# Identifying codes for infinite triangular grids with a finite number of rows

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**Abstract**

Let $\mathcal{G}_T$ be the infinite triangular grid. For any positive integer $k$, we denote by $T_k$ the subgraph of $\mathcal{G}_T$ induced by the vertex set $\{(x, y) \in \mathbb{Z} \times [k]\}$. A set $C \subset V(G)$ is an *identifying code* in a graph $G$ if for all $v \in V(G)$, $N[v] \cap C \neq \emptyset$, and for all $u, v \in V(G)$, $N[u] \cap C \neq N[v] \cap C$, where $N[x]$ denotes the closed neighborhood of $x$ in $G$. The minimum density of an identifying code in $G$ is denoted by $d^*(G)$.

In this paper, we prove that $d^*(T_1) = d^*(T_2) = 1/2$, $d^*(T_3) = d^*(T_4) = 1/3$, $d^*(T_5) = 3/10$, $d^*(T_6) = 1/3$ and $d^*(T_k) = 1/4 + 1/(4k)$ for every $k \geq 7$ odd. Moreover, we prove that $1/4 + 1/(4k) \leq d^*(T_k) \leq 1/4 + 1/(2k)$ for every $k \geq 8$ even.

**Keywords.** identifying codes, triangular grids.

## 1 Introduction

Let $G$ be a graph $G$. The *neighborhood* of a vertex $v$ in $G$, denoted by $N(v)$, is the set of vertices adjacent to $v$ i $G$. It *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$.

A set $C \subseteq V(G)$ is an *identifying code* in $G$ if

(i) for all $v \in V(G)$, $N[v] \cap C \neq \emptyset$, and

(ii) for all $u, v \in V(G)$, $N[u] \cap C \neq N[v] \cap C$.

The *identifier* of $v$ by $C$, denoted by $C[v]$, is the set $N[v] \cap C$. Hence a identifying code is a set such that the vertices have non-empty distinct identifiers.

Let $G$ be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer $r$ and vertex $v$, we denote by $B_r(v)$ the ball of radius $r$ in $G$, that is $B_r(v) = \{x \mid$

$\mathrm{dist}(v,x) \le r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of $C$ in $G$, denoted by $d(C,G)$, is defined by

$$d(C,G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} ,$$

where $v_0$ is an arbitrary vertex in $G$. The infimum of the density of an identifying code in $G$ is denoted by $d^*(G)$. Observe that if $G$ is finite, then $d^*(G) = |C^*|/|V(G)|$, where $C^*$ is a minimum-size identifying code in $G$.

The problem of finding low-density identifying codes was introduced in [9] in relation to fault diagnosis in arrays of processors. Here the vertices of an identifying code correspond to controlling processors able to check themselves and their neighbors. Thus the identifying property guarantees location of a faulty processor from the set of "complaining" controllers. Identifying codes are also used in [10] to model a location detection problem with sensor networks.

Particular interest was dedicated to grids as many processor networks have a grid topology. There are three regular infinite grids in the plane, namely the hexagonal grid, the square grid and the triangular grid.

Regarding the infinite hexagonal grid $\mathcal{G}_H$, the best upper bound on $d^*(\mathcal{G}_H)$ is $3/7$ and comes from two identifying codes constructed by Cohen et al. [4]; these authors also proved a lower bound of $16/39$. This lower bound was improved to $12/29$ by Cranston and Yu [6]. Cukierman and Yu [7] further improved it to $5/12$.

The *infinite square grid* $\mathcal{G}_S$ is the infinite graph with vertices in $\mathbb{Z}\times\mathbb{Z}$ such that $N((x,y)) = \{(x,y\pm1),(x\pm1,y)\}$. Given an integer $k \ge 2$, let $[k] = \{1,\dots,k\}$ and let $S_k$ be the subgraph of $\mathcal{G}_S$ induced by the vertex set $\{(x,y) \in \mathbb{Z}\times[k]\}$. In [3], Cohen et al. gave a periodic identifying code of $\mathcal{G}_S$ with density $7/20$. This density was later proved to be optimal by Ben-Haim and Litsyn [1]. Daniel, Gravier, and Moncel [8] showed that $d^*(S_1) = \frac{1}{2}$ and $d^*(S_2) = \frac{3}{7}$. They also showed that for every $k \ge 3$, $\dfrac{7}{20} - \dfrac{1}{2k} \le d^*(S_k) \le \min\left\{\dfrac{2}{5}, \dfrac{7}{20} + \dfrac{2}{k}\right\}$. These bounds were recently improved by Bouznif et al. [2] who established

$$\frac{7}{20} + \frac{1}{20k} \le d^*(S_k) \le \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}.$$

They also proved $d^*(S_3) = \frac{3}{7}$.

The *infinite triangular grid* $\mathcal{G}_T$ is the infinite graph with vertices in $\mathbb{Z}\times\mathbb{Z}$ such that $N((x,y)) = \{(x,y\pm1),(x\pm1,y),(x-1,y+1),(x+1,y-1)\}$. Given an integer $k \ge 2$, let $[k] = \{1,\dots,k\}$ and let $T_k$ be the subgraph of $\mathcal{G}_T$ induced by the vertex set $\{(x,y) \in \mathbb{Z}\times[k]\}$. Karpovsky et al. [9] showed that $d^*(\mathcal{G}_T) = 1/4$. Trivially, $T_1 = S_1$. Hence $d^*(T_1) = \frac{1}{2}$ In this paper, we prove several results regarding the density of an identifying code of $T_k$, $k > 1$. We prove that $d^*(T_k) = 1/4 + 1/(4k)$ for every odd $k$. Moreover, we prove $d^*(T_2) = 1/2$, $d^*(T_4) = 1/3 = d^*(T_6) = 1/3$, and $1/4 + 1/(4k) \le d^*(T_k) \le 1/4 + 1/(2k)$ for every even $k \ge 8$.

The upper bounds are obtained by showing periodic identifying codes with the desired density.

In general, the lower bounds are obtained via the Discharging Method. The general idea is the following. We consider any identifying code $C$ of $T_k$. The vertices in $C$ receive a certain value $q_k > 0$ of charge and the vertices not in $C$ receive charge 0. Then we apply some local discharging rules. Here local means that there is no charge transfer from a vertex to a vertex at distance more than $d_k$ for some fixed constant $d_k$, and that the total charge sent by a vertex is bounded by some fixed value $m_k$. Finally, we prove that after the discharging, every vertex $v$ has final charge $\text{chrg}^*(v)$ at least $p_k$ for some fixed $p_k > 0$. We claim that it implies $d(C,G) \geq \frac{p_k}{q_k}$. Since a vertex sends charge at most $m_k$ to vertices at distance at most $d_k$, a charge of at most $m_k \cdot |B_{r+s}(v_0) \setminus B_r(v_0)| \leq 2d_k \cdot k \cdot m_k$ enters $B_r(v_0)$ during the discharging phase. Thus

$$\begin{aligned} |C \cap B_r(v_0)| = \frac{1}{q_k} \sum_{v \in B_r(v_0)} \text{chrg}_0(v) &\geq \frac{1}{q_k}\left( \sum_{v \in B_r(v_0)} \text{chrg}^*(v) - m_k \cdot |B_{r+s}(v_0) \setminus B_r(v_0)| \right) \\ &\geq \frac{p_k |B_r(v_0)| - 2d_k \cdot k \cdot m_k}{q_k} . \end{aligned}$$

But $|B_r(v_0)| \geq 2(k+1)r - k^2$, thus $d(C, \mathcal{S}_k) \geq \limsup_{r \to +\infty} \left( \frac{p_k}{q_k} - \frac{1}{q_k} \cdot \frac{2d_k \cdot k \cdot m_k}{2(k+1)r - k^2} \right) = \frac{p_k}{q_k}$. This proves our claim. As the claim holds for any identifying code, we have $d^*(T_k) \geq p_k/q_k$.

Observe that $T_k$ has many isomorphisms. Let $\ell \in \mathbb{Z}$. A *horizontal translation* is a mapping $\tau : (x,y) \mapsto (x+\ell, y)$, a *vertical symmetry* is a mapping $\phi$: $(x,y) \mapsto (x+y+\ell, k+1-y)$, a *horizontal symmetry* is a mapping $\psi : (x,y) \mapsto (-x+2y+\ell, y)$, and all combinations of them are isomorphisms of $T_k$. For example, for every vertex $(x,y)$, by applying a horizontal symmetry, we can obtain an isomorphism $\theta$ such that $\theta((x,y)) = (x,y)$, $\theta((x,y+1)) = (x+1,y+1)$, $\theta((x+1,y+1)) = (x,y+1)$, $\theta((x-1,y)) = (x+1,y)$, $\theta((x+1,y)) = (x-1,y)$, $\theta((x-1,y-1)) = (x,y-1)$, $\theta((x,y-1)) = (x-1,y-1)$. See Figure 1, where the black vertex is $(x,y)$ and $u' = \theta(u)$ for each vertex $u$.

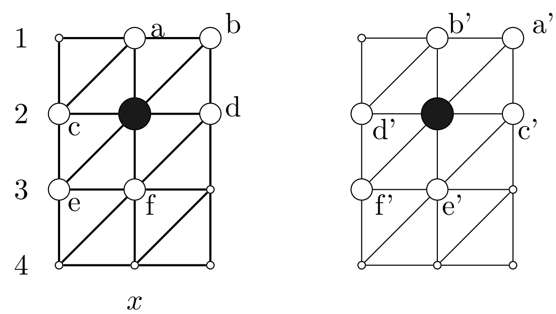


Figure 1: A symmetry of the triangular grids.

