Change detection and isolation in mechanical system parameters based on perturbation analysis

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Abstract: The monitoring of mechanical systems aims at detecting damages at an early stage, in general by using output-only vibration measurements under ambient excitation. In this paper, a method is proposed for the detection and isolation of small changes in the physical parameters of a linear mechanical system. Based on a recent work where the multiplicative change detection problem is transformed to an additive one by means of perturbation analysis, changes in the eigenvalues and eigenvectors of the mechanical system are considered in the first step. In a second step, these changes are related to physical parameters of the mechanical system. Finally, another transformation further simplifies the detection and isolation problem into the framework of a linear regression subject to additive white Gaussian noises, leading to a numerically efficient solution of the considered problems. A numerical example of a simulated mechanical structure is reported for damage detection and localization.

Keywords: Fault detection and isolation, statistical tests, mechanical system, vibrations.

1. INTRODUCTION

The detection and localization of damages based on measured vibration data are fundamental tasks for structural health monitoring to allow an automated damage diagnosis [Farrar and Worden, 2007]. Early sign of damages can be modeled as changes in the parameters of the underlying mechanical system. They affect the dynamic properties of a structure, inducing *small changes* in the eigenstructure (eigenvalues and eigenvectors) of a linear system. It is of interest to *detect* these changes and to decide which of the physical parameters are responsible for the change (*isolation*). A particular difficulty for structural health monitoring is caused by the absence of known system inputs, since the structural excitation is usually only ambient, leading to an output-only monitoring problem.

Among the many model-based or data-based methods for mechanical structure damage detection [Carden and Fanning, 2004], methods based on direct model-data matching are particularly appealing for an automated damage diagnosis, where current measurement data are directly confronted to a reference model. For instance, such methods include non-parametric change detection based on novelty detection [Worden et al., 2000] or whiteness tests on Kalman filter innovations [Bernal, 2013]. Another method within this category, the local asymptotic approach to change detection [Benveniste et al., 1987], has the ability of focusing the detection on some chosen system parameters. Associated to efficient hypothesis testing tools, this method has led to successful applications in the field of vibration monitoring, e.g. in [Jhinaoui et al., 2012, Döhler and Mevel, 2013, Döhler et al., 2014], including fault isolation and estimation [Döhler et al., 2016].

In [Döhler et al., 2015] an alternative method for parametric change detection has been developed, where the *multiplicative* eigenstructure change detection problem is transformed to an *additive* one by means of a perturbation analysis, assuming small parameter changes. Amongst others, this allows addressing random uncertainties more efficiently in associated hypothesis testing tools by avoiding the covariance matrix estimation problem encountered in the local asymptotic approach. In the current paper, we extend this approach from *eigenstructure parameter change detection* to *mechanical system parameter change detection*. By linking changes in the system matrices of a state-space model to mechanical system parameters, the underlying physical problem of *fault isolation*, i.e. of deciding which physical parameters are responsible for the detected changes, is solved.

This paper is organized as follows. In Section 2, the system models and parameters are defined. In Section 3, the perturbation analysis is carried out to transform the system parameter change detection problem into an additive one. In Sections 4 and 5 the respective hypothesis test for change detection and isolation are stated. Finally, an application for vibration-based damage detection and localization is shown in Section 6.

2. PROBLEM STATEMENT

The behavior of mechanical structures subject to unknown ambient excitation can be described by the differential equation

$$\mathcal{M}\ddot{\mathcal{X}}(t) + \mathcal{C}\dot{\mathcal{X}}(t) + \mathcal{K}\mathcal{X}(t) = f(t) \tag{1}$$

where t denotes continuous time; $\mathcal{M}, \mathcal{C}, \mathcal{K} \in \mathbb{R}^{m\times m}$ are mass, damping, and stiffness matrices, respectively; the state vector $\mathcal{X}(t) \in \mathbb{R}^m$ is the displacement vector of the m degrees of freedom of the structure; and $f(t)$ is the external unmeasured force (random disturbance).

Observed at r sensor positions by displacement, velocity or acceleration sensors at discrete time instants $t = k\tau$ (with sampling rate $1/\tau$), system (1) can also be described by a discrete-time state space system model [Juang, 1994]

$$\begin{cases} z_{k+1} = Fz_k + w_k \\ y_k = Hz_k + v_k \end{cases} \tag{2}$$

where the state vector $z_k = \left[\mathcal{X}(k\tau)^T \ \dot{\mathcal{X}}(k\tau)^T\right]^T \in \mathbb{R}^n$ with $n = 2m$, the measured output vector $y_k \in \mathbb{R}^r$ and the system matrices

$$F = \exp(F^c\tau) \in \mathbb{R}^{n\times n}, \quad F^c = \begin{bmatrix} 0 & I \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{C} \end{bmatrix}, \tag{3}$$

$$H = \left[L_d - L_a\mathcal{M}^{-1}\mathcal{K} \quad L_v - L_a\mathcal{M}^{-1}\mathcal{C}\right] \in \mathbb{R}^{r\times n}, \tag{4}$$

with selection matrices $L_d, L_v, L_a \in \{0,1\}^{r\times m}$ indicating the positions of displacement, velocity or acceleration sensors, respectively. The state noise w_k and output noise v_k are unmeasured and assumed to be Gaussian, zero-mean and white.

