HAL
archives-ouvertes.fr

# Note on Curry's style for Linear Call-by-Push-Value

Guillaume Munch-Maccagnoni

▶ **To cite this version:**

Guillaume Munch-Maccagnoni. Note on Curry's style for Linear Call-by-Push-Value. 2017. hal-01528857v3

**HAL Id: hal-01528857**

**https://hal.inria.fr/hal-01528857v3**

Submitted on 10 Sep 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Note on Curry's style for Linear Call-by-Push-Value

Guillaume Munch-Maccagnoni
Team Gallinette, Inria Bretagne Atlantique, Univ. Nantes, LS2N, Nantes, France
Guillaume.Munch-Maccagnoni@Inria.fr

3rd May 2017*

We present Curry-style calculi for intuitionistic (linear) logic with polarised evaluation order [MMS15, CFMM16] and give self-contained proofs of their main properties and of their interpretation into (Linear) Call-by-Push-Value models: subject reduction, confluence, strong normalisation, coherence, soundness, and focusing.

## Contents



*31st August 2017: fixed the measure used in Proposition 67. 10th September 2017: added references in Section 7 and typos.

## List of Figures



## 1 Introduction

The polarised abstract-machine calculi (also called L- or $\mu\tilde{\mu}$-calculi):

- $\mathbf{LJ}_p^\eta$ (*intuitionistic sequent calculus with evaluation order*) for Call-by-Push-Value models by Scherer and the author [MMS15] and

- $\mathbf{ILL}_p^\eta$ (*intuitionistic linear logic with evaluation order*) for Linear Call-by-Push-Value models by Fiore, Curien, and the author [CFMM16]

are introduced à la Curry in Figures 1 and 2.

The goal of this note is to provide self-contained proofs of important properties of these calculi and their interpretations into the models, strengthening results from [CFMM16]: subject reduction, confluence, strong normalisation, coherence, soundness and focusing.

In [MMS15], $\mathbf{LJ}_p^\eta$ is called $\mathbf{L}_{\text{int}}$. We introduce minor improvements in the definition, and (more explicitly than in [CFMM16]) we present it with a "□" modality which we write ! in this note (i.e. $\mathbf{CS4}_p^\eta$ [AMdPR01] without the modality ◇). The naming of $\mathbf{LJ}_p^\eta$ and $\mathbf{ILL}_p^\eta$ follows from that of Danos, Joinet, and Schellinx' classical sequent calculus $\mathbf{LK}_p^\eta$ [DJS97]. Their $\mathbf{LK}_p^\eta$ can be turned into a Curry-style calculus in the style of $\mathbf{LJ}_p^\eta$ following the ideas in [MM09]; in fact all the results proved in this note carry in a straightforward manner over to classical systems $\mathbf{LL}_p^\eta$ and $\mathbf{S4}_p^\eta$ containing $\mathbf{LK}_p^\eta$ (i.e. linear logic **LL** and modal logic **S4** with evaluation order).

Conventions:

- We use the notation $\langle t \,\|\, e\rangle$ for a command when its (unique) polarity can be inferred from the (consistent) polarities of $t$ and $e$.
- A polarity annotation in superscript (e.g. in $\mu\alpha^+.c$) denotes an explicit token in the grammar. A polarity annotation in subscript (e.g. in $A_+$) asserts a polarity, and is not part of the grammar.

Figure 3 recalls how to express the sequent calculi **ILL** and **LJ**. Figure 4 recalls how to express the $\lambda$-calculus with sums and empty type in three different polarisations: call-by-name, call-by-value, and polarised. In these figures, a doubled inference line asserts that the rule is derivable in $\mathbf{ILL}_p^\eta$ and $\mathbf{LJ}_p^\eta$.

The notions of Call-by-Push-Value model [Lev05, EMS12] for $\mathbf{LJ}_p^\eta$, and of Linear Call-by-Push-Value model with resource modality [CFMM16] for $\mathbf{ILL}_p^\eta$, are recalled in Section 2.

### 1.1 The case of the $\lambda$-calculus

The standard method for proving the subject reduction and confluence of the Church-style $\lambda$-calculus is exposed in Barendregt [Bar93]. It consists in proving in turn the following standard properties:

#### Basis lemma

- If $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash t : A$ then $\Gamma' \vdash t : A$,
- If $\Gamma \vdash t : A$ then $\mathbf{fv}\, t \subseteq \operatorname{dom}\Gamma$,
- If $\Gamma \vdash t : A$ then $\Gamma_{\restriction \mathbf{fv}\, t} \vdash t : A$.

#### Generation lemma

- If $\Gamma \vdash x : A$ then $(x : A) \in \Gamma$,
- If $\Gamma \vdash t\, u : B$ then there exists $A$ such that $\Gamma \vdash t : A \to B$ and $\Gamma \vdash u : A$,
- If $\Gamma \vdash \lambda x^B.t : A$ then $A = B \to C$ with $\Gamma, x : B \vdash t : C$.

| | $\mathbf{1}$ | $\otimes$ | $\oplus\,(i\in\{1,2\})$ | $!$ | $\multimap$ | $\&\,(i\in\{1,2\})$ | $\top/\mathbf{0}$ |
|---|---|---|---|---|---|---|---|

Values:

$$V,W \;::=\; x \mid \mu\alpha^{\ominus}.c \mid () \mid V\otimes W \mid \iota_i(V) \mid \mu!\alpha.c \mid \mu(x\cdot\alpha).c \mid \mu{<}\alpha.c\,;\beta.c'{>} \mid \mu{<}V{>}$$

Expressions:

$$t,u \;::=\; V \mid \mu\alpha^{+}.c$$

Stacks:

$$S \;::=\; \alpha \mid \tilde{\mu}x^{+}.c \mid \tilde{\mu}().c \mid \tilde{\mu}(x\otimes y).c \mid \tilde{\mu}[x.c|y.c'] \mid\; !S \mid V\cdot S \mid \pi_i\cdot S \mid \tilde{\mu}[S]$$

Contexts:

$$e \;::=\; S \mid \tilde{\mu}x^{\ominus}.c$$

Commands:

$$c \;::=\; \langle V \,\|\, e\rangle^{\ominus} \mid \langle t \,\|\, S\rangle^{+}$$

$$\varepsilon \;::=\; + \mid \ominus$$

(a) Terms

| | | | |
|---|---|---|---|
| $(R\tilde{\mu}^{\varepsilon})$ : | $\langle V \,\|\, \tilde{\mu}x^{\varepsilon}.c\rangle^{\varepsilon}$ | $\rhd_{\mathrm{R}}$ | $c[V/x]$ |
| $(R\mu^{\varepsilon})$ : | $\langle \mu\alpha^{\varepsilon}.c \,\|\, S\rangle^{\varepsilon}$ | $\rhd_{\mathrm{R}}$ | $c[S/\alpha]$ |
| $(R\mathbf{1})$ : | $\langle () \,\|\, \tilde{\mu}().c\rangle^{+}$ | $\rhd_{\mathrm{R}}$ | $c$ |
| $(R\otimes)$ : | $\langle V\otimes W \,\|\, \tilde{\mu}(x\otimes y).c\rangle^{+}$ | $\rhd_{\mathrm{R}}$ | $c[V/x, W/y]$ |
| $(R\multimap)$ : | $\langle \mu(x\cdot\alpha).c \,\|\, V\cdot S\rangle^{\ominus}$ | $\rhd_{\mathrm{R}}$ | $c[V/x, S/\alpha]$ |
| $(R!)$ : | $\langle \mu!\alpha.c \,\|\, !S\rangle^{+}$ | $\rhd_{\mathrm{R}}$ | $c[S/\alpha]$ |
| $(R\&)$ : | $\langle \mu{<}\alpha_1.c_1\,;\alpha_2.c_2{>} \,\|\, \pi_i\cdot S\rangle^{\ominus}$ | $\rhd_{\mathrm{R}}$ | $c_i[S/\alpha_i]$ |
| $(R\oplus)$ : | $\langle \iota_i(V) \,\|\, \tilde{\mu}[x_1.c_1|x_2.c_2]\rangle^{+}$ | $\rhd_{\mathrm{R}}$ | $c_i[V/x_i]$ |
| | (no rules $R\top$, $R\mathbf{0}$) | | |

(b) Reduction rules

| | | | |
|---|---|---|---|
| $(E\tilde{\mu}^{\varepsilon})$ : | $\tilde{\mu}x^{\varepsilon}.\langle x \,\|\, e\rangle^{\varepsilon}$ | $\rhd_{\mathrm{E}}$ | $e$ |
| $(E\mu^{\varepsilon})$ : | $\mu\alpha^{\varepsilon}.\langle t \,\|\, \alpha\rangle^{\varepsilon}$ | $\rhd_{\mathrm{E}}$ | $t$ |
| $(E\mathbf{1})$ : | $\tilde{\mu}().\langle () \,\|\, S\rangle^{+}$ | $\rhd_{\mathrm{E}}$ | $S$ |
| $(E\otimes)$ : | $\tilde{\mu}(x\otimes y).\langle x\otimes y \,\|\, S\rangle^{+}$ | $\rhd_{\mathrm{E}}$ | $S$ |
| $(E\multimap)$ : | $\mu(x\cdot\alpha).\langle V \,\|\, x\cdot\alpha\rangle^{\ominus}$ | $\rhd_{\mathrm{E}}$ | $V$ |
| $(E!)$ : | $\mu!\alpha.\langle V \,\|\, !\alpha\rangle^{+}$ | $\rhd_{\mathrm{E}}$ | $V$ |
| $(E\&)$ : | $\mu{<}\alpha.\langle V \,\|\, \pi_1\cdot\alpha\rangle^{\ominus};\beta.\langle V \,\|\, \pi_2\cdot\beta\rangle^{\ominus}{>}$ | $\rhd_{\mathrm{E}}$ | $V$ |
| $(E\oplus)$ : | $\tilde{\mu}\big[x.\langle \iota_1(x) \,\|\, S\rangle^{+}\big|y.\langle \iota_2(y) \,\|\, S\rangle^{+}\big]$ | $\rhd_{\mathrm{E}}$ | $S$ |
| $(E\top)$ : | $\mu{<}x_1\otimes\cdots\otimes x_n{>}$ | $\rhd_{\mathrm{E}}$ | $V$ |
| $(E\mathbf{0})$ : | $\tilde{\mu}[x_1\cdots x_n\cdot\alpha]$ | $\rhd_{\mathrm{E}}$ | $S$ |

(c) Extensionality rules

Figure 1: $\mathbf{ILL}_p^{\eta}/\mathbf{LJ}_p^{\eta}$: calculus

$$
\begin{array}{lrl}
\text{types} & A, B, A_\varepsilon & ::= P \mid N \\
\text{positive} & P, Q, A_+ & ::= X^+ \mid \mathbf{1} \mid A \otimes B \mid {!A} \mid A \oplus B \mid \mathbf{0} \\
\text{negative} & N, M, A_\ominus & ::= X^\ominus \mid A \multimap B \mid A \,\&\, B \mid \top
\end{array}
$$

(a) Types

- **Δ** is a pair $\alpha : A$ of a co-variable and a type.
- **Γ** is a map from a finite set of variables to types provided with a total order $\leq_\Gamma$ on its domain, notation $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$.
- Concatenation $(\Gamma, \Gamma')$ is defined when the domains of $\Gamma$ and $\Gamma'$ are disjoint.
- $\mathbf{\Sigma(\Gamma; \Gamma')}$ is the set of maps $\sigma : \operatorname{dom}\Gamma \to \operatorname{dom}\Gamma'$ satisfying $\Gamma'(\sigma(x)) = \Gamma(x)$ for all $x \in \operatorname{dom}\Gamma$.
- $\mathbf{\Sigma^!(\Gamma; \Gamma')}$ is the subset of $\Sigma(\Gamma; \Gamma')$ of maps that are bijective on variables not of the form $!A$.
- Judgements are: $\Gamma \vdash t : A \mid \qquad \Gamma \mid e : A \vdash \Delta \qquad c : (\Gamma \vdash \Delta)$
- $!(x_1 : A_1, \dots, x_n : A_n)$ stands for the typing context $x_1 : !A_1, \dots, x_n : !A_n$.

(b) Judgements

$$
\frac{}{x : A \vdash x : A \mid}\ (\vdash \mathrm{ax})
\qquad
\frac{}{\mid \alpha : A \vdash \alpha : A}\ (\mathrm{ax} \vdash)
$$

$$
\frac{c : (\Gamma, x : A_\varepsilon \vdash \Delta)}{\Gamma \mid \tilde{\mu}x^\varepsilon.c : A_\varepsilon \vdash \Delta}\ (\tilde{\mu}^\varepsilon \vdash)
\qquad
\frac{c : (\Gamma \vdash \alpha : A_\varepsilon)}{\Gamma \vdash \mu\alpha^\varepsilon.c : A_\varepsilon \mid}\ (\vdash \mu^\varepsilon)
$$

$$
\frac{\Gamma \vdash t : A_\varepsilon \mid \qquad \Gamma' \mid e : A_\varepsilon \vdash \Delta}{\langle t \parallel e \rangle^\varepsilon : (\Gamma, \Gamma' \vdash \Delta)}\ (\mathrm{cut}^\varepsilon)
$$

(c) Identity

$$
\frac{\Gamma \vdash t : A \mid}{\Gamma' \vdash t[\sigma] : A \mid}\ (\vdash \sigma)
\qquad
\frac{\Gamma \mid e : A \vdash \Delta}{\Gamma' \mid e[\sigma] : A \vdash \Delta}\ (\sigma \vdash)
\qquad
\frac{c : (\Gamma \vdash \Delta)}{c[\sigma] : (\Gamma' \vdash \Delta)}\ (\sigma)
$$

(d) Structure — $\sigma \in \Sigma(\Gamma; \Gamma')$ ($\mathbf{LJ}^\eta_p$) or $\sigma \in \Sigma^!(\Gamma; \Gamma')$ ($\mathbf{ILL}^\eta_p$)

$$
\frac{\Gamma \vdash V : A \mid \qquad \Gamma' \mid S : B \vdash \Delta}{\Gamma, \Gamma' \mid V \cdot S : A \multimap B \vdash \Delta}\ (\multimap \vdash_f)
\qquad
\frac{c : (\Gamma, x : A \vdash \alpha : B)}{\Gamma \vdash \mu(x \cdot \alpha).c : A \multimap B \mid}\ (\vdash \multimap)
\qquad
\frac{}{\vdash () : \mathbf{1} \mid}\ (\vdash \mathbf{1})
$$

$$
\frac{\Gamma \vdash V : A \mid \qquad \Gamma' \vdash W : B \mid}{\Gamma, \Gamma' \vdash V \otimes W : A \otimes B \mid}\ (\vdash_f \otimes)
\qquad
\frac{c : (\Gamma, x : A, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x \otimes y).c : A \otimes B \vdash \Delta}\ (\otimes \vdash)
\qquad
\frac{c : (\Gamma \vdash \Delta)}{\Gamma \mid \tilde{\mu}().c : \mathbf{1} \vdash \Delta}\ (\mathbf{1} \vdash)
$$

$$
\frac{c : (\Gamma \vdash \alpha : A) \qquad c' : (\Gamma \vdash \beta : B)}{\Gamma \vdash \mu{<}\alpha.c; \beta.c'{>} : A \,\&\, B \mid}\ (\vdash \&)
\qquad
\frac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i \cdot S : A_1 \,\&\, A_2 \vdash \Delta}\ (\&_i \vdash_f)
\qquad
\frac{\Gamma \vdash V : A \mid}{\Gamma \vdash \mu{<}V{>} : \top \mid}\ (\vdash_f \top)
$$

$$
\frac{c : (\Gamma, x : A \vdash \Delta) \qquad c' : (\Gamma, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}[x.c \mid y.c'] : A \oplus B \vdash \Delta}\ (\oplus \vdash)
\qquad
\frac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid}\ (\vdash_f \oplus_i)
\qquad
\frac{\Gamma \mid S : A \vdash \Delta}{\Gamma \mid \tilde{\mu}[S] : \mathbf{0} \vdash \Delta}\ (\mathbf{0} \vdash_f)
$$

$$
\frac{c : (!\Gamma \vdash \alpha : A)}{!\Gamma \vdash \mu!\alpha.c : !A \mid}\ (\vdash !)
\qquad
\frac{\Gamma \mid S : A \vdash \Delta}{\Gamma \mid !S : !A \vdash \Delta}\ (! \vdash_f)
$$

(e) Logic

Figure 2: $\mathbf{ILL}^\eta_p/\mathbf{LJ}^\eta_p$: simple types

$$\frac{\Gamma \vdash t : A \mid \qquad \Gamma' \mid e : B \vdash \Delta}{\Gamma, \Gamma' \mid t \cdot e : A \rightarrow B \vdash \Delta}\,(\rightarrow\vdash) \qquad t_{\varepsilon_1} \cdot e_{\varepsilon_2} \overset{\text{def}}{=} \tilde{\mu} x^{\ominus}. \Big\langle \mu \alpha^{\varepsilon_2}. \big\langle t \,\big\|\, \tilde{\mu} y^{\varepsilon_1}. \langle x \,\|\, y \cdot \alpha \rangle \big\rangle \,\Big\|\, e \Big\rangle$$

$$\frac{\Gamma \vdash t : A \mid \qquad \Gamma' \vdash u : B \mid}{\Gamma, \Gamma' \vdash t \otimes u : A \otimes B \mid}\,(\vdash\otimes) \qquad t_{\varepsilon_1} \otimes u_{\varepsilon_2} \overset{\text{def}}{=} \mu \alpha^{+}. \Big\langle t \,\Big\|\, \tilde{\mu} x^{\varepsilon_1}. \big\langle u \,\big\|\, \tilde{\mu} y^{\varepsilon_2}. \langle x \otimes y \,\|\, \alpha \rangle \big\rangle \Big\rangle$$

$$\frac{\Gamma \mid e : A \vdash \Delta}{\Gamma \mid !e : !A \vdash \Delta}\,(!\vdash) \qquad !e_{\varepsilon} \overset{\text{def}}{=} \tilde{\mu} x^{+}. \big\langle \mu \alpha^{\varepsilon}. \langle x \,\|\, !\alpha \rangle \,\big\|\, e \big\rangle$$

$$\frac{\Gamma \vdash t : (A_i)_{\varepsilon} \mid}{\Gamma \vdash \iota_i(t) : A_1 \oplus A_2 \mid}\,(\vdash\oplus_i) \qquad \iota_i(t_{\varepsilon}) \overset{\text{def}}{=} \mu \alpha^{+}. \big\langle t \,\big\|\, \tilde{\mu} x^{\varepsilon}. \langle \iota_i(x) \,\|\, \alpha \rangle \big\rangle$$

$$\frac{\Gamma \mid e : (A_i)_{\varepsilon} \vdash \Delta}{\Gamma \mid \pi_i \cdot e : A_1 \,\&\, A_2 \vdash \Delta}\,(\&_i\vdash) \qquad (\pi_i \cdot e_{\varepsilon}) \overset{\text{def}}{=} \tilde{\mu} x^{\ominus}. \big\langle \mu \alpha^{\varepsilon}. \langle x \,\|\, \pi_i \cdot \alpha \rangle \,\big\|\, e \big\rangle$$

$$\frac{}{\Gamma \mid \tilde{\mu}[]_{\Gamma,\Delta} : \mathbf{0} \vdash \Delta}\,(\mathbf{0}\vdash) \qquad \tilde{\mu}[]_{\Gamma,\Delta} \overset{\text{def}}{=} \tilde{\mu}[\bullet\, \mathrm{dom}(\Gamma, \Delta)]$$

$$\frac{}{\Gamma \vdash \mu{<>}_{\Gamma} : \top \mid}\,(\vdash\top) \qquad \mu{<>}_{\Gamma} \overset{\text{def}}{=} \mu{<}\otimes \mathrm{dom}\,\Gamma{>}$$

Figure 3: Sequent calculi $\mathbf{ILL}_p$/$\mathbf{LJ}_p$: $\mathbf{LJ}_p^{\eta}$ and $\mathbf{ILL}_p^{\eta}$ without value/stack restriction (called $\eta$-restriction in Danos, Joinet and Schellinx [DJS97]). For $(\rightarrow\vdash)$, $(\vdash\otimes)$, $(\vdash\oplus_i)$, and $(\&_i\vdash)$ this expresses the so-called $\varsigma$-rules [Wad03].

**Typability of subterms** If $\Gamma \vdash t : A$ and $u$ is a subterm of $t$ then there exists $\Gamma'$, $B$ with $\Gamma' \vdash u : B$.

**Substitution lemma**

- If $\Gamma \vdash t : A$ then $\Gamma[B/X] \vdash t[B/X] : A[B/X]$.
- If $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash u : A$ then $\Gamma \vdash t[u/x] : B$.

**Subject reduction** If $t \to u$ (the compatible closure of reduction) and $\Gamma \vdash t : A$ then $\Gamma \vdash u : A$.

**Confluence** Reduction on pseudo-terms is confluent (using the standard Tait-Martin-Löf technique), which implies the confluence of the typed reduction on legal terms via subject reduction.

## 1.2 Contents

This note proves these essential properties of Curry-style $\mathbf{ILL}_p^{\eta}$ and $\mathbf{LJ}_p^{\eta}$, following the outline of the above method, strengthened with corresponding statements about the interpretation. Therefore it is strengthened in various ways.

**Curry's style** The calculi $\mathbf{LJ}_p^{\eta}$ and $\mathbf{ILL}_p^{\eta}$ are variants *à la Curry*, free of typing annotations, of the respective calculi in [CFMM16]. Instead, polarity annotations give the minimal amount of information

$$t, u, v ::= x \mid \lambda x.t \mid t\,u \mid \texttt{<}t;u\texttt{>} \mid \pi_i(t) \mid * \mid \jmath_i(t) \mid \delta(t, x.u, y.v) \mid \mathcal{A}(t)$$

$$A, B ::= X \mid A \to B \mid A \times B \mid 1 \mid A + B \mid 0$$

(a) $\lambda$-calculus with sums and empty types

| | | Call-by-name encoding | Call-by-value encoding |
|---|---|---|---|
| $X$ | $\stackrel{\text{def}}{\equiv}$ | $X^{\ominus}$ | $X^{+}$ |
| $A \to B$ | $\stackrel{\text{def}}{\equiv}$ | $A \multimap B$ | $(A \multimap B) \otimes \mathbf{1}$ |
| $A \times B$ | $\stackrel{\text{def}}{\equiv}$ | $A \,\&\, B$ | $A \otimes B$ |
| $1$ | $\stackrel{\text{def}}{\equiv}$ | $\top$ | $\mathbf{1}$ |
| $A + B$ | $\stackrel{\text{def}}{\equiv}$ | $\mathbf{1} \multimap (A \oplus B)$ | $A \oplus B$ |
| $0$ | $\stackrel{\text{def}}{\equiv}$ | $\mathbf{1} \multimap \mathbf{0}$ | $\mathbf{0}$ |
| $\lambda x.t_{\varepsilon}$ | $\stackrel{\text{def}}{\equiv}$ | $\mu(x\cdot\alpha).\langle t \parallel \alpha\rangle^{\varepsilon}$ | $\big(\mu(x\cdot\alpha).\langle t \parallel \alpha\rangle^{\varepsilon}\big)\otimes()$ |
| $(t\,u_{\varepsilon_1})^{\varepsilon_2}$ | $\stackrel{\text{def}}{\equiv}$ | $\mu\alpha^{\varepsilon_2}.\langle t_{\ominus} \parallel u\cdot\alpha\rangle$ | $\mu\alpha^{\varepsilon_2}.\Big\langle t_{+} \,\Big\Vert\, \tilde{\mu}(y\otimes\_).\big\langle u \,\big\Vert\, \tilde{\mu}x^{\varepsilon_1}.\langle y \parallel x\cdot\alpha\rangle\big\rangle^{\varepsilon_1}\Big\rangle$ |
| $\texttt{<}t_{\varepsilon_1};u_{\varepsilon_2}\texttt{>}$ | $\stackrel{\text{def}}{\equiv}$ | $\mu\texttt{<}\alpha.\langle t_{\varepsilon_1} \parallel \alpha\rangle^{\varepsilon_1};\beta.\langle u_{\varepsilon_2} \parallel \beta\rangle^{\varepsilon_2}\texttt{>}$ | $t_{\varepsilon_1}\otimes u_{\varepsilon_2}$ as per Fig. 3 |
| $\pi_i(t)^{\varepsilon}$ | $\stackrel{\text{def}}{\equiv}$ | $\mu\alpha^{\varepsilon}.\langle t_{\ominus} \parallel \pi_i\cdot\alpha\rangle$ | $\mu\alpha^{\varepsilon}.\big\langle t_{+} \,\big\Vert\, \tilde{\mu}(x_1\otimes x_2).\langle x_i \parallel \alpha\rangle^{\varepsilon}\big\rangle$ |
| $*$ | $\stackrel{\text{def}}{\equiv}$ | $\mu\texttt{<>}$ | $()$ |
| $\jmath_i(t_{\varepsilon})$ | $\stackrel{\text{def}}{\equiv}$ | $\lambda\_.\iota_i(t_{\varepsilon})$ as per Fig. 3 | $\iota_i(t_{\varepsilon})$ as per Fig. 3 |
| $\delta(t, x.u_{\varepsilon}, y.u'_{\varepsilon})$ | $\stackrel{\text{def}}{\equiv}$ | $\mu\alpha^{\varepsilon}.\big\langle t_{\ominus} \,\big\Vert\, ()\cdot\tilde{\mu}\big[x.\langle u \parallel \alpha\rangle^{\varepsilon} \,\big\vert\, y.\langle u' \parallel \alpha\rangle^{\varepsilon}\big]\big\rangle$ | $\mu\alpha^{\varepsilon}.\big\langle t_{+} \,\big\Vert\, \tilde{\mu}\big[x.\langle u \parallel \alpha\rangle^{\varepsilon} \,\big\vert\, y.\langle v \parallel \alpha\rangle^{\varepsilon}\big]\big\rangle$ |
| $\mathcal{A}(t)^{\varepsilon}$ | $\stackrel{\text{def}}{\equiv}$ | $\mu\alpha^{\varepsilon}.\langle t_{\ominus} \parallel ()\cdot\mu\texttt{<}\alpha\texttt{>}\rangle$ | $\mu\alpha^{\varepsilon}.\langle t_{+} \parallel \mu\texttt{<}\alpha\texttt{>}\rangle^{+}$ |

(b) Call-by-name and call-by-value encodings

$\to, 1$ as per the call-by-name encoding $\qquad$ $+, 0$ as per the call-by-value encoding

$$A \times B \begin{cases} \text{as per the call-by-name encoding if } A \text{ and } B \text{ are negative} \\ \text{as per the call-by-value encoding if } A \text{ or } B \text{ is positive} \end{cases}$$

(c) Polarised encoding (Girard [Gir07, 12.B.1])

$$\frac{}{\Gamma, x : A \vdash x : A \mid} \qquad \frac{\Gamma, x : A \vdash t : B \mid}{\Gamma \vdash \lambda x.t : A \to B \mid} \qquad \frac{\Gamma \vdash t : A \to B \mid \quad \Gamma \vdash u : A \mid}{\Gamma \vdash (t\,u)^{\varepsilon} : B_{\varepsilon} \mid}$$

$$\frac{}{\Gamma \vdash * : 1 \mid} \qquad \frac{\Gamma \vdash t : A \mid \quad \Gamma \vdash u : B \mid}{\Gamma \vdash \texttt{<}t;u\texttt{>} : A \times B \mid} \qquad \frac{\Gamma \vdash t : A_1 \times A_2 \mid}{\Gamma \vdash \pi_i(t)^{\varepsilon} : A_{i_{\varepsilon}} \mid}$$

$$\frac{\Gamma \vdash t : 0 \mid}{\Gamma \vdash \mathcal{A}(t)^{\varepsilon} : A_{\varepsilon} \mid} \qquad \frac{\Gamma \vdash t : A_i \mid}{\Gamma \vdash \jmath_i(t) : A_1 + A_2 \mid} \qquad \frac{\Gamma \vdash t : A_1 + A_2 \mid \quad \Gamma, x_i : A_i \vdash u_i : B \mid}{\Gamma \vdash \delta(t, x_1.u_1, x_2.u_2) : B \mid}$$

(d) Expressible rules for the $\lambda$-calculus with sums and empty type in whichever polarisation in $\mathbf{LJ}^{\eta}_{p}$

Figure 4: Expressing the $\lambda$-calculus with sums and empty types

necessary to determine the reduction. In Curry's style we are interested in *untyped terms* related by *type-preserving conversions*. All the properties of the Church-style calculi given in [CFMM16] remain true for their Curry-style variants given here except for the uniqueness of typing, and the proofs are mostly the same except for coherence (and consequently soundness) and for the soundness of type-preserving equivalence of terms, which become a bit more involved.