If $C$ is an identifying code in $S_k$ and $\sigma$ is an isomorphism of $S_k$, then $\sigma(C)$ is clearly an identifying code of $C$ with same density as $C$. Moreover all the discharging rules we consider are stable under isomorphism. Henceforth, in our proof,s when we write without loss of generality, we often mean free to consider $\sigma(C)$, for some isomorphism $\sigma$ of $S_k$.

We also use some common notation in all proofs. Given an identifying code $C$ of $T_k$, we denote by $U$ the set $V(T_k) \setminus C$. For any set $X \subseteq V(T_k)$, we denote $X_k$ (resp. $X_{\geq i}$, $X_{\leq i}$) the set of vertices $x$ in $X$ such that $N[x] \cap C = i$ (resp. $N[x] \cap C \geq i$, $N[x] \cap C \leq i$). let $U_i$ be the set of vertices outside $C$ with $i$ neighbors in $C$.

A vertex of $C$ is *isolated* if all neighbors are not in $C$. If $v \in C$ is isolated, then no neighbor $u$ of $v$ is in $U_1$, since, otherwise, $u$ and $v$ have the same code $\{v\}$. We say that a vertex $v$ is *in the line* $i$ if $v = (x, i)$ for some $x \in \mathbb{Z}$.

In the proof, the discharging rules are applied one after another. at any step of the procedure, we denote by $\mathrm{chrg}(v)$ the charge of a vertex $v$ (usually the step is clear from the context). We say that a vertex is unsatisfied if $\mathrm{chrg}(v) < p_k$. Let $\mathrm{exc}(v) = \mathrm{chrg}(v) - p_k$ be the amount of charge that $v$ can transfer to other vertices. Given a set $X$ of vertices, let $\mathrm{chrg}(X) = \sum_{v \in X} \mathrm{chrg}(v)$ and $\mathrm{exc}(X) = \sum_{v \in X} \mathrm{exc}(v)$. Let $N(X) = \cup_{v \in X} N(v) \setminus X$ be the neighbors not in $X$ of the vertices of $X$. Let $\mathrm{uns}(X)$ be the number of unsatisfied neighbors of $X$, that is, the unsatisfied vertices of $N(X)$.

## 2 The infinite triangular grid with two, three or six rows

In this section, we prove the following theorem.

**Theorem 1.** *$d^*(T_2) = 1/2$, $d^*(T_3) = 1/3$ and $d^*(T_6) = 1/3$.*

From the identifying codes of Figures 2, 3 and 4, we obtain the following upper bounds: $d^*(T_2) \leq 1/2$, $d^*(T_3) \leq 1/3$ and $d^*(T_6) \leq 1/3$. In other words, consider the sets $C_{2,a}, C_{2,b}, C_{3,a}, C_{3,b}, C_6$ given below.

$$\begin{aligned}
C_{2,a} &= \{(x,1) \mid x \equiv 1,3 \mod 5\} \cup \{(x,2) \mid x \equiv 1,2,4 \mod 5\}; \\
C_{2,b} &= \{(x,1) \mid x \equiv 1,2,3,4 \mod 5\} \cup \{(x,2) \mid x \equiv 2 \mod 5\}; \\
C_{3,a} &= \{(x,1) \mid x \equiv 1 \mod 2\} \cup \{(x,3) \mid x \equiv 1 \mod 2\}; \\
C_{3,b} &= \{(x,1) \mid x \equiv 1 \mod 3\} \cup \{(x,2) \mid x \equiv 2 \mod 2\} \cup \{(x,3) \mid x \equiv 3 \mod 3\}; \\
C_6 &= \{(x,1),(x,3) \mid x \text{ odd }\} \cup \{(x,5) \mid x \in \mathbb{Z}\}.
\end{aligned}$$

It is easy to check that $C_{2,a}$ and $C_{2,b}$ are identifying codes of $T_2$ with density $1/2$, $C_{3,a}$ and $C_{3,b}$ are identifying codes of $T_3$ with density $1/3$, and $C_6$ is an identifying code of $T_6$ with density $1/3$.

In order to prove the lower bounds, we introduce the notion of quasi-identifying code of $T_3$. Roughly speaking, a quasi-identifying code of $T_3$ does not care about the last line. Formally, a *quasi-identifying code* $C$ of $T_3$ is a subset $C \subseteq V(T_3)$ such that

(i) for all $v \in V(T_2)$, $N[v] \cap C \neq \emptyset$, and

(ii) for all $u, v \in V(T_2)$, $N[u] \cap C \neq N[v] \cap C$.

The main technical result of this section is the following.

**Lemma 2.** *Every quasi-identifying code $C'$ of $T_3$ has density $d(C', T_3)$ at least 1/3.*

Before proving this lemma, we show that this result implies Theorem 1.

*Proof of Theorem 1.* Notice that every identifying code $C'_3$ of $T_3$ is obviously a quasi-identifying code of $T_3$. Then, from Lemma 2, $d(C'_3, T_3) \geq 1/3$.

Also notice that every identifying code $C'_2$ of $T_2$ is also a quasi-identifying code of $T_3$. Then, since $d(C'_2, T_3) = 2 \cdot d(C'_2, T_2)/3$, we obtain from Lemma 2 that $d(C'_2, T_2) \geq 1/2$.

Finally notice that every identifying code $C'_6$ of $T_6$ induces the following two quasi-identifying codes of $T_3$:

$$\begin{aligned} C'_{3,a} &= \{(x,y) \mid (x,y) \in C'_6 \text{ and } y \in \{1,2,3\}\}; \\ C'_{3,b} &= \{(x,7-y) \mid (x,y) \in C'_6 \text{ and } y \in \{4,5,6\}\}. \end{aligned}$$

Then $d(C'_6, T_6) \geq 1/3$, since, from Lemma 2, $d(C'_{3,a}, T_3) \geq 1/3$ and $d(C'_{3,b}, T_3) \geq 1/3$. $\square$

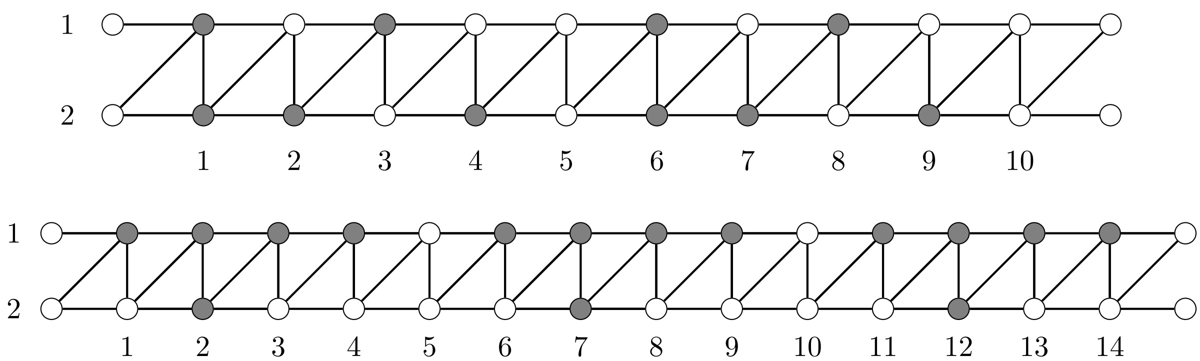


Figure 2: Two optimal identifying codes of $T_2$ : $C_{2,a}$ (top) and $C_{2,b}$ (bottom). The grey vertices are those of the code.

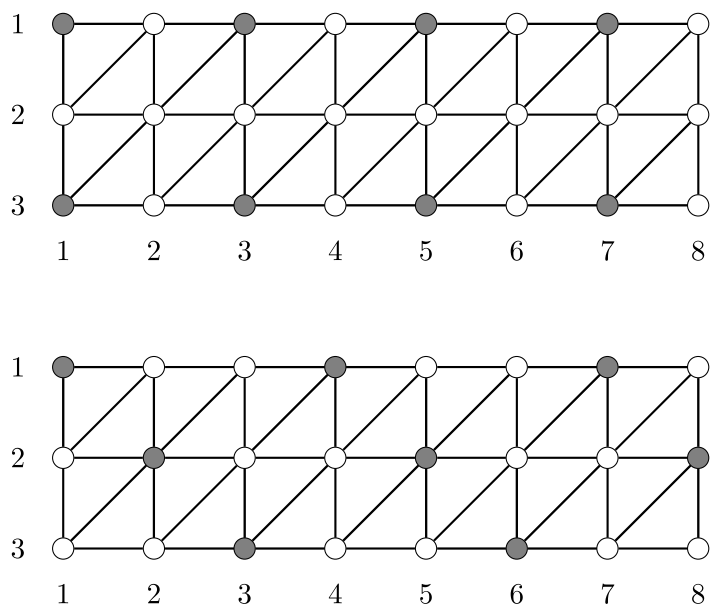


Figure 3: Identifying codes $C_{3,a}$ (top) and $C_{3,b}$ (bottom) of $T_3$ with density 1/3.