In this paper, damages are considered as changes in the structural stiffness properties of system (1), corresponding to changes related to the parameters of structural elements. The corresponding stiffness matrix $\mathcal{K}$ can be parametrized by a vector of independent parameters η with $\mathcal{K} = \mathcal{K}(\eta)$. No changes are assumed in $\mathcal{M}$ and $\mathcal{C}$. Changes in the physical parameter η provoke changes in the eigenstructure of system (1), and consequently of system (2). The related eigenstructure parameter vector θ is defined in the following.

The eigenvalues μ_i and eigenvectors ϕ_i of system (1) satisfy

$$(\mathcal{M}\mu_i^2 + \mathcal{C}\mu_i + \mathcal{K})\phi_i = 0, \quad i = 1, 2, \ldots, 2m.$$

They are related to the eigenvalues and eigenvectors of the matrix F in (2), which satisfy

$$F\psi_i = \lambda_i\psi_i, \quad i = 1, 2, \ldots, n = 2m, \tag{5}$$

through

$$\lambda_i = \mathrm{e}^{\mu_i\tau}, \quad \psi_i = \begin{bmatrix} \phi_i \\ \mu_i\phi_i \end{bmatrix}. \tag{6}$$

Assume that the eigenstructure of the considered system contains only complex modes. This is the typical case for structural health monitoring applications. Then, the eigenvalues μ_i consist of m conjugate complex pairs, so do the eigenvalues λ_i. Let the vectors

$$\mu \triangleq [\mu_1, \mu_2, \ldots, \mu_m]^T \in \mathbb{C}^m, \quad \lambda \triangleq [\lambda_1, \lambda_2, \ldots, \lambda_m]^T \in \mathbb{C}^m$$

contain m of the n eigenvalues, one out of each of the m conjugate complex pairs, and

$$\phi \triangleq [\phi_1, \phi_2, \ldots, \phi_m] \in \mathbb{C}^{m\times m}$$

be composed of the corresponding eigenvectors. The complex eigenvalues and eigenvectors (μ_i, ϕ_i) are then represented by the equivalent real eigenstructure parameter vector $\theta \in \mathbb{R}^{2m+2m^2}$ defined as

$$\theta \triangleq \begin{bmatrix} \Re(\mu) \\ \Im(\mu) \\ \mathrm{vec}(\Re(\phi)) \\ \mathrm{vec}(\Im(\phi)) \end{bmatrix} \tag{7}$$

where $\Re$ and $\Im$ denote respectively the real part and the imaginary part of a complex variable.

Assume that the matrix F in (5) is diagonalizable, then

$$F = T(\theta)A(\theta)T^{-1}(\theta) \tag{8}$$

with real matrices

$$A(\theta) = \begin{bmatrix} \mathrm{diag}(\Re(\lambda)) & \mathrm{diag}(\Im(\lambda)) \\ -\mathrm{diag}(\Im(\lambda)) & \mathrm{diag}(\Re(\lambda)) \end{bmatrix}, \tag{9}$$

$$T(\theta) = \begin{bmatrix} \Re(\phi) & \Im(\phi) \\ \Re(\phi\,\mathrm{diag}(\mu)) & \Im(\phi\,\mathrm{diag}(\mu)) \end{bmatrix}. \tag{10}$$

Similarly, following from (3), (4) and (8), matrix H yields

$$H = [L_d \ L_v] + [0_{r,m} \ L_a]\, T(\theta)A_c(\theta)T^{-1}(\theta) \tag{11}$$

where

$$A_c(\theta) = \begin{bmatrix} \mathrm{diag}(\Re(\mu)) & \mathrm{diag}(\Im(\mu)) \\ -\mathrm{diag}(\Im(\mu)) & \mathrm{diag}(\Re(\mu)) \end{bmatrix}.$$

With the parametrization of F and H with θ expressed in (8) and (11), the state-space model (2) is rewritten as

$$z_{k+1} = T(\theta)A(\theta)T^{-1}(\theta)z_k + w_k \tag{12a}$$

$$y_k = \left([L_d \ L_v] + [0_{r,m} \ L_a]\, T(\theta)A_c(\theta)T^{-1}(\theta)\right) z_k + v_k. \tag{12b}$$

In this paper, it is assumed that the nominal values of the mechanical system matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are available, typically based on a finite element model of the monitored structure. The nominal value θ^0 of the parameter vector θ is then accordingly deduced. Note that the estimation of $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ from output-only sensor data is in general not possible. While the estimation of F and H would be possible, e.g. by subspace system identification [Van Overschee and De Moor, 1996], they can only be estimated up to an unknown similarity transformation, and it is not possible to fully deduce the parameter vector θ from such a result.

Th choice of the parametrization θ is different than in [Döhler et al., 2015], where the eigenvector parts only at the sensor coordinates were part of the parametrization, computed by $H\psi_i$. Though the nominal parameter in [Döhler et al., 2015] can be obtained entirely from measurements without the knowledge of the structural system matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$, it allows only for change *detection*, while no link to the physical properties of the structure was given for fault isolation. In the current paper, a link from θ to the physical parameter set η will be made, which allows for the *isolation* of the physical parameter subset that is responsible for the detected change. Note that this link is necessary for fault isolation, since damages usually correspond to changes in few components of η, whereas in general all components of θ are affected.

Furthermore, monitoring the system in the state basis related to the structural system matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ in (2)–(4) and parameterized in (12) avoids the problem that noise properties are modified by damages when the system is monitored in the modal basis as in [Döhler et al., 2015]. In fact, the state noise term in the modal basis is $T^{-1}(\theta)w_k$ and thus affected by changes in θ, and this dependence in θ was neglected in [Döhler et al., 2015]. In the current paper, change detection and isolation are based on the state-space model (12), formulated in a particular state basis such that the state noise covariance is independent of θ.

