Time-Optimal Trajectories of Generic Control-Affine Systems Have at Worst Iterated Fuller Singularities

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ABSTRACT. We consider in this paper the regularity problem for time-optimal trajectories of a single-input control-affine system on a n-dimensional manifold. We prove that, under generic conditions on the drift and the controlled vector field, any control u associated with an optimal trajectory is smooth out of a countable set of times. More precisely, there exists an integer K, only depending on the dimension n, such that the non-smoothness set of u is made of isolated points, accumulations of isolated points, and so on up to K-th order iterated accumulations.

1. INTRODUCTION

1.1. **Single-input systems and chattering phenomena.** Let M be a smooth[1], connected, n-dimensional manifold and denote by $\mathrm{Vec}(M)$ the space of smooth vector fields on M. Consider the (single-input) control-affine system

$$\dot{q} = f_0(q) + u f_1(q), \quad q \in M, \quad u \in [-1,1], \quad f_0, f_1 \in \mathrm{Vec}(M). \tag{1.1}$$

An *admissible trajectory* of (1.1) is an absolutely continuous curve $q : [0,T] \to M$, $T > 0$, such that there exists $u \in L^\infty([0,T],[-1,1])$ so that $\dot{q}(t) = f_0(q(t)) + u(t) f_1(q(t))$ for almost every $t \in [0,T]$.

For any fixed initial datum $q_0 \in M$, the time-optimal control problem associated with (1.1) consists into looking for admissible trajectories $q : [0,T] \to M$, $T > 0$, that minimize the time needed to steer q_0 to $q(T)$ among all admissible trajectories.

A necessary (but not sufficient) condition for an admissible trajectory to be time-optimal is provided by the Pontryagin maximum principle (PMP, in short) [21]. Introducing the control-dependent Hamiltonian

$$\mathcal{H} : T^*M \times [-1,1] \to \mathbb{R}, \quad \mathcal{H}(\lambda, v) = \langle \lambda, (f_0 + v f_1)(q) \rangle, \quad q = \pi(\lambda), \tag{1.2}$$

the PMP states that if a trajectory $q(\cdot)$ associated with the control $u(\cdot)$ is time-optimal, then it is *extremal*, i.e., there exists an absolutely continuous curve $t \mapsto \lambda(t) \in T^*_{q(\cdot)}M \setminus \{0\}$ such that $\mathcal{H}(\lambda(t), u(t))$ maximizes $\mathcal{H}(\lambda(t), \cdot)$ for a.e. $t \in [0,T]$, and such that $\dot{\lambda}(t) = \overrightarrow{\mathcal{H}}(\lambda(t), u(t))$ a.e. on $[0,T]$. (For the precise definition of the Hamiltonian vector field $\overrightarrow{\mathcal{H}}$ and further details see Section 2.) We call the triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ an *extremal triple*. In particular, the PMP reduces the problem of finding time-optimal trajectories to the study of extremal ones.

The kind of results we are interested in concern the regularity of time-optimal trajectories, even though our techniques handle in fact the broader class of extremal ones. Observe in any case that this is an hopeless task in full generality since, as proved by Sussmann in [29], for any given measurable control $t \mapsto u(t)$, there exist a dynamical system of the form (1.1) and an initial datum $q_0 \in M$ for which the admissible trajectory driven by u and starting at q_0 is time-optimal.

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[1]i.e., C^∞ throughout the whole paper.

It makes then sense to look for better answers imposing some genericity conditions on (f_0, f_1) (with respect to the Whitney topology on the space of pairs of smooth vector fields on M). The question we are then lead to tackle is the following: "What kind of behavior can we expect for time-optimal trajectories of a generic system?" Such a question corresponds to one of the open problems posed by A. Agrachev in [3].

The problem of the regularity of extremal trajectories for control-affine systems of the form (1.1) is known to be delicate. In his striking example, Fuller [13] exhibited a polynomial system of the kind studied here, in which controls associated with optimal trajectories have a converging sequence of isolated discontinuities. Since then the phenomenon of *fast oscillations* (or *chattering*) is also called the Fuller phenomenon, and his presence has important consequences for example on the study of optimal syntheses [9, 11, 18, 20, 28]. Another striking feature of this phenomenon is its stability: if the dimension of M is sufficiently high, then chattering is structurally stable (i.e., it cannot be destroyed by a small perturbation of the initial system). The first result in this direction was presented in [16, Theorem 0] starting from dimension 6, but it was subsequently extensively explored in [31]. It is however worth mentioning the fact that, to the best of our knowledge, none of these extremal trajectories have yet been proved to be time-optimal, nor it is known in lower dimensions (already in the 3D case) whether or not the chattering appears for a generic choice of system (1.1). Finally, we remark that the absence of Fuller phenomena for (1.1) has been proved in dimension 2 for analytic systems and generic smooth systems [17, 19, 26, 27]. A first extensive investigation of the chattering phenomenon for multi-input affine-control systems has been presented in [32].

1.2. **Fuller times along extremals trajectories.** Many contributions have been provided to the description of the structure of optimal trajectories around a given point $q \in M$. The natural setting in which this problem is usually tackled is the study of all possible Lie bracket configurations between f_0 and f_1 at q [4, 7, 10, 15, 22, 23, 25, 30]. This approach, although very precise in its answers, has unfortunately the disadvantage of becoming computationally extremely difficult already for mildly degenerate situations in dimension 3.

Definition 1. Given an admissible trajectory $q : [0, T] \to M$ of (1.1), we denote by O_q (or simply O if no ambiguity is possible) the maximal open subset of $[0, T]$ such that there exists a control $u : [0, T] \to [-1, 1]$, associated with $q(\cdot)$, which is smooth on O. We also define Σ_q (or Σ if no ambiguity is possible) by

$$\Sigma = [0, T] \setminus O.$$

An *arc* is a connected component of O. An arc ω is said to be *bang* if u can be chosen so that $|u| \equiv 1$ along ω, and *singular* otherwise. Two arcs are *concatenated* if they share one endpoint. The time-instant between two arcs is a *switching time*.

The set O_q, defined as above, depends only on the trajectory q in the following sense: as long as $f_1(q(t))$ is different from zero, the control $u(t)$ is uniquely identified up to modification on a set of measure zero, while u can be chosen arbitrarily on $\{t \mid f_1(q(t)) = 0\}$.

Definition 2 (Fuller Times)**.** Let Σ_0 be the set of isolated points in Σ and define the *Fuller times* as the elements of the set $\Sigma \setminus \Sigma_0$. By recurrence, Σ_k is defined as the set of isolated points of $\Sigma \setminus (\cup_{j=0}^{k-1} \Sigma_j)$. If $t \in \Sigma_k$ then t is a *Fuller time of order k*. We say that a Fuller time is of *infinite order* if it belongs to

$$\Sigma_\infty = \Sigma \setminus (\cup_{k \geq 1} \Sigma_k).$$

The leading idea of this paper is to characterize the worst stable behavior for generic single-input systems of the form (1.1), in terms of the maximal order of its Fuller times. The heuristics behind our strategy is the following: thinking of points in $\Sigma\setminus\Sigma_0$ as "accumulations of switchings", points in $\Sigma\setminus(\Sigma_0\cup\Sigma_1)$ as "accumulations of accumulations" and so on, then if t is a Fuller time of sufficiently high order, a large number of relations between $f_0(q(t))$ and $f_1(q(t))$ can be derived. The existence of such a point $q(t)$ can then be ruled out by standard transversality arguments (see, e.g., [1, 14]). In particular, Fuller times are generically of finite order. The main result of this paper is the following.

Theorem 3. *Let M be a n-dimensional smooth manifold. There exist a positive integer $K = K(n)$ and an open and dense set $\mathcal{V} \subset \mathrm{Vec}(M) \times \mathrm{Vec}(M)$ such that, if the pair (f_0, f_1) is in $\mathcal{V}$, then for every extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ of the time-optimal control problem*

$$\dot{q} = f_0(q) + u f_1(q), \quad q \in M, \ u \in [-1, 1],$$

the trajectory $q(\cdot)$ has at most Fuller times of order K, i.e.,

$$\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_K,$$

where Σ and Σ_j are defined as in Definition 2.

Remark 4. Since each Σ_i, for $i = 1, \ldots, K$, is discrete, as a consequence of Theorem 3 we deduce that the control $u(\cdot)$ associated with any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ is smooth out of a finite union of discrete sets (in particular, out of a set of measure zero).

As we already explained, deriving dependence relations directly on f_0 and f_1 is extremely complicated. The PMP naturally suggests to rather search for conditions in the cotangent space T^*M, where they are more easily characterizable, and to subsequently project them down on the level of vector fields. On the other hand, the estimates on the integer K that we can provide by looking at what happens on T^*M and projecting on M are far from being optimal. While our proof of Theorem 3 shows that we can take $K(n) = (n-1)^2$, the computation of the best $K = K(n)$ for which Theorem 3 holds is still an open problem.

1.3. **Structure of the paper.** In Section 2 we introduce the technical tools we need in the rest of the paper and we present a brief survey of related results. Section 3 is the starting point of the novel contributions of the paper: we prove that at Fuller times of order larger than zero, i.e., for $t \in \Sigma\setminus\Sigma_0$, in addition to the conditions $\lambda(t) \perp f_1(q(t)), [f_0, f_1](q(t))$, one also has that either $\lambda(t) \perp [f_0 + f_1, [f_0, f_1]](q(t))$ or $\lambda(t) \perp [f_0 - f_1, [f_0, f_1]](q(t))$. The computations leading to this result do not require any genericity assumption. Section 4, which constitutes the technical core of this work, explains how to derive new conditions at each accumulation step and how to prove their independence. Section 5 concludes the proof of Theorem 3 and, finally, in Section 6, the case of time-optimal trajectories on three dimensional manifolds is analyzed in greater detail.

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2. Previous results and consequences of Theorem 3

2.1. **Notations.** Let us introduce some technical notions which will be extensively used throughout the rest of the paper. Consider the cotangent space T^*M of M, endowed with the canonical symplectic form σ. For any Hamiltonian function $p : T^*M \to \mathbb{R}$, its Hamiltonian lift $\overrightarrow{p} \in \mathrm{Vec}(T^*M)$ is defined using the relation

$$\sigma_\lambda(\cdot, \overrightarrow{p}) = \langle d_\lambda p, \cdot \rangle.$$

For all $T > 0$ and $q_0 \in M$ we define the *attainable set* from q_0 at time T as

$$A(T, q_0) = \{q(t) \mid q : [0, T] \to M \text{ is an admissible trajectory of (1.1) such that } q(0) = q_0\}.$$

The precise content of the PMP, already mentioned at the beginning of Section 1, is then recalled below (see [6, 21]).