**Decompositions** Confluence (with extensionality restricted to the simplest four equations) is obtained by application of a theorem from the literature by carefully designing the calculi as weakly-orthogonal higher-order rewrite systems [vOvR94]. This design reveals decompositions, also informed by the models, of the traditional language constructs that are no longer primitive but become expressible (in the sense of Felleisen [Fel91])—the most striking being the rules for the modalities, sums, and additive units, whose syntactic treatment is usually poor. This is recalled in Figure 3 for the sequent calculi **ILL** and **LJ** and in Figure 4 for the $\lambda$-calculus with sums and empty type in call-by-name and call-by-value. The improvement of the rewriting theory via expressiveness-preserving decompositions is typical of abstract-machine calculi.

**Coherence** The statements involving transformations on derivations are strengthened to state that the transformation leaves the interpretation in any model invariant. This lets us conclude the coherence of the interpretation (that any two typing derivations of a term are equivalent) and its soundness (that it preserves the equivalence). These properties are essential since in multiplicative style (i.e. with split contexts and the presence of structural rules) typing is no longer directed by the syntax (including in the intuitionistic $\mathbf{LJ}_p^\eta$). It is especially useful for the proof of coherence that structural rules are merged into a single rule acting on the context, as demonstrated already by Atkey for a sub-structural calculus [Atk06].

**Coherence, Curry-style** Another remark regarding our proof of coherence is that in Curry's style, cut elimination (strong normalization) still appears to be an essential ingredient. In contrast, once the coherence for structural rules is obtained by relying on the coherence of monoidal categories, strong normalization is not necessary to prove the coherence in Church-style [CFMM16]. Coherence *à la* Curry appears to be a stronger result.

This is where polarised abstract-machine calculi play a further crucial role: they nicely synthesize the proof theory of sequent calculus, solving issues with structural rules and commuting conversions, and enable an elegant proof of strong normalization with polarised orthogonality-based reducibility candidates where $\to_{\mathrm{R}}$-normal forms satisfy an appropriate sub-formula property. As a further corollary we obtain concise presentation and proof of *focusing*, a proof search algorithm leveraging the sub-formula property.

## 2 Models

### 2.1 LCBPV and CBPV models

We now recall the definitions of CBPV and Linear CBPV models.

Recall that every category $\mathcal{V}$ embeds via the Yoneda functor into its category of small presheaves $\mathcal{PV}$, which happens to be its free co-completion [DL07]. For every symmetric monoidal category $\mathcal{V}$, *Day's convolution* [Day70] extends to $\mathcal{PV}$ the symmetric monoidal structure on $\mathcal{V}$, making the Yoneda embedding a symmetric strong monoidal functor [DL07]. We are interested in those $\mathcal{V}$ where the symmetric monoidal structure on $\mathcal{PV}$ is closed, or equivalently where every presheaf $\mathcal{V}(-\otimes P, Q)$ is small [DL07]. This lets us consider $\mathcal{V}$ as a $\mathcal{PV}$-enriched category $\underline{\mathcal{V}}$ with the definition $\underline{\mathcal{V}}_\Gamma(P, Q) = \mathcal{V}(\Gamma \otimes P, Q)$.

**Definition 1.** A monoidal category is *distributive* if it has finite coproducts and if the canonical maps $\coprod_i(P_i \otimes Q) \to (\coprod_i P_i) \otimes Q$ are isomorphisms.

**Definition 2.** For a distributive monoidal category $\mathcal{V}$, a presheaf $\phi : \mathcal{V}^{\mathrm{op}} \to \boldsymbol{Set}$ is *distributive* whenever the canonical maps $\phi\big(\coprod_i P_i\big) \to \prod_i \phi(P_i)$ are isomorphisms.

**Definition 3.** A *Linear Call-by-Push-Value (LCBPV) model* $\underline{\mathcal{V}} \rightleftarrows_{\perp} \underline{\mathcal{S}}$ consists of:

- a distributive symmetric monoidal category $\mathcal{V}$,
- a Cartesian $\mathcal{PV}$-category $\underline{\mathcal{S}}$ in which every hom-presheaf is distributive,
- a $\mathcal{PV}$-adjunction $\underline{\mathcal{V}} \underset{G}{\overset{F}{\rightleftarrows}} \underline{\mathcal{S}}$,
- $\mathcal{V}$-powers in $\underline{\mathcal{S}}$, that is for every $P \in \mathcal{V}$ and $N \in \underline{\mathcal{S}}$ and object $[P \to N]$ together with a natural isomorphism
$$\underline{\mathcal{S}}_{-\otimes P}(=, N) \cong \underline{\mathcal{S}}_{-}(=, [P \to N])\,.$$

A *Call-by-Push-Value* (or $\boldsymbol{LJ}_p^\eta$) *model* is an LCBPV model in which the symmetric monoidal structure is Cartesian.

*Remark* 4. As already mentioned in [CFMM16], for all the results in this note, one can remove the hypothesis "every hom-presheaf in $\underline{\mathcal{S}}$ is distributive" from the definition of LCBPV and CBPV models. It is only expected to have a role for completeness in a way that will be clarified in an ulterior contribution.

**Definition 5.** A (Cartesian) *resource modality* on a symmetric monoidal category $\mathcal{V}$ is a symmetric monoidal adjunction $\mathcal{M} \rightleftarrows_{\perp} \mathcal{V}$ in which a category $\mathcal{M}$ is Cartesian. An $\boldsymbol{ILL}_p^\eta$ *model* is an LCBPV model $\underline{\mathcal{V}} \rightleftarrows_{\perp} \underline{\mathcal{S}}$ with a resource modality $\mathcal{M} \rightleftarrows_{\perp} \mathcal{V}$.

We henceforth consider an $\mathbf{ILL}_p^\eta$ model

$$\Big(\, \mathcal{M} \underset{M}{\overset{L}{\rightleftarrows}} \mathcal{V}\,,\ \underline{\mathcal{V}} \underset{G}{\overset{F}{\rightleftarrows}} \underline{\mathcal{S}}\,\Big)$$

in which we assume (up to monoidal equivalence) that $\mathcal{V}$ is strict monoidal, that is, the associator and the unitors are identities. We write $E$ for the monoidal comonad on $\mathcal{V}$ induced by the resource modality. When it comes to interpreting $\mathbf{LJ}_p^\eta$ (with modality $! = \Box$ as mentioned earlier) we will moreover assume $\mathcal{V}$ to be Cartesian.

### 2.2 Characterisation of Cartesian resource modalities

We now recall Bierman's [Bie95] characterisation of resource modalities as co-monads. An introduction to the categorical concepts appearing in the theorem and a detailed proof are given in Melliès [Mel09]. Other detailed proofs appear in Maneggia [Man04] and Schalk [Sch04]. We refer to Hasegawa [Has16] for a statement of the result for symmetric monoidal adjunctions in general.

**Theorem 6** ([Has16, Theorem 1]). *Any resource modality* $\mathcal{M} \underset{M}{\overset{L}{\rightleftarrows}} \mathcal{V}$ *gives rise to a* linear exponential comonad*, that is a symmetric monoidal comonad*

$$E \stackrel{\text{def}}{=} LM : \mathcal{V} \to \mathcal{V},\ \delta_P \in \mathcal{V}(EP, EEP),\ \epsilon_P \in \mathcal{V}(EP, P),$$
$$m_{P,Q} \in \mathcal{V}(EP \otimes EQ, E(P \otimes Q)),\ m_I \in \mathcal{V}(I, EI)$$

*equipped with monoidal natural transformations*

$$d^2 : E \to E- \otimes E-,\ d^0 : E \to I$$

*such that for each* $P$*:*

- $(EP, d^2_P, d^0_P)$ *forms a commutative monoid,*
- $d^2_P$ *is a coalgebra morphism* $(EP, \delta_P) \to (EP \otimes EP, m_{P,P} \circ (\delta_P \otimes \delta_P))$*,*
- $d^0_P$ *is a coalgebra morphism* $(EP, \delta_P) \to (I, m_I)$*,*
- $\delta_P$ *is a comonoid morphism* $(EP, d^2_P, d^0_P) \to (EEP, d^2_{!P}, d^0_{!P})$*.*

**Corollary 7.** *Extending to the strict monoidal structure, one has monoidal natural transformations*

$$m_{P_1,\ldots,P_n} : \bigotimes_i EP_i \to E \bigotimes_i P_i$$
$$d^n_P : EP \to (EP)^{\otimes n}$$

*and* $E$*-coalgebras* $(\bigotimes_i EP_i, \delta_{P_1,\ldots,P_n})$ *defined with*

$$\delta_{P_1,\ldots,P_n} \stackrel{\text{def}}{=} m_{EP_1,\ldots,EP_n} \circ \bigotimes_i \delta_{P_i} \in \mathcal{V}(\bigotimes_i EP_i, E \bigotimes_i EP_i)$$

*such that for each* $P$*,* $d^n_P$ *is a coalgebra morphism* $(EP, \delta_P) \to ((EP)^{\otimes n}, \delta_{P,\ldots,P})$*.*

**Theorem 8** ([Has16, Theorem 2]). *Given any linear exponential comonad* $E$ *on* $\mathcal{V}$*, the canonical monoidal structure on the Eilenberg-Moore category* $\mathcal{V}^E$ *is Cartesian, and the comonadic adjunction* $\mathcal{V}^E \rightleftarrows \mathcal{V}$ *is symmetric monoidal.*

### 2.3 Interpretations of $\mathbf{ILL}_p^\eta$ and $\mathbf{LJ}_p^\eta$

#### 2.3.1 Interpretation of types

We now describe the interpretation, which is adapted from [CFMM16].

We assume given an assignment of positive type variables $X^+$ to objects of $\mathscr{V}$ and of negative type variables $X^\ominus$ to objects of $\underline{\mathcal{S}}$. This assignment extends into an interpretation of types as follows.

To every type $A$, we associate both a *positive interpretation* $A^+ \in \mathscr{V}$ and a *negative interpretation* $A^\ominus \in \underline{\mathcal{S}}$. These are defined by mutual induction as follows:

$$
\begin{aligned}
\mathbf{1}^+ &\stackrel{\text{def}}{=} I & (A \twoheadrightarrow B)^\ominus &\stackrel{\text{def}}{=} [A^+ \multimap B^\ominus]\\
(A \otimes B)^+ &\stackrel{\text{def}}{=} A^+ \otimes B^+ & (A \mathbin{\&} B)^\ominus &\stackrel{\text{def}}{=} A^\ominus \times B^\ominus\\
(!A)^+ &\stackrel{\text{def}}{=} EGA^\ominus & \top^\ominus &\stackrel{\text{def}}{=} 1\\
(A \oplus B)^+ &\stackrel{\text{def}}{=} A^+ + B^+ & &\\
\mathbf{0}^+ &\stackrel{\text{def}}{=} 0 & &\\
N^+ &\stackrel{\text{def}}{=} GN^\ominus & P^\ominus &\stackrel{\text{def}}{=} FP^+
\end{aligned}
$$

The interpretation of types extends pointwise to typing contexts $\Gamma, \Delta$ as follows:

- $(x_1 : A_1, \dots, x_n : A_n)^+ = {A_1}^+ \otimes \cdots \otimes {A_n}^+$;
- $(\alpha : A)^\ominus = A^\ominus$.

Notice in particular, due to strictness, $(\Gamma, \Gamma')^+ = \Gamma^+ \otimes \Gamma'^+$ and $\mathscr{V}(\Gamma^+, P) = \underline{\mathscr{V}}_{\Gamma^+}(I, P) = \underline{\mathscr{V}}_I(\Gamma^+, P)$.

#### 2.3.2 Interpretation of structure maps

In Figure 2 we give the notion of structure maps between contexts (in $\Sigma(\Gamma;\Gamma')$ for $\mathbf{LJ}_p^\eta$ and in $\Sigma^!(\Gamma;\Gamma')$ for $\mathbf{ILL}_p^\eta$). We recall that $\Gamma, \Gamma'$ is defined only when $\operatorname{dom}\Gamma \cap \operatorname{dom}\Gamma' = \emptyset$. We write $\Gamma \# \Gamma'$ when it is the case; and similarly we define $f \# \Gamma$ when $\mathbf{fv}\, f \cap \operatorname{dom}\Gamma = \emptyset$ and $f \# g$ when $\mathbf{fv}\, f \cap \mathbf{fv}\, g = \emptyset$.

In this section, the properties and definitions for $\Sigma$ instead of $\Sigma^!$ and $\mathbf{LJ}_p^\eta$ instead of $\mathbf{ILL}_p^\eta$ are obtained by removing the restriction to the formulae of the form $!A$.

**Definition 9.** We consider the following category $\Sigma_0^!$:

- objects are tuples of types $(A_1, \dots, A_n)$,
- morphisms $f \in \Sigma_0^!((A_1, \dots, A_n);(B_1, \dots, B_m))$ are functions $\{1, \dots, n\} \to \{1, \dots, m\}$ such that $B_{f(i)} = A_i$, and if $B_i$ is not of the form $!A$ then $f^{-1}(i)$ is a singleton,
- obvious composition and identities.

$\Sigma^!_0$ has an obvious symmetric strict monoidal structure given by concatenation. The functor $U : \Sigma^! \to \Sigma^!_0$ which projects the variables away is full, faithful and surjective on objects. It therefore forms an equivalence in which $U$ is left inverse strictly. This induces a symmetric strict monoidal structure $(\Sigma^!, \uplus, \emptyset)$ making $U$ strict monoidal. Lastly, $U(\Gamma, \Gamma') = U(\Gamma \uplus \Gamma')$ when $\Gamma \# \Gamma'$.

In the case of $\mathbf{LJ}^\eta_p$ the category $\Sigma_0$ similarly defined by removing the clause "then $f^{-1}(i)$ is a singleton" is co-Cartesian.

**Definition 10.** *Renamings* $\sigma \in \Sigma^!(\Gamma; \Gamma')$ are the order-preserving bijections, or equivalently those in $U^{-1}(\mathrm{id}_{U\Gamma})$.

**Lemma 11.** *Any morphism in $\Sigma^!_0(\Gamma; (A_1, \dots, A_n))$ can be written as $\sigma \circ \sigma'$ where $\sigma$ is the concatenation of the unique maps $\Sigma^!_0(\Gamma_{\restriction \sigma^{-1}(i)}; A_i)$ and $\sigma'$ is a bijection in $\Sigma^!_0(\Gamma; (\Gamma_{\restriction \sigma^{-1}(x_1)}, \dots, \Gamma_{\restriction \sigma^{-1}(x_n)}))$.*

*Proof.* By construction. ∎

**Definition 12.** We define $[\![\cdot]\!] : \Sigma^!_0 \to \mathscr{V}^{\mathrm{op}}$ as follows:

- Bijections are interpreted by the canonical isomorphism,
- The unique map in $\Sigma^!_0(A^n; A)$ is interpreted by the identity when $n = 1$ and otherwise (in which case $A = !B$) by $d^n_{GB^\ominus} : EGB^\ominus \to (EGB^\ominus)^{\otimes n}$.
- Concatenation of such maps is obtained by the tensor.

$[\![\cdot]\!]$ extends to $\Sigma^! \to \mathscr{V}^{\mathrm{op}}$ by pre-composition with $U$ which we leave implicit.

Notice that the definition does not depend on the choice of bijection in the decomposition of Lemma 11 because the $\otimes$-comonoid structure is commutative. For $\Sigma/\mathbf{LJ}^\eta_p$, the unique map in $\Sigma_0(A^n; A)$ is instead interpreted by the diagonal in $\mathscr{V}$.

#### 2.3.3 Interpretation of judgements and coercions

We give an interpretation of derivations of values and stacks in the categories $\mathscr{V}$ and $\underline{\mathscr{S}}$, respectively:

- $[\![\Gamma \vdash V : A \,|\!|]\!]_{\mathrm{V}} \in \mathscr{V}(\Gamma^+, A^+)$
- $[\![\Gamma \mid S : A \vdash \Delta]\!]_{\mathrm{S}} \in \underline{\mathscr{S}}_{\Gamma^+}(A^\ominus, \Delta^\ominus)$

The derivations of expressions, contexts, and commands, are interpreted in $\underline{\mathscr{V}}(-, G{=}) \cong \underline{\mathscr{S}}(F-, =)$, more precisely we define interpretations:

- $[\![\Gamma \vdash t : A \,|\!|]\!] \in \underline{\mathscr{S}}_{\Gamma^+}(FI, A^\ominus)$
- $[\![\Gamma \mid e : A \vdash \Delta]\!] \in \underline{\mathscr{V}}_{\Gamma^+}(A^+, G\Delta^\ominus)$
- $[\![c : (\Gamma \vdash \Delta)]\!] \in \mathscr{V}(\Gamma^+, G\Delta^\ominus)$

We will write $[\![c:(\Gamma\vdash\Delta)]\!]^* \in \underline{\mathcal{S}}_{\Gamma^+}(FI, A^\ominus)$ for the transpose with respect to $\Gamma^+$ of $[\![c:(\Gamma\vdash\Delta)]\!]$.

The interpretation is defined by mutual induction on derivations as follows.

- $[\![\Gamma\vdash V:P\mid]\!] \stackrel{\text{def}}{=} F^{\Gamma^+}_{I,P^+}[\![\Gamma\vdash V:P\mid]\!]_{\mathrm{V}} \in \underline{\mathcal{S}}_{\Gamma^+}(FI, FP^+)$
- $[\![\Gamma\mid S:N\vdash\Delta]\!] \stackrel{\text{def}}{=} G^{\Gamma^+}_{N^\ominus,\Delta^\ominus}[\![\Gamma\mid S:N\vdash\Delta]\!]_{\mathrm{S}} \in \underline{\mathcal{V}}_{\Gamma^+}(GN^\ominus, G\Delta^\ominus)$
- The following pairs of interpretations are related by transposition:

$$\begin{array}{cc} [\![\Gamma\vdash V:N\mid]\!]_{\mathrm{V}} \in \underline{\mathcal{V}}_{\Gamma^+}(I, GN^\ominus) & [\![\Gamma\mid S:P\vdash\Delta]\!]_{\mathrm{S}} \in \underline{\mathcal{S}}_{\Gamma^+}(FP^+, \Delta^\ominus) \\ [\![\Gamma\vdash V:N\mid]\!] \in \underline{\mathcal{S}}_{\Gamma^+}(FI, N^\ominus) & [\![\Gamma\mid S:P\vdash\Delta]\!] \in \underline{\mathcal{V}}_{\Gamma^+}(P^+, G\Delta^\ominus) \end{array}$$

- $[\![\Gamma\mid \tilde{\mu}x^\varepsilon.c:A\vdash\Delta]\!] \stackrel{\text{def}}{=} [\![c:(\Gamma,x:A\vdash\Delta)]\!] \in \underline{\mathcal{V}}_{\Gamma^+}(A^+, G\Delta^\ominus)$
- $[\![\Gamma\vdash \mu\alpha^\varepsilon.c:A\mid]\!] \stackrel{\text{def}}{=} [\![c:(\Gamma\vdash\alpha:A)]\!]^* \in \underline{\mathcal{S}}_{\Gamma^+}(FI, A^\ominus)$

We now define from smaller derivations the remaining data, that is $[\![\Gamma\vdash V:P\mid]\!]_{\mathrm{V}}$, $[\![\Gamma\mid S:N\vdash\Delta]\!]_{\mathrm{S}}$, either $[\![\Gamma\vdash V:N\mid]\!]_{\mathrm{V}}$ or $[\![\Gamma\vdash V:N\mid]\!]$, either $[\![\Gamma\mid S:P\vdash\Delta]\!]_{\mathrm{S}}$ or $[\![\Gamma\mid S:P\vdash\Delta]\!]$, and $[\![c:(\Gamma,x:A\vdash\Delta)]\!]$, depending on the last rule.

#### 2.3.4 Identity rules

- $[\![x:A\vdash x:A\mid]\!]_{\mathrm{V}} \stackrel{\text{def}}{=} \mathrm{id}_{A^+} \in \mathcal{V}(A^+, A^+)$
- $[\![\mid\alpha:A\vdash\alpha:A]\!]_{\mathrm{S}} \stackrel{\text{def}}{=} \mathrm{id}_{A^\ominus} \in \underline{\mathcal{S}}_I(A^\ominus, A^\ominus)$
- $[\![\langle V\parallel e\rangle^\ominus:(\Gamma,\Gamma'\vdash\Delta)]\!] \in \mathcal{V}((\Gamma,\Gamma')^+, G\Delta^\ominus)$ is equal to the composition in $\underline{\mathcal{V}}$

$$[\![\Gamma'\mid e:N\vdash\Delta]\!] \circ [\![\Gamma\vdash V:N\mid]\!]_{\mathrm{V}} \in \underline{\mathcal{V}}_{\Gamma^+\otimes\Gamma'^+}(I, G\Delta^\ominus)$$

- $[\![\langle t\parallel S\rangle^+:(\Gamma,\Gamma'\vdash\Delta)]\!] \in \mathcal{V}((\Gamma,\Gamma')^+, G\Delta^\ominus)$ is the transpose of the composition in $\underline{\mathcal{S}}$

$$[\![\Gamma'\mid S:P\vdash\Delta]\!]_{\mathrm{S}} \circ [\![\Gamma\vdash t:P\mid]\!] \in \underline{\mathcal{S}}_{\Gamma^+\otimes\Gamma'^+}(FI, \Delta^\ominus)$$

#### 2.3.5 Structural rules

We now define the interpretation of the structural rules by action of the structure morphism $[\![\sigma]\!]$.

- $[\![\Gamma'\vdash V[\sigma]:A\mid]\!]_{\mathrm{V}} \stackrel{\text{def}}{=} [\![\Gamma\vdash V:A\mid]\!]_{\mathrm{V}} \circ [\![\sigma]\!] \in \mathcal{V}(\Gamma'^+, A^+)$
- $[\![\Gamma'\mid S[\sigma]:A\vdash\Delta]\!]_{\mathrm{S}} \stackrel{\text{def}}{=} \underline{\mathcal{S}}_{[\![\sigma]\!]}([\![\Gamma\mid S:A\vdash\Delta]\!]_{\mathrm{S}}) \in \underline{\mathcal{S}}_{\Gamma'^+}(A^\ominus, \Delta^\ominus)$
- $[\![\Gamma'\vdash t[\sigma]:A\mid]\!] \stackrel{\text{def}}{=} \underline{\mathcal{S}}_{[\![\sigma]\!]}([\![\Gamma\vdash t:A\mid]\!]) \in \underline{\mathcal{S}}_{\Gamma'^+}(FI, A^\ominus)$
- $[\![\Gamma'\mid e[\sigma]:A\vdash\Delta]\!] \stackrel{\text{def}}{=} \underline{\mathcal{V}}_{[\![\sigma]\!]}([\![\Gamma\mid e:A\vdash\Delta]\!]) \in \underline{\mathcal{V}}_{\Gamma'^+}(A^+, G\Delta^\ominus)$

- $[\![c[\sigma] : (\Gamma' \vdash \Delta)]\!] \stackrel{\text{def}}{=} [\![c : (\Gamma \vdash \Delta)]\!] \circ [\![\sigma]\!] \in \mathscr{V}(\Gamma'^{+}, G\Delta^{\ominus})$

Notice that the action of the structure morphism is preserved by the enriched adjunction. In particular, the definition of the interpretation of a value derivation ending in $(\vdash \sigma)$, or of a stack derivation ending in $(\sigma \vdash)$, is unambiguous.