In the following, the change detection in θ will be carried out based on a perturbation analysis. Then, a link to the physical parameterization η will be made, and fault isolation will be presented.

3. PERTURBATION ANALYSIS

In the state-space model (2), parametrized as in (12), the unknown changes in parameter vector θ appear in the system matrices $F(\theta)$ and $H(\theta)$, which are in product with the unknown state vector z_k. In this section, this *multiplicative* change detection problem is transformed to an *additive* one through a perturbation analysis based on the assumption of small changes in θ. Let θ^0 be the nominal value of θ, and assume a small change

$$\theta = \theta^0 + \varepsilon\theta^1. \tag{13}$$

Then accordingly

$$A(\theta) = A(\theta^0) + \varepsilon A(\theta^1) \triangleq A^0 + \varepsilon A^1, \tag{14a}$$

$$A_c(\theta) = A_c(\theta^0) + \varepsilon A_c(\theta^1) \triangleq A_c^0 + \varepsilon A_c^1, \tag{14b}$$

$$T(\theta) = T(\theta^0) + \varepsilon T(\theta^1) \triangleq T^0 + \varepsilon T^1. \tag{14c}$$

Let z_k^0 be the state trajectory estimate assuming $\theta = \theta^0$ that can be obtained from a Kalman filter based on the nominal system (2) with $F = F(\theta^0)$ and $H = H(\theta^0)$, and assume that

$$z_k = z_k^0 + \varepsilon z_k^1. \tag{15}$$

3.1 State equation perturbation

Consider the above perturbation for the state equation (12a). By using the approximation

$$(T^0 + \varepsilon T^1)^{-1} \approx (T^0)^{-1} - \varepsilon (T^0)^{-1} T^1 (T^0)^{-1},$$

consider the matrix $T(\theta)A(\theta)T^{-1}(\theta)$ in (12) as a perturbed matrix $T^0A^0(T^0)^{-1}$, then

$$\begin{aligned} T(\theta)A(\theta)T^{-1}(\theta) &= (T^0 + \varepsilon T^1)(A^0 + \varepsilon A^1)(T^0 + \varepsilon T^1)^{-1} \\ &\approx T^0A^0(T^0)^{-1} + \varepsilon\Big(T^1A^0(T^0)^{-1} + \\ &\quad T^0A^1(T^0)^{-1} - T^0A^0(T^0)^{-1}T^1(T^0)^{-1}\Big) \end{aligned}$$

where the terms involving $\varepsilon^2, \varepsilon^3$ have been omitted. Then, using (15),

$$\begin{aligned} T(\theta)A(\theta)T^{-1}(\theta)z_k &\approx T^0A^0(T^0)^{-1}z_k + \varepsilon\Big(T^1A^0(T^0)^{-1} \\ &+ T^0A^1(T^0)^{-1} - T^0A^0(T^0)^{-1}T^1(T^0)^{-1}\Big)z_k^0. \end{aligned} \tag{16}$$

In the above parenthesis in product with z_k^0, the matrices T^0, A^0 are independent of θ, A^1 and T^1 are linearly parametrized by θ^1, see (14). Let the small change in θ (see (13)) be denoted by

$$\tilde{\theta} = \varepsilon\theta^1.$$

Lemma 1. It holds

$$\begin{aligned} \varepsilon\Big(&T^1A^0(T^0)^{-1} + T^0A^1(T^0)^{-1} \\ &- T^0A^0(T^0)^{-1}T^1(T^0)^{-1}\Big)z_k^0 \approx \Psi_k^\theta\tilde{\theta}, \end{aligned} \tag{17}$$

where matrix Ψ_k^θ is filled with known signals, as shown in Appendix A.

3.2 Output equation perturbation

Analogously to (16), a perturbation in the output equation (12b) yields

$$\begin{aligned} &\big([L_d\ L_v] + [0_{r,m}\ L_a]\,T(\theta)A_c(\theta)T^{-1}(\theta)\big)z_k \\ &\approx H^0z_k + \varepsilon\,[0_{r,m}\ L_a]\,\Big(T^1A_c^0(T^0)^{-1} + T^0A_c^1(T^0)^{-1} \\ &\qquad - T^0A_c^0(T^0)^{-1}T^1(T^0)^{-1}\Big)z_k^0, \end{aligned}$$

where

$$H^0 = [L_d\ L_v] + [0_{r,m}\ L_a]\,T^0A_c^0(T^0)^{-1}.$$

Similarly as in the previous section, the terms A_c^1 and T^1 in the above parenthesis are linearly parametrized by θ^1:

Lemma 2. It holds

$$\begin{aligned} \varepsilon\,[0_{r,m}\ L_a]\,\Big(&T^1A_c^0(T^0)^{-1} + T^0A_c^1(T^0)^{-1} \\ &- T^0A_c^0(T^0)^{-1}T^1(T^0)^{-1}\Big)z_k^0 \approx \Phi_k^\theta\tilde{\theta}, \end{aligned} \tag{18}$$

where matrix Φ_k^θ is filled with known signals, as shown in Appendix B.

Note that $\Phi_k^\theta = 0$ in the case where only displacements or velocities are measured, excluding accelerations.