Theorem (PMP)**.** *Let $q : [0, T] \to M$ be an admissible trajectory of* (1.1)*, associated with a control $u(\cdot)$, such that $q(T) \in \partial A(T, q_0)$. Then there exists $\lambda : [0, T] \to T^*M$ absolutely continuous such that $(q(\cdot), u(\cdot), \lambda(\cdot))$ is an* extremal triple*, i.e., in terms of the control-dependent Hamiltonian $\mathcal{H}$ introduced in* (1.2)*,*

$$\lambda(t) \in T^*_{q(t)}M \setminus \{0\}, \quad \forall t \in [0, T],$$

$$\mathcal{H}(\lambda(t), u(t)) = \max_{v \in [-1,1]} \mathcal{H}(\lambda(t), v), \quad \text{for a.e. } t \in [0, T], \tag{2.1}$$

$$\dot\lambda(t) = \vec{\mathcal{H}}(\lambda(t), u(t)), \quad \text{for a.e. } t \in [0, T]. \tag{2.2}$$

Let $(q(\cdot), u(\cdot), \lambda(\cdot))$ be an extremal triple. The curve $q(\cdot)$ is in particular said to be an *extremal trajectory*. We associate with $(q(\cdot), u(\cdot), \lambda(\cdot))$ the *switching function*

$$h_1(t) = \langle \lambda(t), f_1(q(t)) \rangle.$$

Differentiating a.e. on $[0, T]$, it follows from (2.2) that for every smooth vector field X on M

$$\frac{d}{dt} \langle \lambda(t), X(q(t)) \rangle = \langle \lambda(t), [f_0 + u(t) f_1, X](q(t)) \rangle, \qquad \text{for a.e. } t \in [0, T].$$

In particular, h_1 is of class C^1 and, setting

$$h_{01}(t) = \langle \lambda(t), [f_0, f_1](q(t)) \rangle, \quad \forall t \in [0, T],$$

we have $\dot h_1(t) = h_{01}(t)$ for every $t \in [0, T]$.

Remark 5. The maximality condition (2.1) implies that

$$\mathcal{H}(\lambda(t), u(t)) = \langle \lambda(t), f_0(q(t)) \rangle + \max_{v \in [-1,1]} v \langle \lambda(t), f_1(q(t)) \rangle = \langle \lambda(t), f_0(q(t)) \rangle + |\langle \lambda(t), f_1(q(t)) \rangle|.$$

In particular, $u(t) = \operatorname{sgn}(h_1(t)) \in \{-1, +1\}$ whenever $h_1(t) \neq 0$.

Repeated differentiation shows that h_1 is smooth when the control is. In particular, in terms of the set O introduced in Definition 1, $h_1|_O \in C^\infty(O)$.

A folklore result on bang and singular arcs is the following. Recall that, for every $f \in \mathrm{Vec}(M)$, $\mathrm{ad}_f : \mathrm{Vec}(M) \to \mathrm{Vec}(M)$ denotes the adjoint action defined by $\mathrm{ad}_f g = [f, g]$.

Proposition 6. *Assume that* $\mathrm{span}\{(\mathrm{ad}^k_{f_0+f_1} f_1)(q) \mid k \in \mathbb{N}\} = T_qM$ *and* $\mathrm{span}\{(\mathrm{ad}^k_{f_0-f_1} f_1)(q) \mid k \in \mathbb{N}\} = T_qM$ *for every $q \in M$. Fix an extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ and an arc $\omega \subset O_q$. Then, either $h_1(t) = 0$ for at most finitely many $t \in \omega$ and the arc is bang, or $h_1 \equiv 0$ on ω and the arc is singular.*

Proof. Let us set $Z = \{\tau \in \omega \mid h_1(\tau) = 0\}$ and $F^\pm = \{\tau \in \omega \mid \pm h_1(\tau) > 0\}$. Assume by contradiction that Z has infinitely many points and that it is different from ω. We have from Remark 5 that, up to modifying u on a set of measure zero, $u \equiv 1$ on F^+ and $u \equiv -1$ on F^-. If Z has measure 0, then, by continuity of $u|_\omega$, $u \equiv 1$ or $u \equiv -1$ on ω. In particular, $h_1^{(k)}(t) = \langle \lambda(t), (\mathrm{ad}^k_{f_0+f_1} f_1)(q(t)) \rangle$ or $h_1^{(k)}(t) = \langle \lambda(t), (\mathrm{ad}^k_{f_0-f_1} f_1)(q(t)) \rangle$ on ω. Since between any two vanishing points for $h_1^{(k-1)}$ there is a vanishing point for $h_1^{(k)}$, we deduce that at every clustering point $t \in \bar\omega$ for Z (i.e., the limit of infinitely many distinct points in Z), $\lambda(t)$ annihilates either $(\mathrm{ad}^k_{f_0+f_1} f_1)(q(t))$ for every $k \in \mathbb{N}$ or $(\mathrm{ad}^k_{f_0-f_1} f_1)(q(t))$ for every $k \in \mathbb{N}$, leading to a contradiction.

In the case where the measure of Z is positive, there exist $t \in \omega$ which is both a clustering point for Z and for either F^+ or F^-. By continuity of $h_1^{(k)}|_\omega$ for every $k \in \mathbb{N}$, we deduce that either $h_1^{(k)}(t) = \langle \lambda(t), (\mathrm{ad}^k_{f_0+f_1} f_1)(q(t))\rangle$ for every $k \in \mathbb{N}$ or $h_1^{(k)}(t) = \langle \lambda(t), (\mathrm{ad}^k_{f_0-f_1} f_1)(q(t))\rangle$ for every $k \in \mathbb{N}$ and we conclude as above. □

Notice that the assumption that $\mathrm{span}\{(\mathrm{ad}^k_{f_0+f_1} f_1)(q) \mid k \in \mathbb{N}\} = T_qM$ and $\mathrm{span}\{(\mathrm{ad}^k_{f_0-f_1} f_1)(q) \mid k \in \mathbb{N}\} = T_qM$ for every $q \in M$ holds true generically with respect to $(f_0, f_1) \in \mathrm{Vec}(M)^2$.

Definition 7. Let $\mathfrak{A}$ be the alphabet containing the letters $\{+, -, 0, 1\}$, and let $I = (i_1 \cdots i_d) \in \mathfrak{A}^d$ be a word of length d in $\mathfrak{A}$. Then we employ the shorthand notation

$$f_I = [f_{i_1}, \dots, [f_{i_{d-1}}, f_{i_d}] \dots],$$

with the convention that $f_\pm = f_0 \pm f_1$. Moreover, given an extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0, T]$, we set

$$h_I(t) = \langle \lambda(t), f_I(q(t))\rangle, \qquad t \in [0, T].$$

2.2. **Previous results.** Sussmann proved in [29] that for every $T > 0$ and every control $u \in L^\infty([0, T], [-1, 1])$ there exists a control system of the type (1.1) and an initial datum q_0 such that the trajectory starting at q_0 and corresponding to $u(\cdot)$ is time-optimal. In generic situations, however, some further regularity can be expected, as recalled in the following three results.

Theorem 8 ([8, Theorem 0], [12, Theorem 2.6])**.** *Generically with respect to $(f_0, f_1) \in \mathrm{Vec}(M)^2$, for any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0, T]$ such that $h_1|_{[0,T]} \equiv 0$, the set $\Omega = \{t \in [0, T] \mid h_{101}(t) \neq 0\}$ is of full measure in $[0, T]$ and $u(t) = -h_{001}(t)/h_{101}(t)$ almost everywhere on Ω.*

Theorem 9 ([2, Proposition 1])**.** *Let $I(f_1) \subset \mathrm{Lie}(f_0, f_1)$ denote the ideal generated by f_1. If $I_q(f_1) = T_qM$ for every $q \in M$, then, for every extremal trajectory $q : [0, T] \to M$, the set O_q is open and dense in $[0, T]$.*

Theorem 10 ([7, Proposition 2])**.** *Assume that $\mathrm{span}\{(\mathrm{ad}^k_{f_0+f_1} f_1)(q) \mid k \in \mathbb{N}\} = T_qM$ and $\mathrm{span}\{(\mathrm{ad}^k_{f_0-f_1} f_1)(q) \mid k \in \mathbb{N}\} = T_qM$ for every $q \in M$. Consider an extremal trajectory $q : [0, T] \to M$ such that the union of all bang arcs is open and dense in $[0, T]$. Then either $\Sigma = \Sigma_0$ or there exists an infinite sequence of concatenated bang arcs.*

Theorem 3 can be seen as an extension of Theorem 9 in the sense that it guarantees that, generically with respect to (f_0, f_1), the open set O_q is not only dense but also of countable complement and hence of full measure in $[0, T]$ (see Remark 4). A similar observation can be done for Theorem 10, which is generalized by Theorem 9 as follows: generically, for every $k \geq 0$, either $\Sigma = \cup_{j=0}^k \Sigma_j$ or there exists a subinterval I of $[0, T]$ such that $I \cap \Sigma_k$ is a converging sequence.

Concerning Theorem 8, we can strengthen its conclusion as stated in Corollary 11 below. The corollary is a direct consequence of Proposition 28, which is a step of the proof of Theorem 3 contained in Section 5.

Corollary 11. *Generically with respect to the pair $(f_0, f_1) \in \mathrm{Vec}(M)^2$, for any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0, T]$ such that $h_1|_{[0,T]} \equiv 0$, the set $\Omega = \{t \in [0, T] \mid h_{101}(t) \neq 0\}$ has countable complement in $[0, T]$ and $u(t) = -h_{001}(t)/h_{101}(t)$ almost everywhere on Ω.*

2.3. **Chattering and singular extremals.** Classical instances of the chattering phenomenon occur when trying to join singular and bang arcs along time-optimal trajectories of control systems as in (1.1). Legendre condition [6, Theorem 20.16] holds along singular extremal triples, and imposes the inequality $h_{101}(t) \geq 0$. If the inequality is strict, then the control $u(t)$ is characterized as in Theorem 8, but there are significant examples of mechanical problems in which the third bracket f_{101} vanishes identically (e.g. Dubin's car with acceleration [6, Section 20.6]). This case has been intensively studied in [31], and the situation that forces the chattering can be essentially summarized as follows.

Theorem 12 ([6, Proposition 20.23])**.** *Assume that the vector fields f_0 and f_1 satisfy the identity $f_{101} \equiv 0$. Let $q : [0,T] \to M$ be a time-optimal trajectory of system* (1.1) *which is the projection of a unique (up to a scalar factor) curve $\lambda : [0,T] \to T^*M$ such that $(q(\cdot), u(\cdot), \lambda(\cdot))$ is an extremal triple. Assume moreover that $h_{10001}(t) \neq 0$ on $[0,T]$. Then $q(\cdot)$ cannot contain a singular arc concatenated with a bang arc.*

In particular, under the hypotheses of the theorem, the only possibility for an optimal trajectory to exit a singular arc is through chattering.

