end{pmatrix} \tag{23}$$

Equations (21), (22) then become:

$$\frac{d}{dt}\chi_t = w_t \chi_t \tag{24}$$
$$Y_n = \chi_{t_n}^{-1} V_n \tag{25}$$

Error equation Note that, the output here is slightly different from (7) as V_n (and thus Y_n) is here a matrix. But the IEKF theory can be easily generalized to this case. Indeed, the IEKF equations are unchanged, apart from the linearization process that is slightly modified as explained in the sequel. It is easily seen the theorem still holds (as proved in the first version of the preprint of Barrau and Bonnabel (2017) available on Arxiv). The RIEKF (8)-(9) for the system (24)-(25) is:

$$\frac{d}{dt}\hat{\chi}_t = 0_{4,4}, \quad \hat{\chi}_{t_n}^+ = \exp(L_n[\exp^{-1}(\hat{\chi}_{t_n} Y_n)])\hat{\chi}_{t_n}$$

The right-invariant error is $\eta_t = \hat{\chi}_t \chi_t^{-1}$. Using the general equality $\frac{d}{dt}\chi_t^{-1} = -\chi_t^{-1}(\frac{d}{dt}\chi_t)\chi_t^{-1}$ its evolution reads:

$$\frac{d}{dt}\eta_t = -\eta_t w_t, \quad \eta_{t_n}^+ = \exp(L_n[\exp^{-1}(\eta_{t_n} V_n)])\eta_{t_n} \tag{26}$$

Linearized error equation and gain tuning To linearize this equation we introduce the linearized error ξ_t through the following approximation $\eta_t = I_4 + \mathcal{L}_{\mathfrak{se}(3)}(\xi_t)$. The observation noise now lies on the group and not in a vector space. The IEKF gain acts on the Lie algebra of the group, so that the noise must be defined in the Lie algebra as well. To do so, we assume it be not too large and use the following approximation $V_n = I_4 + \mathcal{L}_{\mathfrak{se}(3)}(v_n)$. Introducing $\eta_t^+ = I_4 + \mathcal{L}_{\mathfrak{se}(3)}(\xi_t^+)$, $\exp(u) = I_4 + \mathcal{L}_{\mathfrak{se}(3)}(u)$, $\eta_t^{-1} = I_4 - \mathcal{L}_{\mathfrak{se}(3)}(\xi_t)$, $\exp^{-1}[I_4 + \mathcal{L}_{\mathfrak{se}(3)}(u)] = u$ and the approximation $\exp^{-1}(ab) = \exp^{-1}(a) + \exp^{-1}(b)$ for small a, b in (26), and removing the second-order terms in ξ_t, $\exp^{-1}(V_n)$ and w_t we obtain:

$$\frac{d}{dt}\xi_t = -w_t, \quad \xi_{t_n}^+ = \xi_{t_n} - L_n(-\xi_{t_n} - v_n)$$

The gains L_n are computed using the Riccati equation (13) where matrices $A_t, H, Q(\chi_t)$ and $N(\chi_{t_n})$ are defined according to the linearized equation above by $A_t = 0_{3,3}$, $H = -I_6$, and

$$Q(\chi_t) = Cov(w_t), \; N(\chi_{t_n}) = Cov(v_n) = Cov(\exp^{-1}(V_n))$$

The stability properties of the IEKF imply (see the remark above about the validity of the theorem in that case):

Proposition 5. The IEKF for the movement estimation problem (24), (25) is an asymptotically stable observer as long as all the eigenvalues of $Cov(\exp^{-1}(V_n))$, and at least one eigenvalue of $Cov(w_t)$ are lower bounded by some fixed $\alpha > 0$.

REFERENCES

Aghannan, N. and Rouchon, P. (2003). An intrinsic observer for a class of lagrangian systems. *IEEE Trans. on Automatic Control*, 48(6), 936–945.

Barczyk, M., Bonnabel, S., Deschaud, J.E., and Goulette, F. (2014). Experimental implementation of an invariant extended kalman filter-based scan matching slam. In *Proceedings of the American Control Conference*, 4121–4126.

Barrau, A. and Bonnabel, S. (2017). The invariant extended Kalman filter as a stable observer. *IEEE-TAC in May 2017. DOI: 10.1109/TAC.2016.2594085. arXiv preprint arXiv:1410.1465.*

Batista, P., Silvestre, C., and Oliveira, P. (2014). Attitude and earth velocity estimation-part i: Globally exponentially stable observer. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, 121–126. IEEE.

