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On idempotent discrete uninorms

Miguel Couceiro, Jimmy Devillet, and Jean-Luc Marichal

Abstract In this paper we provide two axiomatizations of the class of idempotent discrete uninorms as conservative binary operations, where an operation is conservative if it always outputs one of its input values. More precisely we first show that the idempotent discrete uninorms are exactly those operations that are conservative, symmetric, and nondecreasing in each variable. Then we show that, in this characterization, symmetry can be replaced with both bisymmetry and existence of a neutral element.

1 Introduction

Aggregation functions defined on linguistic scales (i.e., finite chains) have been intensively investigated for about two decades; see, e.g., [1–4, 6–11, 13, 14]. Among these functions, discrete fuzzy connectives (such as discrete uninorms) are associative binary operations that play an important role in fuzzy logic.

This short paper focuses on characterizations of the class of idempotent discrete uninorms. Recall that a discrete uninorm is a binary operation on a finite chain that is associative, symmetric, nondecreasing in each variable, and has a neutral element.

A first characterization of the class of idempotent discrete uninorms was given by De Baets et al. [1, Theorem 3]. This characterization reveals that any idempotent

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discrete uninorm is a combination of the minimum and maximum operations. In particular, such an operation is *conservative* in the sense that it always outputs one of the input values.

The outline of this paper is as follows. After presenting some preliminary results on conservative operations in Section 2, we show in Section 3 that the idempotent discrete uninorms are exactly those operations that are conservative, symmetric, and nondecreasing in each variable. This new characterization is very simple and requires neither associativity nor the existence of a neutral element. In Section 4 we provide an alternative characterization of this class in terms of the bisymmetry property. More specifically, we show that the idempotent discrete uninorms are exactly those operations that are conservative, bisymmetric, nondecreasing in each variable, and have neutral elements.

2 Preliminaries

In this section we present some basic definitions and preliminary results.

Let X be an arbitrary nonempty set and let $\Delta_X = \{(x,x) \mid x \in X\}$.

Definition 1. An operation $F\colon X^2 \to X$ is said to be

- *idempotent* if $F(x,x) = x$ for all $x \in X$.
- *conservative* if $F(x,y) \in \{x,y\}$ for all $x,y \in X$.
- *associative* if $F(F(x,y),z) = F(x,F(y,z))$ for all $x,y,z \in X$.

An element $e \in X$ is said to be a *neutral element* of F (or simply a *neutral element*) if $F(x,e) = F(e,x) = x$ for all $x \in X$. In this case we easily show by contradiction that such a neutral element is unique. The points (x,y) and (u,v) of X^2 are said to be *connected for F* (or simply *connected*) if $F(x,y) = F(u,v)$. We observe that "being connected" is an equivalence relation. The point (x,y) of X^2 is said to be *isolated for F* (or simply *isolated*) if it is not connected to another point in X^2.

Remark 1. Conservativeness was introduced in Pouzet et al. [12]. This condition is also called "local internality" in Martín et al. [5].

Lemma 1. *Let $F\colon X^2 \to X$ be an idempotent operation. If the point $(x,y) \in X^2$ is isolated, then it lies on Δ_X, that is, $x = y$.*

Remark 2. We observe that idempotency is necessary in Lemma 1. Indeed, consider the operation $F\colon X^2 \to X$, where $X = \{a,b\}$, defined as $F(x,y) = a$, if $(x,y) = (a,b)$, and $F(x,y) = b$, otherwise. Then (a,b) is isolated and $a \neq b$. The contour plot of F is represented in Figure 1. Here and throughout, connected points are joined by edges. To keep the figures simple we sometimes omit the edges obtained by transitivity.

The following lemma provides an easy test for the existence of a neutral element of a conservative operation.

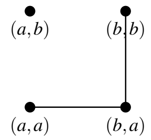

Fig. 1 A non-idempotent operation

Lemma 2. *Let $F:X^2 \to X$ be a conservative operation and let $e \in X$. Then e is a neutral element if and only if (e,e) is isolated.*

Corollary 1. *Any isolated point (x,y) of a conservative operation $F:X^2 \to X$ is unique and lies on Δ_X. Moreover, $x = y$ is a neutral element.*

Remark 3. Lemma 2 no longer holds if conservativeness is relaxed into idempotency. Indeed, by simply taking $X = \{a,b,c\}$ we can easily construct an idempotent operation with an isolated point on Δ_X and no neutral element (see Figure 2). Also, it is easy to construct an idempotent operation with a neutral element and no isolated point (see Figure 3). It is also noteworthy that there are idempotent operations with more than one isolated point (see Figure 4).