3. Annihilation conditions at Fuller times of an extremal trajectory

Let us fix an extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0,T]$. The goal of this section is to prove some useful annihilation conditions of functions of the form h_I, with I a word in $\mathfrak{A}$ (compare with Definition 7), at Fuller times, i.e., on $\Sigma \setminus \Sigma_0$.

Since h_1 is (absolutely) continuous and $u(t) = \operatorname{sgn}(h_1(t))$ for almost every t such that $h_1(t) \neq 0$, then

$$h_1\big|_{\Sigma} \equiv 0.$$

Moreover, between two zeroes of h_1, $h_1^{(1)} = h_{01}$ has at least one zero, which yields

$$h_{01}\big|_{\Sigma \setminus \Sigma_0} \equiv 0.$$

The following proposition states that both h_{001} and h_{101} vanish at every $t \in \Sigma$ which is at positive distance from $\{t \mid h_1(t) \neq 0\}$.

Proposition 13. *Let $t \in \Sigma$ be such that h_1 is identically equal to zero on a neighborhood of t. Then $h_{101}(t) = h_{001}(t) = 0$.*

Proof. Let V be a neighborhood of t such that $h_1|_V \equiv 0$. Therefore, the same is true for $h_{01}|_V$ and

$$h_{001}(\tau) + u(\tau)h_{101}(\tau) = 0 \quad \text{for almost every } \tau \in V. \tag{3.1}$$

Let us first prove that $h_{101}(t) = 0$. By contradiction and up to reducing V, we have that $h_{101}(\tau) \neq 0$ for every $\tau \in V$. By (3.1), moreover, $u(\tau) = -\frac{h_{001}(\tau)}{h_{101}(\tau)}$ for almost every $\tau \in V$.

Notice that the differential system generated by the smooth autonomous Hamiltonian

$$H(p) = \langle p, f_0(\pi(p))\rangle - \frac{\langle p, f_{001}(\pi(p))\rangle}{\langle p, f_{101}(\pi(p))\rangle}\langle p, f_1(\pi(p))\rangle,$$

is well-defined on $\{p \in T^*M \mid \langle p, f_{101}(\pi(p))\rangle \neq 0\}$ and all its trajectories are smooth. Since, moreover, the absolutely continuous curve $(\lambda(t), q(t))$ satisfies $\dot{p} = \overrightarrow{H}(p)$ almost everywhere on V, we deduce that $V \ni t \mapsto (\lambda(t), q(t))$ is a solution of the Hamiltonian system generated by H and that the control u is smooth on V, contradicting the fact that $t \in \Sigma$.

We conclude by showing that also $h_{001}(t) = 0$. Following (3.1), we have

$$|h_{001}(\tau)| = |u(\tau)||h_{101}(\tau)| \leq |h_{101}(\tau)| \quad \text{for almost every } \tau \in V$$

and then we conclude by continuity of h_{101} and h_{001}. □

Proposition 14. *Assume that there exists an infinite sequence of concatenated bang arcs converging to $t \in [0,T]$. Then either $h_{+01}(t) = 0$ or $h_{-01}(t) = 0$.*

Proof. First notice that $t \in \Sigma \setminus \Sigma_0$. Assume by contradiction that neither $h_{+01}(t)$ nor $h_{-01}(t)$ is equal to zero. Then, up to restricting the interval $[0,T]$, we may assume that there exists a positive constant $C > 0$ such that

$$\frac{1}{C} \leq |h_{+01}(s)|, |h_{-01}(s)| \leq C, \quad \forall\, s \in [0,T].$$

By assumption, there exist a sequence of concatenated bang arcs, whose lengths we denote by $\{\sigma_i\}_{i\in\mathbb{N}} \cup \{\tau_i\}_{i\in\mathbb{N}} \subset (0,+\infty)$, with the agreement that $u \equiv 1$ (respectively, $u \equiv -1$) on the intervals of length σ_i (respectively, τ_i) and that the arc of length σ_i is concatenated with the arc of length τ_i, which is concatenated with the arc of length σ_{i+1} and so on. Without loss of generality, the bang arcs converge towards t from the left, so that we can further assume that the arc of length σ_i is concatenated at its right with the arc of length τ_i (see Figure 1).

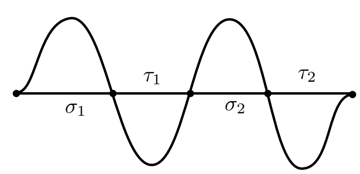

FIGURE 1. A concatenation of bang arcs

By convention, let 0 be the starting time of the sequence in Figure 1. Taylor's formula yields that

$$\sigma_1 = -\frac{2h_{01}(0)}{h_{+01}(0)} + O(\sigma_1^2), \tag{3.2}$$

where the notation $O(\sigma_1^2)$ has the following meaning: using an analogous Taylor expansion for each positive bang arc of length σ_k, we obtain a reminder ρ_k such that $\frac{\rho_k^2}{\sigma_k^2}$ is uniformly bounded. We deduce from (3.2) that $|h_{01}(0)| \approx \sigma_1$, where this notation is used to indicate that $|h_{01}(0)| = O(\sigma_1)$ and $\frac{1}{|h_{01}(0)|} = O(\sigma_1)$. Moreover, from the expansion $h_{01}(\sigma_1) = h_{01}(0) + \sigma_1 h_{+01}(0) + O(\sigma_1^2)$, we get

$$h_{01}(\sigma_1) = -h_{01}(0) + O(\sigma_1^2).$$

Combining these two relations we obtain

$$\tau_1 \approx |h_{01}(\sigma_1)| \approx \sigma_1.$$

The same computations also imply that

$$\sigma_2 \approx h_{01}(\sigma_1 + \tau_1) = -h_{01}(\sigma_1) + O(\tau_1^2) = h_{01}(0) + O(\sigma_1^2).$$

In particular, the sequence σ_i satisfies the relation $\sigma_{i+1} = \sigma_i + O(\sigma_i^2)$. The contradiction is then a consequence of Lemma 15 below. □

Lemma 15. *Let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of positive numbers satisfying the relation*

$$t_{i+1} = t_i + O(t_i^2).$$

Then, $\sum_{i=1}^{\infty} t_i = +\infty$.

Proof. Let $c > 0$ be such that

$$t_{i+1} \geq t_i(1 - ct_i), \quad \forall\, i \in \mathbb{N}. \tag{3.3}$$

Assume by contradiction that the statement is false. In particular, $t_i \to 0$.

We claim that, without loss of generality, $t_i \searrow 0$, that is, there exists a decreasing subsequence of $\{t_i\}_{i\in\mathbb{N}}$, still denoted by $\{t_i\}_{i\in\mathbb{N}}$, satisfying (3.3). In order to check that the claim is true, first notice that, up to removing the first elements of the sequence, we can assume that $t_i < \frac{1}{2c}$ for every i. Notice, moreover, that $\tau \mapsto \tau(1 - c\tau)$ is increasing on the interval $(0, \frac{1}{2c})$.

Take now two indices $k \in \mathbb{N}$ such that $\sup_{h>k} t_h > t_k$. Let $h > k$ be the largest index such that $t_h > t_k$. Hence, $t_{h+1} \leq t_k < t_h$. Let us show that the subsequence obtained by removing $t_{k+1}, t_{k+2}, \dots, t_h$ still satisfies (3.3). Clearly, the only inequality to be satisfied is the one in which we replace the indices i and $i+1$ by, respectively, k and $h+1$ in (3.3). By hypothesis, $t_{h+1} \geq t_h(1 - ct_h)$. Since, moreover, $t_k < t_h$, we deduce by monotonicity that $t_k(1 - ct_k) < t_h(1 - ct_h)$ and the proof of the claim follows.

Next, let us prove by induction on j that $t_{i+j} \geq t_i(1 - cjt_i)$ for every $i, j \in \mathbb{N}$. The initial step $j = 0$ is tautological. Moreover,

$$\begin{aligned} t_{i+j+1} &\geq t_{i+j}(1 - ct_{i+j}) \\ &\geq t_i(1 - cjt_i)(1 - ct_i) \\ &= t_i(1 - c(j+1)t_i + \underbrace{c^2 jt_i^2}_{\geq 0}). \end{aligned}$$

Let $\{t_{i_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{t_i\}_{i\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$ and every $j \in [i_k, i_{k+1}]$, $t_j \geq \frac{t_{i_k}}{2}$ and $t_{i_{k+1}+1} < \frac{t_{i_k}}{2}$.

Then

$$\sum_{i=1}^{\infty} t_k \geq \sum_{i=1}^{i_1} t_i + \sum_{k=1}^{\infty} \sum_{j=i_k+1}^{i_{k+1}} t_j \geq \sum_{k=1}^{\infty} (i_{k+1} - i_k)\frac{t_{i_k}}{2}, \tag{3.4}$$

and moreover the previous argument implies that $t_{i_k+j} \geq t_{i_k}(1 - cjt_{i_k})$. Finally we notice that

$$\frac{t_{i_k}}{2} > t_{i_{k+1}+1} \geq t_{i_k}(1 - c(i_{k+1} - i_k)t_{i_k}) = t_{i_k} - c(i_{k+1} - i_k)t_{i_k}^2$$

which means that $(i_{k+1} - i_k)t_{i_k} > \frac{1}{2c}$. This completes the proof, according to (3.4). □

We say that an arc is *bi-concatenated* if it is concatenated both at its right and at its left with other arcs.

Proposition 16. *Let I be a bang arc compactly contained in $(0, T)$ and which is not bi-concatenated. Then there exists $t \in \bar{I}$ such that either $h_{+01}(t) = 0$ or $h_{-01}(t) = 0$.*

Proof. Without loss of generality, assume that $u \equiv 1$ on $I = (t_1, t_2)$ and that I is not concatenated with any other arc at t_2. In particular, t_2 is a clustering point for $\Sigma \cap (t_2, T]$. If $h_1 \equiv 0$ on a right neighborhood of t_2, then the conclusion follows from Proposition 13 and the continuity of h_{+01} and h_{-01}.

We can then assume that there exists a sequence of times converging from above to t_2 and at which h_1 is not zero. Then, necessarily, there exist a sequence of arcs I_n converging to t_2. Pick, for every $n \in \mathbb{N}$ a time $\tau_n \in I_n$ such that $h_{01}(\tau_n) = 0$. By construction, the sequence $(\tau_k)_{k\in\mathbb{N}}$ converges to t_2 and, by continuity, we deduce that also $h_{01}(t_2) = 0$.

Since $h_1(t_1) = h_1(t_2) = 0$, then by the mean value theorem h_{01} vanishes at an interior point of I, and this in turns implies that $\frac{d}{dt}h_{01}|_I = h_{+01}|_I$ also vanishes somewhere on I. □

The main result of the section is the following theorem.