*Proof of Lemma 2.* Let $C$ be a quasi-identifying code of $T_3$ and let $U = V(T_3) \setminus C$. We shall prove that the density of $C$ in $V(T_3)$ is at least 1/3. Given $x \in \mathbb{Z}$, the *quasi-column* $Q_x$ is the set $\{(x,1), (x,2), (x-1,3)\}$. We shall use the Discharging Method on the quasi-column. Initially every quasi-column $Q_x$ receives a charge $\text{chrg}_0(Q_x) = |Q(x) \cap C|$. A quasi-column is said to be *full*, (resp. *heavy*, *average*, *empty*) if it initial charge is 3 (resp. 2, 1, 0). We apply the following rules:

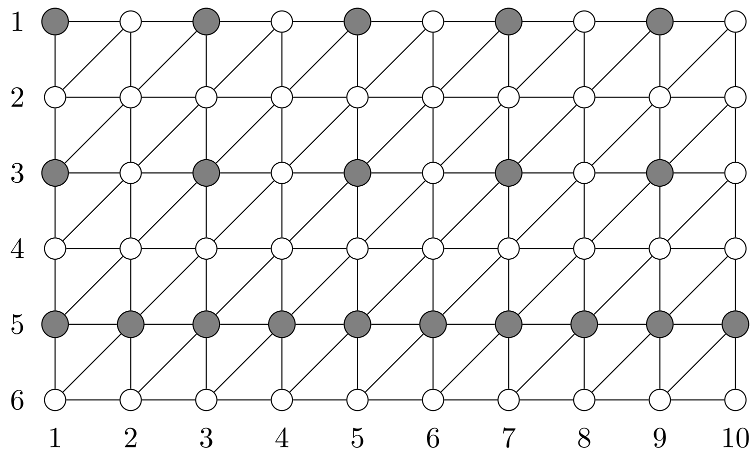


Figure 4: Identifying code $\mathcal{C}_6$ of $T_6$ with density 1/3

- (R1) Every full quasi-column sends 1 to the two closest empty columns to its right if they are at distance less than 5.
- (R2) Every heavy quasi-column whose right column is not heavy sends 1 to the closest empty column to its right if it is at distance less than 5.
- (R3) Every heavy quasi-column whose right column is heavy sends 1 to the second closest empty column to its right if it is at distance less than 5.

Note that the condition of distance less than 5 is here to guarantee that charge is only sent locally. We shall prove that the final charge of every quasi-column is at least 1. Since a quasi-column contains three vertices, this implies that the density of $C$ is clearly at least 1/3.

A full column sends (at most) 2 by (R1), so its final charge is at least 1. A heavy column sends (at most) 1 by (R2) or (R3), so its final charge is at least 1. An average column sends and receives nothing so its final charge receives 1. Hence it only remains to prove that the final charge of every empty quasi-column is at least 1.

Let $Q(x)$ be an empty quasi-column.

Assume first that $Q(x-1)$ is also empty. Then $(x-2,2),(x-2,1) \in C$, because $C[(x-1,2)] \neq \emptyset$ and $C[(x-1,1)] \neq C[(x-1,2)]$. If $(x-3,3) \in C$, then $Q(x-2)$ is full and both $Q(x-1)$ and $Q(x)$ receives 1 from $Q(x-2)$ by (R1).If $(x-3,3) \notin C$, then $(x-3,2),(x-3,1) \in C$, since $C[(x-2,2)] \neq C[(x-1,1)]$ and $C[(x-2,1)] \neq C[(x-2,2)]$. Hence $Q(x-1)$ receives 1 from $Q(x-2)$ by (R1) and $Q(x)$ receives 1 from $Q(x-3)$ by (R1) or (R3).

If $Q(x-1)$ is heavy or full, then $Q(x)$ receives 1 from it by (R1) or (R2). Henceforth we may assume that $Q_{x-1}$ is average.

If $Q_{x-1} \cap C = \{(x-1,1)\}$, then $(x-2,1) \in C$, because $C[(x-1,1)] \neq C[(x-1,2)]$. If $Q_{x-2}$ is full or heavy, then $Q(x)$ receives 1 from it by (R1) or (R2). Hence we may assume $Q_{x-2} \cap C = \{(x-2,1)\}$. Then $(x-3,2),(x-3,1) \in C$, since $C[(x-2,2)] \neq C[(x-1,1)]$ and $C[(x-2,1)] \neq C[(x-2,2)]$. Hence $Q_{x-3}$ sends 1 to $Q_x$, by (R1) or (R2).

Assume now $Q_{x-1} \cap C = \{(x-1,2)\}$. Then $(x-2,1) \in C$, since $C[(x-1,1)] \neq C[(x-1,2)]$. If $Q_{x-2}$ is full or heavy, then $Q(x)$ receives 1 from it by (R1) or (R2). Hence we may assume $Q_{x-2} \cap C = \{(x-2,1)\}$. Furthermore $(x-3,2) \in C$, since $C[(x-2,2)] \neq C[(x-1,1)]$. If

$Q_{x-3}$ is full or heavy, then $Q(x)$ receives 1 from it by (R1) or (R2). Hence we may assume $Q_{x-3} \cap C = \{(x-3,2)\}$. Now $(x-4,2),(x-4,1) \in C$, since $C[(x-3,2)] \neq C[(x-2,1)]$ and $C[(x-3,1)] \neq C[(x-3,2)]$. Thus $Q_{x-3}$ sends 1 to $Q_x$ by (R1) or (R2).

Finally suppose that $Q_{x-1} \cap C = \{(x-2,3)\}$. Since $C[(x-1,1)] \neq \emptyset$, then either $(x-2,1) \in C$ or $(x-2,2) \in C$. If $Q_{x-2}$ is full or heavy, then $Q(x)$ receives 1 from it by (R1) or (R2). Hence we may assume that $Q_{x-2}$ is average.

- Firstly assume that $(x-2,1) \in C$. Since $C[(x-2,1)] \neq C[(x-1,1)]$, then either $(x-3,1) \in C$ or $(x-3,2) \in C$. If $Q_{x-3}$ is full or heavy, then $Q(x)$ receives 1 from it by (R1) or (R2). Hence we may assume that $Q_{x-3} \cap C = \{(x-3,1)\}$ or $Q_{x-3} \cap C = \{(x-3,2)\}$. In both cases $(x-4,2),(x-4,1) \in C$, since $C[(x-3,2)] \neq C[(x-2,1)]$ and $C[(x-3,1)] \neq C[(x-3,2)]$. Thus $Q_{x-4}$ sends 1 to $Q_x$ by (R1) or (R2).
- Secondly assume that $(x-2,2) \in C$. Then $(x-3,2) \in C$, since $C[(x-2,2)] \neq C[(x-1,2)]$. If $Q_{x-3}$ is full or heavy, then $Q(x)$ receives 1 from it by (R1) or (R2). Hence we may assume that $Q_{x-3} \cap C = \{(x-3,2)\}$. Now $(x-4,2) \in C$, since $C[(x-3,2)] \neq C[(x-2,1)]$. Moreover since $C[(x-3,1)] \neq C[(x-4,3)]$, either $(x-4,1)$ or $(x-5,2)$ is in $C$. Therefore $Q(x-4)$ is heavy or full, so $Q_{x-4}$ sends 1 to $Q_x$ by (R1) or (R2).

$\square$

## 3 The infinite triangular grid with four rows

In this section, we prove the following theorem.

**Theorem 3.**
$$d^*(T_4) = 1/3.$$

*Proof.* Consider the two codes $C_4$ and $C'_4$ defined by

$$\begin{aligned} C_4 &= \{(x,2) \mid x \equiv 0,3 \mod 3\} \cup \{(x,3) \mid x \equiv 0,1 \mod 3\}; \\ C'_4 &= \{(x,1) \mid x \equiv 1 \mod 1\} \cup \{(x,2) \mid x \equiv 2 \mod 3\} \\ &\quad \cup \{(x,3) \mid x \equiv 0 \mod 3\} \cup \{(x,4) \mid x \equiv 1 \mod 3\}. \end{aligned}$$

See Figure 5.

It is easy to see that those codes have density 1/3, so $d^*(T_4) \leq 1/3$.

Let us now prove that $d^*(T_4) \geq 1/3$.

Let $C$ be an identifying code of $T_4$ and let $U = V(T_4) \setminus C$. We want to prove by the discharging method that the density of $C$ in $V(T_4)$ is at least 1/3. Set $q_4 = 3$ and $p_4 = 1$. Every vertex of $C$ begins with charge $q_4$ and every vertex of $U$ begins with charge 0. We say that a vertex $v$ is unsatisfied if $\text{chrg}(v) < p_4$. We will prove that, after the application of some discharging rules, every vertex of $T_4$ is satisfied. This yields $d(C,T_4) \geq p_4/q_4 = 1/3$.

We say that a vertex $(x,y) \in V(T_4)$ is in the *border* if $y \in \{1,4\}$, otherwise it is in the *center*. We will apply the following discharging rules. We apply the following discharging rules, one after another. It is important to note tat the order of the rules is important and that the set of unsatisfied vertices and the excess of every vertex are updated after each rule is applied.

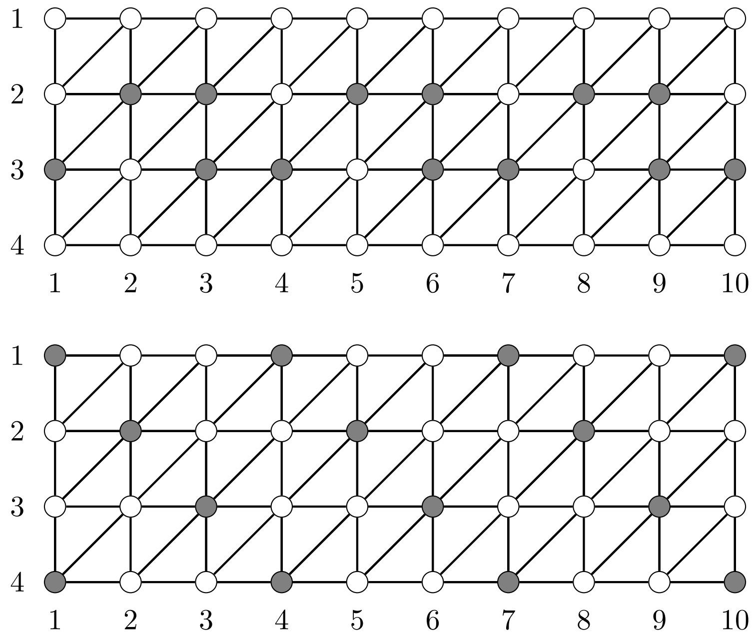


Figure 5: Two identifying codes of $T_4$ with minimum density 1/3.