Following Lemma 1 and 2, system (12) becomes

$$z_{k+1} \approx F^0z_k + \Psi_k^\theta\tilde{\theta} + w_k \tag{19a}$$

$$y_k \approx H^0z_k + \Phi_k^\theta\tilde{\theta} + v_k \tag{19b}$$

with $F^0 = T^0A^0(T^0)^{-1}$. As Ψ_k^θ and Φ_k^θ are filled with known signals, the parameter change $\tilde{\theta}$ is additive in (19).

3.3 Link to physical parametrization

Changes in the structural system (1) are assumed in the stiffness matrix $\mathcal{K} = \mathcal{K}(\eta)$, while the mass and damping matrices $\mathcal{M}$ and $\mathcal{C}$ remain unchanged. This is the typical case for damage detection related to stiffness loss in structural health monitoring applications. The physical parameter vector $\eta = [\eta_1, \eta_2, ..., \eta_p]^T \in \mathbb{R}^p$ constitutes an independent parametrization, which is linked to the structural type and geometry. Usually, damage is related to small changes in few components of η, which affects the entire parameter vector θ. As in (13), the parameter η is decomposed into a nominal value and a small change by

$$\eta = \eta^0 + \varepsilon\eta^1,$$

and $\tilde{\eta} = \varepsilon\eta^1$.

The parameter vector θ in (7) and the physical parameter vector η are related through [Heylen et al., 1998]

$$\frac{\partial\mu_i}{\partial\eta_k} = -\frac{1}{a_i}\phi_i^T\frac{\partial\mathcal{K}(\eta)}{\partial\eta_k}\phi_i, \tag{20}$$

$$\begin{aligned} \frac{\partial\phi_i}{\partial\eta_k} &= \sum_{l=1,l\neq i}^{m}\frac{1}{a_l}\frac{1}{\mu_l - \mu_i}\phi_l^T\frac{\partial\mathcal{K}(\eta)}{\partial\eta_k}\phi_i\phi_l \\ &+ \sum_{l=1}^{m}\frac{1}{a_l^*}\frac{1}{\mu_l^* - \mu_i}\phi_l^H\frac{\partial\mathcal{K}(\eta)}{\partial\eta_k}\phi_i\phi_l^* \end{aligned} \tag{21}$$

where

$$a_i = 2\mu_i\phi_i^T\mathcal{M}\phi_i + \phi_i^T\mathcal{C}\phi_i,$$

"$*$" denotes the complex conjugate and "^{H}" the conjugate transpose. Assembling the real and imaginary parts of (20) and (21) for $i = 1, \ldots, m$ in the rows and for $k = 1, \ldots, p$ in the columns leads to the Jacobian matrix

$$\mathcal{J}_{\theta,\eta} = \frac{\partial\theta}{\partial\eta},$$

which is computed at θ^0 and η^0, and thus

$$\tilde{\theta} \approx \mathcal{J}_{\theta,\eta}\tilde{\eta}.$$

Then, following from (19a) and (19b), the change detection problem in the physical parameter vector η is transformed to the detection of *additive* changes in the system

$$z_{k+1} \approx F^0 z_k + \Psi_k^\eta \tilde{\eta} + w_k \tag{22a}$$
$$y_k \approx H^0 z_k + \Phi_k^\eta \tilde{\eta} + v_k, \tag{22b}$$

where $\Psi_k^\eta = \Psi_k^\theta \mathcal{J}_{\theta,\eta}$ and $\Phi_k^\eta = \Phi_k^\theta \mathcal{J}_{\theta,\eta}$.

4. ADDITIVE CHANGE DETECTION

The additive change detection problem (22) has been studied in [Zhang and Basseville, 2014] by transforming the dynamic system model into an equivalent linear regression model through a particular Kalman filtering. Analogously to [Döhler et al., 2015], this is carried out for problem (22) in the following.

Let the Kalman filter associated to the *nominal system* be given, based on the known nominal system matrices F^0 and H^0 in(3) and the noise covariance matrices

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \mathrm{E}\left(\begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_k^T & v_k^T \end{bmatrix} \right).$$

Let K be the steady state Kalman gain and Σ the innovation covariance matrix with

$$\Sigma = H^0 P (H^0)^T + R, \tag{23}$$

where P is the solution of the algebraic Riccati equation associated to the Kalman filter.

Apply to the *possibly faulty system* (22) the Kalman filter designed for the nominal system (assuming $\tilde{\eta} = 0$). Denote the one step ahead prediction $z_k^0 = z_{k|k-1}(\eta^0)$ and the innovation sequence

$$\zeta_k \triangleq y_k - F^0 z_k^0.$$

If $\tilde{\eta} = 0$, then it is well known that ζ_k is a centered Gaussian white noise, otherwise the innovation sequence is biased and satisfies [Zhang and Basseville, 2014]

$$\zeta_k \approx (H^0 \Gamma_k + \Phi_k^\eta) \tilde{\eta} + e_k \tag{24}$$

where Γ_k is recursively computed as

$$\Gamma_{k+1} = F^0 (I_n - K H^0) \Gamma_k + \Psi_k^\eta - F^0 K \Phi_k^\eta$$
$$\Gamma_0 = 0,$$

and e_k is a white Gaussian noise of zero mean with covariance Σ in (23).