Theorem 17. *Let $t \in \Sigma \setminus \Sigma_0$. Then $h_1(t) = h_{01}(t) = 0$ and, in addition, either $h_{+01}(t) = 0$ or $h_{-01}(t) = 0$.*

Proof. We already noticed that h_1 vanishes on Σ and h_{01} on $\Sigma \setminus \Sigma_0$. We are going to prove the theorem by showing that there exists a sequence of points converging to t at which either h_{+01} or h_{-01} vanishes.

Since $t \notin \Sigma_0$ and thanks to Proposition 13, we can assume without loss of generality that h_1 does not vanish identically on a neighborhood of t. Hence, there exists a sequence $(\tau_n)_{n\in\mathbb{N}} \subset [0,T]$ converging to t such that $h_1(\tau_n) \neq 0$ for every $n \in \mathbb{N}$. Each τ_n is contained in an arc ω_n. If the arc is singular, then it contains a nonempty subinterval on which $h_1 \equiv 0$. Since moreover h_1 has either a positive maximum or a negative minimum on ω_n, we deduce that there exists an inflection point of h_1 on ω_n at which h_{+01} or h_{-01} vanishes.

We can then assume without loss of generality that ω_n is a bang arc for every $n \in \mathbb{N}$. Let us consider the maximal concatenation of bang arcs from ω_n towards t. Three possibilities occur: (i) the concatenation is infinite and converges to a point between τ_n and t, (ii) the concatenation stops with a bang arc which is not bi-concatenated, and (iii) the concatenation stops with a bang arc concatenated with a singular one. In each of the three cases, we prove that there exists a point between ω_n and t at which either h_{+01} or h_{-01} vanishes. In cases (i) and (ii) the conclusion follows from Propositions 14 and 16 respectively. In the case of a bang arc concatenated with a singular one, either h_1 does not vanish everywhere on the singular arc, and we deduce as above that there exists an inflection point of h_1 on the singular arc at which h_{+01} or h_{-01} vanishes, or $h_{01} = 0$ at the junction of the two arcs and then the bang arc contains an inflection point of h_1 at which h_{+01} or h_{-01} vanishes. This concludes the analysis in case (iii) and hence the proof of the theorem. □

4. High-order Fuller points and genericity results

Remark 18. For any given word $J = (j_1, \dots, j_r) \in \mathfrak{A}^r$, with $r \geq 3$, $j_{r-1} = 0$, $j_r = 1$, and at least one j_k in $\{+,-\}$, an easy inductive argument proves that, with the notations of Definition 7, we can decompose f_J as

$$f_J = f_{J_1} + \cdots + f_{J_l},$$

where $J_1, \dots, J_l$ are all words of length r written only with letters in $\{0,1\}$, ending with the string (01) and such that, if $|J_i|_a$ counts the number of occurrences of the letter a in J_i, then

$$|J_1|_0 = \max_{i=1,\dots,l} |J_i|_0, \quad \text{and} \quad |J_2|_1 = \max_{i=1,\dots,l} |J_i|_1.$$

Moreover, J_1 and J_2 are uniquely determined by this requirement.

Definition 19. Let $N \in \mathbb{N}$. A function $S : T^*M \times J^N M \times J^N M \to \mathbb{R}$ is said to be a *simple relation of degree* d if there exists a word $I \in \mathfrak{A}^d$ of length d with $d \leq N$ such that $S = S_I$, where

$$S_I(\lambda, j_q^N(f_0), j_q^N(f_1)) = \langle \lambda, f_I(q) \rangle, \quad q = \pi(\lambda). \tag{4.1}$$

Similarly, we call $Q : T^*M \times J^N M \times J^N M \to \mathbb{R}$ a *polynomial relation* if there exist $l, d_1, \dots, d_l \in \mathbb{N} \setminus \{0\}$ and words $I_1 \in \mathfrak{A}^{d_1}, \dots, I_l \in \mathfrak{A}^{d_l}$ such that

$$Q(\lambda, j_q^N(f_0), j_q^N(f_1)) \in \mathbb{R}[S_{I_1}(\lambda, j_q^N(f_0), j_q^N(f_1)), \dots, S_{I_l}(\lambda, j_q^N(f_0), j_q^N(f_1))]. \tag{4.2}$$

Moreover, we set $\deg(Q) = \max\{d_1, \ldots, d_l\}$.

Finally, given two simple relations S_I, S_J, with a slight abuse of notation we say that the Poisson bracket $\{S_I, S_J\}$ between S_I and S_J is the simple relation S_{IJ}, where IJ is defined by concatenation of words. We extend the Poisson bracket notation to polynomial relations by linearity and the Leibnitz rule.

Lemma 20. *Let $l, d_1, \ldots, d_l \in \mathbb{N} \setminus \{0\}$ and consider l words $I_1 \in \mathfrak{A}^{d_1}, \ldots, I_l \in \mathfrak{A}^{d_l}$ with $d_j < d_l$ for every $j < l$ and $I_l = (+I_{l-1})^2$. Fix an integer $N \geq d_l$ and consider the family of simple relations $S_j = S_{I_j}$, $1 \leq j \leq l$, using the notation introduced in* (4.1). *Define the set $\mathcal{B} \subset T^*M \times J^N M \times J^N M$ by*

$$\mathcal{B} = \Big\{ (\lambda, j_q^N(f_0), j_q^N(f_1)) \mid q = \pi(\lambda),\, (f_0, f_1) \in \mathrm{Vec}(M)^2,$$
$$S_1(\lambda, j_q^N(f_0), j_q^N(f_1)) = \cdots = S_l(\lambda, j_q^N(f_0), j_q^N(f_1)) = 0 \Big\}.$$

If $(q(\cdot), u(\cdot), \lambda(\cdot))$ is an extremal triple on $[0,T]$ for the time-optimal control problem (1.1) *associated with the pair (f_0, f_1), and if the sequence $\{t_i\}_{i \in \mathbb{N}} \subset [0,T]$ is such that*

i) $(\lambda(t_i), j^N_{q(t_i)}(f_0), j^N_{q(t_i)}(f_1)) \in \mathcal{B}$ *for every $i \in \mathbb{N}$,*

ii) *there exists $t_\infty = \lim_{i \to \infty} t_i$,*

then there exists a further simple relation

$$S_{l+1} \in \left\{ S_{(-I_{l-1})}, S_{(-I_l)}, S_{(+I_l)} \right\}$$

such that

$$(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1)) \in \mathcal{B} \cap \{S_{l+1} = 0\}.$$

Finally, defining for every $q \in M$ the set $\mathcal{B}'_q \subset T^*_q M \times J^N_q M \times J^N_q M$ by

$$\mathcal{B}'_q = \Big\{ (\lambda, j_q^N(g_0), j_q^N(g_1)) \mid \lambda \in T_q^*M \setminus \{0\},\ (g_0, g_1) \in \mathrm{Vec}(M)^2,\ g_0(q) \wedge g_1(q) \neq 0,$$
$$S_1(\lambda, j_q^N(g_0), j_q^N(g_1)) = \cdots = S_l(\lambda, j_q^N(g_0), j_q^N(g_1)) = 0 \Big\},$$

if the codimension of $\mathcal{B}'_q$ in $T^*_q M \times J^N_q M \times J^N_q M$ is equal to l, then

$$\mathrm{codim}_{T^*_q M \times J^N_q M \times J^N_q M}(\mathcal{B}'_q \cap \{S_{l+1} = 0\}) = l + 1.$$

Proof. Let $(q(\cdot), u(\cdot), \lambda(\cdot))$ be an extremal triple defined on $[0,T]$ and $\{t_i\}_{i \in \mathbb{N}} \subset [0,T]$ be a sequence of points satisfying i) and ii) in the statement. Then, since for every word $J \in \{I_1, \ldots, I_l\}$ we have that $h_J(t_i) = \langle \lambda(t_i), f_J(q(t_i)) \rangle$ vanishes for every $i \in \mathbb{N}$, by continuity the same is also true for $h_J(t_\infty)$, which implies that the point $(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1))$ belongs to $\mathcal{B}$.

Now, up to the choice of a suitable subsequence of $\{t_i\}_{i \in \mathbb{N}}$, we infer the identity

$$0 = \lim_{i \to \infty} \frac{h_J(t_\infty) - h_J(t_i)}{t_\infty - t_i} = \lim_{i \to \infty} \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} (h_{0J}(\tau) + u(\tau) h_{1J}(\tau)) d\tau \tag{4.3}$$
$$= h_{0J}(t_\infty) + \bar{u} h_{1J}(t_\infty), \quad \bar{u} = \lim_{i \to \infty} \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} u(\tau) d\tau \ \in [-1, 1],$$

which is valid for every $J \in \{I_1, \ldots, I_l\}$. The first of our claims is then proved. Indeed, if $\bar{u} = \pm 1$ we use (4.3) with $J = I_l$ to deduce that

$$\langle \lambda(t_\infty), f_{(\pm I_l)}(q(t_\infty)) \rangle = 0,$$

[2]We denote here by $(+I_{l-1})$ the concatenation of the letter $+$ and the word I_{l-1}

so that S_{l+1} is in the form $S_{(\pm I_l)}$, and we are done. If, on the other hand, $\bar{u} \in (-1,1)$ we apply (4.3) with $J = I_{l-1}$, and we deduce that

$$S_{(\bar{u},l-1)}(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1)) := \langle \lambda(t_\infty), f_{(0I_{l-1})}(q(t_\infty)) + \bar{u} f_{(1I_{l-1})}(q(t_\infty))\rangle = 0.$$

The combination of the relations $S_{(\bar{u},l-1)} = S_l = 0$ at the point $(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1))$ yields

$$\langle \lambda(t_\infty), f_{(0I_{l-1})}(q(t_\infty))\rangle = 0 \quad \text{and} \quad \langle \lambda(t_\infty), f_{(1I_{l-1})}(q(t_\infty))\rangle = 0,$$

which in turn implies that

$$\langle \lambda(t_\infty), f_{(-I_{l-1})}(q(t_\infty))\rangle = 0,$$

so that we conclude by taking $S_{l+1} = S_{(-I_{l-1})}$.