For every $v \in C$ and every unsatisfied neighbor $u \in U$ of $v$:

- Rule 1a: $v$ sends 1 to $u$, if either $u \in U_1$ or $\mathrm{exc}(v) \geq \mathrm{uns}(v)$;
- Rule 1b: $v$ sends 1 to $u$, if $\mathrm{exc}(v) \geq \mathrm{uns}(v)$;
- Rule 2a: $v$ sends 1/2 to $u$, if $v$ is in the border and $\mathrm{exc}(v) \geq \mathrm{uns}(v)/2$;
- Rule 2b: $v$ sends 1/2 to $u$, if $\mathrm{exc}(v) \geq \mathrm{uns}(v)/2$;
- Rule 2c: $v$ sends 1/2 to $u$, if $\mathrm{exc}(v) \geq \mathrm{uns}(v)/2$;
- Rule 2d: $v$ sends 1/2 to $u$, if $\mathrm{exc}(v) \geq \mathrm{uns}(v)/2$;
- Rule 3: $v$ sends 1/2 to $u$, if $u \in U_2$ and $v$ did not send charge to $u$ in the previous rules;
- Rule 4: $v$ sends 1/4 to $u$ if $u \in U_{\geq 3}$ and $v$ did not send charge to $u$ in the previous rules.

We have to prove that the final charge of every vertex is at least 1.

**Claim 3.1.** *(i) At Rule 2a, every border vertex in $C$ sends 1/2 to each of its unsatisfied neighbors.*

*(ii) If a center vertex has at most three neighbors in $U$, or at most four neighbors in $U$, none of them in $U_1$, then at Rule 2b, it sends 1/2 to each of its unsatisfied neighbors.*

*Proof.* (i) Let $v$ be a border vertex in $C$.

If it has at most two neighbors in $U$, then it sends 1 to each by 1a, and all its neighbors satisfied after Rule 1a. Henceforth, we assume that $v$ has at least three neighbors in $U$.

Assume moreover that $v$ has a neighbor $u \in U_1$, then its identifier is $\{v\}$. Hence, any vertex $w \in N[v] \setminus \{u\}$ is adjacent to a vertex of $C \setminus \{v\}$ because $C[w] \neq C[u] = \{v\}$. Hence, $v$ has a neighbor in $C$, and so exactly two neighbors $u_2$ and $u_2'$ in $U_{\geq 2}$. If those are unsatisfied after Rule 1b, then $v$ only sent 1 to its $u_1$ by Rule 1a. Hence after Rule 1b $\text{exc}(v) = 1 = \text{uns}(v)/2$. Hence $u_2$ and $u_2'$ receive 1/2 at Rule 2a.

Assume finally that $v$ has no neighbors in $U_1$, then it has at most four neighbors in $U_{\geq 2}$. If they are unsatisfied after Rule 1b, $v$ did not send anything at Rule 1a and 1b. Therefore, after Rule 1b $\text{exc}(v) = 2 \geq \text{uns}(v)/2$. So each unsatisfied neighbor of $v$ receives 1/2 at Rule 2a.

The proof of (ii) is identical to the one of (i). $\Diamond$

**Claim 3.2.** *Every vertex in $U$ has final charge at least* 1.

*Proof.* Every vertex in $U_1$ receives 1 by Rule 1a, so it is satisfied. Every vertex in $U_2$ receives at least 1/2 from each neighbor in $C$ by Rule 3 or a preceding rule. So it receives at least 1 in total. Every vertex in $U_{\geq 3}$ receives at least 1/4 from each neighbor by the rules. Therefore every vertex in $U_{\geq 4}$ receives at least 1.

It remains to prove that every vertex $u \in U_3$ has final charge at least 1. By the above remark, it suffices to prove that it receives at least 1/2 from one of its three neighbors in $C$. If one of those neighbors is in the border, then we have the result by Claim 3.1-(i). Hence, we may assume that $v$ has no neighbor in $C$ in the border. In particular, $u$ is in the center. Without loss of generality, we may assume $u = (3, 2)$ and that we are in one of the following two cases.

1. $C[(3, 2)] = \{(2, 2), (2, 3), (3, 3)\}$. Since $C[(2, 3)] \neq C[(3, 2)]$, then at least one vertex of $\{(1, 3), (1, 4), (2, 4)\}$ belongs to $C$. Thus, $(2, 3)$ has at most three neighbors in $U$, none of them in $U_1$. Then $(2, 3)$ sends 1/2 to $u = (3, 2)$ at Rule 2b (or 1 earlier).

2. $C[(3, 2)] = \{(2, 2), (4, 2), (2, 3)\}$.

   If $(2, 1) \in C$, then $(2, 2)$ has at most four neighbors in $U$, none of them in $U_1$. Hence, by Claim 3.1-(ii), $(2, 2)$ sends 1/2 to $u = (3, 2)$ at Rule 2b (or 1 earlier), and we are done. Henceforth we assume $(2, 1) \notin C$. Thus $(3, 1) \in U_1$, and $(2, 2)$ sends 1 to it at Rule 1a.

   If $(1, 3) \in C$, then $C[(2, 2)] \neq C[(2, 3)]$ implies that either $(2, 2)$ or $(2, 3)$ has at most three neighbors in $U$ and consequently (by Claim 3.1-(ii)) sends 1/2 to $u = (3, 2)$ at Rule 2b (or 1 earlier). Henceforth, we assume $(1, 3) \notin C$. If either $(1, 4) \in C$ or $(2, 4) \in C$, then $(2, 3)$ has at most four neighbors in $U$, none of them in $U_1$. Hence, by Claim 3.1-(ii), $(2, 3)$ sends 1/2 to $u = (3, 2)$ at Rule 2b. Henceforth, we assume that $(1, 4), (2, 4) \notin C$.

   Thus $(1, 2) \in C$, since $C[(2, 2)] \neq C[(2, 3)] = \{(2, 2), (2, 3)\}$.

   If $(0, 4) \in C$, then, since $C[(1, 2)] \neq C[(2, 1)]$, either $(0, 2) \in C$ or $(0, 3) \in C$. If $(0, 2) \in C$, then $(1, 2)$ has at most four neighbors in $U$, none of them in $U_1$. Thus, by Claim 3.1, $(1, 3)$ is satisfied by $(0, 4)$ and $(1, 2)$ at Rule 2b (or earlier). If $(0, 3) \in C$, then $(0, 3)$ has at most four neighbors in $U$, none of them in $U_1$), so $(1, 3)$ is satisfied by $(0, 4)$ and $(0, 3)$ at Rule 2b (or earlier). In both cases, if $u$ was not satisfied before, at Rule 2c $\text{exc}((2, 2)) = 1 \geq \text{uns}((2, 2))$. So $(2, 2)$ sends 1/2 to $u$. Henceforth we assume $(0, 4) \notin C$.

Consequently, $(0,3) \in C$, since $C[(1,3)] \neq C[(2,2)]$. Now $(3,4) \in C$ because $C[(2,4)] \neq C[(1,4)] = \{(2,3)\}$. Suppose that $(4,3) \in C$. Notice that $(4,3)$ has no vertex in $U_1$, since $C[(4,4)] \neq C[(3,4)]$ implies that either $(5,3) \in C$ or $(5,4) \in C$. Hence by Claim 3.1, $(3,3)$ and $(4,4)$ are satisfied by $(3,4)$ and $(4,3)$ at Rule 2b, and so $(2,4)$ is satisfied by $(3,4)$ at Rules 2b and 2c. Hence, at Rule 2d, $\text{exc}(2,3) = 1$ (because it sends 1 to $(1,4)$ at Rule 1.a) and $\text{uns}(2,3) \leq 1$. Therefore $(2,3)$ sends $1/2$ to $(3,2)$ at Rule 2d. Henceforth, we assume $(4,3) \notin C$.

Then $(5,2), (6,1) \in C$, since $C[(4,2)] \neq C[(4,1)]$ and $C[(5,1)] \neq C[(4,2)]$. Notice that either $(5,3) \in C$ or $(6,2) \in C$, since $C[(5,2)] \neq C[(5,1)]$. In both cases, $(5,2)$ has no vertex in $U_1$. Then, at Rule 2b, $(4,3)$ is satisfied by $(3,4)$ and $(5,2)$, and $(5,1)$ is satisfied by $(5,2)$ and $(6,1)$. Consequently $(4,2)$ sends $1/2$ to $(3,2)$ at Rule 2c. $\Diamond$

**Claim 3.3.** *Every vertex in $C$ has final charge at least* 1.

*Proof.* This is equivalent to prove that every vertex in $C$ has non-negative final excess or that it sends at most 2 in total. Let $v$ be a vertex in $C$. It has initial excess 2 and at most one neighbor in $U_1$ (of such neighbors have identifier $\{v\}$) and so $v$ sends at most 1 to vertices in $U_1$. Hence afer Rule 1a, $v$ has non-negative excess. Therefore it also has non-negative excess after Rule 2d because at each of the Rules 1b to 2d it cannot send more than its excess. Hence a vertex can have negative only if sends charge by Rule 3 or Rule 4.