#### 2.3.6 Multiplicatives

- $[\![\Gamma, \Gamma' \vdash V{\otimes}W : A \otimes B \,|]\!]_{\mathrm{V}} \stackrel{\text{def}}{=} [\![\Gamma \vdash V : A \,|]\!]_{\mathrm{V}} \otimes [\![\Gamma' \vdash W : B \,|]\!]_{\mathrm{V}}$
- $[\![\Gamma \mid \tilde{\mu}(x{\otimes}y).c : A \otimes B \vdash \Delta]\!] \stackrel{\text{def}}{=} [\![c : (\Gamma, x : A, y : B \vdash \Delta)]\!]$
- $[\![\Gamma \vdash \mu(x{\cdot}\alpha).c : A \multimap B \,|]\!] \stackrel{\text{def}}{=} \lambda^{\Gamma^+, FI}_{A^+, B^{\ominus}}([\![c : (\Gamma, x : A \vdash \alpha : B)]\!]^*)$ where

$$\lambda^{\Gamma, M}_{P,N} : \underline{\mathcal{S}}_{\Gamma \otimes P}(M, N) \to \underline{\mathcal{S}}_{\Gamma}(M, [P \multimap N])$$

- $[\![\Gamma, \Gamma' \mid V{\cdot}S : A \multimap B \vdash \Delta]\!]_{\mathrm{S}}$ is equal to

$$[\![\Gamma' \mid S : B \vdash \Delta]\!]_{\mathrm{S}} \circ \underline{\mathcal{S}}_{[\![\Gamma \vdash V : A|]\!]_{\mathrm{V}}}(\mathrm{ev}^{A^+, B^{\ominus}}) \in \underline{\mathcal{S}}_{\Gamma^+ \otimes \Gamma'^+}([A^+ \multimap B^{\ominus}], \Delta^{\ominus})$$

  where $\mathrm{ev}^{P,N} = (\lambda^{I,FI}_{P,N})^{-1}(\mathrm{id}_{[P \multimap N]}) \in \underline{\mathcal{S}}_P([P \multimap N], N)$ is the evaluation map.

- $[\![\vdash () : \mathbf{1} \,|]\!]_{\mathrm{V}} \stackrel{\text{def}}{=} \mathrm{id}_I$
- $[\![\Gamma \mid \tilde{\mu}().c : \mathbf{1} \vdash \Delta]\!] \stackrel{\text{def}}{=} [\![c : (\Gamma \vdash \Delta)]\!]$

#### 2.3.7 Exponentials

- $[\![!\Gamma \vdash \mu!\alpha.c : !A \,|]\!]_{\mathrm{V}} \stackrel{\text{def}}{=} E[\![c : (!\Gamma \vdash \alpha : A)]\!] \circ \delta_\Gamma$ where

$$\delta_\Gamma \stackrel{\text{def}}{=} \delta_{GB_1{}^{\ominus}, \ldots, GB_n{}^{\ominus}}$$

  for $\Gamma = (x_i : B_i)_i$.

- $[\![\Gamma \mid !S : !A \vdash \Delta]\!] \stackrel{\text{def}}{=} G^{\Gamma^+}_{A^{\ominus}, \Delta^{\ominus}} [\![\Gamma \mid S : A \vdash \Delta]\!]_{\mathrm{S}} \circ (\Gamma^+ \otimes \epsilon_{GA^{\ominus}})$

#### 2.3.8 Additives

- $[\![\Gamma \mid \pi_i{\cdot}S : A_1 \,\&\, A_2 \vdash \Delta]\!]_{\mathrm{S}} = [\![\Gamma \mid S : A_i \vdash \Delta]\!]_{\mathrm{S}} \circ \pi_i$
- $[\![\Gamma \vdash \mu{<}\alpha.c ; \beta.c'{>} : A \,\&\, B \,|]\!]$ is the pairing in $\underline{\mathcal{S}}$

$$< [\![c : (\Gamma \vdash \alpha : A)]\!]^* ; [\![c' : (\Gamma \vdash \alpha : B)]\!]^* >$$

- $[\![\Gamma \vdash \mu{<}V{>} : \top \,|]\!] \in \underline{\mathcal{S}}_{\Gamma^+}(FI, 1)$ is defined uniquely.

- $[\![\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid]\!]_{\mathrm{V}} \stackrel{\text{def}}{=} \iota_i \circ [\![\Gamma \vdash V : A_i \mid]\!]_{\mathrm{V}}$
- $[\![\Gamma \mid \tilde{\mu}[x.c|y.c'] : A \oplus B \vdash \Delta]\!]$ is the co-pairing in $\underline{\mathscr{V}}$

$$\left[[\![c : (\Gamma \mid x \,:\, A \vdash \Delta)]\!]\,;\, [\![c' : (\Gamma \mid y \,:\, B \vdash \Delta)]\!]\right]_{\Gamma^+}$$

given by the isomorphism

$$\underline{\mathscr{V}}_{\Gamma^+}(A^+, G\Delta^\ominus) \times \underline{\mathscr{V}}_{\Gamma^+}(B^+, G\Delta^\ominus) \cong \underline{\mathscr{V}}_{\Gamma^+}(A^+ + B^+, G\Delta^\ominus)\,. \tag{1}$$

- $[\![\Gamma \mid \tilde{\mu}[S] : \mathbf{0} \vdash \Delta]\!] \in \underline{\mathscr{V}}_{\Gamma^+}(0, G\Delta^\ominus)$ is defined uniquely.

# 3 Conversions

## 3.1 Untyped conversion

**Definition 13** (Compatible closure on pseudo-terms)**.** The *compatible closure* $\to$ of a relation $\rhd$ on pseudo-terms that preserves syntactic categories $t, e, c$ is defined by induction on the structure of pseudo-terms as follows: one has $f \to g$ whenever $g$ is obtained from $f$ by applying $\rhd$ on one of its sub-term.

Equivalently, in the case of $\mathbf{IMLL}_p^\eta$ and $\mathbf{MLJ}_p^\eta$, the relation $\to$ is the smallest one satisfying:

- if $f \rhd g$ then $f \to g$,
- if $V \to V'$, then:
  - for all $W$ one has $V \otimes W \to V' \otimes W$ and $W \otimes V \to W \otimes V'$,
  - for all $S$ one has $V \cdot S \to V' \cdot S$,
  - for all $e$ one has $\langle V \parallel e \rangle^\ominus \to \langle V' \parallel e \rangle^\ominus$,
- if $S \to S'$, then:
  - for all $V$ one has $V \cdot S \to V \cdot S'$,
  - for all $t$ one has $\langle t \parallel S \rangle^+ \to \langle t \parallel S' \rangle^+$,
- if $t \to t'$ then for all $S$ one has $\langle t \parallel S \rangle^+ \to \langle t' \parallel S \rangle^+$,
- if $e \to e'$ then for all $V$ one has $\langle V \parallel e \rangle^\ominus \to \langle V \parallel e' \rangle^\ominus$,
- if $c \to c'$ then:

$$\begin{aligned}
\mu\alpha^\varepsilon.c &\to \mu\alpha^\varepsilon.c' \\
\mu(x\cdot\alpha).c &\to \mu(x\cdot\alpha).c' \\
\tilde{\mu}x^\varepsilon.c &\to \tilde{\mu}x^\varepsilon.c' \\
\tilde{\mu}().c &\to \tilde{\mu}().c' \\
\tilde{\mu}(x\otimes y).c &\to \tilde{\mu}(x\otimes y).c'
\end{aligned}$$

Explicit descriptions $\to$ for $\mathbf{IMALL}^{\eta}_{p}$, $\mathbf{IMELL}^{\eta}_{p}$, $\mathbf{ILL}^{\eta}_{p}$, and $\mathbf{LJ}^{\eta}_{p}$ are along the same lines.

By definition, the compatible closure preserves syntactic categories. The equivalence closure of the compatible closure is written $\simeq$.

### 3.2 Typed conversion

**Definition 14** (Typed conversion)**.** Given a relation $\rhd$ on pseudo-terms, we define its restriction to (a collection $\rhd^{\vdash}$ of) *typed relations* as follows:

$$c \rhd c' : (\Gamma \vdash \Delta) \iff \begin{cases} c \rhd c' \\ c : (\Gamma \vdash \Delta) \\ c' : (\Gamma \vdash \Delta) \end{cases}$$

and similarly for $t, e$ replacing $c$.

**Definition 15.** Given typed relations $\rhd^{\vdash}$, we define their *typed compatible closure* $\to^{\vdash}$ by an induction on derivations as follows: one has $c \to c' : (\Gamma \vdash \Delta)$ (and accordingly for $t, e$ replacing $c$) whenever one can derive $c : (\Gamma \vdash \Delta)$ and $c' : (\Gamma \vdash \Delta)$ from one application of $\rhd$ followed by applications of identical typing rules.

Equivalently, in the case of $\mathbf{IMLL}^{\eta}_{p}$, the typed relations $\to^{\vdash}$ are defined in Figure 5. Explicit descriptions $\to^{\vdash}$ for $\mathbf{MLJ}^{\eta}_{p}$, $\mathbf{ILL}^{\eta}_{p}$, and $\mathbf{LJ}^{\eta}_{p}$ can be given in a straightforward manner. The typed relations $\simeq^{\vdash}$ are defined by the symmetric and transitive closures of the typed relations $=^{\vdash} \cup \to^{\vdash}$.

**Lemma 16.** *Given a relation $\rhd$ on pseudo-terms that preserves syntactic categories, if $c \to c' : (\Gamma \vdash \Delta)$ (respectively $c \simeq c' : (\Gamma \vdash \Delta)$) then $c : (\Gamma \vdash \Delta)$ and $c' : (\Gamma \vdash \Delta)$ (and similarly for $t, e$ replacing $c$). If moreover $\rhd$ is closed under the action of $\Sigma^{!}$ then $c \to c'$ (resp. $c \simeq c'$).*

*Proof.* For $\to^{\vdash}$ this follows by induction on the definition of the typed closure: that both terms are well-typed is obtained by projection of the derivation, and that the two terms are related by $\to$ follows, for the structural rules, from the fact that $\to$ is closed under the action of $\Sigma^{!}$, and, for the other rules, by definition. For $\simeq^{\vdash}$ this follows immediately. ∎

*Remark* 17. One can have $c \to c'$ and $c, c' : (\Gamma \vdash \Delta)$ without $c \to c' : (\Gamma \vdash \Delta)$. The following counter-examples involve the expansion of additive units with arbitrary $|\ S : P \vdash \alpha : A$ and $x : A \vdash V : N\ |$:

$$\left\langle \mu\alpha^{+}.\langle x \parallel \tilde{\mu}[\alpha]\rangle \,\middle\|\, \tilde{\mu}[\alpha] \right\rangle \to_{\mathrm{E}} \left\langle \mu\alpha^{+}.\langle x \parallel \tilde{\mu}[\alpha]\rangle \,\middle\|\, S \right\rangle$$
$$\left\langle \mu{<}x{>} \,\middle\|\, \tilde{\mu}x^{\ominus}.\langle \mu{<}x{>} \parallel \alpha\rangle \right\rangle \to_{\mathrm{E}} \left\langle V \,\middle\|\, \tilde{\mu}x^{\ominus}.\langle \mu{<}x{>} \parallel \alpha\rangle \right\rangle$$

One has respectively $c, c' : (x : \mathbf{0} \vdash \alpha : A)$ and $c, c' : (x : A \vdash \alpha : \top)$, but unless $S$ is typable with $P = \mathbf{0}$, and $V$ with $N = \top$, then one does not have $c \to c' : (\Gamma \vdash \Delta)$.

Another main goal will be to prove that nevertheless, the notion of typed conversion (a notion defined on derivations) and typed restriction of untyped conversion (a notion defined on terms) coincide (Theorem 61).

$$\frac{c \rhd c' : (\Gamma \vdash \Delta)}{c \to c' : (\Gamma \vdash \Delta)} \qquad \frac{c \to c' : (\Gamma, x : A_\varepsilon \vdash \Delta)}{\Gamma \mid \tilde{\mu}x^\varepsilon.c \to \tilde{\mu}x^\varepsilon.c' : A_\varepsilon \vdash \Delta} \qquad \frac{c \to c' : (\Gamma \vdash \alpha : A_\varepsilon)}{\Gamma \vdash \mu\alpha^\varepsilon.c \to \mu\alpha^\varepsilon.c' : A_\varepsilon \mid}$$

$$\frac{\Gamma \vdash t \rhd t' : A \mid}{\Gamma \vdash t \to t' : A \mid} \qquad \frac{\Gamma \vdash t \to t' : A_\varepsilon \mid \quad \Gamma' \mid e : A_\varepsilon \vdash \Delta}{\langle t \parallel e \rangle^\varepsilon \to \langle t' \parallel e \rangle^\varepsilon : (\Gamma, \Gamma' \vdash \Delta)}$$

$$\frac{\Gamma \mid e \rhd e' : A \vdash \Delta}{\Gamma \mid e \to e' : A \vdash \Delta} \qquad \frac{\Gamma \vdash t : A_\varepsilon \mid \quad \Gamma' \mid e \to e' : A_\varepsilon \vdash \Delta}{\langle t \parallel e \rangle^\varepsilon \to \langle t \parallel e' \rangle^\varepsilon : (\Gamma, \Gamma' \vdash \Delta)}$$

(a) Inclusion (b) Identity

$$\frac{\Gamma \vdash t \to t' : A \mid}{\Gamma' \vdash t[\sigma] \to t'[\sigma] : A \mid} \qquad \frac{\Gamma \mid e \to e' : A \vdash \Delta}{\Gamma' \mid e[\sigma] \to e'[\sigma] : A \vdash \Delta} \qquad \frac{c \to c' : (\Gamma \vdash \Delta)}{c[\sigma] \to c'[\sigma] : (\Gamma' \vdash \Delta)}$$

(c) Structure

$$\frac{\Gamma \vdash V \to V' : A \mid \quad \Gamma' \vdash W : B \mid}{\Gamma, \Gamma' \vdash V \otimes W \to V' \otimes W : A \otimes B \mid} \qquad \frac{\Gamma \vdash V \to V' : A \mid \quad \Gamma' \mid S : B \vdash \Delta}{\Gamma, \Gamma' \mid V \cdot S \to V' \cdot S : A \multimap B \vdash \Delta}$$

$$\frac{\Gamma \vdash V : A \mid \quad \Gamma' \vdash W \to W' : B \mid}{\Gamma, \Gamma' \vdash V \otimes W \to V \otimes W' : A \otimes B \mid} \qquad \frac{\Gamma \vdash V : A \quad \Gamma' \mid S \to S' : B \vdash \Delta}{\Gamma, \Gamma' \mid V \cdot S \to V \cdot S' : A \multimap B \vdash \Delta}$$

$$\frac{c \to c' : (\Gamma, x : A, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x \otimes y).c \to \tilde{\mu}(x \otimes y).c' : A \otimes B \vdash \Delta} \qquad \frac{c \to c' : (\Gamma, x : A \vdash \alpha : B)}{\Gamma \vdash \mu(x \cdot \alpha).c \to \mu(x \cdot \alpha).c' : A \multimap B \mid}$$

$$\frac{c \to c' : (\Gamma \vdash \Delta)}{\Gamma \mid \tilde{\mu}().c \to \tilde{\mu}().c' : \mathbf{1} \vdash \Delta}$$

(d) Multiplicatives

Figure 5: Typed compatible closure $\to^\vdash$ of typed relations $\rhd^\vdash$ in $\mathbf{IMLL}_p^\eta$

**Definition 18.** A relation $\rhd$ *preserves typing* (respectively *preserves typing compatibly*) if whenever $c:(\Gamma\vdash\Delta)$ and $c\rhd c'$ (resp. $c\to c'$) one has $c':(\Gamma\vdash\Delta)$ (resp. $c\to c':(\Gamma\vdash\Delta)$) (and similarly for $t,e$ replacing $c$).

We will show as a corollary of the generation lemma that whenever a relation preserves typing, it does so compatibly.

**Definition 19.** Typed relations $R^{\vdash}$ *preserve the interpretation* (in all models of $\mathbf{ILL}_p^\eta$) if whenever $c\ R\ c':(\Gamma\vdash\Delta)$, for any derivation of $c:(\Gamma\vdash\Delta)$ there exists a derivation of $c':(\Gamma\vdash\Delta)$ with the same denotation (and similarly for $t,e$ replacing $c$).

It is immediate that whenever typed relations $\rhd^{\vdash}$ preserve the interpretation, it is also the case of $\to^{\vdash}$.

### 3.3 Confluence

**Definition 20.** We define $\rhd_{\mathrm{Re}}$ the union of $\rhd_{\mathrm{R}}$ and of the rules $(E\tilde{\mu})$ and $(E\mu)$.

We show that the typed compatible closures $\to_{\mathrm{Re}}^{\vdash}$ of the typed restriction of $\rhd_{\mathrm{Re}}$ are confluent.

**Theorem 21** (Confluence of $\to_{\mathrm{Re}}$)**.** *Let $f,g,h$ be three terms such that $g\ {}^*\!\!\leftarrow_{\mathrm{Re}}\ f\ \to_{\mathrm{Re}}^*\ h$. Then there exists a term $i$ such that $g\to_{\mathrm{Re}}^*\ i\ {}^*\!\!\leftarrow_{\mathrm{Re}}\ h$.*

*Proof.* The language of pseudo-terms of $\mathbf{ILL}_p^\eta$ together with the relation $\rhd_{\mathrm{Re}}$ constitutes a weakly orthogonal higher-order rewriting system: $\rhd_{\mathrm{Re}}$ is left-linear and its critical pairs are trivial. This implies confluence (van Oostrom and van Raamsdonk [vOvR94]). ∎

We wish to write in detail the higher-order rewriting system in the notations of van Raamsdonk [vR99]. It suffices to our point that we limit the exercise to the pseudo terms of $\mathbf{IMLL}_p^\eta/\mathbf{MLJ}_p^\eta$.

*Base types* are the syntactic categories:

$$V\qquad,\qquad t\qquad,\qquad S\qquad,\qquad e\qquad,\qquad c$$

We interpret the variables of $\mathbf{IMLL}_p^\eta$ with variables of type $V$ and the co-variables of $\mathbf{IMLL}_p^\eta$ with variables of type $S$, and we keep the notation $x,\alpha$.

The *signature* defines the constructs. Distinct symbols are written the same for conciseness when there is no ambiguity.

***V*:**

- $() : V$
- $\otimes : V\times V\to V$
- $\mu^\ominus : \begin{cases}(S\to c)\to V\\(V\times S\to c)\to V\end{cases}$

***t*:**

- $\diamond : V \to t$
- $\mu^{+} : (S \to c) \to t$

***S*:**

- $\cdot : V \times S \to S$
- $\tilde{\mu}^{+} : \begin{cases} (V \to c) \to S \\ c \to S \\ (V \times V \to c) \to S \end{cases}$

***e*:**

- $\diamond : S \to e$
- $\tilde{\mu}^{\ominus} : (S \to c) \to e$

***c*:**

- $\langle \parallel \rangle^{+} : t \times S \to c$
- $\langle \parallel \rangle^{\ominus} : V \times e \to c$

*Rewrite rules* are as follows:

$$\begin{array}{ll}
V\gamma^{V\to c}.\langle V \parallel \tilde{\mu}^{\ominus}(x.\gamma(x))\rangle^{\ominus} & \rhd_{\mathrm{R}} V\gamma.\gamma(V) \\
V\gamma^{V\to c}.\langle V^{\diamond} \parallel \tilde{\mu}^{+}(x.\gamma(x))\rangle^{+} & \rhd_{\mathrm{R}} V\gamma.\gamma(V) \\
S\gamma^{S\to c}.\langle \mu^{+}(\alpha.\gamma(\alpha)) \parallel S\rangle^{+} & \rhd_{\mathrm{R}} S\gamma.\gamma(S) \\
S\gamma^{S\to c}.\langle \mu^{\ominus}(\alpha.\gamma(\alpha)) \parallel S^{\diamond}\rangle^{\ominus} & \rhd_{\mathrm{R}} S\gamma.\gamma(S) \\
VS\gamma^{V\times S\to c}.\langle \mu^{\ominus}(x\alpha.\gamma(x,\alpha)) \parallel (V\cdot S)^{\diamond}\rangle^{\ominus} & \rhd_{\mathrm{R}} VS\gamma.\gamma(V,S) \\
VW\gamma^{V\times V\to c}.\langle (V\otimes W)^{\diamond} \parallel \tilde{\mu}^{+}(xy.\gamma(x,y))\rangle^{+} & \rhd_{\mathrm{R}} VW\gamma.\gamma(V,W) \\
c.\langle ()^{\diamond} \parallel \tilde{\mu}c\rangle^{+} & \rhd_{\mathrm{R}} c.c \\
V.\mu^{\ominus}(\alpha.\langle V \parallel \alpha^{\diamond}\rangle^{\ominus}) & \rhd_{\mathrm{e}} V.V \\
t.\mu^{+}(\alpha.\langle t \parallel \alpha\rangle^{+}) & \rhd_{\mathrm{e}} t.t \\
e.\tilde{\mu}^{\ominus}(x.\langle x \parallel e\rangle^{\ominus}) & \rhd_{\mathrm{e}} e.e \\
S.\tilde{\mu}^{+}(x.\langle x^{\diamond} \parallel S\rangle^{+}) & \rhd_{\mathrm{e}} S.S
\end{array}$$

It is immediate to check that these rewrite rules are *well-formed*, that is to say the left-hand side is a *rule-pattern*; it is also immediate to see that they are left-linear. In addition, the only critical pairs are trivial:

$$\begin{array}{rl}
\left\langle V \,\middle\|\, \tilde{\mu}^{\ominus}(x.\langle x \parallel e\rangle^{\ominus})\right\rangle^{\ominus} & \rightrightarrows_{\mathrm{Re}} \langle V \parallel e\rangle^{\ominus}, \langle V \parallel e\rangle^{\ominus} \\
\tilde{\mu}^{\ominus}\big(x.\langle x \parallel \tilde{\mu}^{\ominus}(x.\gamma(x))\rangle^{\ominus}\big) & \rightrightarrows_{\mathrm{Re}} \tilde{\mu}^{\ominus}(x.\gamma(x)), \tilde{\mu}^{\ominus}(x.\gamma(x)) \\
\left\langle V^{\diamond} \,\middle\|\, \tilde{\mu}^{+}(x.\langle x^{\diamond} \parallel S\rangle^{+})\right\rangle^{+} & \rightrightarrows_{\mathrm{Re}} \langle V^{\diamond} \parallel S\rangle^{+}, \langle V^{\diamond} \parallel S\rangle^{+} \\
\tilde{\mu}^{+}\big(x.\langle x^{\diamond} \parallel \tilde{\mu}^{+}(x.\gamma(x))\rangle^{+}\big) & \rightrightarrows_{\mathrm{Re}} \tilde{\mu}^{+}(x.\gamma(x)), \tilde{\mu}^{+}(x.\gamma(x))
\end{array}$$

(and symmetrically with $\mu$ replacing $\tilde{\mu}$). We have therefore defined a weakly orthogonal higher-order rewriting system.

**Corollary 22** (Confluence of $\to^{\vdash}_{\mathrm{Re}}$)**.** *Let $f, g, h$ be three terms such that $g \;{}^*\!\!\leftarrow^{\vdash}_{\mathrm{Re}} f \to^{\vdash\,*}_{\mathrm{Re}} h$ (for a same typing context). Then there exists a term $i$ such that $g \to^{\vdash\,*}_{\mathrm{Re}} i \;{}^*\!\!\leftarrow^{\vdash}_{\mathrm{Re}} h$ (for the previous typing context).*

*Proof.* By Theorem 40 and Theorem 21. ∎

# 4 Coherence and soundness lemmas

## 4.1 Coherence and soundness of the structure maps

**Lemma 23.** $[\![\cdot]\!] : \Sigma^!_0 \to \mathscr{V}^{\mathrm{op}}$ *is a symmetric strict monoidal functor.*

*Proof.* Preservation of composition is a consequence of associativity of the ⊗-comonoid structure:

$$d^{n+1}_P \circ (EP \otimes \cdots \otimes d^m_P \otimes \cdots \otimes EP) = d^{n+m}_P \,:\, EP \to (EP)^{\otimes(n+m)}$$

As for the symmetric strict monoidal property, the result is by construction. ∎

In particular, $[\![\cdot]\!] \circ U : \Sigma^! \to \mathscr{V}^{\mathrm{op}}$ is symmetric strict monoidal, with moreover $[\![\sigma, \sigma']\!] = [\![\sigma]\!] \otimes [\![\sigma']\!]$.

**Lemma 24.** *Let $\sigma \in \Sigma^!(!\Gamma; !\Gamma')$. Its interpretation $[\![\sigma]\!] \in \mathscr{V}((!\Gamma)^+, (!\Gamma')^+)$ is a morphism of $E$-coalgebras $((!\Gamma')^+, \delta_{\Gamma'}) \to ((!\Gamma)^+, \delta_\Gamma)$.*

*Proof.* The case of the unique map in $\Sigma^!_0((!A)^n; !A)$ is treated in Corollary 7. We show that the property is closed under tensor. For $\sigma_1, \sigma_2$ two morphisms of coalgebras as above, one has from the monoidal structure on the category of coalgebras

$$[\![\sigma_1]\!] \otimes [\![\sigma_2]\!] \,:\, ((!\Gamma')^+, \delta_{(\Gamma'_1)^+,(\Gamma'_2)^+}) \to ((!\Gamma)^+, \delta_{(\Gamma_1)^+,(\Gamma_2)^+})$$

from which one obtains

$$[\![\sigma_1, \sigma_2]\!] \,:\, ((!\Gamma')^+, \delta_{\Gamma'_1,\Gamma'_2}) \to ((!\Gamma)^+, \delta_{\Gamma_1,\Gamma_2})$$

from Lemma 23 and from

$$\delta_{\Gamma,\Gamma'} = \delta_{(\Gamma)^+,(\Gamma')^+}$$

that is,

$$\delta_{P_1,\ldots,P_n,Q_1,\ldots,Q_m} = m_{\otimes_i EP_i, \otimes_i EQ_i} \circ (\delta_{P_1,\ldots,P_n} \otimes \delta_{Q_1,\ldots,Q_m})$$

which is by the associativity axiom of the monoidal functor. Similarly, the property is closed under unit and symmetry. ∎

This extends to $\mathbf{LJ}^\eta_p$ straightforwardly by replacing $\Sigma^!/\Sigma^!_0$ with $\Sigma/\Sigma_0$. Note that in this case $[\![\cdot]\!] : \Sigma^{\mathrm{op}}_0 \to \mathscr{V}$ is Cartesian.