To prove the second claim of the statement, it is not restrictive to work within a coordinate neighborhood $(U,x) \subset \mathbb{R}^n$ centered at the origin (identified with q), the whole argument being local. Then $g_i(x) = \sum_{j=1}^n \alpha_i^j(x)\partial_{x_j}$ on U, for $i = 0,1$. On $J^N_0 M \times J^N_0 M$, $J^N_0 g_0$ and $J^N_0 g_1$ are given in local coordinates respectively by

$$(\alpha_0^j(0), \nabla\alpha_0^j(0), \dots, \nabla^{(N)}\alpha_0^j(0), 0, \dots, 0)_{j=1}^n \in \left(\mathbb{R} \times \mathbb{R}^n \times \dots \times \mathbb{R}^{n^N}\right)^{2\times n} \quad \text{and}$$
$$(0, \dots, 0, \alpha_1^j(0), \nabla\alpha_1^j(0), \dots, \nabla^{(N)}\alpha_1^j(0))_{j=1}^n \in \left(\mathbb{R} \times \mathbb{R}^n \times \dots \times \mathbb{R}^{n^N}\right)^{2\times n}.$$

Moreover, since $g_0(q) \wedge g_1(q) \neq 0$, without loss of generality we can assume that

$$\alpha_0^1(0) = \alpha_1^2(0) = 1, \quad \alpha_0^j(0) = 0 \text{ if } j \neq 1, \quad \alpha_1^j(0) = 0 \text{ if } j \neq 2.$$

Let the codimension of $\mathcal{B}_0'$ in $T^*_0 M \times J^N_0 M \times J^N_0 M$ be equal to l, and assume that S_{l+1} is of the form $S_{(\pm I_l)}$. In particular, the degree of S_{l+1} is maximal among $\{\deg(S_1), \dots, \deg(S_{l+1})\}$. Following Remark 18, let us write the decomposition

$$g_{I_{l+1}} = g_{I_{l+1,1}} + \dots + g_{I_{l+1,k}},$$

where we recall that $I_{l+1,1}$ is uniquely identified by the requirement that it contains the maximal number, say s, of occurrences of the letter 0. Writing the analogous decomposition for simple relations

$$S_{I_{l+1}} = S_{I_{l+1,1}} + \dots + S_{I_{l+1,k}},$$

we see that the coordinate expression of $S_{I_{l+1,1}}$ at $(\lambda, j^N_0(g_0), j^N_0(g_1)) \in T^*_0 M \setminus \{0\} \times J^N_0 M \times J^N_0 M$ takes the form

$$0 = \langle \lambda, g_{I_{l+1,1}}(0)\rangle = \sum_{j=1}^n \lambda_j \partial^s_{x_1}\partial^{\deg(S_{l+1})-s-1}_{x_2}\alpha_1^j(0) + P_{I_{l+1,1}}(\lambda, j^N_0(g_0), j^N_0(g_1)),$$

where $P_{I_{l+1,1}}$ is a polynomial expression in the coordinates of λ, $j^N_0(g_0)$ and $j^N_0(g_1)$ that does not contain any term of the form $\partial^s_{x_1}\partial^{\deg(S_{l+1})-s-1}_{x_2}\alpha_1^j(0)$, for $1 \leq j \leq n$. By construction, these terms do not appear in any of the other summands $\langle \lambda, g_{I_{l+1,i}}\rangle$, for $i \neq 1$, neither among all other simple relations $S_1, \dots, S_l$. Therefore, as $\lambda \neq 0$, we infer the existence of a further independent relation, and we conclude that

$$\operatorname{codim}_{T^*_0 M \times J^N_0 M \times J^N_0 M}(\mathcal{B}_0' \cap \{S_{l+1} = 0\}) = l + 1.$$

The case in which $S_{l+1} = S_{(-I_{l-1})}$ can be tackled similarly. In this situation $\deg(S_l) = \deg(S_{l+1}) > \deg(S_i)$ for every $i < l$. We may again exploit Remark 18, and isolate the terms $g_{I_{l,1}}, g_{I_{l,2}}$ and $g_{I_{l+1,1}}, g_{I_{l+1,2}}$ in the decompositions of g_{I_l} and $g_{I_{l+1}}$ respectively. Observe that, by definition of I_l and I_{l+1}, one has $g_{I_{l+1,1}} = g_{I_{l,1}}$ and $g_{I_{l+1,2}} = -g_{I_{l,2}}$. Moreover, 0 appears s times in $I_{l,1}$, while 1 appears t times in $I_{l,2}$, and both s and t are maximal among their corresponding decompositions, so that we can write

$$\begin{aligned}
0 =& \langle \lambda, g_{I_l}(0)\rangle = \langle \lambda, g_{I_{l,1}}(0)\rangle + \langle \lambda, g_{I_{l,2}}(0)\rangle + P_{I_l}(\lambda, j_0^N(g_0), j_0^N(g_1)) \\
=& \sum_{j=1}^n \lambda_j \left(\partial_{x_1}^s \partial_{x_2}^{\deg(S_l)-s-1}\alpha_1^j(0) + \partial_{x_1}^{\deg(S_l)-t-1}\partial_{x_2}^t \alpha_0^j(0)\right) + Q_{I_l}(\lambda, j_0^N(g_0), j_0^N(g_1)), \\
0 =& \langle \lambda, g_{I_{l+1}}(0)\rangle = \langle \lambda, g_{I_{l,1}}(0)\rangle - \langle \lambda, g_{I_{l,2}}(0)\rangle + P_{I_{l+1}}(\lambda, j_0^N(g_0), j_0^N(g_1)) \\
=& \sum_{j=1}^n \lambda_j \left(\partial_{x_1}^s \partial_{x_2}^{\deg(S_l)-s-1}\alpha_1^j(0) - \partial_{x_1}^{\deg(S_l)-t-1}\partial_{x_2}^t \alpha_0^j(0)\right) + Q_{I_{l+1}}(\lambda, j_0^N(g_0), j_0^N(g_1)),
\end{aligned}$$

where $P_{I_l}, P_{I_{l+1}}, Q_{I_l}, Q_{I_{l+1}}$ are polynomial expressions in the coordinates of λ, $j_0^N(g_0)$ and $j_0^N(g_1)$ that do not contain any term of the form $\partial_{x_1}^s \partial_{x_2}^{\deg(S_l)-s-1}\alpha_1^j(0)$ and $\partial_{x_1}^{\deg(S_l)-t-1}\partial_{x_2}^t \alpha_0^j(0)$, for $1 \le j \le n$. In addition, these two terms are neither found among all other simple relations $S_1, \dots, S_{l-1}$. Thus, as $\lambda \neq 0$, the relations $\langle \lambda, g_{I_l}(0)\rangle = 0$ and $\langle \lambda, g_{I_{l+1}}(0)\rangle = 0$ are mutually independent (since their gradients are not parallel) and also independent from $\langle \lambda, g_{I_k}(0)\rangle = 0$, $k = 1, \dots, l-1$. □

Lemma 21. *Let $l, d_1, \dots, d_l \in \mathbb{N}\setminus\{0\}$ and consider l words $I_1 \in \mathfrak{A}^{d_1}, \dots, I_l \in \mathfrak{A}^{d_l}$ with $d_j < d_{l-1}$ for every $j < l-1$ and $d_{l-1} = d_l$. Suppose that there exists $j < l-1$ such that $I_{l-1} = (0\, I_j)$ and $I_l = (1\, I_j)$. Using the notations introduced in* (4.1) *and* (4.2)*, consider the family of polynomial relations Q_r, $r \in \mathbb{N}\setminus\{0\}$, constructed inductively using the simple relations $S_{I_1}, \dots, S_{I_l}$ as follows*

$$Q_1 = \det \begin{pmatrix} \{S_0, S_{I_l}\} & \{S_1, S_{I_l}\} \\ \{S_0, S_{I_{l-1}}\} & \{S_1, S_{I_{l-1}}\} \end{pmatrix}, \quad Q_r = \det \begin{pmatrix} \{S_0, S_{I_l}\} & \{S_1, S_{I_l}\} \\ \{S_0, Q_{r-1}\} & \{S_1, Q_{r-1}\} \end{pmatrix} \quad \textit{for } r \ge 2.$$

Fix $h \in \mathbb{N}$, an integer $N \ge d_l + h$, and define the set $\mathcal{B} \subset T^*M \times J^N M \times J^N M$ by

$$\mathcal{B} = \left\{ (\lambda, j_q^N(f_0), j_q^N(f_1)) \;\middle|\; q = \pi(\lambda), \begin{array}{l} S_{I_1}(\lambda, j_q^N(f_0), j_q^N(f_1)) = \cdots = S_{I_l}(\lambda, j_q^N(f_0), j_q^N(f_1)) = 0 \\ Q_1(\lambda, j_q^N(f_0), j_q^N(f_1)) = \cdots = Q_h(\lambda, j_q^N(f_0), j_q^N(f_1)) = 0 \end{array} \right\}.$$

If $(q(\cdot), u(\cdot), \lambda(\cdot))$ is an extremal triple on $[0,T]$, and if the sequence $\{t_i\}_{i\in\mathbb{N}} \subset [0,T]$ is such that

i) *$(\lambda(t_i), j_{q(t_i)}^N(f_0), j_{q(t_i)}^N(f_1)) \in \mathcal{B}$ for every $i \in \mathbb{N}$,*
ii) *there exists $t_\infty = \lim_{i\to\infty} t_i$,*

then, setting

$$I_{l+1} = (0, I_l), \quad I_{l+2} = (1, I_l),$$

either

$$(\lambda(t_\infty), j_{q(t_\infty)}^N(f_0), j_{q(t_\infty)}^N(f_1)) \in \mathcal{B} \cap \{S_{I_{l+2}} \neq 0\} \cap \{Q_{h+1} = 0\},$$

or

$$(\lambda(t_\infty), j_{q(t_\infty)}^N(f_0), j_{q(t_\infty)}^N(f_1)) \in \mathcal{B} \cap \{S_{I_{l+1}} = 0\} \cap \{S_{I_{l+2}} = 0\}.$$

Finally, defining for every $q \in M$ the set $\mathcal{B}'_q \subset T_q^*M \times J_q^N M \times J_q^N M$ by

$$\mathcal{B}'_q = \Big\{ (\lambda, j_q^N(g_0), j_q^N(g_1)) \mid \lambda \in T_q^*M \setminus \{0\},\ (g_0, g_1) \in \mathrm{Vec}(M)^2,\ g_0(q) \wedge g_1(q) \neq 0,$$
$$S_{I_1}(\lambda, j_q^N(g_0), j_q^N(g_1)) = \cdots = S_{I_l}(\lambda, j_q^N(g_0), j_q^N(g_1)) = 0 \Big\},$$

if the codimension of $\mathcal{B}'_q$ in $T^*_qM \times J^N_qM \times J^N_qM$ is equal to l, then

$$\operatorname{codim}_{T^*_qM\times J^N_qM\times J^N_qM}(\mathcal{B}'_q \cap \{S_{I_{l+2}} \neq 0\} \cap \{Q_1 = 0\} \cap \cdots \cap \{Q_{h+1} = 0\}) = l + h + 1,$$
$$\operatorname{codim}_{T^*_qM\times J^N_qM\times J^N_qM}(\mathcal{B}'_q \cap \{S_{I_{l+1}} = 0\} \cap \{S_{I_{l+2}} = 0\}) = l + 2.$$

Proof. The proof of the first part of the statement follows along the same lines of Lemma 21, using equation (4.3) both on S_{I_l} and on Q_h, with the convention that $Q_0 = S_{I_{l-1}}$. We prove in this way that the relations

$$\{S_0, S_{I_l}\} + \bar{u}\{S_1, S_{I_l}\} = 0 \quad \text{and} \quad \{S_0, Q_h\} + \bar{u}\{S_1, Q_h\} = 0 \tag{4.4}$$

hold at $(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1))$, where the value $\bar{u}$ is the same in both identities, since it is computed as the limit of a common sequence. If $S_{I_{l+2}} = \{S_1, S_{I_l}\}$ vanishes on the triple $(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1))$, then so does $S_{I_{l+1}} = \{S_0, S_{I_l}\}$. From equation (4.4) we also deduce that $(1, \bar{u})$ is in the kernel of

$$\begin{pmatrix} \{S_0, S_{I_l}\} & \{S_1, S_{I_l}\} \\ \{S_0, Q_h\} & \{S_1, Q_h\} \end{pmatrix},$$

and therefore that its determinant Q_{h+1} vanishes at $(\lambda(t_\infty), j^N_{q(t_\infty)}(f_0), j^N_{q(t_\infty)}(f_1))$.