In the algebraic equation (24), the parameter increment $\tilde{\eta}$ is the only unknown, apart from the white Gaussian noise e_k. In this *linear Gaussian framework*, it is well known that the generalized likelihood ratio (GLR) test [Basseville and Nikiforov, 1993] for $\tilde{\eta} \neq 0$ against $\tilde{\eta} = 0$ amounts to

$$\Omega = \sum_{k=1}^{N} (H^0 \Gamma_k + \Phi_k^\eta)^T \Sigma^{-1} (H^0 \Gamma_k + \Phi_k^\eta) \tag{25a}$$
$$\beta = \sum_{k=1}^{N} (H^0 \Gamma_k + \Phi_k^\eta)^T \Sigma^{-1} \zeta_k \tag{25b}$$
$$s = \beta^T \Omega^{-1} \beta. \tag{25c}$$

The resulting statistics s follows a χ^2 distribution of $\dim(\tilde{\eta}) = p$ degrees of freedom, central if $\tilde{\eta} = 0$, otherwise non-central with its non-centrality parameter equal to $\tilde{\eta}^T \Omega \tilde{\eta}$. The decision for change detection is thus made by comparing s to a positive threshold.

5. FAULT ISOLATION

To determine the parameter subset of η that is responsible for a detected change, fault isolation tests based on minmax tests are appropriate in the Gaussian framework above [Zhang and Basseville, 2014, Döhler et al., 2016]. By considering partitions

$$\tilde{\eta} = \begin{bmatrix} \tilde{\eta}_a \\ \tilde{\eta}_b \end{bmatrix} \tag{26}$$

of the parameter change vector, the hypothesis $\tilde{\eta}_a = 0$ is tested against $\tilde{\eta}_a \neq 0$, in order to decide if there is a change in parameter component η_a, for any partition.

The minmax test is carried out as follows, considering the computations in (25). Accordingly to (26), matrix Ω and vector β are partitioned as

$$\Omega = \begin{bmatrix} \Omega_{aa} & \Omega_{ab} \\ \Omega_{ba} & \Omega_{bb} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_a \\ \beta_b \end{bmatrix}.$$

Define the robust (minmax) residual that is blind to possible changes $\tilde{\eta}_b$ as

$$\beta_a^* = \beta_a - \Omega_{ab} \Omega_{bb}^{-1} \beta_b$$

and its covariance

$$\Omega_a^* = \Omega_{aa} - \Omega_{ab} \Omega_{bb}^{-1} \Omega_{ba}.$$

Then, the corresponding GLR test statistic for $\tilde{\eta}_a = 0$ against $\tilde{\eta}_a \neq 0$ writes as

$$s_a = \beta_a^{*T} \Omega_a^{*-1} \beta_a^*,$$

which is χ^2-distributed with $\dim(\tilde{\eta}_a)$ degrees of freedom and non-centrality parameter $\tilde{\eta}_a^T \Omega_a^* \tilde{\eta}_a$, independently of $\tilde{\eta}_b$. For a decision, s_a is compared to a threshold.

6. NUMERICAL EXAMPLE

A simulated mass-spring chain with eight elements (Fig. 1) is considered as an application similarly as in [Döhler et al., 2015], while now considering the physical parametrization η of the system for *damage detection* instead of the eigenstructure parametrization. In addition, *damage localization* is performed through fault isolation.

The matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ in (1) of the nominal structural model are defined based on the masses $m_1 = m_3 = m_5 = m_7 = 1, m_2 = m_4 = m_6 = m_8 = 2$, stiffnesses $k_1 = k_3 = k_5 = k_7 = 1000, k_2 = k_4 = k_6 = k_8 = 500$ and a damping ratio of 2% for all modes. The physical parameter vector is defined by $\eta = [k_1, k_2, \ldots, k_8]$.

Datasets containing output-only time series of accelerations with time step $\tau = 0.05$ s are simulated for different structural states at four sensor coordinates at masses 1, 3, 5 and 7 from white noise excitation at all structural elements. White measurement noise is added with a magnitude of 5% of each generated output signal.

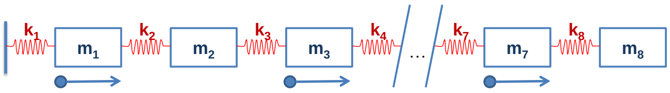

Fig. 1. Mass-spring chain with four sensors.

6.1 *Damage detection*

Three different structural states are considered, namely the nominal state and two faulty (damaged) states with 2% and 4% stiffness decrease in spring 2, respectively. For each structural state, datasets of length $N = 10{,}000$ are simulated and the test statistics s in (25) is computed.

The test values of the nominal state are used to set up an empirical threshold to decide between $\tilde{\theta} \neq 0$ against $\tilde{\theta} = 0$. The resulting test values for 1000 datasets are shown in the histogram in Fig. 2, where a threshold (red line) is drawn from the nominal state for a 1% type I error. At this type I error, the power of the test for the 2% damage is 72%, and for the 4% damage the power of the test is 100%.

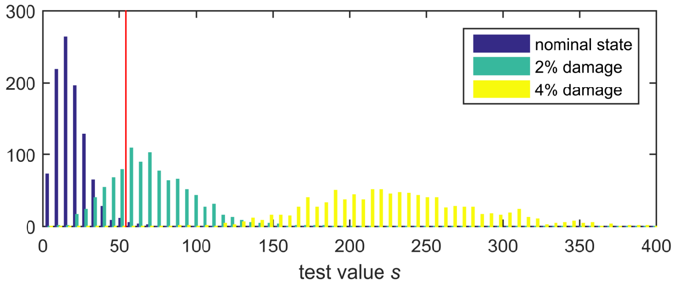

Fig. 2. Histogram of test values for the mass-spring chain in the nominal state, and 2% and 4% damage in spring 2.