Bonnabel, S., Martin, P., and Rouchon, P. (2009a). Nonlinear symmetry-preserving observers on Lie groups. *IEEE Transactions on Automatic Control*, 54(7), 1709–1713.

Bonnabel, S., Martin, P., and Salaun, E. (2009b). Invariant Extended Kalman Filter: Theory and application to a velocity-aided attitude estimation problem. In *IEEE Conference on Decision and Control.*

Bonnabel, S. (2007). Left-invariant extended kalman filter and attitude estimation. In *IEEE conference on decision and control*, 1027–1032.

Bonnabel, S. and Slotine, J.J. (2015). A contraction theory-based analysis of the stability of the deterministic extended kalman filter. *IEEE Transactions on Automatic Control*, 60(2), 565–569.

Bourmaud, G., Mégret, R., Giremus, A., and Berthoumieu, Y. (2013). Discrete extended Kalman filter on Lie groups. In *Signal Processing Conference (EUSIPCO), 2013 Proceedings of the 21st European*, 1–5. IEEE.

Censi, A. (2007). An accurate closed-form estimate of icp's covariance. In *Robotics and Automation, 2007 IEEE International Conference on*, 3167–3172. IEEE.

de Ruiter, A.H. and Forbes, J.R. (2016). Discrete-time so (n)-constrained kalman filtering. *Journal of Guidance, Control, and Dynamics.*

Hervier, T., Bonnabel, S., and Goulette, F. (2012). Accurate 3D maps from depth images and motion sensors via nonlinear kalman filtering. *CoRR*, abs/1205.3727.

Hua, M.D., Ducard, G., Hamel, T., Mahony, R., and Rudin, K. (2014). Implementation of a nonlinear attitude estimator for aerial robotic vehicles. *Control Systems Technology, IEEE Transactions on*, 22(1), 201–213.

Hua, M.D., Martin, P., and Hamel, T. (2016). Stability analysis of velocity-aided attitude observers for accelerated vehicles. *Automatica*, 63, 11–15.

Izadi, M. and Sanyal, A.K. (2014). Rigid body attitude estimation based on the lagrange–dalembert principle. *Automatica*, 50(10), 2570–2577.

Johansen, T.A. and Fossen, T.I. (2016). The exogenous kalman filter (XKF). *The International Journal of Control*.

Krener, A.J. (2003). The convergence of the extended kalman filter. In *Directions in mathematical systems theory and optimization*, 173–182. Springer.

Lee, T., Leok, M., McClamroch, N.H., and Sanyal, A. (2007). Global attitude estimation using single direction measurements. In *2007 American Control Conference*, 3659–3664. IEEE.

Mahony, R., Hamel, T., and Pflimlin, J.M. (2008). Nonlinear complementary filters on the special orthogonal group. *IEEE-Trans. on Automatic Control*, 53(5), 1203–1218.

Martin, P. and Salaün, E. (2010). Generalized multiplicative extended kalman filter for aided attitude and heading reference system. In *Proc. AIAA Guid., Navigat., Control Conf*, 1–13.

Sanyal, A.K. and Nordkvist, N. (2012). Attitude state estimation with multirate measurements for almost global attitude feedback tracking. *Journal of Guidance, Control, and Dynamics*, 35(3), 868–880.

Song, Y. and Grizzle, J. (1995). The extended Kalman filter as a local asymptotic observer. *Estimation and Control*, 5, 59–78.

Stengel, R.F. (1986). *Stochastic optimal control*. John Wiley and Sons New York, New York.

Tayebi, A., McGilvray, S., Roberts, A., and Moallem, M. (2007). Attitude estimation and stabilization of a rigid body using low-cost sensors. In *Decision and Control, 2007 46th IEEE Conference on*, 6424–6429. IEEE.

Wolfe, K.C., Mashner, M., and Chirikjian, G.S. (2011). Bayesian fusion on Lie groups. *Journal of Algebraic Statistics*, 2(1), 75–97.

Zamani, M., Trumpf, J., and Mahony, R. (2014). On the distance to optimality of the geometric approximate minimum-energy attitude filter. In *American Control Conference (ACC), 2014*, 4943–4948. IEEE.