In order to prove the second part of the statement, as in Lemma 21 the idea is to express all relations in local coordinates around q on the product space $T^*_qM \times J^N_qM \times J^N_qM$, with the non-restrictive hypothesis that $g_0(0) = \partial_{x_1}$ and $g_1(0) = \partial_{x_2}$. Notice that for what concerns the codimension of $\mathcal{B}'_q \cap \{S_{I_{l+1}} = 0\} \cap \{S_{I_{l+2}} = 0\}$ we can reason exactly as in Lemma 21, since we deal in fact only with simple relations. Thus we are left with the task of proving that, if $S_{I_{l+2}} \neq 0$, each polynomial relation Q_r provides a condition independent from $S_{I_1}, \dots, S_{I_l}$ and $Q_1, \dots, Q_{r-1}$.

It is not hard to show, by induction, that

$$Q_r = (-1)^r (S_{I_{l+2}})^r \operatorname{ad}^r_{S_0}(S_{I_{l-1}}) + Q'_r,$$

where $\operatorname{ad}^r_{S_0}$ denotes the iterated Poisson bracket with S_0 and Q'_r is a polynomial relation that does not contain the term $\operatorname{ad}^r_{S_0}(S_{I_{l-1}})$. Following Remark 18, we further decompose $f_{I_{l-1}}$ as $f_{I_{l-1},1} + \cdots + f_{I_{l-1},k}$, where the letter 0 appears in $I_{l-1,1}$ the maximal number of times, say s, among the collection $\{I_{l-1,i}\}_{i=1}^k$. In coordinates we then write

$$\operatorname{ad}^r_{S_0}(S_{I_{l-1}})(\lambda(t_\infty)) = \sum_{j=1}^n \lambda_j \partial^{r+s}_{x_1} \partial^{\deg(S_{I_{l-1}})-s-1}_{x_2} \alpha^j_1(0) + P_{I_{l-1}}(\lambda, j^N_0(g_0), j^N_0(g_1)),$$

where $P_{I_{l-1}}$ is a polynomial expression in λ, $j^N_0(g_0)$ and $j^N_0(g_1)$ that does not contain any term of the form $\partial^{r+s}_{x_1} \partial^{\deg(S_{I_{l-1}})-s-1}_{x_2} \alpha^j_1(0)$. Since $\lambda \neq 0$ and the above is true for any $r \in \mathbb{N}$, we conclude that, as soon as $S_{I_{l+2}} \neq 0$, each Q_r gives a new independent condition, and the claim on the codimension follows. ☐

4.1. **Collinear case.** We associate with the pair $(f_0, f_1) \in \operatorname{Vec}(M)^2$ the collinearity set

$$\mathcal{C} = \{q \in M \mid f_0(q) \wedge f_1(q) = 0\}. \tag{4.5}$$

Lemma 22. *Let $u \in L^\infty([0,T], [-1,1])$ and $q : [0,T] \to M$ be a trajectory of the control system (1.1) associated with the control u. Assume that $t_\infty \in [0,T]$ is such that $q(t_\infty) \in \{q \in M \mid f_1(q) \wedge [f_0, f_1](q) \neq 0\}$ and that there exists a sequence $\{t_i\}_{i\in\mathbb{N}} \subset [0,T]$ converging to t_∞ such that $q(t_i) \in \mathcal{C}$ for every $i \in \mathbb{N}$. Then there exists*

$$\bar{u} := \lim_{i\to\infty} \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} u(\tau) d\tau \in [-1, 1]$$

and $f_0(q(t_\infty)) + \bar{u} f_1(q(t_\infty)) = 0$.

Proof. First notice that, by continuity, $f_0(q(t_\infty)) \wedge f_1(q(t_\infty)) = 0$ and that, since $f_1(q(t_\infty)) \wedge [f_0, f_1](q(t_\infty)) \neq 0$, the set $\mathcal{C}$ is, locally around $q(t_\infty)$, contained in an embedded $(n-1)$-dimensional manifold $\hat{\mathcal{C}}$ transversal to the vector field f_1.

Let us take any coordinate system around $q(t_\infty)$. Notice that any converging subsequence of $\frac{q(t_\infty)-q(t_i)}{t_\infty - t_i}$ is tangent to $\hat{\mathcal{C}}$. Writing

$$\frac{q(t_\infty) - q(t_i)}{t_\infty - t_i} = \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} (f_0(q(\tau)) + u(\tau) f_1(q(\tau))) d\tau,$$

we deduce that for every converging subsequence of $\{\frac{1}{t_\infty - t_i}\int_{t_i}^{t_\infty} u(\tau)d\tau\}_{i\in\mathbb{N}}$, its limit $\tilde{u}$ is such that $f_0(q(t_\infty)) + \tilde{u} f_1(q(t_\infty))$ is tangent to $\hat{\mathcal{C}}$. The proof is concluded by noticing that, by transversality of $\hat{\mathcal{C}}$ and f_1, the only vector of the form $f_0(q(t_\infty)) + \tilde{u} f_1(q(t_\infty)) \in \operatorname{span}(f_1(q(t_\infty)))$ which is tangent to $\hat{\mathcal{C}}$ is zero. $\square$

Remark 23. The lemma says in particular that for every $(f_0, f_1) \in \operatorname{Vec}(M)^2$ and every trajectory $q : [0, T] \to M$ of (1.1), almost everywhere on $\{t \in [0, T] \mid q(t) \in \mathcal{C},\ f_1(q) \wedge [f_0, f_1](q) \neq 0\}$ we have $\dot{q} = 0$. This result is in the same spirit as [12, Theorem 2.1], where the multi-input case in considered.

Definition 24. For any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0, T]$ of the time-optimal control problem (1.1), we call $\Omega = \{t \in [0, T] \mid q(t) \in \mathcal{C},\ h_1(t) = 0\}$. Moreover, we denote by Ω_0 the set of all isolated points in Ω, and inductively we declare Ω_k to be the set of isolated points in $\Omega \setminus (\bigcup_{j=0}^{k-1} \Omega_j)$.

Theorem 25. *Let* $(f_0, f_1) \in \operatorname{Vec}(M)^2$ *and let* $(q(\cdot), u(\cdot), \lambda(\cdot))$ *be any extremal trajectory on* $[0, T]$ *of the time-optimal control problem* (1.1). *Assume that there exist a sequence* $\{t_i\}_{i\in\mathbb{N}} \subset [0, T]$ *and an integer* $k \geq 0$ *such that*

a) $t_i \in \Omega \setminus (\bigcup_{j=0}^{k} \Omega_j)$ *for every* $i \in \mathbb{N}$,
b) *there exists* $t_\infty = \lim_{i\to\infty} t_i$ *and* $q(t_\infty) \in \{q \in M \mid f_1(q) \wedge [f_0, f_1](q) \neq 0\}$.

Then there exists $a \in [-1, 1]$ *such that, with the notation* $s_a(\lambda) := \langle \lambda, (f_0 + a f_1)(\pi(\lambda)) \rangle$, *we have*

$$\operatorname{ad}_{s_a}^j (s_1)(\lambda(t_\infty)) = 0, \quad \text{for every } 0 \leq j \leq k+2.$$

Proof. We proceed by induction on k, and we begin with the case $k = 0$. First notice that for $t \in \Omega \setminus \Omega_0$ both $s_1(\lambda(t)) = h_1(t) = 0$ and $\{s_0, s_1\}(\lambda(t)) = h_{01}(t) = 0$ by continuity and by Rolle's theorem. Also notice that $\{s_0, s_1\} = \operatorname{ad}_{s_a} s_1$ for every $a \in [-1, 1]$. Moreover, by item b) and Lemma 22, there exists

$$a = \lim_{i\to\infty} \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} u(\tau) d\tau \in [-1, 1]$$

and $f_0(q(t_\infty)) + a f_1(q(t_\infty)) = 0$.

From the identity

$$\begin{aligned} 0 &= \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} \frac{d}{d\tau} \{s_0, s_1\}(\lambda(\tau)) d\tau \\ &= \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} (\{s_0, \{s_0, s_1\}\}(\lambda(\tau)) + u(\tau)\{s_1, \{s_0, s_1\}\}(\lambda(\tau)))\, d\tau, \end{aligned}$$

which is valid for every $i \in \mathbb{N}$, passing to the limit as $i \to \infty$ we deduce the further relation $\operatorname{ad}_{s_a}^2 s_1(\lambda(t_\infty)) = \operatorname{ad}_{s_a}\{s_0, s_1\}(\lambda(t_\infty)) = 0$.

Assume now that the theorem holds for some $k \in \mathbb{N}$, and consider any sequence of points $\{t_i\}_{i\in\mathbb{N}} \in \Omega \setminus (\bigcup_{j=0}^{k+1} \Omega_j)$ satisfying items a) and b). Apply Lemma 22 and define a as above. The conclusion comes from noticing that

$$\begin{aligned}
0 &= \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} \frac{d}{d\tau} \mathrm{ad}_{s_a}^{k+1}(s_1)(\lambda(\tau)) d\tau \\
&= \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} (\mathrm{ad}_{s_0} \mathrm{ad}_{s_a}^{k+1}(s_1) + u(\tau)\mathrm{ad}_{s_1} \mathrm{ad}_{s_a}^{k+1}(s_1))(\lambda(\tau)) d\tau \\
&\to \mathrm{ad}_{s_a}^{k+2}(s_1)(\lambda(t_\infty)) \text{ as } i \to \infty.
\end{aligned}$$

□

Inspired by the arguments of [8, Definition 4 and Lemma 4], we are now in the position of deducing quantitative estimates on the possible accumulations of points of Ω within the collinearity set $\mathcal{C}$.