6.2 Damage localization

The test statistic s_a in Section 5 is computed for each physical parameter $k_1, k_2, \ldots, k_8$ in η for two damage cases, in which datasets of length $N = 10{,}000$ are simulated, respectively. In the first case the stiffness k_2 is reduced by 2%, and in the second case the stiffnesses k_2 and k_4 are reduced by 2% each. The resulting tests are shown as bar plots in Figure 3. It can be seen that the tests for the damaged elements react and are significantly higher than for the undamaged elements.

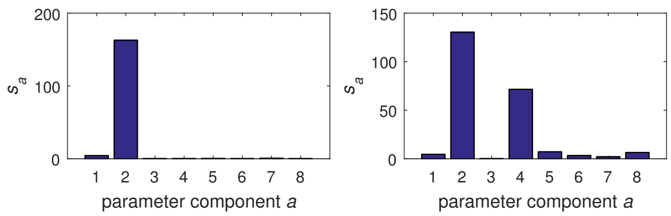

Fig. 3. Tests for each stiffness parameter $k_1, \ldots, k_8$. Damage is in spring 2 (left) and springs 2 and 4 (right).

7. CONCLUSIONS

In this work, we have derived tests for the detection and isolation of small changes in mechanical system parameters, based on a linear mechanical model (e.g. finite element model) and vibration measurements of the system. After transformations of the original problem, the GLR test is applied in the framework of a simple linear regression. Numerical applications to damage detection and localization in a mechanical structure illustrate the performance of the approach.

Appendix A. PROOF OF LEMMA 1

Consider $\phi = \phi^0 + \varepsilon\phi^1$ and $\mu = \mu^0 + \varepsilon\mu^1$ in parameter θ in (7), corresponding to (13). Then,

$$\phi\mathrm{diag}(\mu) \approx \phi^0\mathrm{diag}(\mu^0) + \varepsilon\phi^1\mathrm{diag}(\mu^0) + \varepsilon\phi^0\mathrm{diag}(\mu^1).$$

Since $T(\theta) = T^0 + \varepsilon T^1$, it follows from (10)

$$T^1 = \left[\begin{array}{c|c} \Re(\phi^1) & \Im(\phi^1) \\ \hline \Re(\phi^1\mathrm{diag}(\mu^0) & \Im(\phi^1\mathrm{diag}(\mu^0) \\ \quad + \phi^0\mathrm{diag}(\mu^1)) & \quad + \phi^0\mathrm{diag}(\mu^1)) \end{array}\right], \quad \text{(A.1)}$$

where

$$\begin{aligned}
\Re(\phi^1\mathrm{diag}(\mu^0)) &= \Re(\phi^1)\mathrm{diag}(\Re(\mu^0)) - \Im(\phi^1)\mathrm{diag}(\Im(\mu^0)) \\
\Re(\phi^0\mathrm{diag}(\mu^1)) &= \Re(\phi^0)\mathrm{diag}(\Re(\mu^1)) - \Im(\phi^0)\mathrm{diag}(\Im(\mu^1)) \\
\Im(\phi^1\mathrm{diag}(\mu^0)) &= \Im(\phi^1)\mathrm{diag}(\Re(\mu^0)) + \Re(\phi^1)\mathrm{diag}(\Im(\mu^0)) \\
\Im(\phi^0\mathrm{diag}(\mu^1)) &= \Im(\phi^0)\mathrm{diag}(\Re(\mu^1)) + \Re(\phi^0)\mathrm{diag}(\Im(\mu^1)).
\end{aligned}$$

Similarly, it follows from (9) with $\lambda = \lambda^0 + \varepsilon\lambda^1$

$$A^1 = \begin{bmatrix} \mathrm{diag}(\Re(\lambda^1)) & \mathrm{diag}(\Im(\lambda^1)) \\ -\mathrm{diag}(\Im(\lambda^1)) & \mathrm{diag}(\Re(\lambda^1)) \end{bmatrix}. \quad \text{(A.2)}$$

Consider now the terms in (17). Define

$$h_k \triangleq (T^0)^{-1} z_k^0, \quad h_k = \begin{bmatrix} \overline{h}_k \\ \underline{h}_k \end{bmatrix}, \quad \text{(A.3)}$$

where h_k is divided into two sub-vectors $\overline{h}_k \in \mathbb{R}^m$ and $\underline{h}_k \in \mathbb{R}^m$. Define $\overline{H}_k \triangleq \mathrm{diag}(\overline{h}_k)$, $\underline{H}_k \triangleq \mathrm{diag}(\underline{h}_k)$. Similarly,

$$l_k \triangleq A^0(T^0)^{-1} z_k^0 = A^0 h_k, \quad l_k = \begin{bmatrix} \overline{l}_k \\ \underline{l}_k \end{bmatrix} \quad \text{(A.4)}$$

with $\overline{l}_k, \underline{l}_k \in \mathbb{R}^m$, and define $\overline{L}_k \triangleq \mathrm{diag}(\overline{l}_k)$, $\underline{L}_k \triangleq \mathrm{diag}(\underline{l}_k)$.

To make the relation of the terms in (17) to $\tilde{\theta}$ explicit, we use the relationship $\mathrm{diag}(a)b = \mathrm{diag}(b)a$ for any vectors $a, b \in \mathbb{R}^m$, as well as $Ba = (a^T \otimes I_m)\mathrm{vec}(B)$ for any matrix $B \in \mathbb{R}^{m \times m}$ in the following, where $\otimes$ denotes the Kronecker product.