If $v$ is in the border, then by Claim 3.1, it sends at least $1/2$ to each unsatisfied neighbor at Rule 2a. Therefore, Rule 3 and Rule 4 never applies to it, and so its final charge is at least 1. Henceforth we assume that $v$ is in the center. Without loss of generality, $v = (2,2)$.

Furthermore, if $v$ sends charge to a neighbor by Rule 3 or Rule 4, then its has not sent any charge by Rules 1b to 2d, because it always send to all its unsatisfied neighbors. Let $\text{exc}'(v)$ be the excess of $v$ before Rule 3: $\text{exc}'(v) = 2$ if $v$ has no neighbor in $U_1$ and $\text{exc}'(v) = 1$ if it has a neighbor in $U_1$. For any $j$, let $u_j(v)$ (resp. $u_{\geq j}(v)$) be the number of neighbors of $v$ in $U_j$ (resp. $U_{\geq j}$), and let $u'_j(v)$ (resp. $u'_{\geq j}(v)$) be the number of neighbors of $v$ in $U_j$ that are unsatisfied after Rule 2. We need to prove

$$\frac{1}{2}u'_2(v) + \frac{1}{4}u'_3(v) \leq \text{exc}'(v). \tag{1}$$

This equation holds trivially when $u_{\geq 2}(v) \leq 2$ and $u_1(v) = 1$, or $u_{\geq 2}(v) \leq 4$ and $u_1(v) = 0$. Henceforth we assume that $v$ has either one neighbor in $U_1$ and at least three neighbors in $U_{\geq 2}$, or no neighbor in $U_1$ and at least five neighbors in $U_{\geq 2}$. We will consider all possibilities for the neighborhood of $v$ under this condition. We distinguish the three following cases : $v$ has a neighbor in $U_1$ which is in the border (Case 1); $v$ has a neighbor in $U_1$ which is in the center (Case 2); $v$ has no neighbor in $U_1$ (Case 3).

**Case 1:** $v$ has a neighbor in $U_1$ which is in the border. By symmetry, we may assume that this neighbor is $(3,1)$. Since $u_{\geq 2}(v) \geq 3$, at least one vertex in $\{(1,2),(1,3),(2,3)\}$ is in $U_{\geq 2}$.

We consider five subcases: $(1,2),(1,3) \in C$ and $(2,3) \in U$ (Subcase 1.1), $(1,3),(2,3) \in C$ and $(1,2) \in U$ (Subcase 1.2), $(2,3) \in C$ and $(1,3) \in U$ (Subcase 1.3), $v$ has only a horizontal neighbor in $C$ (Subcase 1.4), $v$ has only a diagonal neighbor in $C$ (Subcase 1.5).

Subcase 1.1: $(1,2),(1,3) \in C$ and $(2,3) \in U$. If $(1,1) \in C$, then $u_1((1,2)) = 0$ and $u_{\geq 2}((1,2)) \leq 4$, so $(1,1)$ and $(1,2)$ send $1/2$ each to $(2,1)$ by Rule 2b by Claim 3.1. Hence

$u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). Similarly, we get the result if $(1,4) \in C$. Henceforth we assume that $(1,1),(1,4) \notin C$.

If $(0,4) \in C$, then $u_1((1,3)) = 0$ and $u_{\geq 2}((1,3)) \leq 3$. Consequently $(0,3)$ and $(1,4)$ are satisfied by $(0,4)$ and $(1,3)$ at Rule 2b, $(2,3)$ is satisfied by $(1,3)$ at Rules 2b and 2c, and so $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). Henceforth we assume that $(0,4) \notin C$.

Then $(0,3) \in C$ (since $C[(1,3)] \neq C[(2,2)]$) and consequently $(0,2) \in C$ (since $C[(1,2)] \neq C[(1,3)]$). Hence $(1,2)$ satisfies $(2,1)$ at Rule 1a, and $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1).

Subcase 1.2: $(1,3),(2,3) \in C$ and $(1,2) \in U$. Because $C[(2,1)] \neq C[(3,1)]$, $(1,1) \in C$. Moreover, $(1,3)$ has at most three neighbors in $U$ (since $C[(1,3)] \neq C[(2,2)]$). Then $(1,2)$ is satisfied by $(1,1)$ and $(1,3)$ at Rule 2b. Hence $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1).

Subcase 1.3: $(2,3) \in C$ and $(1,3) \in U$.

Suppose that $(1,4) \in C$. If either $(0,4) \in C$ or $(2,4) \in C$, then $(1,4)$ satisfies $(1,3)$ at Rule 1a, $(2,4)$ is either in $C$ or $(2,4)$ satisfied by $(1,4)$ at Rule 1a, and $(3,2)$ and $(3,3)$ are satisfied by $(2,3)$ at Rule 1b. Consequently $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). If $(0,4),(2,4) \notin C$, then either $(3,3) \in C$ or $(3,4) \in C$ (since $C[(2,4)] \neq C[(1,4)] = \{(1,4),(2,3)\}$). Therefore, $(2,3)$ has at most four neighbors in $U$, none of them in $U_1$. If $(3,3) \in C$, then $(1,3)$ and $(2,4)$ are satisfied by $(1,4)$ and $(2,3)$ at Rule 2b, $(3,2)$ is satisfied by $(2,3)$ at Rules 2b and 2c, so $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). If $(3,4) \in C$, then $(2,4)$ is satisfied by $(1,4)$ and $(3,4)$ at Rule 2a, $(1,3)$ is satisfied by $(1,4)$ and $(2,3)$, $(3,3)$ is satisfied by $(2,3)$ and $(3,4)$, $(3,2)$ is satisfied by $(2,3)$ at Rules 2b and 2c, so $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). Henceforth we assume that $(1,4) \notin C$.

Therefore either $(-2,4) \in C$ or $(-1,3) \in C$, since $C[(-1,4)] \neq C[(0,4)]$.

Suppose that $(2,4) \in C$. If $(0,4) \in C$, then $(1,4)$ is satisfied by $(0,4)$ and $(2,4)$ at Rule 2a, $(1,3)$ is satisfied by $(0,4)$ and $(2,3)$ at Rule 2b, $(3,3)$ is satisfied by $(2,4)$ and $(2,3)$, $(3,2)$ is satisfied by $(2,3)$ at Rules 2b and 2c, so $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). If $(0,4) \notin C$, then either $(3,3) \in C$ or $(3,4) \in C$ (since $C[(2,4)] \neq C[(1,4)] = \{(2,3)\}$), and consequently $(1,4)$ is satisfied by $(2,4)$ at Rule 1a, $(3,3)$ is either in $C$ or satisfied by $(2,4)$ at Rule 1a, and $(1,3)$ and $(3,2)$ are satisfied by $(2,3)$ at Rules 2b and 2c, so $u'_{\geq 2}(v) \leq 2$ and $v$ satisfies Eq. (1). Henceforth, we assume that $(2,4) \notin C$.

It follows that $(4,2) \in C$, since $C[(3,2)] \neq C[(2,3)]$.

If $(1,2) \in C$, then $u_2(v) \leq 1$ and $u_{\geq 3}(v) \leq 2$, so $v$ satisfies Eq. (1). Henceforth we assume that $(1,2) \notin C$.

It follows that $(1,1) \in C$ because $C[(2,1)] \neq C[(3,1)] = \{v\}$, and $(3,3) \in C$ because $C[(2,2)] \neq C[(2,3)]$. Hence $((3,2) \in U_3$. Furthermore $(1,2),(1,3) \in U_{\geq 3}$, since $C[(1,2)] \supset C[(2,1)]) = \{(1,1),(2,2)\}$ and $C[(1,3)] \supset C[(2,2)] = \{(2,2),(2,3)\}$. Hence $u_2(v) = 1$ and $u_{\geq 3}(v) = 3$.

If $(0,4) \in C$, then $(1,3)$ is satisfied by $(0,4)$ and $(2,3)$ at Rule 2b. So $u'_2(v) \leq 1$ and $u'_{\geq 3}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(0,4) \notin C$.

Consequently $(0,3) \in C$ because $C[(1,3)] \neq C[(2,2)]$.

If $(0,2) \in C$, then $(0,2)$ has no vertex in $U_1$, since $C[(0,1)] \neq C[(1,1)]$ implies that either $(-1,1) \in C$ or $(-1,2) \in C$. Consequently $(1,2)$ is satisfied by $(1,1)$ and $(0,2)$ at Rule 2b, so $u'_2(v) \leq 1$ and $u'_{\geq 3}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(0,2) \notin C$.

Therefore $(-1,3) \in C$ since $C[(0,3)] \neq C[(0,4)]$, and $(-2,4) \in C$ since $C[(-1,4)] \neq$

$C[(0,3)]$. Notice that $(-1,3)$ has at most four neighbors in $U$, none of them in $U_1$, since $C[(0,1)] \neq C[(1,1)]$ implies that either $(-1,1) \in C$ or $(-1,2) \in C$. Therefore $(-1,4)$ is satisfied by $(-2,4)$ and $(-1,3)$ at 2b, $(0,2)$ is satisfied by $(-1,3)$ and $(1,1)$ at Rule 2b, and consequently $(1,2)$ is satisfied by $(1,1)$ and $(0,3)$ at Rules 2b and 2c. Hence $u'_2(v) \leq 1$ and $u'_{\geq 3}(v) \leq 2$, and $v$ satisfies Eq. (1).