Lemma 26. *Let $q \in M$ and $N = n-1$. With the convention that $f_a := f_0 + af_1$ for any $a \in \mathbb{R}$, let us define the following two subsets of $J_q^N M \times J_q^N M$:*

$$\begin{aligned}
\mathcal{L}' &= \left\{ (j_q^N(f_0), j_q^N(f_1)) \in J_q^N M \times J_q^N M \mid \dim(\mathrm{span}\{f_0(q), f_1(q), [f_0, f_1](q)\}) \le 1 \right\}, \\
\mathcal{L}'' &= \Big\{ (j_q^N(f_0), j_q^N(f_1)) \in J_q^N M \times J_q^N M \mid f_1(q) \neq 0,\ \exists\, a \in \mathbb{R} \textit{ such that } f_0(q) = af_1(q) \\
&\qquad \textit{and } \dim\left(\mathrm{span}\{\mathrm{ad}_{f_a}^i(f_1)(q) \mid 0 \le i \le n-1\}\right) < n \Big\}.
\end{aligned}$$

Then

$$\mathrm{codim}_{J_q^N M \times J_q^N M} \mathcal{L}' = 2n-2 \quad \textit{and} \quad \mathrm{codim}_{J_q^N M \times J_q^N M} \mathcal{L}'' = n.$$

Proof. The first assertion is clear. For the second one just notice that for every $a \in \mathbb{R}$, the dimension of $\mathrm{span}\{\mathrm{ad}_{f_a}^i(f_1)(q) \mid 0 \le i \le n-1\}$ is smaller than n if and only if, in coordinates,

$$\det(H) = 0, \qquad \text{with} \qquad H = \left(f_1, \dots, \mathrm{ad}_{f_a}^{n-1}(f_1) \right).$$

The latter condition, taking as a the unique scalar such that $f_0(q) + af_1(q) = 0$, identifies a set of codimension one inside

$$\mathcal{D} := \left\{ (j_q^N(f_0), j_q^N(f_1)) \in J_q^N M \times J_q^N M \mid f_1(q) \neq 0, f_0(q) \wedge f_1(q) = 0 \right\}.$$

Summing it up, we deduce that

$$\begin{aligned}
\mathrm{codim}_{J_q^N M \times J_q^N M} \mathcal{L}'' &= \mathrm{codim}_{J_q^N M \times J_q^N M} \mathcal{D} \\
&\quad + \mathrm{codim}_{\mathcal{D}} \left\{ (j_q^N(f_0), j_q^N(f_1)) \in J_q^N M \times J_q^N M \mid \det(H) = 0 \right\} \\
&= (n-1) + 1 = n.
\end{aligned}$$

□

Corollary 27. *Let $n \geq 2$. For a generic pair $(f_0, f_1) \in \mathrm{Vec}(M)^2$ and for every extremal trajectory of the time-optimal control problem* (1.1)*, we have $\Omega = \Omega_0 \cup \cdots \cup \Omega_{n-2}$, where Ω and Ω_j are defined as in Definition 24.*

Proof. If along any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ there exists $t \in \Omega \setminus (\bigcup_{j=0}^{n-3} \Omega_j)$, which is not isolated in this set and such that $q(t) \in \{q \in M \mid f_1(q) \wedge f_{01}(q) \neq 0\}$, then by Theorem 25 $\lambda(t)$ annihilates $\mathrm{ad}_{f_a}^i(f_1)(q(t))$ for every $0 \le i \le n-1$, where a is the proportionality coefficient between $-f_0(q(t))$ and $f_1(q(t))$. By Lemma 26, for a generic pair $(f_0, f_1) \in \mathrm{Vec}(M)^2$ this is possible only at isolated points of M, which is equivalent to say that for any such pair $\Omega_{n-2} = \Omega \setminus (\bigcup_{j=0}^{n-3} \Omega_j)$. On the other hand, since $2n-2 \geq n$ if $n \geq 2$, the usual transversality theorem says that for $n \geq 2$ and for a generic choice of (f_0, f_1) the points $q \in M$ such that $f_0(q) \wedge f_1(q) = 0$ and $f_1(q) \wedge f_{01}(q) = 0$ are isolated. This concludes the proof. $\square$

5. Proof of Theorem 3

Theorem 3 directly follows from Theorem 17 and Proposition 28 below.

Proposition 28. *There exist a positive integer $K = K(n)$ and an open and dense set $\mathcal{V} \subset \mathrm{Vec}(M)^2$ such that, for any pair $(f_0, f_1) \in \mathcal{V}$ and for any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0,T]$ of the time-optimal control problem* (1.1)*, the set*

$$\Xi = \{t \in [0,T] \mid h_1(t) = h_{01}(t) = h_{+01}(t) = 0 \text{ or } h_1(t) = h_{01}(t) = h_{-01}(t) = 0\}$$

satisfies $\Xi = \Xi_1 \cup \cdots \cup \Xi_K$, where Ξ_1 denotes the set of isolated points of Ξ and Ξ_{j+1} denotes the set of isolated points of $\Xi \setminus \cup_{i=1}^{j} \Xi_i$ for $j \geq 1$.

Proof. Let $k \in \mathbb{N}$, $(f_0, f_1) \in \mathrm{Vec}(M)^2$ and $(q(\cdot), u(\cdot), \lambda(\cdot))$ be a time-extremal trajectory of the time-optimal control problem (1.1). Let $t \in \Xi \setminus (\bigcup_{j=1}^{k} \Xi_j)$ and assume for now that $f_0(q(t)) \wedge f_1(q(t)) \neq 0$. Owing to the fact that t is an accumulation point for $\Xi \setminus (\bigcup_{j=1}^{k-1} \Xi_j)$ and reasoning iteratively, we identify a set $\{t_{n_1,\ldots,n_r} \mid r = 1, \ldots, k,\ n_1, \ldots, n_r \in \mathbb{N}\}$ such that

$$\begin{aligned}
&\lim_{n_1 \to \infty} t_{n_1} = t, && \\
&\lim_{n_r \to \infty} t_{n_1,\ldots,n_r} = t_{n_1,\ldots,n_{r-1}}, && \text{for } r = 2, \ldots, k \text{ and } n_1, \ldots, n_{r-1} \in \mathbb{N}, \\
&t_{n_1,\ldots,n_r} \in \Xi \setminus \bigcup_{j=1}^{k-r} \Xi_j, && \text{for } r = 2, \ldots, k \text{ and } n_1, \ldots, n_r \in \mathbb{N}, \\
&t_{n_1,\ldots,n_k} \in \Xi, && \text{for } n_1, \ldots, n_k \in \mathbb{N}.
\end{aligned}$$

Using repeatedly Lemmas 20 and 21 and exploiting the fact that each application of one of the two lemmas yields a finite number of alternatives, we deduce from a diagonal extraction argument that, up to taking suitable subsequences,

i) There exist $k+1$ sets $\mathcal{B}_0, \ldots, \mathcal{B}_k \subset T^*M \times J^{k+2}M \times J^{k+2}M$ such that
$$(\lambda(t), j^{k+2}_{q(t)}(f_0), j^{k+2}_{q(t)}(f_1)) \in \mathcal{B}_k$$
and
$$(\lambda(t_{n_1,\ldots,n_r}), j^{k+2}_{q(t_{n_1,\ldots,n_r})}(f_0), j^{k+2}_{q(t_{n_1,\ldots,n_r})}(f_1)) \in \mathcal{B}_{k-r}$$
for every $r = 1, \ldots, k$, $n_1, \ldots, n_r \in \mathbb{N}$.

ii) For every $0 \leq r \leq k$, $\mathcal{B}_r$ is defined by the vanishing of, say, l_r simple relations and m_r polynomial relations (using the terminology of Definition 19). Moreover, denoting $\mathcal{B}_r(q) = \mathcal{B}_r \cap T^*_q M \times J^{k+2}_q M \times J^{k+2}_q M$, we have
$$\mathrm{codim}_{T^*_{q(t)}M \times J^{K+2}_{q(t)}M \times J^{K+2}_{q(t)}M} \mathcal{B}_k(q(t)) = l_k + m_k,$$
$$\mathrm{codim}_{T^*_{q(t_{n_1,\ldots,n_r})}M \times J^{K+2}_{q(t_{n_1,\ldots,n_r})}M \times J^{K+2}_{q(t_{n_1,\ldots,n_r})}M} \mathcal{B}_{k-r}(q(t_{n_1,\ldots,n_r})) = l_{k-r} + m_{k-r},$$
for every $r = 1, \ldots, k$, $n_1, \ldots, n_r \in \mathbb{N}$.

By construction, the set $\mathcal{B}_k(q)$ is homogeneous with respect to the first component. To prove the proposition in the set $\{q \in M \mid f_0(q) \wedge f_1(q) \neq 0\}$ it is then sufficient to show that there exists $K = K(n)$ such that if $k \geq K$, then there exists $r \in \{0, \ldots, k\}$ such that the codimension $l_{k-r} + m_{k-r}$ of $\mathcal{B}_{k-r}(q(t_{n_1,\ldots,n_r}))$ in $T^*_{q(t_{n_1,\ldots,n_r})}M \times J^{K+2}_{q(t_{n_1,\ldots,n_r})}M \times J^{K+2}_{q(t_{n_1,\ldots,n_r})}M$ is strictly larger than $2n-1$. Indeed, if this were true, then denoting by $\pi : T^*M \times J^{K+2}M \times J^{K+2}M \to J^{K+2}M \times J^{K+2}M$

the canonical projection, we would conclude by standard transversality arguments [14] combined with the inequality

$$\begin{aligned}&\operatorname{codim}_{J^{K+2}_{q(t_{n_1,\dots,n_r})}M\times J^{K+2}_{q(t_{n_1,\dots,n_r})}M}\pi\left(\mathcal{B}_{k-r}(q(t_{n_1,\dots,n_r}))\right)\\&\qquad\geq \operatorname{codim}_{T^*_{q(t_{n_1,\dots,n_r})}M\times J^{K+2}_{q(t_{n_1,\dots,n_r})}M\times J^{K+2}_{q(t_{n_1,\dots,n_r})}M}\mathcal{B}_{k-r}(q(t_{n_1,\dots,n_r}))-n+1>n,\end{aligned}$$

where the term $+1$ is due to the homogeneity of $\mathcal{B}_{k-r}(q)$ with respect to the first component.