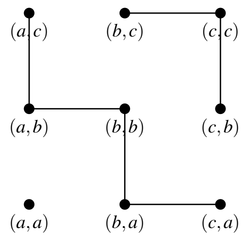

Fig. 2 An operation with no neutral element

3 Main results

We now focus on characterizations of the class of idempotent discrete uninorms. These operations are defined on finite chains. Without loss of generality we will only consider the n-element chains $L_n = \{1,\ldots,n\}$, $n \geq 1$, endowed with the usual ordering relation $\leq$.

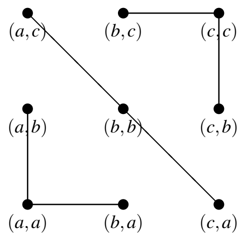

Fig. 3 An operation with no isolated point

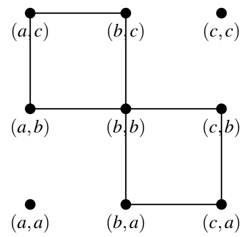

Fig. 4 An operation with two isolated points

Recall that an operation $F\colon L_n^2 \to L_n$ is said to be *nondecreasing in each variable* if $F(x,y) \leq F(x',y')$ whenever $x \leq x'$ and $y \leq y'$.

Definition 2 (see, e.g., [1]). A *discrete uninorm* on L_n is an operation $U\colon L_n^2 \to L_n$ that is associative, symmetric, nondecreasing in each variable, and has a neutral element.

A characterization of the class of idempotent discrete uninorms is given in the following theorem. Although this characterization is somewhat intricate, it shows, together with Lemma 3 below, that any idempotent discrete uninorm is conservative.

Theorem 1 (see [1, Theorem 3]). *An operation $F\colon L_n^2 \to L_n$ with a neutral element $1 < e < n$ is an idempotent discrete uninorm if and only if there exists a nonincreasing map $g\colon [1,e] \to [e,n]$ (nonincreasing means that $g(x) \geq g(y)$ whenever $x \leq y$), with $g(e) = e$, such that*

$$F(x,y) \;=\; \begin{cases} \min\{x,y\}, & \text{if } y \leq \overline{g}(x) \text{ and } x \leq \overline{g}(1), \\ \max\{x,y\}, & \text{otherwise}, \end{cases}$$

where $\overline{g}\colon L_n \to L_n$ is defined by

$$\overline{g}(x) \;=\; \begin{cases} g(x), & \text{if } x \leq e, \\ \max\{z \in [1,e] \mid g(z) \geq x\}, & \text{if } e \leq x \leq g(1), \\ 1, & \text{if } x > g(1). \end{cases}$$

We now show that the idempotent discrete uninorms are exactly those operations that are conservative, symmetric, and nondecreasing in each variable (see Theorem 2).

First consider the following lemma, which actually holds on arbitrary, not necessarily finite, chains.

Lemma 3. *If $F\colon L_n^2 \to L_n$ is idempotent, nondecreasing in each variable, and has a neutral element $e \in L_n$, then $F|_{[1,e]^2} = \min$ and $F|_{[e,n]^2} = \max$.*

Proposition 1. *If $F\colon L_n^2 \to L_n$ is conservative, symmetric, and nondecreasing in each variable, then it is associative and it has a neutral element.*

For $n = 2$ and $n = 3$, the possible operations $F\colon L_n^2 \to L_n$ that are conservative, symmetric, and nondecreasing in each variable have contour plots depicted in Figures 5 and 6, respectively.