Subcase 1.4: $(1,2) \in C$, $(1,3),(2,3) \notin C$. Because $C[(2,1)] \neq C[(2,2)] = \{(1,2),(2,2)\}$, $(1,1) \in C$.

If $(0,2) \in C$, then $(2,1)$ is satisfied by $(1,1)$ at Rule 1a, $(1,3)$ is satisfied by $(1,2)$ at Rule 1b, and so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth, we assume $(0,2) \notin C$.

Now $(0,3) \in C$, since $C[(1,2)] \neq C[(2,1)] = \{(1,1),(1,2),(2,2)\}$. Therefore, $(0,2)$ and $(2,1)$ are satisfied by $(1,1)$ and $(1,2)$ at Rule 2b, $(1,3)$ is satisfied by $(1,2)$ at Rules 2b and 2c, and so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1).

Subcase 1.5: $(1,3) \in C$, $(1,2),(2,3) \notin C$. Because $C[(2,1)] \neq C[(3,1)] = \{(2,2)\}$, vertex $(1,1)$ is in $C$.

Suppose that $(1,4) \in C$. If $(0,4) \in C$, then $(2,3)$ is satisfied by $(1,4)$ at Rule 1a, $(1,2)$ is satisfied by $(1,3)$ at Rule 1b, and so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). If $(2,4) \in C$ and $(0,4) \notin C$, then $(0,4)$ and $(2,3)$ are satisfied by $(1,4)$ at Rule 1a, $(1,2)$ is satisfied by $(1,3)$ at Rule 1b, and so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). If $(0,4),(2,4) \notin C$, then $(1,3)$ has at most four neighbors in $U$, none of them in $U_1$ (since $C[(0,4)] \neq C[(1,4)]$ implies that either $(-1,4) \in C$ or $(0,3) \in C$). Consequently $(1,2)$ is satisfied by $(1,3)$ and $(1,1)$ at Rule 2a, $(2,3)$ is satisfied by $(1,3)$ and $(1,4)$ at Rule 2a, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(1,4) \notin C$.

Suppose $(0,4) \in C$. If $(2,4) \in C$, then $(1,2)$ is satisfied by $(1,3)$ and $(1,1)$ at Rule 2a, $(2,3)$ is satisfied by $(1,3)$ and $(2,4)$ at Rule 2a, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). If $(2,4) \notin C$, then $C[(0,4)] \neq C[(1,4)]$ implies that either $(-1,4) \in C$ or $(0,3) \in C$. In both cases, $(1,4)$ is satisfied by $(0,4)$ at Rule 1a, $(0,3)$ is either in $C$ or satisfied by $(0,4)$ at Rule 1a, $(1,2)$ and $(2,3)$ are satisfied by $(1,3)$ at Rule 1b, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(0,4) \notin C$.

It follows that $(0,3) \in C$, because $C[(1,3)] \neq C[(2,2)]$.

Since $C[(0,3)] \neq C[(0,4)]$, then either $(-1,3) \in C$ or $(0,2) \in C$. If $(-1,3) \in C$, then $(1,2)$ is satisfied by $(1,1)$ and $(0,3)$ at Rule 2a (since $(0,3)$ has at most four neighbors in $U$, none of them in $U_1$). If $(0,2) \in C$, then $(1,2)$ is satisfied by $(1,1)$ and $(0,2)$ at Rule 2a (since $C[(0,1)] \neq C[(1,1)]$ implies that either $(-1,1) \in C$ or $(-1,2) \in C$ and consequently $(0,2)$ has at most four neighbors in $U$, none of them in $U_1$) and $(2,3)$ receives charge $1/2$ from $(1,3)$ at Rule 2c. In both cases, $(1,2)$ is satisfied at Rule 2b by $(1,1)$ and either $(0,3)$ or $(0,2)$, and $(2,3)$ receives charge $1/2$ from $(1,3)$ at Rule 2b or 2c. Therefore, if we prove that $(2,3)$ receives charge $1/2$ at Rule 2a from a vertex distinct of $(1,3)$, then $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1).

If $(2,4) \in C$, then $(2,3)$ receives charge $1/2$ from $(2,4)$, and we are done. Henceforth, we assume that $(2,4) \notin C$.

Consequently $(3,3) \in C$, since $C[(2,3)] \neq C[(2,2)]$.

If $(4,2) \in C$, then $u_2(v) \leq 1$ and $u_{\geq 3}(v) \leq 2$, so $v$ satisfies Eq. (1). Henceforth we assume $(4,2) \notin C$.

It follows that $(5,1) \in C$ (since $C[(4,1)] \neq \emptyset$), $(4,3) \in C$ (since $C[(3,3)] \neq C[2,3] =$

$\{(3,3)\}$) and $(4,4) \in C$ (since $C[(3,4)] \neq C[(3,3)] = \{(3,3),(4,3)\}$). If $(3,4) \in C$, then $(2,4)$ is satisfied by $(3,4)$ at Rule 1a, and $(2,3)$ receives charge $1/2$ from $(3,3)$, since $(3,3)$ has at most three unsatisfied neighbors. Henceforth we assume that $(3,4) \notin C$.

Now $(3,4)$ is satisfied by $(4,4)$ and $(4,3)$ at Rule 2b, $(4,2)$ is satisfied by $(4,3)$ and $(5,1)$ at Rule 2b, and $(2,3)$ receives charge $1/2$ from $(3,3)$ at Rule 2c, as desired.

**Case 2:** $v$ has a neighbor $u$ in $U_1$ which is in the center.

Without loss of generality, we may assume that $u = (3,2)$ (Subcase 2.1) or $u = (2,3)$ (Subcase 2.2).

Subcase 2.1: $u = (3.2)$. Since $C[(3,1)] \neq C[(3,2)] = \{v\}$, vertex $(2,1)$ is in $C$.

If $(1,1) \in C$, then $(3,1)$ and $(1,2)$ are satisfied by $(2,1)$ at Rule 1a, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(1,1) \notin C$.

It follows that $(1,2) \in C$, since $C[(2,1)] \neq C[(3,1)] = \{(2,1),(2,2)\}$. Furthermore, $(1,3) \in C$, since $C[(2,2)] \neq C[(2,1)] = \{(1,2),(2,1),(2,2)\}$. Hence $u_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1).

Subcase 2.2: $u = (2,3)$. Necessarily, $(0,4),(3,4) \in C$, since $C[(1,4)] \neq \emptyset$ and $C[(2,4)] \neq \emptyset$. Moreover, $(1,1) \in C$, since $C[(2,1)] \neq C[(2,2)]$.

If $(2,1) \in C$, then $(1,2)$ and $(3,1)$ are satisfied by $(2,1)$ at Rule 1a, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(2,1) \notin C$.

If $(3,1) \in C$, then $(2,1)$ is satisfied by $(1,1)$ and $(3,1)$ at Rule 2a, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1). Henceforth we assume $(3,1) \notin C$.

It follows that $(4,1) \in C$, since $C[(3,1)] \neq C[(2,3)] = \{(2,2)\}$. Moreover $(1,2) \in C$, since $C[(2,2)] \neq C[(2,3)]$. Thus $(2,1)$ is satisfied by $(1,1)$ and $(1,2)$ at Rule 2b, $(1,3)$ is satisfied by $(1,2)$ and $(0,4)$ at Rule 2b, so $u'_{\geq 2}(v) \leq 2$, and $v$ satisfies Eq. (1).

**Case 3 :** $v$ has no neighbor in $U_1$.

Recall that in that case $\mathrm{exc}'(v) = 2$, and so we assumed that $u_{\geq 2}(v) \geq 5$. In particular, $v$ has at most one neighbor in $C$. Furthermore if $u_2(v) \leq 2$, then $u'(v) \leq 2$ and so $v$ satisfies Eq. (1). Henceforth we assume that $u_2(v) \geq 3$. Thus, without loss of generality, we must be in one of the following cases: $(2,1) \in C$ (Subcase 3.1), $(1,2) \in C$ (Subcase 3.2), $(1,3) \in C$ (Subcase 3.3), $v$ has no neighbor in $C$ (Subcase 3.4).

Subcase 3.1: $(2,1) \in C$. Because $C[(2,1)]$ and $C[(3,1)]$ are distinct from $C[v] = \{v,(2,1)\}$, both $(1,1)$ and $(4,1)$ are in $C$. Then $(3,1)$ is satisfied by $(2,1)$ and $(4,1)$ at Rule 2a, so $u'_{\geq 2}(v) \leq 4$ and $v$ satisfies Eq. (1).

Subcase 3.2: $(1,2) \in C$. Since $C[(2,1)] \neq C[(2,2)]$, $(1,1) \in C$, and since $C[(1,2)] \neq C[(2,1)] = \{(1,1),(1,2),(2,2)\}$, then $(1,2) \in U_{\geq 3}$. Hence $(2,1)$ is satisfied by $(1,1)$ and $(1,2)$ at Rule 2b, so $u'_{\geq 2}(v) \leq 4$ and $v$ satisfies Eq. (1).

Subcase 3.3: $(1,3) \in C$. Observe that $(1,1)$ and $(4,1)$ are in $C$, beacuse $(2,1)$ and $(3,1)$ are in $U_{\geq 2}$. Moreover, $(2,3)$ and $(3,2)$ are in $U_{\geq 3}$ because $C[(2,3)] \neq C[v]$ and $C[(3,2)] \neq C[(3,1)]$. This contradicts the assumption $u_2(v) \geq 3$.