We introduce now a discrete dynamics on $\mathbb{N}^2$, which describes the admissible patterns of $r\mapsto(l_r,m_r)$. Define three mappings $F_0,F_1,F_2:\mathbb{N}^2\to\mathbb{N}^2$ by

$$F_0(x_1,x_2)=(x_1,x_2)+(1,0),\quad F_1(x_1,x_2)=(x_1,x_2)+(0,1),\quad F_2(x_1,x_2)=(x_1,0)+(2,0).$$

We say that an admissible curve γ of length $p\in\mathbb{N}$ for this dynamical system is a map $\gamma:\{0,\dots,p\}\to\mathbb{N}^2$ such that

i) $\gamma(0)=(3,0)$,
ii) there exists $j\in\{1,\dots,p\}$ such that $\gamma(i)=F_0(\gamma(i-1))$ for $i=1,\dots,j$,
iii) $\gamma(i)=F_{\sigma_i}(\gamma(i-1))$, $\quad\sigma_i\in\{1,2\}$, $\forall\, i=j+1,\dots,p$.

Observe that the initial condition fixed in i) reflects the definition of Ξ, F_0 describes the creation of a new simple relation (Lemma 20), while F_1 and F_2 encode the occurrence of, respectively, a new polynomial relation and two new simple relations (Lemma 21).

We are going to compute the minimal K so that, for $k\geq K$, any admissible curve γ of length k exits the region $T:=\{(x_1,x_2)\in\mathbb{N}^2\mid x_1+x_2\leq 2n-1\}$. It is not difficult to see that the longest admissible curve γ staying in T is as indicated in Figure 2, that is, we apply once F_0, then $2n-5$ times F_1, then once F_2, then $2n-7$ times F_1, once F_2, and so on. The length of such curve γ is equal to

$$\operatorname{length}(\gamma)=1+(2n-5)+1+(2n-7)+1+\cdots+(2n-(2n-1))=(n-2)(n-1),$$

which implies that $K=1+(n-2)(n-1)$.

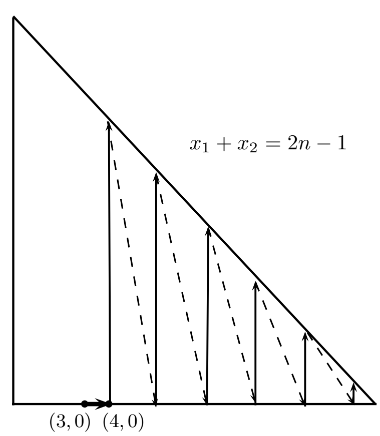

FIGURE 2. The longest admissible curve γ

It just remains to explain what can happen inside the collinearity set $\mathcal{C}$ introduced in (4.5): for a generic choice of (f_0,f_1), along any extremal trajectory the points of Ω can accumulate at most $n-2$ times according to Corollary 27. On the other hand any point of Ω is itself an element of $\Xi\setminus(\bigcup_{j=1}^K\Xi_j)$ at worst, which implies that K can increase at most by $n-2$ within $\mathcal{C}$. This concludes the proof of Proposition 28. □

Remark 29. As a byproduct of the proof proposed above, we get that the sharpest choice of the integer $K(n)$ in the statement of Theorem 3 satisfies $K(n) \leq 1+(n-2)(n-1)+n-2 = (n-1)^2$.

6. Time-optimal trajectories in dimension $n = 3$

We devote this section to a more careful analysis of Fuller times for *time-optimal* (and not only extremal) trajectories, in the case of a three dimensional manifold $M = M^3$. In fact, for a time-optimal trajectory there are powerful second-order techniques [5] that permit us to be a bit sharper in our estimate of $K(3)$, at least if we just focus on this smaller class of curves. By Remark 29, we already know the upper bound $K(3) \leq 4$. The main result of this section is the following.

Theorem 30. *For a generic pair* $(f_0, f_1) \in \mathrm{Vec}(M)^2$*, none of the time-optimal trajectories of the control system* (1.1) *has Fuller times of order greater than two.*

For the rest of this section we adopt the following convention: for any subset $\Theta \subset [0,T]$, we denote by $q(\Theta)$ its image along the trajectory $q(\cdot)$.

Let us fix then a time-optimal trajectory. We collect previous results from [7, 15, 24] in the following statement.

Proposition 31. *Let* $(f_0, f_1) \in \mathrm{Vec}(M)^2$ *and* $q(\cdot)$ *be any time-optimal trajectory of the control system* (1.1)*. Let us consider, with the notations of Definition 7, the subsets*

$$\begin{aligned}
A_1 &= \{q \in M \mid f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) \neq 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) \neq 0\},\\
A_2 &= \{q \in M \mid f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) = 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{++01}(q) \neq 0,\\
&\qquad f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) \neq 0\},\\
A_3 &= \{q \in M \mid f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) = 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{--01}(q) \neq 0,\\
&\qquad f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) \neq 0\},\\
A_4 &= \{q \in M \mid f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) = 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{++01}(q) = 0,\\
&\qquad f_1(q) \wedge f_{01}(q) \wedge f_{+++01}(q) \neq 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) \neq 0\},\\
A_5 &= \{q \in M \mid f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) = 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{--01}(q) = 0,\\
&\qquad f_1(q) \wedge f_{01}(q) \wedge f_{---01}(q) \neq 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) \neq 0\},\\
A_6 &= \{q \in M \mid f_1(q) \wedge f_{01}(q) = 0,\ f_1(q) \wedge f_{+01}(q) \wedge f_{-01}(q) \neq 0,\\
&\qquad f_1(q) \wedge f_{+01}(q) \wedge f_{++01}(q) \neq 0,\ f_1(q) \wedge f_{-01}(q) \wedge f_{--01}(q) \neq 0\}.
\end{aligned}$$

If $q(t) \in \bigcup_{i=1}^{6} A_i$*, then* $t \notin \Sigma \setminus \Sigma_0$*.*

Define now the set

$$W = \{q \in M \mid f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) = 0,\ f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) = 0,\ f_1(q) \wedge f_{01}(q) \neq 0\}.$$

As a consequence of Proposition 31, we can infer the following result.

Lemma 32. *For a generic pair* $(f_0, f_1) \in \mathrm{Vec}(M)^2$ *and for every time-optimal trajectory* $q(\cdot)$ *of the control system* (1.1)*,* $q(\Sigma \setminus \Sigma_0) \setminus W$ *is made of isolated points only.*

Proof. The result is proved by using the same computational approach based on transversality theory as in the proof of Lemma 20. Instead of working in T^*M as in Lemma 20, it is actually sufficient to prove that

$$\mathrm{codim}_{J_q^N M \times J_q^N M} \left(\bigcap_{i=1}^{6} \mathcal{A}_i^c \cap \mathcal{W}^c \right) \geq 3, \qquad q \in M,$$

where $\mathcal{A}_1, \ldots, \mathcal{A}_6$ and $\mathcal{W}$ are the subsets of $J^N M \times J^N M$ defined implicitly by the relations

$$A_i = \{q \in M \mid (j_q^N(f_0), j_q^N(f_1)) \in \mathcal{A}_i\}, \quad i = 1, \ldots, 6, \quad W = \{q \in M \mid (j_q^N(f_0), j_q^N(f_1)) \in \mathcal{W}\}.$$

Pick then any point $q \in W^c$ that satisfies $f_1(q) \wedge f_{01}(q) = 0$. Then $\mathcal{W} \cap J_q^N M \times J_q^N M$ is already a set of codimension two in $J_q^N M \times J_q^N M$. Moreover, if $q \in A_6^c$, then necessarily the jets of f_0, f_1 at q satisfy another nontrivial dependence relation, and we can conclude.

On the other hand, suppose that $q \in \cap_{i=1}^6 A_i^c$ and that $f_1(q) \wedge f_{01}(q) \wedge f_{+01}(q) \neq 0$, the remaining case being identical. Then since $q \in A_1^c$ we infer the relation $f_1(q) \wedge f_{01}(q) \wedge f_{-01}(q) = 0$. We pass now to the condition $q \in A_3^c$, and we see that this obliges $f_1(q) \wedge f_{01}(q) \wedge f_{--01}(1) = 0$. Finally, the relation $q \in A_5^c$ forces $f_1(q) \wedge f_{01}(q) \wedge f_{---01}(q) = 0$, which in turn provides us with a third dependence relation at q, and therefore once again we conclude. □

6.1. **Proof of Theorem 30.** Lemma 32 states, in particular, that for a generic choice of the pair (f_0, f_1) and for every time-optimal trajectory $q(\cdot)$ we have that $q(\Sigma \setminus \Sigma_0) \setminus W \subset q(\Sigma_1)$, or equivalently that

$$q(\Sigma \setminus (\Sigma_0 \cup \Sigma_1)) \subset W.$$

We are left to prove that the density points of $q(\Sigma \setminus (\Sigma_0 \cup \Sigma_1)) = q(\Sigma \setminus (\Sigma_0 \cup \Sigma_1)) \cap W$ are isolated.

We have already shown that along any time-extremal $(q(\cdot), u(\cdot), \lambda(\cdot))$, whenever $t \in \Sigma \setminus \Sigma_0$ the relations

$$h_1(\lambda(t)) = \langle \lambda(t), f_1(q(t)) \rangle = 0 \quad \text{and} \quad h_{01}(\lambda(t)) = \langle \lambda(t), f_{01}(q(t)) \rangle = 0$$

hold true. Since, by definition, for every point $q \in W$ both $f_{+01}(q)$ and $f_{-01}(q)$ belong to the two-dimensional space $\operatorname{span}\{f_1(q), f_{01}(q)\}$, then for every $t \in \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ also $h_{+01}(\lambda(t)) = h_{-01}(\lambda(t)) = 0$. If t_∞ is an accumulation point of $\Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, then, by Lemma 21 and using the Jacobi identity, either $h_{0101}(t_\infty) = 0$ or $h_{0101}(t_\infty) \neq 0$ and

$$h_{0001}(t_\infty) h_{1101}(t_\infty) - h_{0101}(t_\infty)^2 = 0. \tag{6.1}$$

When $h_{0101}(t_\infty) = 0$, we conclude by transversality, noticing that

$$f_{0101}(q(t_\infty)) \in \lambda(t_\infty)^\perp = \operatorname{span}\{f_1(q(t_\infty)), f_{01}(q(t_\infty))\}$$

provides a third independent condition on the jet of the pair (f_0, f_1) at $q(t_\infty)$. In the case $h_{0101}(t_\infty) \neq 0$, let us define in a neighborhood of $q(t_\infty)$ a system of coordinates (x_1, x_2, x_3) so that (dx_1, dx_2, dx_3) is dual to (f_1, f_{01}, f_{0101}). Then (6.1) says that the product of the third components of $f_{0001}(q(t_\infty))$ and $f_{1101}(q(t_\infty))$ is equal to one, which corresponds to a third independent condition on the jet of the pair (f_0, f_1) at $q(t_\infty)$.

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