Appendix A. LIE GROUPS BACKGROUND

A matrix Lie group G is a subset of square invertible $N\times N$ matrices $\mathcal{M}_N(\mathbb{R})$ verifying the following properties:

$$Id \in G, \qquad \forall g\in G, g^{-1}\in G, \qquad \forall a,b\in G, ab\in G$$

If $\gamma(t)$ is a curve over G with $\gamma(0)=I_N$, then its derivative at $t=0$ necessarily lies in a subset $\mathfrak{g}$ of $\mathcal{M}_N(\mathbb{R})$. $\mathfrak{g}$ is a vector space and it is called the Lie algebra of G and has same dimension as G. Thanks to a *linear* map denoted by $\mathcal{L}_\mathfrak{g}:\mathbb{R}^{\dim\mathfrak{g}}\to\mathfrak{g}$, one can advantageously identify $\mathfrak{g}$ to $\mathbb{R}^q$ where $q=\dim\mathfrak{g}$. Besides, the vector space $\mathfrak{g}$ can be mapped to the matrix Lie group G through the classical matrix exponential $\exp_m$. Thus, $\mathbb{R}^q$ can be mapped to G through the Lie exponential map defined by $\exp(\xi):=\exp_m(\mathcal{L}_\mathfrak{g}(\xi))$ for $\xi\in\mathbb{R}^q$. We have thus $\exp(\xi)=I_N+\mathcal{L}_\mathfrak{g}(\xi)+O(\xi^2)\in\mathcal{M}_N(\mathbb{R})$. For all $\zeta\in\mathbb{R}^q$ the adjoint matrix $ad_\zeta\in\mathbb{R}^{q\times q}$ is very useful to linearize equations over Lie groups. It satisfies $ad_\zeta\xi=-ad_\xi\zeta$ and

$$\mathcal{L}_\mathfrak{g}(\xi)\mathcal{L}_\mathfrak{g}(\zeta)-\mathcal{L}_\mathfrak{g}(\zeta)\mathcal{L}_\mathfrak{g}(\xi)=\mathcal{L}_\mathfrak{g}(-ad_\zeta\xi) \qquad \text{(A.1)}$$

A.1 Group of rotation matrices $SO(3)$

We have here $G=SO(3)=\{R\in\mathcal{M}_3(\mathbb{R}), RR^T=Id\}$. The tangent space around identity is $\mathfrak{so}(3)=\{A\in\mathcal{M}_3(\mathbb{R}), A=-A^T\}$, the space of skew-symetric matrices. An isomorphism between $\mathbb{R}^3$ and $\mathfrak{g}$ is given by $\mathcal{L}_{\mathfrak{so}(3)}\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix}=\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix}_\times=\begin{pmatrix}0&-\xi_3&\xi_2\\\xi_3&0&-\xi_1\\-\xi_2&\xi_1&0\end{pmatrix}$. The exponential mapping is given by the formula:

$$\exp(\xi)=I_3+\frac{\sin(||\xi||)}{||\xi||}(\xi)_\times+2\frac{\sin(||\xi||/2)^2}{||\xi||^2}(\xi)_\times^2$$

A.2 Group of direct spatial isometries $SE(3)$

We have here $G=SE(3)=\{\begin{pmatrix}R&x\\0_{1,3}&1\end{pmatrix}, R\in SO(3), x\in\mathbb{R}^3\}$. The tangent space around identity is $\mathfrak{se}(3)=\{\begin{pmatrix}(\xi)_\times&x\\0_{1,3}&1\end{pmatrix}, \xi,x\in\mathbb{R}^3\}$. An isomorphism between $\mathbb{R}^6$ and $\mathfrak{se}(3)$ is given by $\mathcal{L}_{\mathfrak{se}(3)}\begin{pmatrix}\xi\\x\end{pmatrix}=\begin{pmatrix}(\xi)_\times&x\\0_{1,3}&0\end{pmatrix}$. The exponential mapping is given by the formula: $\exp\begin{pmatrix}\xi\\x\end{pmatrix}=I_4+S+\frac{1-\cos(||\xi||)}{||\xi||^2})S^2+\frac{||\xi||-\sin(||\xi||)}{||\xi||^3}S^3$, where $S=\mathcal{L}_{\mathfrak{se}(3)}\begin{pmatrix}\xi\\x\end{pmatrix}$.