Subcase 3.4: $v$ has no neighbor in $C$. Both $(1,1)$ and $(4,1)$ are in $C$ because $C[(2,1)] \neq C[(2,2)]$ and $C[(3,1)] \neq C[(2,2)]$. Now $(1,2)$ and $(3,2)$ are in $U_{\geq 3}$ because $C[(1,2)] \neq C[(2,1)]$ and $C[(3,2)] \neq C[(3,1)]$. Recall that $u_2(v) \geq 3$. So $(1,3) \in U_2$ or $(2,3) \in U_2$. Without loss of generality, we may assume that $(1,3) \in U_2$.

Suppose that $(0,3) \notin C$. Then, $(0,2) \in C$ because $(1,2)$ is in $U_{\geq 3}$. If $(-1,3) \in C$, then $(0,2)$ has no $U_1$-vertex, so $(0,1)$ and $(1,2)$ are satisfied by $(0,2)$ and $(1,1)$ at Rule 2b, $(2,1)$ is satisfied by $(1,1)$ at Rules 2b and 2c; hence $u'_{\geq 2}(v) \leq 4$ so $v$ satisfies Eq. (1). Henceforth we assume that $(-1,3) \notin C$. Consequently $(-1,2) \in C$ because $C[(0,2)] \neq C[(1,1)]$. As above, if $(0,3) \notin U_1$, we get that $u'_{\geq 2}(v) \leq 4$ and $v$ satisfies Eq. (1). Thus we assume that $(0,3) \in U_1$ and consequently $(-1,4), (0,4) \notin C$. Therefore $(1,4), (2,4) \in C$, since $C[(0,4)] \neq \emptyset$ and $C[(1,4)] \neq C[(0,4)] = \{(1,4)\}$. Since $C[(2,4)] \supset C[(1,4)] = \{(1,4),(2,4)\}$, then $(2,3)$ is satisfied by $(2,4)$ at Rule 1a, $(1,3)$ is satisfied by $(1,4)$ at Rule 1b, so $u'_{\geq 2}(v) \leq 4$ so $v$ satisfies Eq. (1). Henceforth we assume that $(0,3) \in C$.

It follows that $(0,4), (1,4) \notin C$, since $(1,3) \in U_2$. Consequently, $(2,4) \in C$ because $C[(1,4)] \neq \emptyset$.

If $(3,4)$ is in $C$, then it has a neighbor in $C \setminus \{(2,4)$ because $C[(3,4)] \neq C[(2,4)]$. Hence either $(3,3) \in C$ or $(3,3)$ is satisfied by $(3,4)$ at Rule 1a, $(2,3)$ is satisfied by $(2,4)$ at Rule 1b, so $u'_2(v) \leq 3$ and $u'_3(v) \leq 2$ and $v$ satisfies Eq. (1). Henceforth we assume that $(3,4) \notin C$.

It follows that $(3,3) \in C$, since $C[(2,4)] \neq C[(1,4)] = \{(2,4)\}$. Consequently, $(2,3)$ and $(3,2)$ are in $U_{\geq 3}$. Because $C[(3,3)] \neq C[(2,4)]$, either $(4,2) \in C$ or $(4,3) \in C$. Thus $(3,3)$ has at most four neighbors in $U$, none of them in $U_1$. Hence $(2,3)$ is satisfied by $(2,4)$ and $(3,3)$ at Rule 2a and 2b. Therefore $u'_2(v) \leq 3$ and $u'_3(v) \leq 2$ and $v$ satisfies Eq. (1). ♢

□

## 4 The infinite triangular grid with five rows or more

**Theorem 4.** *Let $k \geq 5$ be an integer. Then $d^*(T_k) \geq 1/4 \ + \ 1/4k$.*

*Proof.* Let $C$ be an identifying code of $T_k$ and $U = V(T_k) \setminus C$. We want to prove by the discharging method that the density of $C$ in $V(T_k)$ is at least $1/4 + 1/4k$. Every vertex of $C$ begins with charge 1 and every vertex of $U$ begins with charge 0. We say that a vertex $(x,y) \in V(T_k)$ is in the *border* if $y \in \{1,k\}$; otherwise it is in the *center*. If $v$ is in the center, $\mathrm{exc}(v) = \mathrm{chrg}(v) - 1/4$. If $v$ is in the border, $\mathrm{exc}(v) = \mathrm{chrg}(v) - 3/8$. We say that a vertex $v$ is satisfied if $\mathrm{exc}(v) \geq 0$. We will prove that, after the application of some discharging rules, every vertex of $T_k$ will be satisfied, and we are done, since $((k-2)\cdot(1/4) + 2\cdot(3/8))/k = 1/4 + 1/4k$.

Given a vertex $v$ of $T_k$, let $B_1(v)$ be the set of neighbors of $v$ in the border and let $B_2(v)$ be the set of vertices in the border that are at distance 2 of $v$. When we say that $v$ sends *full-excess* (resp. *half-excess*) to $B_i(v)$ ($i \in \{1,2\}$), this means that $v$ sends $\mathrm{exc}(v)/b$ (resp. $\mathrm{exc}(v)/(2b)$) to every unsatisfied vertex of $B_i(v)$, where $b$ is the number of unsatisfied vertices of $B_i(v)$.

We will apply the following discharging rules.

- Rule 1: Every vertex in $C$ sends $1/4$ to its neighbor in $U_1$ (if one exists) and $1/8$ to each of its neighbors in $U_{\geq 2}$;
- Rule 2: Every vertex $v$ in the border with positive excess sends full-excess to $B_1(v)$;
- Rule 3: Every vertex $v$ in the row 2 or $k-1$ with positive excess sends full-excess to $B_1(v)$;

- Rule 4: Every vertex $v$ in the border with positive excess sends full-excess to $B_2(v)$;
- Rule 5: Every vertex $v$ in the row 2 or $k-1$ with positive excess sends full-excess to $B_2(v)$;
- Rule 6: Every vertex $v$ in the row 3 or $k-2$ with positive excess sends half-excess to $B_2(v) \cap (\mathbb{Z} \times \{1\})$ and half-excess to $B_2(v) \cap (\mathbb{Z} \times \{k\})$.

Notice that, after the application of Rule 1, every vertex has charge at least $1/4$. Indeed every vertex $u \in U$ receives either charge $1/4$, if $u \in U_1$, or charge at least $2 \cdot (1/8) = 1/4$, if $u \in U_{\geq 2}$, and every vertex $v \in C$ has either charge at least $1 - 6 \cdot (1/8) = 1/4$, if $v$ has no $U_1$-neighbor, or charge $1 - (1/4) - 4 \cdot (1/8) = 1/4$, if $v$ has a $U_1$-neighbor, since this implies that $v$ has a $C$-neighbor.

In particular, all vertices in the center are already satisfied. Moreover, some of them have positive excess:

(a) if $v \in C$ is in the border, then $\mathrm{chrg}_1(v) \geq 1 - 4 \cdot (1/8) = 1/2$ so $\mathrm{exc}_1(v) \geq 1/8$;

(b) if $v \in C_3$ is in the center, then $\mathrm{chrg}_1(v) \geq 1 - (1/4) - 3 \cdot (1/8) = 3/8$ so $\mathrm{exc}_1(v) \geq 1/8$;

(c) if $v \in C_{\geq 4}$ is in the center, then $\mathrm{chrg}_1(v) \geq 1 - (1/4) - 2 \cdot (1/8) = 1/2$ so $\mathrm{exc}_1(v) \geq 1/4$;

(d) if $u \in U_{\geq 3}$, then $\mathrm{chrg}_1(u) \geq 3 \cdot (1/8) = 3/8$ and $\mathrm{exc}_1(u) \geq 1/8$;

(e) if $u \in U_{\geq 4}$, then $\mathrm{chrg}_1(u) \geq 4 \cdot (1/8) = 1/2$ and $\mathrm{exc}_1(u) \geq 1/4$.

In particular, the vertices both in $C$ and the border are also satisfied.

Observe that at Rules 2 to 6 a vertex with positive excess divides its excess and sends it to its unsatisfied neighbors. Hence once a vertex is it never becomes unsatisfied after the application of those rules. Thus, we only have to prove that every vertex $u$ in $U$ and in the border is satisfied after some rule. It is equivalent to show that $u$ receives at lest $1/8$ after Rule 1. Without loss of generality, let $u = (1,1)$. $u$ has a neighbor in $C$, since $C[u] \neq \emptyset$.

If both $(0,1)$ and $(2,1)$ are in $C$, then $\mathrm{exc}_1((0,1)) \geq 1/8$ and $\mathrm{exc}_1((0,1)) \geq 1/8$. Therefore, $(0,1)$ and $(2,1)$ send charge at least $1/16$ each to $u$ at Rule 2, and $u$ is satisfied.

Suppose that exactly one of $\{(0,1),(2,1)\}$ is in $C$. Without loss of generality, assume that $(0,1) \notin C$ and $(2,1) \in C$. If $(3,1) \in C$, then $(2,1)$ sends $1/8$ to $(1,1)$ at Rule 2, and we are done. Hence assume that $(3,1) \notin C$. Because the three sets $C[(1,1)]$, $C[(2,1)]$, and $C[(1,2)]$ are distinct, $|C[(1,2)]| \geq 3$. Consequently by (b) and (d), $\mathrm{exc}_1((1,2)) \geq 1/8$ and $(1,2)$ sends $1/8$ to $u$ at Rule 3, and we are done. Henceforth we assume $(0,1),(2,1) \notin C$.