### 4.2 Coherent basis lemma

**Definition 25.** Two derivations are *equivalent* if they have the same denotation in any interpretation.

One of the main goals will be to prove that any two derivations of a judgement $c:(\Gamma \vdash \Delta)$ are equivalent (Theorem 56).

**Lemma 26.** *For any derivation there is an equivalent derivation ending with a structural rule preceded by a non-structural rule.*

*Proof.* We treat the case of a derivation of $c : (\Gamma \vdash \Delta)$. The cases of terms and contexts are identical. Among the equivalent derivations of $c : (\Gamma \vdash \Delta)$, we consider one with the smallest height. If it is of the form:

$$\dfrac{\dfrac{c' : (\Gamma'' \vdash \Delta)}{c'[\sigma] : (\Gamma' \vdash \Delta)}\,(\sigma)}{c'[\tau\circ\sigma] : (\Gamma \vdash \Delta)}\,(\tau)$$

with $\sigma \in \Sigma^!(\Gamma''; \Gamma')$, $\tau \in \Sigma^!(\Gamma'; \Gamma)$, and $c = c'[\sigma\circ\tau]$, then one has $\tau\circ\sigma \in \Sigma^!(\Gamma''; \Gamma)$ and:

$$\dfrac{c' : (\Gamma'' \vdash \Delta)}{c'[\tau\circ\sigma] : (\Gamma \vdash \Delta)}\,(\tau\circ\sigma)$$

is a smaller derivation which is equivalent by functoriality of $[\![\cdot]\!]$, which is impossible. On the other hand, if it ends with a non-structural rule, then it can be completed into an equivalent derivation ending with $(\mathrm{id}_{\mathrm{dom}\,\Gamma})$. ∎

**Definition 27.** For any typing context $\Gamma$ and $X \subseteq \mathrm{dom}\,\Gamma$, the typing context $\Gamma_{\restriction X}$ is defined on $X$ with $\Gamma_{\restriction X}(x) = \Gamma(x)$ and the induced total order. Also, the notation $!(x_1 : A_1, \dots, x_n : A_n)$ is for the typing context $x_1 : !A_1, \dots, x_n : !A_n$.

**Lemma 28** (Coherent basis)**.** *For any derivation $c : (\Gamma \vdash \Delta)$:*

- *one has $\boldsymbol{fv}\, c \subseteq \mathrm{dom}\,\Gamma \cup \mathrm{dom}\,\Delta$,*
- *$\Gamma_{\restriction \mathrm{dom}\,\Gamma \setminus \boldsymbol{fv}\, c}$ is of the form $!\Gamma'$,*
- *there exists an equivalent derivation ending with:*

$$\dfrac{c : (\Gamma_{\restriction \boldsymbol{fv}\, c} \vdash \Delta)}{c : (\Gamma \vdash \Delta)}\,(\sigma)$$

*where $\sigma \in \Sigma^!(\Gamma_{\restriction \boldsymbol{fv}\, c}; \Gamma)$ is the unique substitution given by the identity on $\boldsymbol{fv}\, c$,*

*and similarly for $t, e$ replacing $c$.*

For $\mathbf{LJ}^{\eta}_{p}$ one replaces $\Sigma^!$ with $\Sigma$ and removes the clause "of the form $!\Gamma'$".

*Proof.* First notice that for any $\Gamma, \Gamma'$ such that $\operatorname{dom}\Gamma' \subseteq \operatorname{dom}\Gamma$, the morphism $\sigma \in \Sigma^!(\Gamma';\Gamma)$ extending the identity $\mathrm{id}_{\Gamma'} \in \Sigma^!(\Gamma';\Gamma')$ is obviously unique and $\Gamma_{\restriction \operatorname{dom}\Gamma \setminus \operatorname{dom}\Gamma'}$ is of the form $!\Gamma''$. Since the action of the structure morphism is preserved by the adjunction, it amounts to the same to interpret the derivation in $\mathscr{V}$ or in $\underline{\mathscr{S}}$. We now proceed by induction on the size of the term by analysing the first non-structural rule using Lemma 26. Indeed, if the structure map that follows is $\sigma$, then any $\Gamma_{\restriction \mathbf{fv}\, f} \xrightarrow{\sigma'} \Gamma \xrightarrow{\sigma} \Gamma'$ factors as in Lemma 11; in particular it factors into $\Gamma_{\restriction \mathbf{fv}\, f} \xrightarrow{\sigma''} \Gamma'_{\restriction \mathbf{fv}\, f[\sigma]} \xrightarrow{\sigma'''} \Gamma'$ where $\sigma''$ is the composite of a bijection and of a tensor of identities and contractions, and $\sigma'''$ a tensor of identities and weakenings. The interpretation is preserved by functoriality. This lets us conclude after the following.

($\vdash$ ax) The non-structural rule is necessarily an axiom rule:

$$\frac{}{y : A \vdash y : A \mid}$$

for some $y$, and the structural rule is $(\vdash_f \sigma)$ with $\sigma \in \Sigma^!((y:A);\Gamma)$ and $\sigma(y) = x$. Same reasoning for (ax $\vdash$). $(\tilde{\mu}^\varepsilon \vdash)$ is by definition, $(\vdash \mu^\varepsilon)$ by naturality of the transposition.

$(\vdash_f \otimes)$, $(\vdash_f \oplus_i)$, $(\vdash \mathbf{1})$, $(\otimes \vdash)$, and $(\mathbf{1} \vdash)$: immediate.

$(\to \vdash_f)$: by induction $[\![V \cdot S]\!]_{\mathrm{S}}$ is of the form

$$\begin{aligned}
&\underline{\mathscr{S}}_{\sigma'} [\![\Gamma'_{\restriction \mathbf{fv}\, S} \mid S : B \vdash \Delta]\!]_{\mathrm{S}} \circ \underline{\mathscr{S}}_{[\![\Gamma_{\restriction \mathbf{fv}\, V} \vdash V : A \mid]\!]_{\mathrm{V}} \circ \sigma}(\mathrm{ev}^{A^+, B^\ominus}) \\
&= \underline{\mathscr{S}}_{\sigma'} [\![\Gamma'_{\restriction \mathbf{fv}\, S} \mid S : B \vdash \Delta]\!]_{\mathrm{S}} \circ \underline{\mathscr{S}}_{[\![\Gamma_{\restriction \mathbf{fv}\, V} \vdash V : A \mid]\!]_{\mathrm{V}} \circ \sigma}(\mathrm{ev}^{A^+, B^\ominus}) \\
&= \underline{\mathscr{S}}_{\sigma \otimes \sigma'}([\![\Gamma'_{\restriction \mathbf{fv}\, S} \mid S : B \vdash \Delta]\!]_{\mathrm{S}} \circ \underline{\mathscr{S}}_{[\![\Gamma_{\restriction \mathbf{fv}\, V} \vdash V : A \mid]\!]_{\mathrm{V}}}(\mathrm{ev}^{A^+, B^\ominus}))
\end{aligned}$$

by naturality of composition in $\underline{\mathscr{S}}$, hence the result. Same reasoning for (cut$^+$), $(! \vdash_f)$, and $(\&_i \vdash_f)$ in $\underline{\mathscr{S}}$, and (cut$^\ominus$) in $\underline{\mathscr{V}}$.

$(\mathbf{0} \vdash_f)$ and $(\vdash_f \top)$: immediate.

$(\vdash \&)$: from the induction hypotheses

$$[\![c_i : (\Gamma \vdash \alpha : A_i)]\!] = [\![c_i : (\Gamma_{\restriction \mathbf{fv}\, c_i} \vdash \alpha : A_i)]\!] \circ [\![\sigma_i]\!]$$

one factors each $[\![\sigma_i]\!]$ as $[\![\sigma'_i]\!] \circ [\![\sigma]\!]$ with the unique map $\sigma \in \Sigma^!_0(\Gamma_{\restriction \mathbf{fv}\,\{c,c'\}};\Gamma)$. Then one has

$$\begin{aligned}
&[\![\mu{<}\alpha.c;\beta.c'{>}]\!] \\
&= {<}([\![c_1]\!] \circ [\![\sigma'_1]\!] \circ [\![\sigma]\!])^*;([\![c_2]\!] \circ [\![\sigma'_2]\!] \circ [\![\sigma]\!])^*{>} \\
&= {<}\underline{\mathscr{S}}_{[\![\sigma]\!]}([\![c_1]\!] \circ [\![\sigma'_1]\!])^*;\underline{\mathscr{S}}_{[\![\sigma]\!]}([\![c_2]\!] \circ [\![\sigma'_2]\!])^*{>} \\
&= \underline{\mathscr{S}}_{[\![\sigma]\!]}({<}([\![c_1]\!] \circ [\![\sigma'_1]\!])^*;([\![c_2]\!] \circ [\![\sigma'_2]\!])^*{>})
\end{aligned}$$

by naturality of transposition and pairing. Same reasoning for $(\oplus \vdash)$.

$(\vdash \to)$ By induction hypothesis $[\![c : (\Gamma, x : A \vdash \alpha : B)]\!]$ has an equivalent derivation $[\![c : ((\Gamma, x : A)_{\restriction \mathbf{fv}\, c} \vdash \alpha : B)]\!] \circ [\![\sigma]\!]$ in which $\sigma \in \Sigma^!_0((\Gamma, x : A)_{\restriction \mathbf{fv}\, c};\Gamma)$ decomposes as $\sigma_A \circ \sigma_\Gamma$ where $\sigma_A = {\Gamma_{\restriction \mathbf{fv}\, c}}^+ \otimes f$ where $f$ is either $\mathrm{id}_{A^+}$ or $d^0_{A^+}$ depending on whether $x \in \mathbf{fv}\, c$, and $\sigma_\Gamma \in \Sigma^!_0(\Gamma_{\restriction \mathbf{fv}\, c}, x : A;\Gamma)$ is of the form $\sigma' \otimes A^+$. We write $\sigma_A = \Gamma$:

$$([\![c]\!] \circ [\![\sigma_A]\!] \circ [\![\sigma_\Gamma]\!])^* = \underline{\mathscr{S}}_{\sigma'^+ \otimes A^+}(([\![c]\!] \circ [\![\sigma_A]\!])^*)$$

by naturality of transposition, from which the result follows by naturality of the power adjunction.

($\vdash$ !) follows by induction and with $[\![\sigma]\!]$ being a morphism of coalgebras $((!\Gamma)^+, \delta_\Gamma) \to ((!\Gamma_{\restriction \mathbf{fv}\, c})^+, \delta_{\Gamma_{\restriction \mathbf{fv}\, c}})$ by Lemma 24. ∎

### 4.3 Coherent generation lemma

**Proposition 29** (Coherent generation)**.**

**($\vdash$ ax)** *Any derivation of* $\Gamma \vdash x : A \mid$ *satisfies* $\Gamma(x) = A$ *and is equivalent to the derivation:*

$$\dfrac{\dfrac{}{x : A \vdash x : A \mid}\,(\vdash ax)}{\Gamma \vdash x : A \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!((x : A); \Gamma)$ *is the unique substitution such that* $\sigma(x) = x$ *(implying that* $\Gamma(y)$ *for* $y \neq x$ *is of the form* $!B$*).*

**(ax$\vdash$)** *Any derivation of* $\Gamma \mid \alpha : A \vdash \Delta$ *satisfies* $\Delta = \alpha : A$ *and is equivalent to the derivation:*

$$\dfrac{\dfrac{}{\mid \alpha : A \vdash \alpha : A}\,(ax\vdash)}{\Gamma \mid \alpha : A \vdash \alpha : A}\,(\sigma \vdash_f)$$

*where* $\sigma$ *is the unique substitution in* $\Sigma^!(\emptyset; \Gamma)$ *(implying that* $\Gamma$ *is of the form* $!\Gamma'$*).*

**($\vdash \mu^\varepsilon$)** *For any derivation of* $\Gamma \vdash \mu\alpha^\varepsilon.c : A \mid$ *one has* $A$ *of polarity* $\varepsilon$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c : (\Gamma_{\restriction \mathit{fv}\, c} \vdash \alpha : A)}{\Gamma_{\restriction \mathit{fv}\, c} \vdash \mu\alpha^\varepsilon.c : A \mid}\,(\vdash \mu^\varepsilon)}{\Gamma \vdash \mu\alpha^\varepsilon.c : A \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, c}; \Gamma)$ *is the unique substitution given by the identity on* $\mathit{fv}\, c$*.*

**($\tilde{\mu}^\varepsilon \vdash$)** *For any derivation of* $\Gamma \mid \tilde{\mu}x^\varepsilon.c : A \vdash \Delta$ *(*$x \notin \operatorname{dom}\Gamma$*) one has* $A$ *of polarity* $\varepsilon$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c : (\Gamma_{\restriction \mathit{fv}\, c \setminus \{x\}}, x : A \vdash \Delta)}{\Gamma_{\restriction \mathit{fv}\, c \setminus \{x\}} \mid \tilde{\mu}x^\varepsilon.c : A \vdash \Delta}\,(\tilde{\mu}^\varepsilon \vdash)}{\Gamma \mid \tilde{\mu}x^\varepsilon.c : A \vdash \Delta}\,(\sigma \vdash_f)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, c \setminus \{x\}}; \Gamma)$ *is the unique substitution given by the identity on* $\mathit{fv}\, c \setminus \{x\}$*.*

**(cut$^\varepsilon$)** *For any derivation of* $\langle t \parallel e \rangle^\varepsilon : (\Gamma \vdash \Delta)$ *there exists* $A$ *with polarity* $\varepsilon$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\dfrac{!\Gamma_0, \Gamma_1 \vdash t : A \mid}{\Gamma_1' \vdash t[\sigma_1] : A \mid}\,(\sigma_1) \quad \dfrac{!\Gamma_0, \Gamma_2 \mid e : A \vdash \Delta}{\Gamma_2' \mid e[\sigma_2] : A \vdash \Delta}\,(\sigma_2)}{\langle t[\sigma_1] \parallel e[\sigma_2] \rangle^\varepsilon : (\Gamma_1', \Gamma_2' \vdash \Delta)}\,(cut)}{\langle t \parallel e \rangle^\varepsilon : (\Gamma \vdash \Delta)}\,(\sigma)$$

*where:*

$$!\Gamma_0 = \Gamma_{\restriction \mathit{fv}\, t \cap \mathit{fv}\, e} \qquad , \qquad \Gamma_1 = \Gamma_{\restriction \mathit{fv}\, V \setminus \mathit{fv}\, e} \qquad , \qquad \Gamma_2 = \Gamma_{\restriction \mathit{fv}\, e \setminus \mathit{fv}\, t}$$

*where $\sigma_1$ and $\sigma_2$ are any two renamings such that $\Gamma'_1 \# \Gamma'_2$, and where $\sigma \in \Sigma^!((\Gamma'_1, \Gamma'_2); \Gamma)$ is the unique substitution that coincides with $\sigma_1^{-1}$ and $\sigma_2^{-1}$ on their domains of definition.*

**($\vdash_f \otimes$)** *For any derivation of $\Gamma \vdash V{\otimes}W : A \mid$ one has $A$ of the form $B \otimes C$ and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\dfrac{!\Gamma_0, \Gamma_1 \vdash V : B \mid}{\Gamma'_1 \vdash V[\sigma_1] : B \mid}\,(\sigma_1) \quad \dfrac{!\Gamma_0, \Gamma_2 \vdash W : C \mid}{\Gamma'_2 \vdash W[\sigma_2] : C \mid}\,(\sigma_2)}{\Gamma'_1, \Gamma'_2 \vdash V[\sigma_1]{\otimes}W[\sigma_2] : B \otimes C \mid}\,(\vdash_f \otimes)}{\Gamma \vdash V{\otimes}W : B \otimes C \mid}\,(\vdash_f \sigma)$$

*where:*

$$!\Gamma_0 = \Gamma_{\restriction \mathit{fv}\, V \cap \mathit{fv}\, W} \qquad , \qquad \Gamma_1 = \Gamma_{\restriction \mathit{fv}\, V \setminus \mathit{fv}\, W} \qquad , \qquad \Gamma_2 = \Gamma_{\restriction \mathit{fv}\, W \setminus \mathit{fv}\, V}$$

*where $\sigma_1$ and $\sigma_2$ are any two renamings such that $\Gamma'_1 \# \Gamma'_2$, and where $\sigma \in \Sigma^!((\Gamma'_1, \Gamma'_2); \Gamma)$ is the unique substitution that coincides with $\sigma_1^{-1}$ and $\sigma_2^{-1}$ on their domains of definition.*

**($\multimap \vdash_f$)** *For any derivation of $\Gamma \mid V{\cdot}S : A \vdash \Delta$ one has $A$ of the form $B \multimap C$ and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\dfrac{!\Gamma_0, \Gamma_1 \vdash V : B \mid}{\Gamma'_1 \vdash V[\sigma_1] : B \mid}\,(\sigma_1) \quad \dfrac{!\Gamma_0, \Gamma_2 \mid S : C \vdash \Delta}{\Gamma'_2 \mid S[\sigma_2] : C \vdash \Delta}\,(\sigma_2)}{\Gamma'_1, \Gamma'_2 \mid V{\cdot}S : B \multimap C \vdash \Delta}\,(\multimap \vdash_f)}{\Gamma \mid V{\cdot}S : B \multimap C \vdash \Delta}\,(\sigma \vdash_f)$$

*where:*

$$!\Gamma_0 = \Gamma_{\restriction \mathit{fv}\, V \cap \mathit{fv}\, S} \qquad , \qquad \Gamma_1 = \Gamma_{\restriction \mathit{fv}\, V \setminus \mathit{fv}\, S} \qquad , \qquad \Gamma_2 = \Gamma_{\restriction \mathit{fv}\, S \setminus \mathit{fv}\, V}$$

*where $\sigma_1$ and $\sigma_2$ are any two renamings such that $\Gamma'_1 \# \Gamma'_2$, and where $\sigma \in \Sigma^!((\Gamma'_1, \Gamma'_2); \Gamma)$ is the unique substitution that coincides with $\sigma_1^{-1}$ and $\sigma_2^{-1}$ on their domains of definition.*

**($\vdash \mathbf{1}$)** *Any derivation of $\Gamma \vdash () : A \mid$ satisfies $A = \mathbf{1}$ and is equivalent to the derivation:*

$$\dfrac{\dfrac{}{\vdash () : \mathbf{1} \mid}\,(\vdash_f 1)}{\Gamma \vdash () : \mathbf{1} \mid}\,(\vdash_f \sigma)$$

*where $\sigma$ is the unique substitution in $\Sigma^!(\emptyset; \Gamma)$.*

**($\otimes \vdash$)** *For any derivation of $\Gamma \mid \tilde{\mu}(x{\otimes}y).c : A \vdash \Delta$ one has $A$ of the form $B \otimes C$, and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c : (\Gamma_{\restriction \mathit{fv}\, c \setminus \{x,y\}}, x : B, y : C \vdash \Delta)}{\Gamma_{\restriction \mathit{fv}\, c \setminus \{x,y\}} \mid \tilde{\mu}(x{\otimes}y).c : B \otimes C \vdash \Delta}\,(\otimes \vdash)}{\Gamma \mid \tilde{\mu}(x{\otimes}y).c : B \otimes C \vdash \Delta}\,(\sigma \vdash_f)$$

*where $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, c \setminus \{x,y\}}; \Gamma)$ is the unique substitution given by the identity on $\mathit{fv}\, c \setminus \{x, y\}$.*

**($\vdash \multimap$)** *For any derivation of* $\Gamma \vdash \mu(x\cdot\alpha).c : A \mid$ *one has* $A$ *of the form* $B \multimap C$*, and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c : (\Gamma_{\restriction fv\, c\setminus\{x\}}, x : B \vdash \alpha : C)}{\Gamma_{\restriction fv\, c\setminus\{x\}} \vdash \mu(x\cdot\alpha).c : B \multimap C \mid}\,(\vdash \multimap)}{\Gamma \vdash \mu(x\cdot\alpha).c : B \multimap C \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction fv\, c\setminus\{x\}}; \Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, c \setminus \{x\}$*.*

**($\mathbf{1} \vdash$)** *For any derivation of* $\Gamma \mid \tilde{\mu}().c : A \vdash \Delta$ *one has* $A = \mathbf{1}$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c : (\Gamma_{\restriction fv\, c} \vdash \Delta)}{\Gamma_{\restriction fv\, c} \mid \tilde{\mu}().c : \mathbf{1} \vdash \Delta}\,(\mathbf{1} \vdash)}{\Gamma \mid \tilde{\mu}().c : \mathbf{1} \vdash \Delta}\,(\sigma \vdash_f)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction fv\, c}; \Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, c$*.*

**($\vdash\, !$)** *For any derivation of* $\Gamma \vdash \mu!\alpha.c : A \mid$ *one has* $A$ *of the form* $!B$*, one has* $\Gamma_{\restriction fv\, c}$ *of the form* $!\Gamma'$*, and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c : (!\Gamma' \vdash \alpha : B)}{!\Gamma' \vdash \mu!\alpha.c : !B \mid}\,(\vdash\, !)}{\Gamma \vdash \mu!\alpha.c : !B \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!(!\Gamma'; \Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, c$*.*

**($!\vdash_f$)** *For any derivation of* $\Gamma \mid\, !S : A \vdash \Delta$ *one has* $A$ *of the form* $!B$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_{\restriction fv\, S} \mid S : B \vdash \Delta)}{\Gamma_{\restriction fv\, S} \mid\, !S : !B \vdash \Delta}\,(! \vdash_f)}{\Gamma \mid\, !S : !B \vdash \Delta}\,(\sigma \vdash_f)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction fv\, S}; \Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, S$*.*

**($\vdash_f \oplus_i$)** *For any derivation of* $\Gamma \vdash \iota_i(V) : A \mid$ *one has* $A$ *of the form* $B_1 \oplus B_2$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_{\restriction fv\, V} \vdash V : B_i \mid}{\Gamma_{\restriction fv\, V} \vdash \iota_i(V) : B_1 \oplus B_2 \mid}\,(\vdash_f \oplus)}{\Gamma \vdash \iota_i(V) : B_1 \oplus B_2 \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction fv\, V}; \Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, V$*.*

**($\&_i \vdash_f$)** *For any derivation of* $\Gamma \mid \pi_i\cdot S : A \vdash \Delta$ *one has* $A$ *of the form* $B_1 \,\&\, B_2$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_{\restriction fv\, S} \mid S : B_i \vdash \Delta)}{\Gamma_{\restriction fv\, S} \mid \pi_i\cdot S : B_1 \,\&\, B_2 \vdash \Delta}\,(\& \vdash_f)}{\Gamma \mid \pi_i\cdot S : B_1 \,\&\, B_2 \vdash \Delta}\,(\sigma \vdash_f)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction fv\, S}; \Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, S$*.*

**($\vdash$ &)** *For any derivation of* $\Gamma \vdash \mu{<}\beta.c_1\,;\gamma.c_2{>} : A \mid$ *one has* $A = B \mathbin{\&} C$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c_1 : (\Gamma_{\restriction \mathit{fv}\, c_1,c_2} \vdash \beta : B) \qquad c_2 : (\Gamma_{\restriction \mathit{fv}\, c_1,c_2} \vdash \gamma : C)}{\Gamma_{\restriction \mathit{fv}\, c_1,c_2} \vdash \mu{<}\beta.c_1\,;\gamma.c_2{>} : B \mathbin{\&} C \mid}\,(\vdash \&)}{\Gamma \vdash \mu{<}\beta.c_1\,;\gamma.c_2{>} : B \mathbin{\&} C \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, c_1,c_2};\Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, c_1, c_2$*.*

**($\oplus \vdash$)** *For any derivation of* $\Gamma \mid \tilde{\mu}[x.c_1|y.c_2] : A \vdash \Delta$ *one has* $A = B \oplus C$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{c_1 : (\Gamma_{\restriction \mathit{fv}\, c_1,c_2}, x : B \vdash \Delta) \qquad c_2 : (\Gamma_{\restriction \mathit{fv}\, c_1,c_2}, y : C \vdash \Delta)}{\Gamma_{\restriction \mathit{fv}\, c_1,c_2} \mid \tilde{\mu}[x.c_1|y.c_2] : B \oplus C \vdash \Delta}\,(\oplus \vdash)}{\Gamma \mid \tilde{\mu}[x.c_1|y.c_2] : B \oplus C \vdash \Delta}\,(\sigma \vdash_f)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, c_1,c_2};\Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, c_1, c_2$*.*

**($\vdash_f \top$)** *For any derivation of* $\Gamma \vdash \mu{<}V{>} : A \mid$ *one has* $A = \top$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_{\restriction \mathit{fv}\, V} \vdash V : B \mid}{\Gamma_{\restriction \mathit{fv}\, V} \vdash \mu{<}V{>} : \top \mid}\,(\vdash_f \top)}{\Gamma \vdash \mu{<}V{>} : \top \mid}\,(\vdash_f \sigma)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, V};\Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, V$*.*

**($\mathbf{0} \vdash_f$)** *For any derivation of* $\Gamma \mid \tilde{\mu}[S] : A \vdash \Delta$ *one has* $A = \mathbf{0}$ *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_{\restriction \mathit{fv}\, S} \mid S : B \vdash \Delta}{\Gamma_{\restriction \mathit{fv}\, S} \mid \tilde{\mu}[S] : \mathbf{0} \vdash \Delta}\,(\mathbf{0} \vdash_f)}{\Gamma \mid \tilde{\mu}[S] : \mathbf{0} \vdash \Delta}\,(\sigma \vdash_f)$$

*where* $\sigma \in \Sigma^!(\Gamma_{\restriction \mathit{fv}\, S};\Gamma)$ *is the unique substitution given by the identity on* $\mathbf{fv}\, S$*.*

For $\mathbf{LJ}_p^\eta$ one replaces $\Sigma^!$ with $\Sigma$ throughout and $!\Gamma_0$ with $\Gamma_0$ in the rules (cut$^\varepsilon$), ($\vdash_f \otimes$), ($\multimap \vdash_f$).

*Proof.* Using Lemma 28 and noticing that being the unique substitution given by identity is a property closed under composition, we restrict to the case where $\mathrm{dom}\,\Gamma = \mathbf{fv}\, f$ for $f$ the term under consideration, and analyse the first non-structural rule.