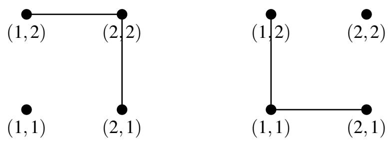

Fig. 5 Possible operations when $n = 2$

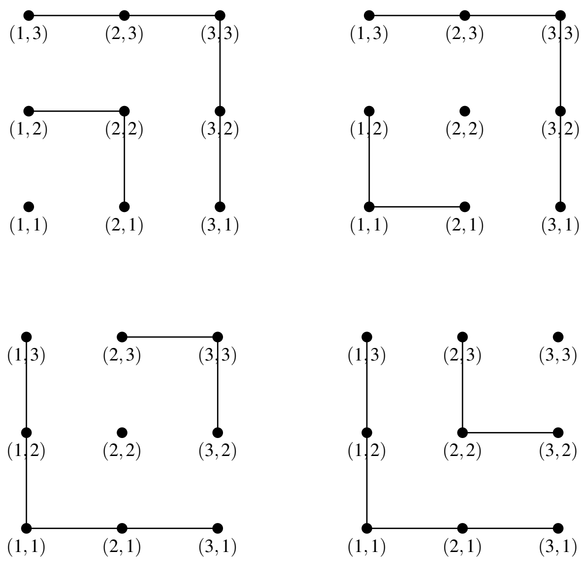

Fig. 6 Possible operations when $n = 3$

Remark 4. (a) The existence of a neutral element in Proposition 1 is no longer guaranteed if the chain is not finite. For instance, the real operation $F\colon [0,1]^2 \to [0,1]$ defined by $F(x,y) = \min\{x,y\}$, if $x, y \in [0, \frac{1}{2})^2$, and $F(x,y) = \max\{x,y\}$, otherwise, is conservative, symmetric, and nondecreasing in each variable, but it does not have a neutral element.

(b) We observe that conservativeness cannot be relaxed into idempotency in Proposition 1. For instance the operation $F\colon L_3^2 \to L_3$ whose contour plot is depicted in Figure 2 is idempotent, symmetric, and nondecreasing in each variable, but one can show that it is not associative and it has no neutral element.
(c) We also observe that each of the conditions of Proposition 1 is necessary. Indeed, we give in Figure 7 an operation that is conservative and symmetric but that is not nondecreasing in each variable. We also give in Figure 8 an operation that is conservative and nondecreasing in each variable but not symmetric. Finally, we give in Figure 9 an operation that is symmetric and nondecreasing in each variable but not conservative. None of these three operations is associative and none has a neutral element.

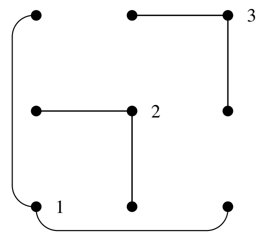

Fig. 7 An operation that fails to be nondecreasing in each variable

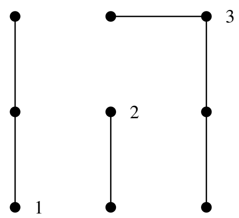

Fig. 8 An operation that fails to be symmetric

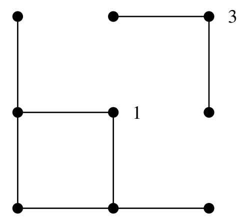

Fig. 9 An operation that fails to be conservative

Theorem 2. *An operation $F\colon L_n^2 \to L_n$ is conservative, symmetric, and nondecreasing in each variable if and only if it is an idempotent discrete uninorm. Moreover, there are exactly 2^{n-1} such operations.*

Remark 5. Theorem 2 enables us to provide a graphical characterization of the idempotent discrete uninorms in terms of their contour plots. Indeed, denoting by L an arbitrary n-element chain, we observe that the restriction $F|_{L'}$ of any idempotent discrete uninorm $F\colon L^2 \to L$ to any subchain L' obtained by removing one of the endpoints of L is also an idempotent discrete uninorm. Moreover, the operation F (or equivalently its contour plot) can be retrieved from $F|_{L'}$ by connecting all the points of $L^2 \setminus L'^2$. It follows that all the idempotent discrete uninorms can be constructed recursively in terms of their contour plots.

4 Bisymmetric operations

In this section we provide a characterization of the class of idempotent discrete uninorms in terms of the bisymmetry (or mediality) property.

Definition 3. An operation $F\colon X^2 \to X$ is said to be *bisymmetric* if

$$F(F(x,y),F(u,v)) \;=\; F(F(x,u),F(y,v))$$

for all $x,y,u,v \in X$.

Proposition 2. *Let $F\colon X^2 \to X$ be a conservative operation that has a neutral element. Then F is bisymmetric if and only if it is associative and symmetric.*

Combining Proposition 2 with Theorem 2, we can easily derive the following alternative characterization of idempotent discrete uninorms.

Theorem 3. *An operation $F\colon L_n^2 \to L_n$ is conservative, bisymmetric, nondecreasing in each variable, and has a neutral element if and only if it is an idempotent discrete uninorm.*

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