If both $(0,2)$ and $(1,2)$ are in $C$, then they both are in $C_{\geq 3}$ because $C[(1,1)]$, $C[(0,2)]$, and $C[(1,2)]$ are all distinct. Therefore by (b), $\mathrm{exc}_1((0,2)) \geq 1/8$ and $\mathrm{exc}_1((1,2)) \geq 1/8$. So $(0,2)$ and $(1,2)$ send at least $1/16$ each to $u$ at Rule 3, and so $u$ is satisfied. Henceforth exactly one vertex among $(0,2)$ and $(1,2)$ is in $C$. Without loss of generality, we may assume that $(1,2) \in C$ and $(0,2) \notin C$.

Suppose that $(3,1) \notin C$. Then $(2,2) \in C$, since $C[(2,1)] \neq C[(1,1)] = \{(1,2)\}$. Moreover both $(1,2)$ and $(2,2)$ are in $C_{\geq 3}$ because their identifier is different from $C[(2,1)] = \{(1,2),(2,2)\}$. Thus according to (b), $\mathrm{exc}_1((1,2)) \geq 1/8$ and $\mathrm{exc}_1((1,2)) \geq 1/8$. Therefore at Rule 3, $(1,2)$ sends $1/16$ to $(1,1)$ and $(2,1)$, and $(2,2)$ sends $1/16$ to $(2,1)$ and $(3,1)$. Hence, after Rule 3, $(2,1)$ is satisfied and the possibly unsatisfied vertices of $B_2((1,3))$ are $(1,1)$ and $(3,1)$. Now $C[(1,3)]$, $C[(1,2)]$ and $C[(2,1)]$ are all distinct and contain $\{(1,2),(2,2)\} = C[(2,1)]$. Hence $(1,3)$ is either in $C_{\geq 4}$ or in $U_{\geq 4}$. In both cases, according to (c) or (e), we have $\text{exc}_1((1,3)) \geq 1/4$. Thus $(1,3)$ sends charge $1/16$ to $u$ (and $(3,1)$) at Rule 6, and so $u$ is satisfied. Henceforth we assume $(3,1) \in C$.

If $(2,2) \in C$, then $(2,1) \in U_3$ and by (d) $\text{exc}_1((2,1)) \geq 1/8$. Moreover $B_1((2,1)) = \{u\}$ so $(2,1)$ sends $1/8$ to $u$ at Rule 2, and $u$ is satisfied. Henceforth, we assume $(2,2) \notin C$.

$C[(2,2)] \neq C[(2,1)] = \{(1,2),(3,1)\}$, so $(2,2) \in U_{\geq 3}$ and $\text{exc}_1((2,2)) \geq 1/8$ by (d).

If $(4,1) \in C$, then $(2,1)$ is satisfied by $(3,1)$ at Rule 2. So Before Rule 5, the only unsatisfied vertex in $B_2(2,2)$ is $u$. Thus at Rule 5, $(1,1)$ is satisfied by $(2,2)$. Henceforth we assume that $(4,1) \notin C$.

Observe that $(1,3)$ is either in $U_{\geq 3}$ or is in $C_{\geq 3}$ because $C[(1,3)]$, $C[(1,2)]$ and $C[(1,1)] = \{(1,2)\}$ are all distinct. Therefore $\text{exc}_1((1,2)) \geq 1/8$ by (b) and (d). Hence half-excess of $(1,2)$ at Rule 6 is at least $1/16$. We claim that $(2,1)$ is satisfied before Rule 6 and that $(1,1)$ receives charge at least $1/16$ from a vertex other than $(1,3)$ before Rule 6. This claim implies that $u$ is satisfied because it will be the only unsatisfied vertex of $B_2((1,3))$ before Rule 6 and so it will receive half-excess (i.e at least $1/16$) from $(1,3)$ at Rule 6. It remains to prove the claim.

If $(3,2)$ is in $C$, then it has at least two neighbors in $C$ because $C[(3,2)] \neq C[(3,1)] = \{(3,1),(3,2)\}$. Therefore $\text{exc}_1((3,2)) \geq 1/8$. Thus $(4,1)$ is satisfied by $(3,1)$ and $(3,2)$ at Rules 2 and 3, $(2,1)$ is satisfied by $(3,1)$ and $(2,2)$ at Rules 2 and 3, and $(2,2)$ sends $1/16$ to $u$ at Rule 5; so the claim holds. Henceforth we assume that $(3,2) \notin C$.

If $(5,1) \in C$, then $(4,1)$ is satisfied by $(3,1)$ and $(5,1)$ at Rule 2, $(2,1)$ is satisfied by $(3,1)$ and $(2,2)$ at Rules 2 and 3, and $(2,2)$ sends $1/16$ to $u$ at Rule 5; so the claim holds. Henceforth we assume that $(5,1) \notin C$.

It follows $(4,2) \in C$ because $C[(4,1)] \neq C[(3,1)] = \{(3,1)\}$. Consequently $(3,2) \in U_{\geq 3}$ because $C[(3,2)] \neq C[(4,1)] = \{(3,1),(4,2)\}$). Hence $(4,1)$ is satisfied by $(3,1)$ and $(3,2)$ at Rules 2 and 3, $(2,1)$ is satisfied by $(3,1)$ and $(2,2)$ at Rules 2 and 3, and $(2,2)$ sends $1/16$ to $u$ at Rule 5; this finishes the proof of the claim and of the whole proof. □

**Theorem 5.** *Let $k \geq 5$ be an integer. If $k$ is odd, then $d^*(T_k) = \frac{1}{4} + \frac{1}{4k}$. If $k$ is even, then $\frac{1}{4} + \frac{1}{4k} \leq d^*(T_k) \leq \frac{1}{4} + \frac{1}{2k}$.*

*Proof.* Assume first that $k$ is odd. Let $C_k = \{(x,y) : x \in \mathbb{Z}, y \in [k], x \text{ and } y \text{ are odd}\}$ (see Figure 6 for an example with $k = 5$). It is not difficult to see that $C_k$ is an identifying code : indeed $C[(x,y)] = \{(x,y)\}$, if $x,y$ odd, $C[(x,y)] = \{(x-1,y),(x+1,y)\}$, if $x$ even and $y$ odd, $C[(x,y)] = \{(x,y-1),(x,y+1)\}$, if $x$ odd and $y$ even, and $C[(x,y)] = \{(x-1,y+1),(x+1,y-1)\}$, if $x,y$ even.

Since $C_k$ has $(k+1)/2$ rows with density $1/2$ and $(k-1)/2$ rows with density 0, Hence $d(C_k, T_k) = \frac{1}{k} \cdot \frac{1}{2} \cdot \frac{k+1}{2} = 1/4 + 1(4k)$. Thus by Theorem 4, we have $d^*(T_k) = 1/4 + 1/(4k)$.

Assume now that $k \geq 6$ is even. Let $C_2$ be the second identifying code of $T_2$ in Figure 2 and let $C_2' = \{(x+k-2, y) : (x,y) \in C_2\}$. Let $C_k = C_{k-3} \cup C_2'$, where $C_{k-3}$ is the code defined above in the odd case. It is not difficult to see that $C_k$ is an identifying code of $T_k$: suppose for a contradiction that two vertices $(x,y)$ and $(x',y')$ have the same identifier. Since $C_{k-3}$ is an identifying code of $T_{k-3}$, we cannot have both $x \leq k-3$ and $x' \leq k-3$, and

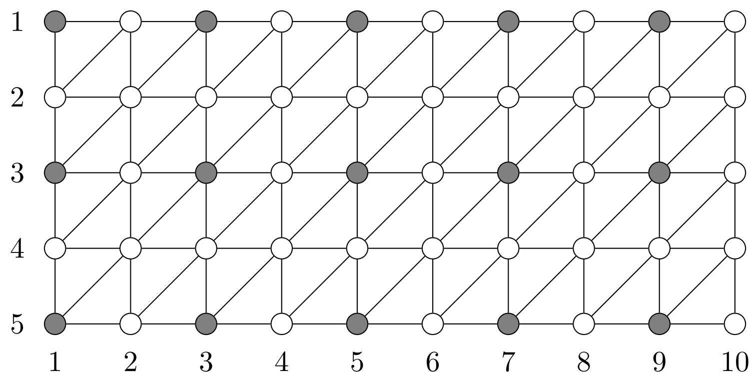


Figure 6: Identifying code $C_5$ of $T_5$ with density 3/10.

since $C_2$ is an identifying code of $T_2$ we cannot have both $x \geq k-1$ and $x' \geq k-1$. Observe moreover that every vertex in rows $k-3$ and $k-2$ has a neighbour in row $k-3$ and that every vertex in rows $k-2$ and $k-1$ has a neighbour in row $k-1$. Hence necessarily $y = y' = k-2$. Moreover, $(x, y)$ and $(x', y')$ are at distance at most two. Hence, without loss of generality, we may assume that $x' =\in x-1, x-2$. Now observe that either $(x-2, k-1)$ or $(x, k-1)$ is in $C_k$, by definition of $C_k$ and $C_2'$. So $C_k[(x, y)] \neq C_k[(x', y')]$, a contradiction.

Since $C_k$ has $(k-2)/2$ rows with density 1/2, $(k-2)/2$ rows with density 0, and two rows with average density $d(C_2, T_2) = 1/2$, $d(C_k, T_k) = \frac{1}{k}\left(\frac{1}{2}\left(\frac{k-2}{2}+2\right)\right) = \frac{1}{4} + \frac{1}{2k}$. Thus by Theorem 4, we have $\frac{1}{4} + \frac{1}{4k} \leq d^*(T_k) \leq \frac{1}{4} + \frac{1}{2k}$.

□

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