($\vdash$ ax) The non-structural rule is necessarily an axiom rule:

$$\dfrac{}{y : A \vdash y : A \mid}$$

for some $y$, and the structural rule is ($\vdash_f$ $\sigma$) with $\sigma \in \Sigma^!((y : A);(x : A))$ and $\sigma(y) = x$. Consequently one has $\Gamma(x) = A$, and the derivation differ by a renaming whose interpretation is the identity. Same reasoning for (ax $\vdash$) and ($\vdash \mathbf{1}$).

(cut$^\varepsilon$) The non-structural rule is necessarily a cut rule for some type $A$ with polarity $\varepsilon$:

$$\frac{\overset{\vdots}{\Gamma_1' \vdash t' : A \mid} \qquad \overset{\vdots}{\Gamma_2' \mid e' : A \vdash \Delta}}{\langle t' \parallel e' \rangle^\varepsilon : (\Gamma_1', \Gamma_2' \vdash \Delta)}$$

with $\mathrm{dom}\,\Gamma_1' = \mathbf{fv}\,t'$ and $\mathrm{dom}\,\Gamma_2' = \mathbf{fv}\,e'$, and the structural rule is $(\sigma')$ with $\sigma' \in \Sigma^!((\Gamma_1', \Gamma_2'); \Gamma)$, $t'[\sigma'] = t$, and $e'[\sigma'] = e$. By definition, $\Gamma_{\restriction \mathbf{fv}\,t \cap \mathbf{fv}\,e}$ is of the form $!\Gamma_0$. ($\Gamma_0$, $\Gamma_1$, and $\Gamma_2$ are defined in the statement.) We write the command as

$$\langle t \parallel e \rangle^\varepsilon = \langle t[\sigma_1] \parallel e[\sigma_2] \rangle^\varepsilon[\sigma] = \langle t'[\sigma_1 \circ \sigma_1'] \parallel e'[\sigma_2 \circ \sigma_2'] \rangle^\varepsilon[\sigma]$$

where $\sigma_1' \in \Sigma^!(\Gamma_1'; (!\Gamma_0, \Gamma_1))$ and $\sigma_2' \in \Sigma^!(\Gamma_2'; (!\Gamma_0, \Gamma_2))$ are obtained from $\sigma'$ by restriction, and $\sigma_1$ and $\sigma_2$ are two renamings with disjoint co-domain. One has $t'[\sigma_1'] = t$ because $\mathbf{fv}\,t = \mathrm{dom}(\Gamma_0, \Gamma_1)$, and $e'[\sigma_2'] = e$ because $\mathbf{fv}\,e = \mathrm{dom}(\Gamma_0, \Gamma_2)$. To ensure the above equality we define $\sigma \in \Sigma^!((\Gamma_1'', \Gamma_2''); \Gamma)$ with $\sigma(x) = \sigma_i^{-1}(x)$ for all $x \in \mathrm{dom}\,\Gamma_i''$. By definition one has $\sigma \circ (\sigma_1, \sigma_2) = \mathrm{id}$, hence by functoriality (Lemma 23) one has the equivalent derivation:

$$\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Gamma_1' \vdash t' : A \mid} \qquad \overset{\vdots}{\Gamma_2' \mid e' : A \vdash \Delta}}{\langle t' \parallel e' \rangle^\varepsilon : (\Gamma_1', \Gamma_2' \vdash \Delta)}\,(\mathrm{cut})}{\langle t'[\sigma_1 \circ \sigma_1'] \parallel e'[\sigma_2 \circ \sigma_2'] \rangle^\varepsilon : (\Gamma_1'', \Gamma_2'' \vdash \Delta)}\,((\sigma_1 \circ \sigma_1'), (\sigma_2 \circ \sigma_2'))}{\langle t \parallel e \rangle^\varepsilon : (\Gamma \vdash \Delta)}\,(\sigma)$$

Therefore one has the equivalent derivation we sought:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Gamma_1' \vdash t' : A \mid}}{!\Gamma_0, \Gamma_1 \vdash t : A \mid}\,(\vdash \sigma_1')}{\Gamma_1'' \vdash t[\sigma_1] : A \mid}\,(\vdash \sigma_1) \qquad \dfrac{\dfrac{\overset{\vdots}{\Gamma_2' \mid e' : A \vdash \Delta}}{!\Gamma_0, \Gamma_2 \mid e : A \vdash \Delta}\,(\sigma_2' \vdash)}{\Gamma_2'' \mid e[\sigma_2] : A \vdash \Delta}\,(\sigma_2 \vdash)}{\langle t[\sigma_1] \parallel e[\sigma_2] \rangle^\varepsilon : (\Gamma_1', \Gamma_2' \vdash \Delta)}\,(\mathrm{cut})}{\langle t \parallel e \rangle^\varepsilon : (\Gamma \vdash \Delta)}\,(\sigma)$$

by naturality of composition in $\underline{\mathscr{V}}$ when $\langle t \parallel e \rangle^\varepsilon = \langle V \parallel e \rangle^\ominus$ and $t' = V'$:

$$\begin{aligned}
&\underline{\mathscr{V}}_{[\![(\sigma_1 \circ \sigma_1'), (\sigma_2 \circ \sigma_2')]\!]}([\![e']\!] \circ [\![V']\!]_{\mathrm{V}}) \\
&= \underline{\mathscr{V}}_{[\![\sigma_1 \circ \sigma_1']\!] \otimes [\![\sigma_2 \circ \sigma_2']\!]}([\![e']\!] \circ [\![V']\!]_{\mathrm{V}}) \\
&= \underline{\mathscr{V}}_{[\![\sigma_2 \circ \sigma_2']\!]}([\![e']\!]) \circ \underline{\mathscr{V}}_{[\![\sigma_1 \circ \sigma_1']\!]}([\![V']\!]_{\mathrm{V}})
\end{aligned}$$

and similarly in $\underline{\mathscr{S}}$ when $\varepsilon = +$. Same reasoning for $(\vdash_f \otimes)$, $(\multimap \vdash_f)$, $(! \vdash_f)$, $(\vdash_f \oplus_i)$, and $(\&_i \vdash_f)$.

($\vdash \&$) The non-structural rule is necessarily of the following form:

$$\frac{\overset{\vdots}{c_1' : (\Gamma' \vdash \beta : B)} \qquad \overset{\vdots}{c_2' : (\Gamma' \vdash \gamma : C)}}{\Gamma' \vdash \mu{<}\beta.c_1' ; \gamma.c_2'{>} : B \,\&\, C \mid}$$

and the structural rule is $(\vdash_f \sigma)$ with $\sigma \in \Sigma^!(\Gamma';\Gamma)$, $c'_1[\sigma] = c_1$, and $c'_2[\sigma] = c_2$. The equivalent derivation we seek is therefore:

$$\dfrac{\dfrac{\vdots \atop c'_1 : (\Gamma' \vdash \beta : B)}{c_1 : (\Gamma \vdash \beta : B)}\,(\vdash_f \sigma) \qquad \dfrac{\vdots \atop c'_2 : (\Gamma' \vdash \gamma : C)}{c_2 : (\Gamma \vdash \gamma : C)}\,(\vdash_f \sigma)}{\Gamma \vdash \mu{<}\beta.c_1 ; \gamma.c_2{>} : B \,\&\, C \mid}\,(\vdash \&)$$

by commutation with structural rules stemming from naturality of transposition and pairing as in Lemma 28. Similar reasoning for $(\vdash \mu^\varepsilon)$, $(\tilde{\mu}^\varepsilon \vdash)$, $(\otimes \vdash)$, $(\vdash \multimap)$, $(\mathbf{1} \vdash)$, $(\vdash !)$, $(\oplus \vdash)$, $(\vdash_f \top)$, and $(\mathbf{0} \vdash_f)$. In the special case of $(\vdash !)$ we use Lemma 24. ∎

### 4.3.1 Applications

Coherent generation lets us reason on derivations and their interpretation by induction on the term. For instance:

**Proposition 30** (Typability of subterms)**.** *If $f$ is typable and $g$ is a sub-term of $f$ then $g$ is typable as well.*

*Proof.* By induction on the definition of being a sub-term, by applying Proposition 29. ∎

**Proposition 31** (Compatibility of typing preservation)**.** *Consider $\rhd$ a relation on terms. Whenever $\rhd$ preserves typing, it does so compatibly.*

*Proof.* By induction on the definition of $\to$. We assume $f \to g$ and $f$ typable. If $f \to g$ comes from $f \rhd g$, the result follows from the hypothesis that $\rhd$ preserves typing. Otherwise $f \to g$ is obtained from $\to$ applied to one of its immediate subterms. One reduces the problem to the subterm by applying Proposition 29. If $f = \langle t \,\|\, e\rangle \to g = \langle t' \,\|\, e\rangle$ comes from $t \to t'$, and if $\langle t \,\|\, e\rangle : (\Gamma \vdash \Delta)$, then by this lemma there exists $A$ such that:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \vdash t : A \mid}{\Gamma'_1 \vdash t[\sigma_1] : A \mid}\,(\sigma_1) \qquad \dfrac{\Gamma_2 \mid e : A \vdash \Delta}{\Gamma'_2 \mid e[\sigma_2] : A \vdash \Delta}\,(\sigma_2)}{\langle t[\sigma_1] \,\|\, e[\sigma_2]\rangle : (\Gamma'_1, \Gamma'_2 \vdash \Delta)}\,(\text{cut})}{\langle t \,\|\, e\rangle : (\Gamma \vdash \Delta)}\,(\sigma)$$

and by induction hypothesis one has $\Gamma_1 \vdash t \to t' : A \mid$. Therefore:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \vdash t \to t' : A \mid}{\Gamma'_1 \vdash t[\sigma_1] \to t'[\sigma_1] : A \mid}\,(\sigma_1) \qquad \Gamma'_2 \mid e[\sigma_2] : A \vdash \Delta}{\langle t[\sigma_1] \,\|\, e[\sigma_2]\rangle \to \langle t'[\sigma_1] \,\|\, e[\sigma_2]\rangle : (\Gamma'_1, \Gamma'_2 \vdash \Delta)}\,(\text{cut})}{\langle t \,\|\, e\rangle \to \langle t' \,\|\, e\rangle : (\Gamma \vdash \Delta)}\,(\sigma)$$

The other cases are treated identically. ∎

### 4.3.2 Cut-free derivations

The case ($\text{cut}^\varepsilon$) from Proposition 29 introduces a new formula in the judgement over which there is no control. This will prevent us from proving certain properties by a straightforward induction on the terms, such as coherence (Theorem 56). In this case, the technique is to reduce the problem to $\to_{\mathrm{R}}$-normal terms using strong normalization (Theorem 55). For such terms, the case ($\text{cut}^\varepsilon$) strengthens as follows.

**Proposition 32** (Coherent generation for normal forms)**.** *For any $\rhd_{\mathrm{R}}$-normal command $c$ and any derivation of $c:(\Gamma\vdash\Delta)$ one has either of two cases:*

***(deactivation⊢)*** *$c$ is of the form $\langle x\,\|\,S\rangle$ with $S\neq\tilde{\mu}x.c'$ and one has an equivalent derivation ending with:*

$$\dfrac{\dfrac{\dfrac{}{y:\Gamma(x)\vdash y:\Gamma(x)\mid}\,(\vdash ax)\qquad \Gamma_1\mid S:\Gamma(x)\vdash\Delta}{\langle y\,\|\,S\rangle:(y:\Gamma(x),\Gamma_1\vdash\Delta)}\,(cut)}{\langle x\,\|\,S\rangle:(\Gamma\vdash\Delta)}\,(\sigma)$$

*where $\Gamma_1=\Gamma_{\restriction fv\,S}$, or*

***(⊢deactivation)*** *$c$ is of the form $\langle V\,\|\,\alpha\rangle$ with $V\neq\mu\alpha.c$ and one has an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_1\vdash V:\Delta(\alpha)\qquad \dfrac{}{\mid\alpha:\Delta(\alpha)\vdash\Delta\mid}\,(ax\vdash)}{\langle V\,\|\,\alpha\rangle:(\Gamma_1\vdash\Delta)}\,(cut)}{\langle V\,\|\,\alpha\rangle:(\Gamma\vdash\Delta)}\,(\sigma)$$

*where $\Gamma_1=\Gamma_{\restriction fv\,V}$.*

*Proof.* By case analysis on the shape of the well-typed commands using Proposition 29. ∎

Wadler made a similar suggestion that a command involving a variable should not be seen as a cut in the sense of sequent calculus [Wad03]. In particular, this generalises the sub-formula property:

**Proposition 33** (Sub-formula property)**.** *If $c:(\Gamma\vdash\Delta)$ is $\to_{\mathrm{R}}$-normal then it possesses a derivation in which only sub-formulae of $\Gamma$ and $\Delta$ occur.*

*Proof.* By a straightforward induction on $c$ using Propositions 29 and 32. ∎

## 4.4 Sound substitution lemma

**Lemma 34.** *For any derivation as follows:*

$$\dfrac{\Gamma_1\vdash V:A_\varepsilon\mid\qquad \Gamma_2\mid S:A_\varepsilon\vdash\Delta}{\langle V\,\|\,S\rangle^\varepsilon:(\Gamma_1,\Gamma_2\vdash\Delta)}\,(cut)$$

*one has depending on $\varepsilon$:*

$$\begin{aligned}[\![\langle V\,\|\,S\rangle^\ominus]\!]^* &= [\![S]\!]_S\circ^{\mathscr{S}}[\![V]\!]\\ [\![\langle V\,\|\,S\rangle^+]\!] &= [\![S]\!]\circ^{\mathscr{V}}({\Gamma_2}^+\otimes[\![V]\!]_V)\,.\end{aligned}$$

*Proof.* One has when $\varepsilon = \ominus$:

$$\begin{aligned} & [\![\langle V \parallel S\rangle^{\ominus}]\!]^{*} \\ &= \left(G^{\Gamma_2}_{A^{\ominus},\Delta^{\ominus}}([\![S]\!]_{\mathrm{S}}) \circ^{\mathscr{V}} [\![V]\!]^{*}\right)^{*} && (2) \\ &= [\![S]\!]_{\mathrm{S}} \circ^{\underline{\mathscr{S}}} [\![V]\!] && (3) \end{aligned}$$

(2): by definition. (3): by naturality of transposition. The case $\varepsilon = +$ is similar. ∎

**Lemma 35.** *Let* $\Gamma \vdash V : !A \mid$ *such that* $\mathrm{dom}\,\Gamma = \boldsymbol{fv}\,V$*. Then* $\Gamma$ *is of the form* $!\Gamma_0$*, and for all interpretations the morphism:*

$$[\![\Gamma \vdash V : !A \mid]\!] \in \mathscr{V}(\Gamma^{+}, EGA^{\ominus})$$

*is a morphism of* $E$*-coalgebras* $(\Gamma^{+}, \delta_{\Gamma_0}) \to (EGA^{\ominus}, \delta_{GA^{\ominus}})$ *(where* $\delta_{\Gamma_0}$ *is defined in Section 2.3.7).*

The statement is the same between $\mathbf{ILL}^{\eta}_{p}$ and $\mathbf{LJ}^{\eta}_{p}$.

*Proof.* By Proposition 29, $[\![\Gamma \vdash V : !A \mid]\!]$ is given by the interpretation of a derivation:

$$\dfrac{\dfrac{\vdots}{\Gamma' \vdash V : !A \mid}\,(R)}{\Gamma \vdash V : !A \mid}\,(\vdash_f \sigma)$$

where $(R)$ is either rule $(\vdash \mathrm{ax})$ or $(\vdash !)$, and where necessarily $\Gamma' = \Gamma_{\restriction \mathbf{fv}\,V} = \Gamma$ and $\sigma = \mathrm{id}_{\Gamma}$. In particular, $V$ is of either form $x$ or $\mu!\alpha.c$ and $\Gamma$ is of the form $!\Gamma_0$. Then, $[\![\Gamma \vdash V : !A \mid]\!]$ is given as follows: if $V = x$, by the identity, and if $V = \mu!\alpha.c$, by the composite $Ef \circ \delta_{\Gamma_0}$, both of which are homomorphisms. ∎

**Lemma 36** (Sound value substitution)**.** *Let a derivation of* $\Gamma \vdash V : A \mid$ *where* $\mathrm{dom}\,\Gamma = \boldsymbol{fv}\,V$*, and let* $\Gamma' \# \Gamma$*. We consider* $[\![V]\!]_V \in \mathscr{V}(\Gamma^{+}, A^{+})$ *its interpretation.*

1. *For any derivation of* $c : (x : A, \Gamma' \vdash \Delta)$ *and for* $\varepsilon$ *the polarity of* $A$ *there exists a derivation of* $c[V/x] : (\Gamma, \Gamma' \vdash \Delta)$ *such that*

$$\begin{aligned} [\![c[V/x]]\!] &= [\![c]\!] \circ ([\![V]\!]_V \otimes \Gamma'^{+}) \\ &= [\![\langle V \parallel \tilde{\mu}x^{\varepsilon}.c\rangle^{\varepsilon}]\!] \in \mathscr{V}((\Gamma, \Gamma')^{+}, G\Delta^{\ominus}) \end{aligned}$$

   *where* $[\![c]\!]$ *is the interpretation of the derivation.*

2. *For any derivation of* $x : A, \Gamma' \vdash t : B$ *there exists a derivation of* $\Gamma, \Gamma' \vdash t[V/x] : B$ *such that*

$$[\![t[V/x]]\!] = \underline{\mathscr{S}}_{[\![V]\!]_V \otimes \Gamma'^{+}}([\![t]\!]) \in \underline{\mathscr{S}}_{(\Gamma,\Gamma')^{+}}(FI, B^{\ominus})$$

   *where* $[\![t]\!] \in \underline{\mathscr{S}}_{A^{+} \otimes \Gamma'^{+}}(FI, B^{\ominus})$ *interprets the derivation.*

3. *For any derivation of* $x : A, \Gamma' \mid e : B \vdash \Delta$ *there exists a derivation of* $\Gamma, \Gamma' \mid e[V/x] : B \vdash \Delta$ *such that*

$$[\![e[V/x]]\!] = \underline{\mathscr{V}}_{[\![V]\!]_V \otimes \Gamma'}([\![e]\!]) \in \underline{\mathscr{V}}_{\Gamma^{+} \otimes \Gamma'^{+}}(B^{\ominus}, G\Delta^{\ominus})$$

   *where* $[\![e]\!] \in \underline{\mathscr{V}}_{A^{+} \otimes \Gamma'^{+}}(B^{\ominus}, G\Delta^{\ominus})$ *interprets the derivation.*

*Proof.* We first treat the case $x \notin \mathbf{fv}\, c$ (resp. $\mathbf{fv}\, t, \mathbf{fv}\, e$). Necessarily $A$ is of the form $!A'$. By Lemma 35 $\Gamma$ is therefore of the form $(x_1 : !A_1, \dots, x_n : !A_n)$ and $[\![V]\!]$ is a morphism of $E$-coalgebras $(\Gamma^+, \delta_{A_1,\dots,A_n}) \to (EGA'^{\ominus}, \delta_{GA'^{\ominus}})$. By Lemma 28, the derivations are equivalent to ones ending with the unique substitutions in $\Sigma^!((y : A); (x : A, \Gamma'))$ and $\Sigma^!((y : A); (\Gamma, \Gamma'))$ whose interpretations are respectively of the form $d^0_{GA'^{\ominus}} \otimes f$ and $d^0_{GA_1^{\ominus}} \otimes \cdots \otimes d^0_{GA_n^{\ominus}} \otimes f$. The result is then a consequence of $(I, m_I)$ being terminal in $\mathscr{V}^E$ by Theorem 8 which enforces $d^0_{GA'^{\ominus}} \circ [\![V]\!]_{\mathrm{V}} = d^0_{GA_1^{\ominus}} \otimes \cdots \otimes d^0_{GA_n^{\ominus}}$. We prove the result for $x \in \mathbf{fv}\, c$ by induction on $c, t, e$, using Proposition 29. In each case one can $\mathbf{fv}\, c = \{x\} \cup \operatorname{dom} \Gamma'$ (resp. $\mathbf{fv}\, t, \mathbf{fv}\, e$) using Lemma 28.

($\vdash$ ax): One has $[\![x[V/x]]\!] = [\![\Gamma \vdash V : A \mid]\!]$ which we show equal to $\underline{\mathscr{S}}_{[\![V]\!]_{\mathrm{V}}}([\![x]\!])$. In the case $A$ positive one has

$$\underline{\mathscr{S}}_{[\![V]\!]_{\mathrm{V}}}([\![x]\!]) = \underline{\mathscr{S}}_{[\![V]\!]_{\mathrm{V}}}(F^{A^+}_{I,A^+}\mathrm{id}_{A^+}) = F^{\Gamma^+}_{I,A^+}\underline{\mathscr{V}}_{[\![V]\!]_{\mathrm{V}}}(\mathrm{id}_{A^+})$$

by naturality of $F_{I,A^+}$, which is indeed equal to $F^{\Gamma^+}_{I,A^+}[\![V]\!]_{\mathrm{V}} = [\![\Gamma \vdash V : A \mid]\!]$. In the case $A$ negative the equation $\underline{\mathscr{S}}_{[\![V]\!]_{\mathrm{V}}}([\![x]\!]) = [\![\Gamma \vdash V : A \mid]\!]$ is equivalent to $\underline{\mathscr{V}}_{[\![V]\!]_{\mathrm{V}}}(\mathrm{id}_{GN^{\ominus}}) = [\![V]\!]_{\mathrm{V}}$ by transposing each member, which is by definition.

($\mathrm{cut}^{\varepsilon}$): One distinguishes the cases $x \in \mathbf{fv}\, t \cup \mathbf{fv}\, e$, $x \in \mathbf{fv}\, t \setminus \mathbf{fv}\, e$, and $x \in \mathbf{fv}\, e \setminus \mathbf{fv}\, t$. In the first case one has $A$ of the form $!B$ and we consider the equivalent derivation:

$$\dfrac{\dfrac{\dfrac{x : !B, !\Gamma_0, \Gamma_1 \vdash t : A \mid}{\Gamma_1'' \vdash t[\sigma_1] : A \mid}\,(\sigma_1) \qquad \dfrac{x : !B, !\Gamma_0, \Gamma_2 \mid e : A \vdash \Delta}{\Gamma_2'' \mid e[\sigma_2] : A \vdash \Delta}\,(\sigma_2)}{\langle t[\sigma_1] \,\|\, e[\sigma_2]\rangle^{\varepsilon} : (\Gamma_1'', \Gamma_2'' \vdash \Delta)}\,(\mathrm{cut})}{\langle t \,\|\, e\rangle^{\varepsilon} : (x : !B, \Gamma' \vdash \Delta)}\,(\sigma)$$

and similarly in the two other cases.

In the case $x \in \mathbf{fv}\, t \cup \mathbf{fv}\, e$ by Lemma 35 $\Gamma$ is of the form $(x_1 : !A_1, \dots, x_n : !A_n)$ and $[\![V]\!]_{\mathrm{V}}$ is a morphism of $E$-coalgebras $(\Gamma^+, \delta_{A_1,\dots,A_n}) \to (EGB^{\ominus}, \delta_{GB^{\ominus}})$. Furthermore, by Theorem 8 one has

$$([\![V]\!]_{\mathrm{V}} \otimes [\![V]\!]_{\mathrm{V}}) \circ d^2_{EGB^{\ominus}} = d \circ [\![V]\!]_{\mathrm{V}}$$

where $d^2_{EGB^{\ominus}}$ and $d \in \mathscr{V}(\Gamma^+, \Gamma^+ \otimes \Gamma^+)$ are the diagonal morphisms in the Cartesian $\mathscr{V}^E$. We first establish

$$d = \gamma \circ \textstyle\bigotimes_i d^2_{EGA_i^{\ominus}}$$

where $\gamma \in \mathscr{V}(\bigotimes_i (!A_i{}^+)^{\otimes 2}, \Gamma^+ \otimes \Gamma^+)$ is the obvious map obtained by symmetry. This is established by showing

$$\pi_1 \circ \gamma \circ \textstyle\bigotimes_i d^2_{EGA_i^{\ominus}} = \pi_2 \circ \gamma \circ \textstyle\bigotimes_i d^2_{EGA_i^{\ominus}} = \mathrm{id}_{\Gamma^+}$$

where $\pi_1, \pi_2 \in \mathscr{V}(\Gamma^+ \otimes \Gamma^+, \Gamma^+)$ are defined with

$$\pi_1 = \mathrm{id}_{\Gamma^+} \otimes (d^0_{\Gamma^+} \circ \delta_{\Gamma})$$
$$\pi_2 = (d^0_{\Gamma^+} \circ \delta_{\Gamma}) \otimes \mathrm{id}_{\Gamma^+}$$

One notices

$$d^0_{\Gamma^+} \circ \delta_\Gamma = \bigotimes_i d^0_{GA_i{}^\ominus} : \bigotimes_i EGA_i{}^\ominus \to I$$

by virtue of $(I, m_I)$ being terminal in $\mathscr{V}^E$. Consequently one has $\pi_1 \circ \gamma = \bigotimes_i(\mathrm{id}_{EGA_i{}^\ominus} \otimes d^0_{GA_i{}^\ominus})$ and symmetrically for $\pi_2 \circ \gamma$; hence the result by comonoid laws. This establishes that $d$ is the interpretation of the obvious substitution in $\Sigma^!(\Gamma \uplus \Gamma; \Gamma)$.

From there, all three cases $x \in \mathbf{fv}\, t \cup \mathbf{fv}\, e$, $x \in \mathbf{fv}\, t \setminus \mathbf{fv}\, e$, and $x \in \mathbf{fv}\, e \setminus \mathbf{fv}\, t$ are reasonably straightforward, although tedious to write, and follow from naturality of composition (in $\underline{\mathscr{V}}$ or $\underline{\mathscr{S}}$ depending on $\varepsilon$) and induction hypothesis. The cases $(\vdash_f \otimes)$ and $(\multimap \vdash_f)$ are treated similarly.