Development of $T^1A^0(T^0)^{-1}z_k^0$: From (A.1) and (A.4) it follows

$$\begin{aligned}
T^1A^0(T^0)^{-1}z_k^0 &= T^1 l_k \\
&= \left[\begin{array}{r} \Re(\phi^1)\overline{l}_k + \Im(\phi^1)\underline{l}_k \\ \hline \Re(\phi^1)\overline{L}_k\Re(\mu^0) - \Im(\phi^1)\overline{L}_k\Im(\mu^0) \\ + \Re(\phi^0)\overline{L}_k\Re(\mu^1) - \Im(\phi^0)\overline{L}_k\Im(\mu^1) \\ + \Im(\phi^1)\underline{L}_k\Re(\mu^0) + \Re(\phi^1)\underline{L}_k\Im(\mu^0) \\ + \Im(\phi^0)\underline{L}_k\Re(\mu^1) + \Re(\phi^0)\underline{L}_k\Im(\mu^1) \end{array}\right].
\end{aligned}$$

Sorting the terms with respect to $\Re(\mu^1)$, $\Im(\mu^1)$, $\Re(\phi^1)$ and $\Im(\phi^1)$ leads to

$$\begin{aligned}
\varepsilon T^1A^0(T^0)^{-1}z_k^0 &= \begin{bmatrix} 0 & 0 & \overline{l}_k^T \otimes I_m & \underline{l}_k^T \otimes I_m \\ \Psi_k^{(1,1)} & \Psi_k^{(1,2)} & \Psi_k^{(1,3)} & \Psi_k^{(1,4)} \end{bmatrix} \tilde{\theta} \\
&\triangleq \Psi_k^{(1)}\tilde{\theta} \qquad \text{(A.5)}
\end{aligned}$$

where

$$\begin{aligned}
\Psi_k^{(1,1)} &= \Re(\phi^0)\overline{L}_k + \Im(\phi^0)\underline{L}_k \\
\Psi_k^{(1,2)} &= -\Im(\phi^0)\overline{L}_k + \Re(\phi^0)\underline{L}_k \\
\Psi_k^{(1,3)} &= (\overline{L}_k\Re(\mu^0) + \underline{L}_k\Im(\mu^0))^T \otimes I_m \\
\Psi_k^{(1,4)} &= (-\overline{L}_k\Im(\mu^0) + \underline{L}_k\Re(\mu^0))^T \otimes I_m.
\end{aligned}$$

Development of $-T^0A^0(T^0)^{-1}T^1(T^0)^{-1}z_k^0$: Following from (A.3), the development of $T^1(T^0)^{-1}z_k^0 = T^1h_k$ is analogous to the previous case, yielding

$$\varepsilon T^1(T^0)^{-1}z_k^0 = P_k\tilde{\theta}, \quad \text{(A.6)}$$

where matrix P_k is defined analogously to $\Psi_k^{(1)}$ in (A.5) when replacing $\overline{l}_k$, $\underline{l}_k$, $\overline{L}_k$ and $\underline{L}_k$ by $\overline{h}_k$, $\underline{h}_k$, $\overline{H}_k$ and $\underline{H}_k$, respectively. It follows

$$\begin{aligned}
-\varepsilon T^0A^0(T^0)^{-1}T^1(T^0)^{-1}z_k^0 &= -T^0A^0(T^0)^{-1}P_k\tilde{\theta} \\
&\triangleq \Psi_k^{(2)}\tilde{\theta}. \qquad \text{(A.7)}
\end{aligned}$$

Development of $T^0 A^1 (T^0)^{-1} z_k^0$**:** From (A.2) and (A.3) it follows

$$\begin{aligned} T^0 A^1 (T^0)^{-1} z_k^0 &= T^0 A^1 h_k \\ &= T^0 \begin{bmatrix} \mathrm{diag}(\Re(\lambda^1))\overline{h}_k + \mathrm{diag}(\Im(\lambda^1))\underline{h}_k \\ -\mathrm{diag}(\Im(\lambda^1))\overline{h}_k + \mathrm{diag}(\Re(\lambda^1))\underline{h}_k \end{bmatrix} \\ &= T^0 \begin{bmatrix} \overline{H}_k & \underline{H}_k \\ \underline{H}_k & -\overline{H}_k \end{bmatrix} \begin{bmatrix} \Re(\lambda^1) \\ \Im(\lambda^1) \end{bmatrix} \end{aligned} \tag{A.8}$$

Based on (6), the derivatives of an eigenvalue λ_i of the discrete-time system with respect to an eigenvalue μ_i of the continuous-time system writes as [Basseville et al., 2004]

$$\begin{bmatrix} \partial\Re(\lambda_i)/\partial\Re(\mu_i) & \partial\Re(\lambda_i)/\partial\Im(\mu_i) \\ \partial\Im(\lambda_i)/\partial\Re(\mu_i) & \partial\Im(\lambda_i)/\partial\Im(\mu_i) \end{bmatrix} = \tau \begin{bmatrix} \Re(\lambda_i) & -\Im(\lambda_i) \\ \Im(\lambda_i) & \Re(\lambda_i) \end{bmatrix},$$

hence

$$\begin{bmatrix} \Re(\lambda^1) \\ \Im(\lambda^1) \end{bmatrix} \approx \tau \begin{bmatrix} \mathrm{diag}(\Re(\lambda^0)) & -\mathrm{diag}(\Im(\lambda^0)) \\ \mathrm{diag}(\Im(\lambda^0)) & \mathrm{diag}(\Re(\lambda^0)) \end{bmatrix} \begin{bmatrix} \Re(\mu^1) \\ \Im(\mu^1) \end{bmatrix}.$$