$(\vdash !)$: Necessarily $A$ is of the form $!A'$, and again by Lemma 35, $\Gamma$ is of the form $!\Gamma_0$ and $[\![V]\!]$ is a morphism of $E$-coalgebras $(\Gamma^+, \delta_{\Gamma_0}) \to (EGA'^\ominus, \delta_{GA'^\ominus})$. An equivalent derivation is

$$\frac{c : (x : !A', !\Gamma' \vdash \alpha : B)}{x : !A', !\Gamma' \vdash \mu!\alpha.c : !B \mid}\ (\vdash !)$$

and one has therefore a derivation

$$\frac{c[V/x] : (!\Gamma_0, !\Gamma' \vdash \alpha : B)}{!\Gamma_0, !\Gamma' \vdash \mu!\alpha.c[V/x] : !B \mid}\ (\vdash !)$$

whose interpretation is

$$E[\![c[V/x]]\!] \circ \delta_{\Gamma_0,\Gamma'} = E[\![c]\!] \circ E([\![V]\!]_{\mathrm{V}} \otimes \mathrm{id}_{!\Gamma'^+}) \circ \delta_{\Gamma_0,\Gamma'}$$

by induction hypothesis. To establish that the latter is equal to $[\![\mu!\alpha.c]\!] \circ ([\![V]\!]_{\mathrm{V}} \otimes \mathrm{id}_{!\Gamma'^+})$ we mention that $[\![V]\!]_{\mathrm{V}} \otimes \mathrm{id}_{!\Gamma'^+}$ is a homomorphism

$$((!\Gamma_0, !\Gamma')^+, \delta_{\Gamma_0,\Gamma'}) \to ((!A', !\Gamma')^+, \delta_{A',\Gamma'})$$

due to $[\![V]\!]_{\mathrm{V}}$ being a homomorphism as above and to the definition of the monoidal structure on $\mathscr{V}^E$.

$(\mathrm{ax} \vdash)$, $(\vdash \mathbf{1})$, $(\vdash_f \top)$, $(\mathbf{0} \vdash_f)$, $(! \vdash_f)$, $(\&_i \vdash_f)$, and $(\vdash_f \oplus_i)$ are trivial or immediate from induction hypothesis.

$(\vdash \mu^\varepsilon)$, $(\tilde{\mu}^\varepsilon \vdash)$, $(\otimes \vdash)$, $(\vdash \multimap)$, $(\mathbf{1} \vdash)$, $(\oplus \vdash)$ and $(\vdash \&)$ are straightforward by identity or naturality and induction hypothesis. ∎

For $\mathbf{LJ}^\eta_p$ instead of $\mathbf{ILL}^\eta_p$, the statement is the same, and in the proof one replaces the cartesian structure on $\mathscr{V}^E$ with the one on $\mathscr{V}$.

**Lemma 37** (Sound stack substitution)**.** *Let a derivation of* $\Gamma \mid S : A \vdash \Delta$ *and* $\Gamma' \# \Gamma$. *We consider* $[\![S]\!]_S \in \underline{\mathscr{S}}_{\Gamma^+}(A^\ominus, \Delta^\ominus)$ *its interpretation.*

1. *For any derivation of* $c : (\Gamma' \vdash \alpha : A)$ *and for* $\varepsilon$ *the polarity of* $A$ *there exists a derivation of* $c[S/\alpha] : (\Gamma, \Gamma' \vdash \Delta)$ *such that*

$$\begin{aligned}[\![c[S/\alpha]]\!] &= G^{\Gamma^+}_{A^\ominus,\Delta^\ominus}[\![S]\!]_S \circ (\Gamma^+ \otimes [\![c]\!]) \\ &= [\![\langle \mu\alpha^\varepsilon.c \parallel S\rangle^\varepsilon]\!] \in \mathscr{V}(\Gamma^+ \otimes \Gamma'^+, \Delta^\ominus)\end{aligned}$$

2. *For any derivation of* $\Gamma' \mid e : B \vdash \alpha : A$ *there exists a derivation of* $\Gamma, \Gamma' \mid e[S/\alpha] : B \vdash \Delta$ *such that*

$$[\![e[S/\alpha]]\!] = G^{\Gamma^+}_{A^\ominus,\Delta^\ominus}[\![S]\!]_S \circ (\Gamma^+ \otimes [\![e]\!]) \in \mathscr{V}(\Gamma^+ \otimes \Gamma'^+ \otimes GB^\ominus, \Delta^\ominus)\,.$$

*Proof.* By induction on $c, e$ using Proposition 29. We first notice that the equations can equivalently be written as $[\![c[S/\alpha]]\!]^* = [\![S]\!]_S \circ [\![c]\!]^*$ and similarly for $e$. We first notice that composition with $[\![S]\!]_S$ commutes with structural rules up to renaming of $\Gamma$ by naturality of composition. In particular in each case we can assume $\mathbf{fv}\, c = \mathrm{dom}\,\Gamma'$ (resp. $\mathbf{fv}\, e$) by Lemma 28.

(ax $\vdash$): One has $[\![\alpha[S/\alpha]]\!] = [\![\Gamma \mid S : A \vdash \Delta]\!]$ and we show $[\![\Gamma \mid S : A \vdash \Delta]\!]^* = [\![S]\!]_S \circ [\![\alpha]\!]^*$. This is by definition in the case $A$ is positive since $[\![\Gamma \mid S : A \vdash \Delta]\!]^* = [\![S]\!]_S$ and $[\![\alpha]\!]^* = \mathrm{id}_{FA^+}$. When $A$ is negative both sides are equal to $[\![S]\!]_S \circ \varepsilon_{A^\ominus}$ by immediate calculation.

($\tilde{\mu}^\varepsilon \vdash$), (cut$^\varepsilon$), ($\to \vdash_f$), ($\otimes \vdash$), ($\mathbf{1} \vdash$), ($! \vdash_f$), ($\&_i \vdash_f$), and ($\mathbf{0} \vdash_f$) are straightforward by induction hypothesis and either identity or naturality of transposition.

($\oplus \vdash$): by naturality in $N$ of the isomorphism:

$$\underline{\mathscr{S}}_\Gamma(FP, N) \times \underline{\mathscr{S}}_\Gamma(FQ, N) \cong \underline{\mathscr{S}}_\Gamma(F(P+Q), N)$$

stemming from naturality of transposition and from the natural isomorphism:

$$\underline{\mathscr{V}}_\Gamma(P, GN) \times \underline{\mathscr{V}}_\Gamma(Q, GN) \cong \underline{\mathscr{V}}_\Gamma(P+Q, GN)\,. \qquad \blacksquare$$

### 4.5 Sound subject reduction

**Definition 38.** The relation $\rhd_{\mathrm{E}'}$ is defined as $\rhd_{\mathrm{E}}$ at the exclusion of the rules ($E\mathbf{0}$) and ($E\top$).

**Lemma 39.** $\rhd_{\mathrm{RE}'}$ *preserves typing and its typed restrictions* $\rhd_{\mathrm{RE}'}^{\vdash}$ *preserve the interpretation.*

*Proof.* By case analysis. In each case we apply Proposition 29 to reduce the problem to applications of Lemmas 36 and 37.

We treat in detail the cases ($R\to$) and ($E\to$). For any $c = \langle \mu(x\cdot\alpha).c \parallel V\cdot S \rangle \rhd_{\mathrm{R}\to} c[V/x, S/\alpha]$ and $c : (\Gamma \vdash \Delta)$, by Proposition 29 there exists a derivation of $c : (\Gamma \vdash \Delta)$ that has the form:

$$\dfrac{\dfrac{\dfrac{\dfrac{c[\sigma] : (\Gamma'_1, x : A \vdash \alpha : B)}{\Gamma'_1 \vdash \mu(x\cdot\alpha).c[\sigma] : A \to B \mid}\,(\vdash \to)}{\Gamma_1 \vdash \mu(x\cdot\alpha).c[\sigma] : A \to B \mid}\,(\vdash_f \sigma) \qquad \dfrac{\dfrac{\Gamma'_2 \vdash V[\sigma_2] : A \mid \qquad \Gamma'_3 \mid S[\sigma_3] : B \vdash \Delta}{\Gamma'_2, \Gamma'_3 \mid V[\sigma_2]\cdot S[\sigma_3] : A \to B \vdash \Delta}\,(\to \vdash_f)}{\Gamma_2 \mid (V\cdot S)[\sigma'] : A \to B \vdash \Delta}\,(\sigma)}{\langle \mu(x\cdot\alpha).c[\sigma_1] \parallel (V\cdot S)[\sigma_2] \rangle : (\Gamma_1, \Gamma_2 \vdash \Delta)}\,(\mathrm{cut})}{\langle \mu(x\cdot\alpha).c \parallel V\cdot S \rangle : (\Gamma \vdash \Delta)}\,(\sigma)$$

and where $\Gamma'_1 = \Gamma_{1\restriction \mathbf{fv}\, c[\sigma_1] \setminus \{x\}}$, $\mathrm{dom}\,\Gamma'_2 = \mathbf{fv}\, V[\sigma_2]$, $\mathrm{dom}\,\Gamma'_3 = \mathbf{fv}\, S[\sigma_3]$ and $\Gamma_1 \# \Gamma_2$ and $\Gamma'_2 \# \Gamma'_3$, and where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are renamings chosen such that $\Gamma'_1 \# \Gamma'_2$ and $\Gamma'_1 \# \Gamma'_3$.

By naturality of composition and monoidality of the interpretation of structure maps, there is an equivalent derivation as follows:

$$\dfrac{\dfrac{\dfrac{c' : (\Gamma'_1, x : A \vdash \alpha : B)}{\Gamma'_1 \vdash \mu(x\cdot\alpha).c' : A \to B \mid} \qquad \dfrac{\Gamma'_2 \vdash V' : A \mid \qquad \Gamma'_3 \mid S' : B \vdash \Delta}{\Gamma'_2, \Gamma'_3 \mid V'\cdot S' : A \to B \vdash \Delta}}{\langle \mu(x\cdot\alpha).c' \parallel V'\cdot S' \rangle : (\Gamma'_1, \Gamma'_2, \Gamma'_3 \vdash \Delta)}\,(\mathrm{cut})}{\langle \mu(x\cdot\alpha).c \parallel V\cdot S \rangle : (\Gamma \vdash \Delta)}\,(\sigma)$$

where $c' \overset{\text{def}}{=} c[\sigma_1]$, $V' \overset{\text{def}}{=} V[\sigma_2]$, $S' \overset{\text{def}}{=} S[\sigma_3]$.

Since the supports are disjoint one has $c'[V'/x, S'/\alpha] = c'[S'/\alpha][V'/x]$ and $(\Gamma'_1, \Gamma'_3) \# \Gamma'_2$. Lemmas 36 and 37 therefore apply to show that there exists, modulo an additional exchange, a derivation of $c'[V'/x, S'/\alpha] : (\Gamma'_1, \Gamma'_2, \Gamma'_3 \vdash \Delta)$ whose interpretation satisfies:

$$[\![c'[V'/x, S'/\alpha]]\!]^* = \underline{\mathcal{S}}_{(\Gamma_1'^+ \otimes [\![V']\!]_{\mathrm{V}} \otimes \Gamma_3'^+)}([\![S']\!]_{\mathrm{S}} \circ [\![c']\!]^*)$$

We show that it is equal to the interpretation of the derivation of

$$\langle \mu(x{\cdot}\alpha).c' \parallel V'{\cdot}S' \rangle : (\Gamma'_1, \Gamma'_2, \Gamma'_3 \vdash \Delta)$$

appearing above.

$$\begin{aligned}
&[\![\langle \mu(x{\cdot}\alpha).c' \parallel V'{\cdot}S' \rangle]\!]^* \\
&= [\![S']\!]_{\mathrm{S}} \circ \underline{\mathcal{S}}_{[\![V']\!]_{\mathrm{V}}}(\mathrm{ev}^{A^+,B^\ominus}) \circ \lambda^{\Gamma_1'^+}_{A^+,B^\ominus}([\![c']\!]^*) && (4)\\
&= \underline{\mathcal{S}}_{(\Gamma_1'^+ \otimes [\![V']\!]_{\mathrm{V}} \otimes \Gamma_3'^+)}\big([\![S']\!]_{\mathrm{S}} \circ \mathrm{ev}^{A^+,B^\ominus} \circ \lambda^{\Gamma_1'^+}_{A^+,B^\ominus}([\![c']\!]^*)\big) && (5)\\
&= \underline{\mathcal{S}}_{(\Gamma_1'^+ \otimes [\![V']\!]_{\mathrm{V}} \otimes \Gamma_3'^+)}\big([\![S']\!]_{\mathrm{S}} \circ [\![c']\!]^*\big) && (6)
\end{aligned}$$

(4): by definition and Lemma 34. (5): by naturality of composition in $\underline{\mathcal{S}}$. (6): by definition of ev and $\lambda$.

This establishes the result for $(R\multimap)$. As for $(E\multimap)$, any derivation of $\Gamma \vdash \mu(x{\cdot}\alpha).\langle V \parallel x{\cdot}\alpha \rangle^\ominus : A \mid$ with $x, \alpha \notin \mathbf{fv}\, V$ is equivalent to a derivation of the following form:

$$\dfrac{\dfrac{\Gamma_{\restriction \mathbf{fv}\, V} \vdash V : A \multimap B \mid \qquad \dfrac{\overline{x : B \vdash x : B \mid} \qquad \overline{\mid \alpha : C \vdash \alpha : C}}{x : B \mid x{\cdot}\alpha : B \multimap C \vdash \alpha : C \mid}\,(\multimap\vdash)}{\dfrac{\langle V \parallel x{\cdot}\alpha \rangle : (\Gamma_{\restriction \mathbf{fv}\, V}, x : B \vdash \alpha : C \mid)}{\Gamma_{\restriction \mathbf{fv}\, V} \vdash \mu(x{\cdot}\alpha).\langle V \parallel x{\cdot}\alpha \rangle : B \multimap C \mid}\,(\vdash\multimap)}\,(\mathrm{cut}^\ominus)}{\Gamma \vdash \mu(x{\cdot}\alpha).\langle V \parallel x{\cdot}\alpha \rangle : B \multimap C \mid}\,(\sigma)$$

where $A = B \multimap C$ and $\Gamma' = \Gamma_{\restriction \mathbf{fv}\, V}$. We now show that the interpretations of $\Gamma' \vdash \mu(x{\cdot}\alpha).\langle V \parallel x{\cdot}\alpha \rangle : A \mid$ and $\Gamma' \vdash V : A \mid$ are equivalent. One has:

$$\begin{aligned}
&[\![\mu(x{\cdot}\alpha).\langle V \parallel x{\cdot}\alpha \rangle]\!] \\
&= \lambda^{\Gamma'^+}_{B^+,C^\ominus}([\![\alpha]\!]_{\mathrm{S}} \circ \underline{\mathcal{S}}_{[\![x]\!]_{\mathrm{V}}}(\mathrm{ev}^{B^+,C^\ominus}) \circ [\![V]\!]) && (7)\\
&= \lambda^{\Gamma'^+}_{B^+,C^\ominus}(\mathrm{ev}^{B^+,C^\ominus} \circ [\![V]\!]) \\
&= [\![V]\!] && (8)
\end{aligned}$$

(7): by definition and Lemma 34. (8): by definition of ev and $\lambda$. This establishes the result for $(E\multimap)$.

Above we have reduced the result for the cases $(R\multimap)$ and $(E\multimap)$ to principal cuts, by which we mean derivations involving pairwise fresh subterms and without structural rules. It is straightforward to see that all the other cases reduce to principal cuts in the same manner. We leave this rearranging of derivations implicit.

($R\mu^+$): For

$$c : (\Gamma \vdash \alpha : P)$$
$$\Gamma' \mid S : P \vdash \Delta$$

one has:

$$[\![\langle \mu\alpha^+.c \parallel S\rangle^+]\!]^* = [\![S]\!]_{\mathrm{S}} \circ^{\mathcal{S}} [\![c]\!]^* = [\![c[S/\alpha]]\!]^*$$

by definition and Lemma 37.

($R\tilde{\mu}^{\ominus}$): For

$$c : (\Gamma \mid x : N \vdash \Delta)$$
$$\Gamma' \vdash V : N \mid$$

one has:

$$[\![\langle V \parallel \tilde{\mu}x^{\ominus}.c\rangle^{\ominus}]\!] = [\![c]\!] \circ^{\mathcal{V}} [\![V]\!]_{\mathrm{V}} = [\![c[V/x]]\!]$$

by definition, Lemma 36 and exchange.

($R\otimes$): For:

$$c : (\Gamma_1, x : A, y : B \vdash \Delta)$$
$$\Gamma_2 \vdash V : A \mid$$
$$\Gamma_3 \vdash W : B \mid$$

one has:

$$\begin{aligned} &[\![\langle V \otimes W \parallel \tilde{\mu}(x{\otimes}y).c\rangle]\!] \\ &= [\![c]\!] \circ^{\mathcal{V}} ({\Gamma_1}^+ \otimes [\![V]\!]_{\mathrm{V}} \otimes [\![W]\!]_{\mathrm{V}}) && (9) \\ &= [\![c[V/x]]\!] \circ^{\mathcal{V}} ({\Gamma_1}^+ \otimes {\Gamma_2}^+ \otimes [\![W]\!]_{\mathrm{V}}) && (10) \\ &= [\![c[V/x][W/y]]\!] && (11) \end{aligned}$$

and $c[V/x][W/y] = c[V/x, W/y]$. (9): by definition and Lemma 34. (10) and (11): by Lemma 36.

From there, the following cases are straightforward: ($R\mu^{\ominus}$), ($R\tilde{\mu}^+$), ($E\mu^{\varepsilon}$), ($E\tilde{\mu}^{\varepsilon}$), ($E\otimes$), ($R\mathbf{1}$), and ($E\mathbf{1}$).

($R!$): For:

$$c : (!\Gamma_1 \vdash \alpha : A)$$
$$\Gamma_2 \mid S : A \vdash \Delta$$

one has:

$$
\begin{aligned}
& [\![\langle \mu !\alpha.c \parallel !S\rangle]\!] \\
& = G[\![S]\!]_{\mathrm{S}} \circ ({\Gamma_2}^{+} \otimes \epsilon_{GA^{\ominus}}) \circ ({\Gamma_2}^{+} \otimes (E[\![c]\!] \circ \delta_{\Gamma_1})) && (12)\\
& = G[\![S]\!]_{\mathrm{S}} \circ [\![c]\!] && (13)\\
& = [\![c[S/\alpha]]\!] && (14)
\end{aligned}
$$

(12): by definition and Lemma 34. (13): $\epsilon_{!{\Gamma_1}^{+}} \circ \delta_{\Gamma_1} = \mathrm{id}_{!{\Gamma_1}^{+}}$ by Corollary 7. (14): by Lemma 37.

($E!$): For $!\Gamma \vdash V : !A \mid$ such that $\mathrm{dom}\,\Gamma = \mathbf{fv}\,V$ one has:

$$
\begin{aligned}
& [\![\mu !\alpha.\langle V \parallel !\alpha\rangle^{+}]\!] \\
& = E(\epsilon_{GA^{\ominus}} \circ [\![V]\!]_{\mathrm{V}}) \circ \delta_{\Gamma} && (15)\\
& = E\epsilon_{GA^{\ominus}} \circ \delta_{GA^{\ominus}} \circ [\![V]\!]_{\mathrm{V}} = [\![V]\!]_{\mathrm{V}} && (16)
\end{aligned}
$$

(15): by definition and Lemma 34. (16): since $[\![V]\!]_{\mathrm{V}}$ is a morphism of $E$-coalgebras $(!\Gamma^{+}, \delta_{\Gamma}) \to (EGA^{\ominus}, \delta_{GA^{\ominus}})$ by (35).

($R\&$): For:

$$
\begin{aligned}
& c_1 : (\Gamma \vdash \alpha : A_1) \\
& c_2 : (\Gamma \vdash \alpha : A_2) \\
& \Gamma' \mid S : A_i \vdash \Delta
\end{aligned}
$$

one has:

$$
\begin{aligned}
& [\![\langle \mu {<}\alpha.c_1 ; \alpha.c_2{>} \parallel \pi_i \cdot S\rangle]\!]^{*} \\
& = [\![S]\!]_{\mathrm{S}} \circ \pi_i \circ {<}[\![c_1]\!]^{*} ; [\![c_2]\!]^{*}{>} && (17)\\
& = [\![S]\!]_{\mathrm{S}} \circ [\![c_i]\!]^{*} \\
& = [\![c_i[S/\alpha]]\!] && (18)
\end{aligned}
$$

(17): by definition and Lemma 34. (18): by Lemma 37.

($E\&$): For $\Gamma \vdash V : A \,\&\, B \mid$

$$
\begin{aligned}
& [\![\mu {<}\alpha.\langle V \parallel \pi_1 \cdot \alpha\rangle ; \beta.\langle V \parallel \pi_2 \cdot \beta\rangle{>}]\!] \\
& = {<}\pi_1 \circ^{\mathcal{S}} [\![V]\!] ; \pi_1 \circ^{\mathcal{S}} [\![V]\!]{>} && (19)\\
& = [\![V]\!]
\end{aligned}
$$

(19): by definition and Lemma 34.

($R\oplus$): For:

$$
\begin{aligned}
& c_1 : (\Gamma \mid x : A_1 \vdash \Delta) \\
& c_2 : (\Gamma \mid x : A_2 \vdash \Delta) \\
& \Gamma' \vdash V : A_i \mid
\end{aligned}
$$

one has:

$$
\begin{aligned}
&[\![\langle \iota_i(V) \parallel \tilde{\mu}[x.c_1 | x.c_2]\rangle]\!] \\
&= \left[[\![c_1]\!] ; [\![c_2]\!]\right]_{\Gamma^+} \circ (\Gamma^+ \otimes (\iota_i \circ [\![V]\!]_{\mathrm{V}})) && (20)\\
&= [\![c_i]\!] \circ (\Gamma^+ \otimes [\![V]\!]_{\mathrm{V}}) && (21)\\
&= [\![c_i[V/x]]\!] && (22)
\end{aligned}
$$

(20): by definition and Lemma 34. (21): since by definition the inverse map in the isomorphism (1) is given by $f \mapsto <\Gamma^+ \otimes \iota_1 ; \Gamma^+ \otimes \iota_2>$. (22): by Lemma 36 and exchange.

($E\oplus$): For $\Gamma \mid S : A \oplus B \vdash \Delta$:

$$
\begin{aligned}
&[\![\tilde{\mu}\left[x.\langle \iota_1(x) \parallel S\rangle \middle| y.\langle \iota_2(y) \parallel S\rangle\right]]\!] \\
&= \left[[\![S]\!] \circ (\Gamma^+ \otimes \iota_1); [\![S]\!] \circ (\Gamma^+ \otimes \iota_2)\right]_{\Gamma^+} && (23)\\
&= [\![S]\!] && (24)
\end{aligned}
$$

(23): by definition and Lemma 34. (24): same argument as for (21). ∎

**Theorem 40.** $\rhd_{\mathrm{RE'}}$ *preserves typing compatibly.*

*Proof.* By Proposition 31 and Lemma 39. ∎

**Corollary 41** (Subject reduction)**.** $\to_{\mathrm{RE'}}$ *preserves typing.*

*Proof.* By Theorem 40 and Lemma 16. ∎

**Lemma 42.** ${\to_{\mathrm{RE}}}^{\vdash}$ *preserves the interpretation.*

*Proof.* The typed restriction of ${\rhd_{\mathrm{RE}}}^{\vdash} = ({\rhd_{\mathrm{RE'}}}^{\vdash} \cup {\rhd_{\mathrm{E0T}}}^{\vdash})$ preserves the interpretation by Lemma 39 and because ${\rhd_{\mathrm{E0T}}}^{\vdash}$ trivially preserves the interpretation given that the interpretation of **0** is initial and the interpretation of $\top$ is terminal. This immediately carries over to ${\to_{\mathrm{RE}}}^{\vdash}$ by the definition of ${\to_{\mathrm{RE}}}^{\vdash}$. ∎

## 5 Strong normalisation

This section is adapted from [MM09, MM11], themselves inspired by Krivine's adaptation [Kri93, Kri09] of Girard's reducibility candidates [Gir72, Gir87].

Judgements can equivalently refer to typability in $\mathbf{ILL}_p^\eta$ or in $\mathbf{LJ}_p^\eta + \square$ since the restriction on the structural rules is not accounted for by the model.

**Proposition 43** (Weak standardization)**.** *If* $c\ (\to_{\mathrm{R}} \setminus \rhd_{\mathrm{R}})^*\ c'\ \rhd_{\mathrm{R}}\ c''$ *then there exists* $c'''$ *such that* $c \rhd_{\mathrm{R}} c''' \to_{\mathrm{R}}^* c''$.

*Proof.* Suppose one has $c\ (\to_R \setminus \rhd_R)^*\ c' \rhd_R c''$. We consider the rule involved in the reduction $c' \rhd_R c''$: if it is $(R\otimes)$ then $c'$ is of the form $\langle V'\otimes W' \,\|\, \tilde{\mu}(x\otimes y).c'_0\rangle$ and $c'' = c'_0[V'/x, W'/y]$. Now $\to_R \setminus \rhd_R$ can only create strict subcommands, therefore $c$ is of the form $\langle V\otimes W \,\|\, \tilde{\mu}(x\otimes y).c_0\rangle$ where $V \to_R^* V'$, $W \to_R^* W'$, and $c_0 \to_R^* c'_0$. Therefore by taking $c''' = c_0[V/x, W/y]$ one has $c \rhd_R c''' \to_R^* c'_0$. Similarly for all other reduction rules: no redex for $\rhd_R$ can be created by $\to_R \setminus \rhd_R$ and one can apply the same reasoning. ∎

**Definition 44.** A command $c$ is strongly normalizing ($c \in \perp\!\!\!\perp$) if any $\to_R$-sequence starting from $c$ is finite. An expression $t$ (respectively a context $e$) is strongly normalizing ($t \in \mathbb{T}$, resp. $e \in \mathbb{E}$) if any $\langle t \,\|\, \alpha\rangle^\varepsilon$ (resp. any $\langle x \,\|\, e\rangle^\varepsilon$) is in $\perp\!\!\!\perp$.

**Lemma 45.** *Let $c$ be a command. If all strict sub-commands of $c$ are in $\perp\!\!\!\perp$, and if either $c \not\rhd_R$ or $c \rhd_R c' \in \perp\!\!\!\perp$ for some command $c'$, then $c \in \perp\!\!\!\perp$.*

*Proof.* Assume that $c$ admits an infinite $\to_R$-sequence. If this sequence is an infinite $(\to_R \setminus \rhd_R)$-sequence, then at least one immediate sub-command of $c$ (there are finitely many so) admits an infinite $\to_R$-sequence. Otherwise, let us write $c' \rhd_R c''$ the first occurrence of $\rhd_R$ in the sequence. By Proposition 43 there exists $c'''$ with $c \rhd_R c''' \to_R^* c''$ and therefore $c'''$ admits an infinite $\to_R$-sequence. In this case, since $\rhd_R$ is deterministic, this means that all $\rhd_R$-reducts admit an infinite $\to_R$-sequence. ∎

**Lemma 46.** *One has $t \in \mathbb{T}$ (respectively $e \in \mathbb{E}$) if and only if all sub-commands of $t$ (resp. $e$) are in $\perp\!\!\!\perp$.*

*Proof.* ($\Rightarrow$) Immediate. ($\Leftarrow$) Assume that all sub-commands of $t$ are in $\perp\!\!\!\perp$. We apply Lemma 45 on the command $\langle t \,\|\, \alpha\rangle^\varepsilon$. If there exists $c$ with $\langle t \,\|\, \alpha\rangle^\varepsilon \rhd_R c$, then one has $t = \mu\beta^\varepsilon.c'$ with $c = c'[\alpha/\beta]$. Now $c'$, being a sub-command of $t$, is in $\perp\!\!\!\perp$, and since the rewriting rules are left-linear, variable substitution cannot unlock new reductions. Therefore $c \in \perp\!\!\!\perp$ as required. The proof is similar for $e$. ∎

Notice that the direction ($\Leftarrow$) relies on destructors (e.g. function application $tu$) being expressed with a $\mu$ binder. In calculi where $tu$ is given as a primitive, it is not possible to state that $\langle t \,\|\, u\cdot\alpha\rangle$ is a sub-command of $tu$!

**Proposition 47** (Saturation)**.** *One has $\langle t \,\|\, e\rangle \in \perp\!\!\!\perp$ if and only if $t \in \mathbb{T}$, $e \in \mathbb{E}$, and either $\langle t \,\|\, e\rangle \not\rhd_R$ or $\langle t \,\|\, e\rangle \rhd_R c \in \perp\!\!\!\perp$.*

*Proof.* By Lemma 45 and Lemma 46. ∎

As usual, we let $\perp\!\!\!\perp$ define an antitone Galois correspondence $\cdot^\perp \dashv \cdot^\perp : \mathcal{P}(\mathbb{T}) \to \mathcal{P}(\mathbb{E})^{\mathrm{op}}$. In fact, we define two such Galois correspondences:

$$(\cdot^{\perp_+}) \dashv (\cdot^{\perp_+}) : \mathcal{P}(\mathbb{T}) \to \mathcal{P}(\mathbb{S})^{\mathrm{op}}$$
$$(\cdot^{\perp_\ominus}) \dashv (\cdot^{\perp_\ominus}) : \mathcal{P}(\mathbb{V}) \to \mathcal{P}(\mathbb{E})^{\mathrm{op}}$$

by the following two poles:

$$t \perp\!\!\!\perp^+ S \iff \langle t \,\|\, S\rangle^+ \in \perp\!\!\!\perp$$
$$V \perp\!\!\!\perp^\ominus e \iff \langle V \,\|\, e\rangle^\ominus \in \perp\!\!\!\perp$$

**Definition 48.**

- $\mathbb{V} \subseteq \mathbb{T}$ is the set of strongly normalising values.
- $\mathbb{S} \subseteq \mathbb{E}$ is the set of strongly normalising stacks.
- We write $\mathcal{X}_{\mathbb{V}} \stackrel{\text{def}}{=} \mathcal{X} \cap \mathbb{V}$ and $\mathcal{X}_{\mathbb{S}} \stackrel{\text{def}}{=} \mathcal{X} \cap \mathbb{S}$.
- We consider the following maps:

$$\begin{aligned}
\mathcal{V} \otimes \mathcal{W} &\stackrel{\text{def}}{=} \{V \otimes W \mid V \in \mathcal{V} \text{ and } W \in \mathcal{W}\} \\
\mathcal{V}_1 \oplus \mathcal{V}_2 &\stackrel{\text{def}}{=} \{\iota_i(V) \mid V \in \mathcal{V}_i\} \\
!\mathcal{S} &\stackrel{\text{def}}{=} \{!S \mid S \in \mathcal{S}\} \\
\mathcal{V} \bullet \mathcal{S} &\stackrel{\text{def}}{=} \{V \cdot S \mid V \in \mathcal{V} \text{ and } S \in \mathcal{S}\} \\
\mathcal{S}_1 \mathbin{\&} \mathcal{S}_2 &\stackrel{\text{def}}{=} \{\pi_i \cdot S \mid S \in \mathcal{S}_i\}
\end{aligned}$$

**Lemma 49.** *These maps preserve strong normalisation.*

*Proof.* This follows immediately from Lemma 46. ∎

**Definition 50.** We define by mutual induction interpretations of types:

$$\|P\| \subseteq \mathbb{V} \qquad \|N\| \subseteq \mathbb{S}$$
$$\mathbb{T}(A) \subseteq \mathbb{T} \qquad \mathbb{E}(A) \subseteq \mathbb{E}$$

with:

$$\begin{aligned}
\mathbb{T}(P) &\stackrel{\text{def}}{=} \|P\|^{\perp_+\perp_+} & \mathbb{T}(N) &\stackrel{\text{def}}{=} \|N\|^{\perp_\ominus} \\
\mathbb{E}(P) &\stackrel{\text{def}}{=} \|P\|^{\perp_+} & \mathbb{E}(N) &\stackrel{\text{def}}{=} \|N\|^{\perp_\ominus\perp_\ominus} \\
\|X^+\| &\stackrel{\text{def}}{=} \mathbb{V} & \|X^\ominus\| &\stackrel{\text{def}}{=} \mathbb{S} \\
\|\mathbf{1}\| &\stackrel{\text{def}}{=} \{()\} & \|A \multimap B\| &\stackrel{\text{def}}{=} \mathbb{T}(A)_{\mathbb{V}} \bullet \mathbb{E}(B)_{\mathbb{S}} \\
\|A \otimes B\| &\stackrel{\text{def}}{=} \mathbb{T}(A)_{\mathbb{V}} \otimes \mathbb{T}(B)_{\mathbb{V}} & \|A \mathbin{\&} B\| &\stackrel{\text{def}}{=} \mathbb{E}(A)_{\mathbb{S}} \mathbin{\&} \mathbb{E}(B)_{\mathbb{S}} \\
\|!A\| &\stackrel{\text{def}}{=} (!\mathbb{E}(B)_{\mathbb{S}})^{\perp_+}{}_{\mathbb{V}} & \|\top\| &\stackrel{\text{def}}{=} \emptyset \\
\|A \oplus B\| &\stackrel{\text{def}}{=} \mathbb{T}(A)_{\mathbb{V}} \oplus \mathbb{T}(B)_{\mathbb{V}} & & \\
\|\mathbf{0}\| &\stackrel{\text{def}}{=} \emptyset & &
\end{aligned}$$

From now on it will be clear from the polarity of the formula whether $\cdot^{\perp}$ means $\cdot^{\perp_+}$ or $\cdot^{\perp_\ominus}$.

**Lemma 51.** *For all A one has:*

$$\mathbb{T}(A) = \mathbb{E}(A)_{\mathbb{S}}{}^{\perp} \qquad \mathbb{E}(A) = \mathbb{T}(A)_{\mathbb{V}}{}^{\perp}.$$

*Proof.* If $A$ is positive, then $\mathbb{E}(A_+)_\mathbb{S} = \mathbb{E}(A_+)$ and $\mathbb{T}(A) = \mathbb{E}(A)^\perp$ by definition; if it is negative then we prove $\mathbb{T}(A_\ominus) = \mathbb{E}(A_\ominus)_\mathbb{S}{}^\perp$ by inclusion. ($\supseteq$) One has $\|A_\ominus\| \subseteq \mathbb{E}(A_\ominus)_\mathbb{S}$, hence $\mathbb{T}(A_\ominus) = \|A_\ominus\|^\perp \supseteq \mathbb{E}(A_\ominus)_\mathbb{S}{}^\perp$. ($\subseteq$) One has $\mathbb{E}(A_\ominus)_\mathbb{S} \subseteq \mathbb{E}(A_\ominus) = \|A_\ominus\|^{\perp\perp}$, hence $\mathbb{T}(A_\ominus) = \|A_\ominus\|^\perp \subseteq \mathbb{E}(A_\ominus)_\mathbb{S}{}^\perp$. The reasoning for $\mathbb{E}(A) = \mathbb{T}(A)_\mathbb{V}{}^\perp$ is symmetric. ∎

**Lemma 52.** *Let $A$ be a formula. For all $x$ one has $x \in \mathbb{T}(A)_\mathbb{V}$ and for all $\alpha$ one has $\alpha \in \mathbb{E}(A)_\mathbb{S}$.*

*Proof.* This is immediate from the definitions of $\mathbb{T}$ and $\mathbb{E}$. ∎

**Definition 53.** For $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ and $\Delta = (\alpha : B)$ (possibly empty), and any substitution $\sigma : \{x_1, \ldots, x_n, \alpha\} \to \mathbb{V} \cup \mathbb{S}$, we write $\sigma \Vdash \Gamma, \Delta$ whenever $\forall i, \sigma(x_i) \in \mathbb{T}(A_i)_\mathbb{V}$ and $\sigma(\alpha) \in \mathbb{E}(A)_\mathbb{S}$.

**Lemma 54** (Adequacy[1])**.** *Consider $\sigma \Vdash \Gamma, \Delta$. One has:*

- *if $c : (\Gamma \vdash \Delta)$ then $c[\sigma] \in \perp\!\!\!\perp$,*
- *if $\Gamma \vdash t : A$ then $t[\sigma] \in \mathbb{T}(A)$,*
- *if $\Gamma \mid e : A \vdash \Delta$ then $e[\sigma] \in \mathbb{E}(A)$.*

*Proof.* This is proved by induction on the derivations.

*Rules ($\vdash$ ax) and (ax $\vdash$):* immediate. *Rule ($\vdash \mu^\varepsilon$):* One has to show $\mu\alpha^\varepsilon.c[\sigma] \in \mathbb{T}(A_\varepsilon)$ for some $\sigma \Vdash \Gamma$ with $\alpha \notin \operatorname{dom}\Gamma$. By Lemma 51, it suffices to show $\langle \mu\alpha^\varepsilon.c[\sigma] \,\|\, S\rangle^\varepsilon \in \perp\!\!\!\perp$ for all $S \in \mathbb{E}(A_\varepsilon)_\mathbb{S}$. By Proposition 47, this follows from $c[\sigma, S/\alpha] \in \perp\!\!\!\perp$ which follows from the induction hypothesis. *Rule ($\tilde{\mu}^\varepsilon \vdash$):* same reasoning.

*Rule (cut):* One has to show $\langle t \,\|\, e\rangle^\varepsilon[\sigma] \in \perp\!\!\!\perp$ for any $\sigma \Vdash \Gamma, \Gamma', \Delta$. We consider the restrictions $\sigma' \Vdash \Gamma$ and $\sigma'' \Vdash \Gamma', \Delta$ of $\sigma$. By induction hypothesis, one has $t[\sigma'] \in \mathbb{T}(A_\varepsilon)$ and $e[\sigma''] \in \mathbb{E}(A_\varepsilon)$, hence the result from $\mathbb{T}(A_\varepsilon) \perp\!\!\!\perp^\varepsilon \mathbb{E}(A_\varepsilon)$.

*Rules ($\sigma$), ($\vdash \sigma$) and ($\sigma \vdash$) for $\sigma \in \Sigma^!(\Gamma;\Gamma')$.* For any $\sigma' \Vdash \Gamma'$ one has $\sigma' \circ \sigma \Vdash \Gamma$, from which the result follows by induction.

*Rule ($\multimap \vdash_f$):* One has to show $V \cdot S[\sigma] \in \mathbb{T}(A)_\mathbb{V} \bullet \mathbb{E}(B)_\mathbb{S}$ for any $\sigma \Vdash \Gamma, \Gamma', \Delta$. We consider the restrictions $\sigma' \Vdash \Gamma$ and $\sigma'' \Vdash \Gamma', \Delta$ of $\sigma$. By induction, one has $V[\sigma'] \in \mathbb{T}(A)_\mathbb{V}$ and $S[\sigma''] \in \mathbb{E}(B)_\mathbb{S}$, hence the result. *Same reasoning for the rules ($\vdash_f \otimes$), ($\&_i \vdash_f$), and ($\vdash_f \oplus_i$).*

*Rule ($\vdash \&$):* the goal is to prove $\mu\langle\alpha.c;\beta.c'\rangle \in \mathbb{T}(A \& B)$ for $\sigma \Vdash \Gamma$ with $\alpha, \beta \notin \operatorname{dom}\Gamma$. By definition it is enough to prove $\langle \mu\langle\alpha.c;\beta.c'\rangle[\sigma] \,\|\, \pi_i \cdot S_i\rangle^\ominus \in \perp\!\!\!\perp$ for both $S_1 \in \mathbb{E}(A)_\mathbb{S}$ and $S_2 \in \mathbb{E}(B)_\mathbb{S}$. By Proposition 47, this follows from $\mu\langle\alpha.c;\beta.c'\rangle[\sigma] \in \mathbb{T}$, $c[\sigma, S_1/\alpha] \in \perp\!\!\!\perp$ and $c'[\sigma, S_2/\beta] \in \perp\!\!\!\perp$. The latter two follow from induction hypothesis. The former follows from Lemma 46 and $c[\sigma], c'[\sigma] \in \perp\!\!\!\perp$ obtained by the induction hypothesis by substituting $\alpha$ and $\beta$ by themselves by Lemma 52. *Same reasoning for the rules ($\vdash \multimap$), ($\otimes \vdash$), ($\oplus \vdash$).*

*Rule ($\vdash_f \top$):* one has to show $\mu\langle V\rangle[\sigma] \in \emptyset^{\perp_\ominus} = \mathbb{V}$. By Lemma 46, this follows from $V[\sigma] \in \mathbb{V}$ which follows by the induction hypothesis. *Same reasoning for the rule ($\mathbf{0} \vdash_f$).*

[1] From the French “Lemme d’adéquation”, which does not refer to what is usually called adequacy in logical relations, but rather the main theorem.

*Rule* ($\vdash$ !)*:* we show $\mu!\alpha.c[\sigma] \in \|!A\| = (!\mathbb{E}(A)_{\mathbb{S}})^{\perp_+}{}_{\mathbb{V}}$. We show for any $S \in \mathbb{E}(A)_{\mathbb{S}}$, $\langle \mu!\alpha.c[\sigma] \parallel !S\rangle^+ \in \perp\!\!\!\perp$. By Proposition 47 this follows from $c[\sigma, S/\alpha] \in \perp\!\!\!\perp$ which follows from induction hypothesis.

*Rule* (! $\vdash_f$)*:* the goal is to show $!S[\sigma] \in (!\mathbb{E}(A)_{\mathbb{S}})^{\perp_+}{}_{\mathbb{V}}{}^{\perp_+}$. One has $!S[\sigma] \in !\mathbb{E}(A)_{\mathbb{S}}$ by induction hypothesis, and $!\mathbb{E}(A)_{\mathbb{S}} \subseteq (!\mathbb{E}(A)_{\mathbb{S}})^{\perp_+}{}_{\mathbb{V}}{}^{\perp_+}$ follows from $(!\mathbb{E}(A)_{\mathbb{S}})^{\perp_+}{}_{\mathbb{V}} \subseteq !\mathbb{E}(A)_{\mathbb{S}}{}^{\perp_+}$. ∎

**Theorem 55** (Strong normalization)**.** *Any typable term is strongly normalizing.*

*Proof.* For $c : (\Gamma \vdash \Delta)$, $\Gamma \vdash t : A$ and $\Gamma \mid e : A \vdash \Delta$, we show $c \in \perp\!\!\!\perp$, $t \in \mathbb{T}$ and $e \in \mathbb{E}$. This follows from Lemma 54 applied with the identity substitution, which indeed satisfies $\mathrm{id} \Vdash \Gamma, \Delta$ by Lemma 52. ∎

## 6 Main results

Unless stated otherwise, statements and proofs in this section are for both $\mathbf{ILL}_p^\eta$ and $\mathbf{LJ}_p^\eta$ $(+\,\square)$. They all crucially rely on strong normalisation.

### 6.1 Coherence

**Theorem 56** (Coherence)**.** *For any term and any typing judgement of this term, all the derivations of this judgement are equivalent.*

*Proof.* When the term is $\to_{\mathrm{R}}$-normal, Proposition 29 strengthened with Proposition 32 build a canonical derivation to which any other derivation is equivalent. This proves the result for $\to_{\mathrm{R}}$-normal terms. By Theorem 55 and Lemma 42, any two derivations of a judgement are equivalent to derivations of the $\to_{\mathrm{R}}$-normal form of the term. ∎

One consequence is for the expression of derived rules such as in Figure 3 and 4: it is not important to know how the derivations are initially given, only the definitions of the terms matter.

### 6.2 Soundness

**Theorem 57** (Soundness)**.** *The* $\simeq_{\mathrm{RE}}{}^{\vdash}$ *preserve the interpretation.*

*Proof.* By Lemma 42, the $\to_{\mathrm{RE}}{}^{\vdash}$ preserve the interpretation. It follows from Theorem 56 that the $\leftarrow_{\mathrm{RE}}{}^{\vdash}$ preserve the interpretation as well, by unicity of the interpretation of the left-hand side. Since $\simeq_{\mathrm{RE}}{}^{\vdash} = (\leftarrow_{\mathrm{RE}}{}^{\vdash} \cup \to_{\mathrm{RE}}{}^{\vdash})^+ \cup =^{\vdash}$, the $\simeq_{\mathrm{RE}}{}^{\vdash}$ preserve the interpretation. ∎

### 6.3 Curry-style equivalence

We now prove with Theorem 61 that typed conversions $\simeq_{\mathrm{RE}}{}^{\vdash}$ are the step-by-step-typed restrictions of conversion $\simeq_{\mathrm{RE}}$ (that is, the equivalence closure of the typed restrictions of $\to_{\mathrm{RE}}$), despite Remark 17.

**Lemma 58.** *If $c$ is $\to_{\mathrm{R}}$-normal, if $c \to_{\mathbf{E0}\top} c'$, and if $c, c' : (\Gamma \vdash \Delta)$, then $c \to_{\mathbf{E0}\top} c' : (\Gamma \vdash \Delta)$. (And likewise for expressions and contexts.)*

*Proof.* By a straightforward induction the definition of $f \to_{\mathrm{E0T}} g$ using Proposition 29 strengthened over $\to_{\mathrm{R}}$-normal terms by Proposition 32. ∎

**Lemma 59.** *If $g \leftarrow_{\mathrm{R}} f \to_{\mathrm{E0T}} h$ then there exists $i$ such that $g \to^{*}_{\mathrm{E0T}} i \leftarrow_{\mathrm{R}} h$.*

*Proof.* We first notice that if $c' \lhd_{\mathrm{R}} c \to_{\mathrm{E0T}} c''$ then there exists $c'''$ such that $c' \to^{*}_{\mathrm{E0T}} c''' \lhd_{\mathrm{R}} c''$ due to non-overlapping rewriting rules. The result follows by induction on the definition of $f \to_{\mathrm{R}} g$. ∎

**Lemma 60.** *Let $c, c' : (\Gamma \vdash \Delta)$ such that $c \to_{\mathrm{E0T}} c'$. One has $c \simeq_{\mathrm{RE}} c' : (\Gamma \vdash \Delta)$. (And similarly for expressions and contexts.)*

*Proof.* We consider the general case $f \to_{\mathrm{E0T}} g$. By Theorem 55 there exists a $\to_{\mathrm{R}}$-normal term $h$ such that $f \to^{*}_{\mathrm{R}} h$. There exists a term $i$ such that $h \to^{*}_{\mathrm{E0T}} i \leftarrow^{*}_{\mathrm{R}} g$ by Lemma 59 and induction on the definition of $\to^{*}_{\mathrm{R}}$. Therefore one has:

$$f \to^{*}_{\mathrm{R}} h \simeq_{\mathrm{E0T}} i \leftarrow^{*}_{\mathrm{R}} g : (\Gamma \vdash \Delta)$$

by Theorem 40 and Lemma 58. ∎

**Theorem 61.** *Typed equivalence $\simeq_{\mathrm{RE}}{}^{\vdash}$ is the equivalence closure of the typed restrictions of $\to_{\mathrm{RE}}$.*

*Proof.* Typed equivalence is contained in the equivalence closure of the typed restrictions of $\to_{\mathrm{RE}}$ by Lemma 16. For the converse inclusion, it is enough to show that the typed restriction of $\to_{\mathrm{RE}}$ is included in typed equivalence. This follows from Theorem 40 or Lemma 60 depending on whether $\to_{\mathrm{RE}}$ comes from $\to_{\mathrm{RE'}}$ or $\to_{\mathrm{E0T}}$. ∎

### 6.4 Focusing

Proposition 33 admits a slight generalisation in the form of a proof-search strategy known as *focusing* [And92, Lau04, LM09]. We now have all the ingredients to describe focusing and prove its completeness, strengthened with respect to the interpretation: for any proof there will be an *equivalent* focused proof. In this section,

- sequents are measured by the number of connectives and units,
- terms are measured by the number of subterms of the form $\langle V \,\|\, \alpha\rangle^{+}$, $\langle x \,\|\, S\rangle^{\ominus}$, or $!S$ in their $\to_{\mathrm{R}}$-normal form if it exists (in which case it is unique by Theorem 21), otherwise it is infinite by convention.

**Definition 62.** We define an *inversion* relation $\succ$ between sequents and multisets of sequents:

$$\begin{aligned}
(\Gamma \vdash A \multimap B) &\succ \{(\Gamma, A \vdash B)\}\\
(\Gamma, A \otimes B, \Gamma' \vdash \Delta) &\succ \{(\Gamma, A, B, \Gamma' \vdash \Delta)\}\\
(\Gamma, \mathbf{1}, \Gamma' \vdash \Delta) &\succ \{(\Gamma, \Gamma' \vdash \Delta)\}\\
(\Gamma \vdash A \mathbin{\&} B) &\succ \{(\Gamma \vdash A), (\Gamma \vdash B)\}\\
(\Gamma, A \oplus B, \Gamma' \vdash \Delta) &\succ \{(\Gamma, A, \Gamma' \vdash \Delta), (\Gamma, B, \Gamma' \vdash \Delta)\}\\
(\Gamma \vdash \top) &\succ \emptyset\\
(\Gamma, \mathbf{0}, \Gamma' \vdash \Delta) &\succ \emptyset
\end{aligned}$$

A sequent that is normal for $\succ$ is called *inverted*.

In other words, a sequent $\Gamma \vdash \Delta$ is inverted if:

- $\Gamma$ only contains formulae that are either negative or of the form $!A$ or $X^{+}$; and
- $\Delta$ contains either a positive formula, or a formula of the form $X^{\ominus}$.