It follows

$$\varepsilon T^0 A^1 (T^0)^{-1} z_k^0 \approx [Q_k \ 0_{n,2m^2}]\, \tilde{\theta} \triangleq \Psi_k^{(3)} \tilde{\theta}, \tag{A.9}$$

where

$$Q_k = \tau\, T^0 \begin{bmatrix} \overline{H}_k & \underline{H}_k \\ \underline{H}_k & -\overline{H}_k \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\Re(\lambda^0)) & -\mathrm{diag}(\Im(\lambda^0)) \\ \mathrm{diag}(\Im(\lambda^0)) & \mathrm{diag}(\Re(\lambda^0)) \end{bmatrix}.$$

Combining the results from (A.5), (A.7) and (A.9) concludes the proof with

$$\Psi_k^\theta = \Psi_k^{(1)} + \Psi_k^{(2)} + \Psi_k^{(3)}.$$

Appendix B. PROOF OF LEMMA 2

The constructive proof is analogous to the proof of Lemma 1 in Appendix A. Considering the terms in (18), define first h_k, $\overline{h}_k$, $\underline{h}_k$, $\overline{H}_k$ and $\underline{H}_k$ in (A.3). Similarly,

$$d_k \triangleq A_c^0 (T^0)^{-1} z_k^0 = A_c^0 h_k, \quad d_k = \begin{bmatrix} \overline{d}_k \\ \underline{d}_k \end{bmatrix} \tag{B.1}$$

and $\overline{D}_k \triangleq \mathrm{diag}(\overline{d}_k)$, $\underline{D}_k \triangleq \mathrm{diag}(\underline{d}_k)$, with $\overline{d}_k, \underline{d}_k \in \mathbb{R}^m$. Then, the three terms in (18) are developed:

Development of $[0_{r,m} \ L_a]\, T^1 A_c^0 (T^0)^{-1} z_k^0$**:** From (A.1) and (B.1) it follows $T^1 A_c^0 (T^0)^{-1} z_k^0 = T^1 d_k$, and analogously to (A.5),

$$\begin{aligned} &\varepsilon\, [0_{r,m} \ L_a]\, T^1 A_c^0 (T^0)^{-1} z_k^0 \\ &\quad = L_a \left[\Phi_k^{(1,1)} \ \Phi_k^{(1,2)} \ \Phi_k^{(1,3)} \ \Phi_k^{(1,4)} \right] \tilde{\theta} \triangleq L_a \Phi_k^{(1)} \tilde{\theta} \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} \Phi_k^{(1,1)} &= \Re(\phi^0)\overline{D}_k + \Im(\phi^0)\underline{D}_k \\ \Phi_k^{(1,2)} &= -\Im(\phi^0)\overline{D}_k + \Re(\phi^0)\underline{D}_k \\ \Phi_k^{(1,3)} &= (\overline{D}_k \Re(\mu^0) + \underline{D}_k \Im(\mu^0))^T \otimes I_m \\ \Phi_k^{(1,4)} &= (-\overline{D}_k \Im(\mu^0) + \underline{D}_k \Re(\mu^0))^T \otimes I_m. \end{aligned}$$

Development of $-[0_{r,m} \ L_a]\, T^0 A_c^0 (T^0)^{-1} T^1 (T^0)^{-1} z_k^0$**:** Based on (A.6), it follows directly

$$\begin{aligned} &-\varepsilon\, [0_{r,m} \ L_a]\, T^0 A_c^0 (T^0)^{-1} T^1 (T^0)^{-1} z_k^0 \\ &\quad = -L_a\, [0_{m,m} \ I_m]\, T^0 A_c^0 (T^0)^{-1} P_k \tilde{\theta} \triangleq L_a \Phi_k^{(2)} \tilde{\theta}. \end{aligned} \tag{B.3}$$

Development of $[0_{r,m} \ L_a]\, T^0 A_c^1 (T^0)^{-1} z_k^0$**:** Analogously to (A.8) it follows

$$\begin{aligned} &\varepsilon\, [0_{r,m} \ L_a]\, T^0 A_c^1 (T^0)^{-1} z_k^0 \\ &= \varepsilon\, [0_{r,m} \ L_a]\, T^0 \begin{bmatrix} \overline{H}_k & \underline{H}_k \\ \underline{H}_k & -\overline{H}_k \end{bmatrix} \begin{bmatrix} \Re(\mu^1) \\ \Im(\mu^1) \end{bmatrix} \\ &= L_a\, [0_{m,m} \ I_m]\, T^0 \begin{bmatrix} \overline{H}_k & \underline{H}_k & 0_{m,2m^2} \\ \underline{H}_k & -\overline{H}_k & 0_{m,2m^2} \end{bmatrix} \tilde{\theta} \triangleq L_a \Phi_k^{(3)} \tilde{\theta}. \end{aligned} \tag{B.4}$$

Combining the results from (B.2), (B.3) and (B.4) concludes the proof with

$$\Phi_k^\theta = L_a(\Phi_k^{(1)} + \Phi_k^{(2)} + \Phi_k^{(3)}).$$

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