**Lemma 63.** *The extension of $\succ$ to a relation between multisets of sequents is terminating and confluent.*

*Proof.* The relation is strictly decreasing for the induced multiset order, therefore it is terminating. It is confluent by Newman's lemma because it is locally confluent. ∎

**Lemma 64.** *Let $f$ be any term. Then $f[x{\otimes}y/z]$, $f[\iota_i(x)/y]$, $f[()/x]$, $f[x{\cdot}\alpha/\beta]$, and $f[\pi_i{\cdot}\alpha/\beta]$ are smaller than $f$.*

*Proof.* For any term $f$ let us write $|f|$ the number of subterms of the form $\langle V \parallel \alpha\rangle^{+}$, $\langle x \parallel S\rangle^{\ominus}$, or $!S$ and $\lfloor f \rfloor$ the $\to_{\mathrm{R}}$-normal form of $f$ if it exists. In particular, if $f$ is normalisable then its size is given by $\left|\lfloor f \rfloor\right|$. We now consider $f[p/\kappa]$ as in the statement. One first observes $\left|f[p/\kappa]\right| \leq |f|$ since substitution can only decrease the number of commands of the form $\langle V \parallel \alpha\rangle^{+}$ or $\langle x \parallel S\rangle^{\ominus}$. We first prove the result for $f$ $\to_{\mathrm{R}}$-normal, that is $\left|\lfloor f[p/\kappa] \rfloor\right| \leq |f|$. In this case, redexes in $f[p/\kappa]$ are the ones created by the substitution. Since they all bind (co-)variables to (co-)variables, their parallel reduction decreases $|\cdot|$ and does not create new redexes. (The notion of contracting in parallel a set of redexes is standard for left-linear higher-order rewriting systems and is included in $\to_{\mathrm{R}}^{*}$, see e.g. [vR99].) Hence, the parallel reduct is normal and therefore equal to $\lfloor f[p/\kappa] \rfloor$ by uniqueness of the normal form; moreover it satisfies $\left|\lfloor f[p/\kappa] \rfloor\right| \leq \left|f[p/\kappa]\right| \leq |f|$. Now, in the case of $f$ $\to_{\mathrm{R}}$-normalisable, we have just proved $\left|\lfloor \lfloor f \rfloor [p/\kappa] \rfloor\right| \leq \left|\lfloor f \rfloor\right|$. Then from $f \to_{\mathrm{R}}^{*} \lfloor f \rfloor$ one deduces $f[p/\kappa] \to_{\mathrm{R}}^{*} \lfloor f \rfloor[p/\kappa] \to_{\mathrm{R}}^{*} \lfloor \lfloor f \rfloor [p/\kappa] \rfloor$. Therefore $\lfloor \lfloor f \rfloor [p/\kappa] \rfloor = \lfloor f[p/\kappa] \rfloor$ by uniqueness of the normal form, and one concludes $\left|\lfloor f[p/\kappa] \rfloor\right| \leq \left|\lfloor f \rfloor\right|$. Lastly, if $f$ is not normalisable then the result is obvious. ∎

$$\dfrac{c : (\Gamma, x : A \vdash \alpha : B)}{\langle \mu(x{\cdot}\alpha).c \parallel \beta \rangle : (\Gamma \vdash \beta : A \rightarrow B)} \qquad \dfrac{c : (\Gamma \vdash \alpha : A) \quad c' : (\Gamma \vdash \beta : B)}{\langle \mu{<}\alpha.c ; \beta.c'{>} \parallel \gamma \rangle : (\Gamma \vdash \gamma : A \mathbin{\&} B)}$$

$$\dfrac{c : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \tilde{\mu}(x{\otimes}y).c \rangle : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta)} \qquad \dfrac{c : (\Gamma, x : A, \Gamma' \vdash \Delta) \quad c' : (\Gamma, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \tilde{\mu}[x.c | y.c'] \rangle : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta)}$$

$$\dfrac{c : (\Gamma, \Gamma' \vdash \Delta)}{\langle x \parallel \tilde{\mu}().c \rangle : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta)} \qquad \dfrac{}{\langle x \parallel \tilde{\mu}[]_{\Gamma, \Gamma', \Delta} \rangle : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta)}$$

$$\dfrac{}{\langle \mu{<>}_{\Gamma} \parallel \alpha \rangle : (\Gamma \vdash \alpha : \top)}$$

Figure 6: Inversion

$$\dfrac{}{!\Gamma, x : X^{+} \vdash x : X^{+} \mid} \qquad \dfrac{}{!\Gamma, x : !A, !\Gamma' \vdash x : !A \mid} \qquad \dfrac{}{!\Gamma \mid \alpha : X^{\ominus} \vdash \alpha : X^{\ominus}}$$

$$\dfrac{!\Gamma', \Gamma_{\restriction \mathbf{fv}\, V} \vdash V : A \mid \quad !\Gamma', \Gamma_{\restriction \mathbf{fv}\, W} \vdash W : B \mid}{!\Gamma', \Gamma \vdash V{\otimes}W : A \otimes B \mid} \qquad \dfrac{!\Gamma', \Gamma_{\restriction \mathbf{fv}\, V} \vdash V : A \mid \quad !\Gamma', \Gamma_{\restriction \mathbf{fv}\, S} \mid S : A \vdash \Delta}{!\Gamma', \Gamma \mid V{\cdot}S : A \rightarrow B \vdash \Delta}$$

$$\dfrac{}{!\Gamma \vdash () : \mathbf{1} \mid} \qquad \dfrac{c : (!\Gamma \vdash \alpha : A)}{!\Gamma \vdash \mu!\alpha.c : !A \mid} \qquad \dfrac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid} \qquad \dfrac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i{\cdot}S : A_1 \mathbin{\&} A_2 \vdash \Delta}$$

$$\dfrac{c : (\Gamma \vdash \alpha : N)}{\Gamma \vdash \mu\alpha^{\ominus}.c : N \mid} \qquad \dfrac{c : (\Gamma, x : P \vdash \Delta)}{\Gamma \mid \tilde{\mu}x^{+}.c : P \vdash \Delta}$$

Figure 7: Focusing in $\mathbf{ILL}_p^{\eta}$

$$\dfrac{}{\Gamma, x : X^{+}, \Gamma' \vdash x : X^{+} \mid} \qquad \dfrac{}{\Gamma, x : !A, \Gamma' \vdash x : !A \mid} \qquad \dfrac{}{\Gamma \mid \alpha : X^{\ominus} \vdash \alpha : X^{\ominus}}$$

$$\dfrac{\Gamma \vdash V : A \mid \quad \Gamma \vdash W : B \mid}{\Gamma \vdash V{\otimes}W : A \otimes B \mid} \qquad \dfrac{\Gamma \vdash V : A \mid \quad \Gamma \mid S : A \vdash \Delta}{\Gamma \mid V{\cdot}S : A \rightarrow B \vdash \Delta}$$

$$\dfrac{}{\Gamma \vdash () : \mathbf{1} \mid} \qquad \dfrac{c : (!\Gamma' \vdash \alpha : A)}{\Gamma \vdash \mu!\alpha.c : !A \mid}\,(!\Gamma' \subseteq \Gamma) \qquad \dfrac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid} \qquad \dfrac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i{\cdot}S : A_1 \mathbin{\&} A_2 \vdash \Delta}$$

$$\dfrac{c : (\Gamma \vdash \alpha : N)}{\Gamma \vdash \mu\alpha^{\ominus}.c : N \mid} \qquad \dfrac{c : (\Gamma, x : P, \Gamma' \vdash \Delta)}{\Gamma, \Gamma' \mid \tilde{\mu}x^{+}.c : P \vdash \Delta}$$

Figure 8: Focusing in $\mathbf{LJ}_p^{\eta}$ $(+\,\square)$

**Lemma 65.** *For any $c : \Psi$, and for any $\{\Psi_1, \dots, \Psi_n\} \prec \Psi$, there exists an equivalent command $c' : \Psi$ derived from smaller commands of type $\Psi_1, \dots, \Psi_n$ by application of a rule in Fig. 6.*

*Proof.* The result follows from Lemma 42 by mapping each inversion as follows:

$$\begin{aligned}
c : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta) &\succ \{\langle x \otimes y \,\|\, \tilde{\mu} z.c \rangle : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)\} \\
c : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta) &\succ \{\langle \iota_1(x) \,\|\, \tilde{\mu} z.c \rangle : (\Gamma, x : A, \Gamma' \vdash \Delta), \langle \iota_2(y) \,\|\, \tilde{\mu} z.c \rangle : (\Gamma, y : B, \Gamma' \vdash \Delta)\} \\
c : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta) &\succ \{\langle () \,\|\, \tilde{\mu} x.c \rangle : (\Gamma, \Gamma' \vdash \Delta)\} \\
c : (\Gamma \vdash \beta : A \multimap B) &\succ \{\langle \mu \beta.c \,\|\, x \cdot \alpha \rangle : (\Gamma, x : A \vdash \alpha : B)\} \\
c : (\Gamma \vdash \gamma : A \,\&\, B) &\succ \{\langle \mu \gamma.c \,\|\, \pi_1 \cdot \alpha \rangle : (\Gamma \vdash \alpha : A), \langle \mu \gamma.c \,\|\, \pi_2 \cdot \beta \rangle : (\Gamma \vdash \beta : B)\} \\
c : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta) &\succ \emptyset \\
c : (\Gamma \vdash \alpha : \top) &\succ \emptyset
\end{aligned}$$

The commands on the right-hand side are smaller by application of Lemma 64. ∎

**Proposition 66** (Inversion)**.** *For any sequent $\Psi$, there exist inverted sequents $\Psi_1, \dots, \Psi_n$ such that any command $c : \Psi$ can up to equivalence be derived from smaller commands $c_1 : \Psi_1, \dots, c_n : \Psi_n$ by applying the rules in Fig. 6 in any order.*

*Proof.* By Lemmas 63 and 65. ∎

**Proposition 67** (Focusing)**.** *For any typable command $c : (\Gamma \vdash \Delta)$ where $\Gamma \vdash \Delta$ is inverted, there exist equivalent command and derivation of either of the following forms:*

$$\frac{!\Gamma', \Gamma'' \vdash V : \Delta(\alpha)_+ \mid}{\langle V \,\|\, \alpha \rangle^+ : (\Gamma \vdash \Delta)} \qquad \frac{!\Gamma', \Gamma'' \mid S : \Gamma(x)_\ominus \vdash \Delta}{\langle x \,\|\, S \rangle^\ominus : (\Gamma \vdash \Delta)} \qquad \frac{!\Gamma', \Gamma'' \mid S : \Gamma'(x) \vdash \Delta}{\langle x \,\|\, !S \rangle^+ : (\Gamma \vdash \Delta)}$$

*where $!\Gamma'$ is the restriction of $\Gamma$ to formulae of the form $!A$, and $\Gamma''$ is obtained from $\Gamma$ by removing $!\Gamma'$ and $x$. Moreover, $S$ and $V$ derive from zero or more commands strictly smaller than $c$ by applications of rules in Fig. 7.*

For $\mathbf{LJ}^\eta_p$ one replaces $!\Gamma', \Gamma''$ with $\Gamma$ and Fig. 7 with Fig. 8.

*Proof.* Up to equivalence, one can assume $c$ $\to_{\mathrm{R}}$-normal: indeed, the normal form exists by Theorem 55, its derivation is equivalent by Lemma 42, and it has the same size by definition. Then one has either $c = \langle x \,\|\, S^{\Gamma(x)} \rangle$ or $c = \langle V^{\Delta(\alpha)} \,\|\, \alpha \rangle$ as observed in Proposition 32, with $S$ and $V$ $\to_{\mathrm{R}}$-normal. We sort them into one of three cases: either $\langle V' \,\|\, \alpha \rangle^+$, $\langle x \,\|\, S' \rangle^\ominus$, or $\langle x \,\|\, !S' \rangle^+$, derived as in the above statement. If $\Gamma(x)$ is negative or $\Delta(\alpha)$ is positive, we are in one of the first two cases. Since $\Gamma \vdash \Delta$ is inverted, only three cases remain: $\Gamma(x) = X^+$, $\Gamma(x) = !A$, and $\Delta(\alpha) = X^\ominus$. By Proposition 29, one respectively has $S = \alpha$ (first case), $S = !S'$ (third case), and $V = x$ (second case). Notice that $V'$ and $S'$ are strictly smaller than $c$. Then the result follows by an induction on $S'$ and $V'$ typable in an inverted sequent, by Proposition 29, and applying weakening on the hypotheses. The base cases are

$$\Gamma \mid \alpha : N \vdash \Delta \qquad \Gamma \vdash x : P \mid \qquad !\Gamma \vdash \mu !\alpha.c : !A \mid \qquad \Gamma \mid S_0 : P \vdash \Delta \qquad \Gamma \vdash V_0 : N \mid$$

In the first two cases, one has indeed $P = \Gamma(x)$ of the form $X^+$ or $!A$ and $N = \Delta(\alpha)$ of the form $X^\ominus$ by inversion. In the last three cases, $\mu!\alpha.c$, $V_0$, and $S_0$ are strictly smaller than $c$ by being subterms of $V'$ or $S'$. Then in the last two cases we apply a $\tilde{\mu}^+$- or a $\mu^\ominus$-expansion to obtain equivalent terms $\tilde{\mu}x^+.c'$ and $\mu\alpha^\ominus.c'$. The expansion preserves the size, so that $c'$ is strictly smaller than $c$. ∎

Thus, for any proof, there is an equivalent proof obtained by alternating one inversion phase with one or more[2] focusing phases.

### 6.5 The case of expressible systems

**Corollary 68.** *The interpretations of* ***ILL*** *and* ***LJ*** $+ \square$ *induced by Figure 3, and of the three variants of the Curry-style* $\lambda$*-calculus with sums and empty type induced by Figure 4, into LCBPV and CBPV models, enjoy coherence, soundness and completeness of focusing.*

*Proof.* Soundness and coherence are immediate consequences of the previous results. As for completeness of focusing, it is immediate for **ILL** and **LJ** $+ \square$ by replacing nodes $(\mathbf{0} \vdash_f)$ and $(\vdash_f \top)$ with $(\mathbf{0} \vdash)$ and $(\vdash \top)$. For the $\lambda$-calculi, the normal sequent derivation is transformed into a derivation in natural deduction in the standard way. ∎

## 7 Comparison with other approaches to focusing

The above proof of focusing and inversion follows the conceptual outline given for sequent calculus by Laurent [Lau04]. All the ingredients necessary to the current concise exposition—relationship with categorical semantics left aside—have been around for some time [MM09, CMM10]. As far as proof search is concerned, the result for $\mathbf{LJ}^\eta_p$ and $\mathbf{ILL}^\eta_p$ is equivalent to the standard focusing result (e.g. Liang and Miller [LM07, LM09]) as one could expect.

In the traditional termless approach to focusing inspired by Andreoli [And92], the proof system is reduced to an algorithmic description of canonical forms, itself treated as a proof system. Our approach lets us express instead inversion and focusing as properties of terms in a non-restricted type system where the rules of familiar proof systems are expressible. It is inspired by subsequent works (Girard's [Gir91], Danos, Joinet and Schellinx's [DJS97]) which from the early days suggested that polarisation and focalisation describes more than canonical normal forms. In Munch-Maccagnoni and Scherer [MMS15], we have sketched without proof how $\mathbf{LJ}^\eta_p$ could be used to recover the focused system of Liang and Miller [LM07] by characterising, as per the folklore, the cut-free, $\eta$-long proofs; however it was again stated as a reconstructing the restricted system rather than capturing its essence into properties.

Our approach amounts to separate the proof transformations involved in the completeness of focusing into three familiar categories:

- When the two derivations have the same underlying term, for instance for the permutations involving structural rules and for simplifications of structural rules with themselves. In this case, the terms provide a canonical representative, as coherence establishes (Theorem 56), without restrictions on the proof system being necessary.

[2] in the case where a promotion rule comes from a hypothesis that is already inverted.

- Reductions ($\to_{\mathrm{R}}$), which involve for instance principal cuts, commutations of cuts with main formulae, and the interaction structure / main formulae. It is substantiated by strong normalisation (Theorem 55) and focusing (Proposition 67).

- Expansions ($\to_{\mathrm{E}}$), which involve expansion of the axiom rules but also express commutations with invertible rules. Naturally, expansions are involved in Proposition 66 (inversion).

Among the latter two, the rules $(R\mu)$ and $(R\tilde{\mu})$ play an essential role in leveraging the other conversions, and also in expressing compound constructs as in Figures 3 and 4. They are therefore essential to the conclusion of Corollary 68 which transports the results to conventional systems straightforwardly. The rules $(E\mu)$ and $(E\tilde{\mu})$, which find a strong justification in confluence (Theorem 21), mean in some sense that the constructs $\mu\alpha$ and $\tilde{\mu}x$ come for free.

Other proof transformations built into representations such as proof nets [Gir87], like ones expressing the idempotency of the enriched adjunction [MM14, CFMM16]:

$$\left\langle \mu\alpha.\langle t \,\|\, \tilde{\mu}x.c\rangle \,\middle\|\, e \right\rangle = \left\langle t \,\middle\|\, \tilde{\mu}x.\langle \mu\alpha.c \,\|\, e\rangle \right\rangle$$

or more generally the commutativity of the strong monad:

$$\left\langle t \,\middle\|\, \tilde{\mu}x.\langle u \,\|\, \tilde{\mu}y.c\rangle \right\rangle = \left\langle u \,\middle\|\, \tilde{\mu}y.\langle t \,\|\, \tilde{\mu}x.c\rangle \right\rangle$$

are irrelevant, and are actually detrimental to the generality of the result given that important models invalidate these equations. However, as explained by Scherer et al. [Sch15, MMS15] they play a role in specializations called multi-focusing [CMS08, CHM14], in a way that still has to be fully exploited in our approach.

It appears that the approach to focalisation as a reduction strategy on a calculus presented in [MM09] was misunderstood. According to Brock-Nannestad and Guenot [BNG15]:

> *this system is not focused in general and cuts, performing the selection of a formula to focus on, cannot all be eliminated.*

Simmons [Sim14] demonstrated how working with proof terms can simplify proofs of focalisation:

> *Existing focalization proofs almost all fall prey to the need to prove multiple tedious invertibility lemmas describing the interaction of each rule with every other rule; this results in proofs that are unrealistic to write out, difficult to check, and exhausting to contemplate mechanizing. [...] the approaches we call "tedious" tend to scale quadratically.*

From Simmons' criteria of scaling, the present development falls into the "pleasant" category. The development grows linearly in the number of connectives (for instance starting from $\mathbf{IMLL}_p^\eta/\mathbf{MLJ}_p^\eta$ and adding additives and exponentials). Here, the $\mu\tilde{\mu}$ approach, which quotients over structural rules, solves the problem with commutative cuts, and permits a simple proof of strong normalization, is crucial. A different measure of success than the modularity of proofs is the integration into a broader theory. For focusing alone, this allows us to state and establish clearly and in full generality aspects that are sometimes omitted for simplicity, such as the freedom in choosing the order of inversion [Sim14], or only stated recently such as preservation of the interpretation in any model [BDS16]. The latter states the preservation of the denotation with an appropriate sketch of proof; in our approach we see that it follows naturally and linearly by adapting each lemma into a denotation-preserving version, and also that the range of models can be relaxed from intuitionistic and linear logics to CBPV and Linear CBPV models.

In contrast, Simmons [Sim14], Brock-Nannestad and Guenot [BNG15], and also Brock-Nannestad, Guenot and Gustafsson [BNGG15] proposed calculi whose typing rules reflect restrictions of traditional focused proof systems. As a consequence, instead of having all of the necessary proof transformations described internally, and instead of having conventional proof systems and $\lambda$-calculi directly expressible, they still have to introduce and explain various codings between systems. In this aspect their approaches to focusing are intermediate between the earlier *"tedious"* proofs and ours. And, to us, the benefits of following so closely the shape of traditional focused proof systems, for the calculi that are put forward as *"computational"* accounts of polarities and focalisation [BNGG15, BNG15], have yet to be demonstrated and compared to the earlier approach. In [Sim14], a main interest and contribution is the mechanisation of the statements and proofs. We believe that our technique can be mechanised with few gaps to fill, but it remains to be seen whether this can be accomplished conveniently without the support of "libraries" for higher-order rewriting and for polarised logical relations.

The importance of $\mu\alpha$ for expressiveness has sometimes been misunderstood as well. Accattoli and Guerrieri [AG16] compare four different reduction theories of higher-order computation in call-by-value, among which the positive and $\to_{\mathrm{R}\mu^+}$-normal implicative fragment of $\mathbf{LJ}_p^\eta$ (with weak reduction) which they call $\lambda_{\mathrm{vseq}}$, noting, compared to the classical version $\bar{\lambda}\tilde{\mu}$ [CH00]:

> *$\lambda_{vseq}$ does not need the syntactic category of co-variables $\alpha$, as there can be only one of them*

They proceed with defining by hand a translation of call-by-value $\lambda$-calculus into $\lambda_{\mathrm{vseq}}$ that coincides with the positive encoding followed by $\to_{\mathrm{R}\mu^+}$-normalisation. According to them:

> *The advantage of $\lambda_{vseq}$ is that it avoids both rules at a distance and shuffling rules. The drawback of $\lambda_{vseq}$ is that, syntactically, it requires to step out of the $\lambda$-calculus. We will show in Sect. 4 how to reformulate it as a fragment of $\lambda_{vsub}$, i.e. in natural deduction. However, it will still be necessary to restrict the application constructor, thus preventing the natural way of writing terms.*

But the impossibility of expressing the application constructor comes from the authors' choice to restrict to $\to_{\mathrm{R}\mu^+}$-normal forms.

It is now clear that relying on term syntaxes is an alternative and perhaps more general approach to focusing. On the other hand, the advantage of focusing on a simpler aspect of a result is that, as it often happens, the same standard tools and techniques have been given increasingly varied domains of application. The perspective of a unified theory of focalisation which is general both in scope and in results brings exciting research directions.

## References


[AG16] Beniamino Accattoli and Giulio Guerrieri, *Open call-by-value*, Programming Languages and Systems (2016). 48

[AMdPR01] Natasha Alechina, Michael Mendler, Valeria de Paiva, and Eike Ritter, *Categorical and Kripke Semantics for Constructive S4 modal logic*, Proc. CSL (Laurent Fribourg, ed.), Lecture Notes in Computer Science, vol. 2142, Springer, 2001, pp. 292–307. 3

[And92] Jean-Marc Andreoli, *Logic Programming with Focusing Proof in Linear Logic*, Journal of Logic and Computation **2** (1992), no. 3, 297–347. 42, 46

[Atk06] Robert Atkey, *Substructural simple type theories for separation and in-place update*. 8

[Bar93] Hendrik Pieter Barendregt, *Handbook of Logic in Computer Science*, vol. 2, ch. Lambda Calculi with Types, Oxford University Press, 1993. 3

[BDS16] David Baelde, Amina Doumane, and Alexis Saurin, *Infinitary proof theory: the multiplicative additive case*, Proc. CSL (2016). 48

[Bie95] Gavin Bierman, *What is a categorical model of Intuitionistic Linear Logic?*, Proc. TLCA, Lecture Notes in Computer Science, vol. 902, Springer-Verlag, 1995, pp. 78–93. 10

[BNG15] Taus Brock-Nannestad and Nicolas Guenot, *Focused linear logic and the $\lambda$-calculus*, Mathematical Foundations of Programming Semantics XXXI (MFPS), vol. 319, Elsevier, 2015, pp. 103–119. 47, 48

[BNGG15] Taus Brock-Nannestad, Nicolas Guenot, and Daniel Gustafsson, *Computation in focused intuitionistic logic*, Proc. PPDP, ACM, 2015, pp. 43–54. 48

[CFMM16] Pierre-Louis Curien, Marcelo Fiore, and Guillaume Munch-Maccagnoni, *A Theory of Effects and Resources: Adjunction Models and Polarised Calculi*, Proc. POPL, 2016. 1, 3, 6, 8, 9, 11, 47

[CH00] Pierre-Louis Curien and Hugo Herbelin, *The duality of computation*, ACM SIGPLAN Notices **35** (2000), 233–243. 48

[CHM14] Kaustuv Chaudhuri, Stefan Hetzl, and Dale Miller, *A multi-focused proof system isomorphic to expansion proofs*, Journal of Logic and Computation (2014). 47

[CMM10] Pierre-Louis Curien and Guillaume Munch-Maccagnoni, *The duality of computation under focus*, Proc. IFIP TCS, 2010, Extended version. 46

[CMS08] Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin, *Canonical sequent proofs via multi-focusing*, Fifth Ifip International Conference On Theoretical Computer Science–Tcs 2008, Springer, 2008, pp. 383–396. 47

[Day70] Brian Day, *On closed categories of functors*, Lecture Notes in Mathematics (1970), no. 137, 1–38. 9

[DJS97] Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx, *A New Deconstructive Logic: Linear Logic*, Journal of Symbolic Logic **62 (3)** (1997), 755–807. 3, 6, 46

[DL07] Brian Day and Stephen Lack, *Limits of small functors*, Journal of Pure and Applied Algebra (2007), no. 210, 651–663. 9

[EMS12] Jeff Egger, Rasmus Ejlers Møgelberg, and Alex Simpson, *The enriched effect calculus: syntax and semantics*, Journal of Logic and Computation **24** (2012), no. 3, 615–654. 3

[Fel91] Matthias Felleisen, *On the expressive power of programming languages*, Science of computer programming **17** (1991), no. 1, 35–75. 8

[Gir72] Jean-Yves Girard, *Interprétation fonctionnelle et élimination des coupures dans l'arithmétique d'ordre supérieur*, Ph.D. thesis, Université Paris VII, 1972. 37

[Gir87] ______, *Linear Logic*, Theoretical Computer Science **50** (1987), 1–102. 37, 47

[Gir91] ______, *A new constructive logic: Classical logic*, Math. Struct. Comp. Sci. **1** (1991), no. 3, 255–296. 46

[Gir07] ______, *Le Point Aveugle, Cours de logique, Tome II: Vers l'imperfection*, Visions des Sciences, Hermann, 2007, published subsequently in English [Gir11]. 7

[Gir11] ______, *The Blind Spot: Lectures on Logic*, European Mathematical Society, 2011. 50

[Has16] Masahito Hasegawa, *Linear exponential comonads without symmetry*, Fourth International Workshop on Linearity, vol. abs/1701.04919, 2016. 10

[Kri93] Jean-Louis Krivine, *Lambda-calculus, types and models*, Ellis Horwood, 1993. 37

[Kri09] ______, *Realizability in Classical Logic*, Panoramas et Synthèses **27** (2009), 197–229, Circulated since 2004. 37

[Lau04] Olivier Laurent, *A proof of the focalization property of linear logic*, lecture notes, 2004. 42, 46

[Lev05] Paul Blain Levy, *Adjunction models for call-by-push-value with stacks*, Theory and Application of Categories **14** (2005), no. 5, 75–110. 3

[LM07] Chuck Liang and Dale Miller, *Focusing and Polarization in Intuitionistic Logic*, CSL, 2007, pp. 451–465. 46

[LM09] ______, *Focusing and polarization in linear, intuitionistic, and classical logics*, Theor. Comput. Sci. **410** (2009), no. 46, 4747–4768. 42, 46

[Man04] Paola Maneggia, *Models of linear polymorphism*, 2004. 10

[Mel09] Paul-André Melliès, *Categorical semantics of linear logic*, Panoramas et Synthèses, vol. 27, ch. 1, pp. 15–215, Société Mathématique de France, 2009. 10

[MM09] Guillaume Munch-Maccagnoni, *Focalisation and Classical Realisability*, Proc. CSL, LNCS, Springer-Verlag, 2009. 3, 37, 46, 47

[MM11] ______, *λ-calcul, machines et orthogonalité*, Unpublished manuscript, October 2011. 37

[MM14] ______, *Models of a Non-Associative Composition*, Proc. FoSSaCS (A. Muscholl, ed.), LNCS, vol. 8412, Springer, 2014, pp. 397–412. 47

[MMS15] Guillaume Munch-Maccagnoni and Gabriel Scherer, *Polarised Intermediate Representation of Lambda Calculus with Sums*, Proc. LICS 2015, 2015. 1, 2, 3, 46, 47

[Sch04] Andrea Schalk, *Whats is a categorical model of linear logic*, 2004, Lecture notes. 10

[Sch15] Gabriel Scherer, *Multi-focusing on extensional rewriting with sums*, 13th International Conference on Typed Lambda Calculi and Applications, TLCA 2015, July 1-3, 2015, Warsaw, Poland (Thorsten Altenkirch, ed.), LIPIcs, vol. 38, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015, pp. 317–331. 47

[Sim14] Robert J. Simmons, *Structural Focalization*, ACM Trans. Comput. Log. **15** (2014), no. 3, 21:1–21:33. 47, 48

[vOvR94] Vincent van Oostrom and Femke van Raamsdonk, *Weak Orthogonality Implies Confluence: The Higher Order Case*, Proc. LFCS, 1994, pp. 379–392. 8, 18

[vR99] Femke van Raamsdonk, *Higher-order Rewriting*, Proc. Rewrit. Tech. App., LNCS, vol. 1631, Springer, 1999, pp. 220–239. 18, 43

[Wad03] Philip Wadler, *Call-by-value is dual to call-by-name*, SIGPLAN Not. **38** (2003), no. 9, 189–201. 6, 29