Colouring graphs with constraints on connectivity

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Abstract

A graph G has *maximal local edge-connectivity* k if the maximum number of edge-disjoint paths between every pair of distinct vertices x and y is at most k. We prove Brooks-type theorems for k-connected graphs with maximal local edge-connectivity k, and for any graph with maximal local edge-connectivity 3. We also consider several related graph classes defined by constraints on connectivity. In particular, we show that there is a polynomial-time algorithm that, given a 3-connected graph G with maximal local connectivity 3, outputs an optimal colouring for G. On the other hand, we prove, for $k \geq 3$, that k-COLOURABILITY is NP-complete when restricted to minimally k-connected graphs, and 3-COLOURABILITY is NP-complete when restricted to $(k-1)$-connected graphs with maximal local connectivity k. Finally, we consider a parameterization of k-COLOURABILITY based on the number of vertices of degree at least $k+1$, and prove that, even when k is part of the input, the corresponding parameterized problem is FPT.

Keywords: colouring; local connectivity; local edge-connectivity; Brooks' theorem; minimally k-connected; vertex degree.

1 Introduction

We consider the problem of finding a proper vertex k-colouring for a graph for which, loosely speaking, the "connectivity" is somehow constrained. For example, if we consider the class of

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graphs of degree at most k, then, by Brooks' theorem, it is easy to find if a graph in this class is k-colourable.

Theorem 1.1 (Brooks, 1941)**.** *Let G be a connected graph with maximum degree k. Then G is k-colourable if and only if G is not a complete graph or an odd cycle.*

On the other hand, if we consider the class of graphs with maximum degree 4, then the decision problem 3-COLOURABILITY is well known to be NP-complete, even when restricted to planar graphs [9]. Moreover, for any fixed $k \geq 3$, k-COLOURABILITY is NP-complete.

The classes we consider are defined using the notion of local connectivity. The *local connectivity* $\kappa(x, y)$ of distinct vertices x and y in a graph is the maximum number of internally vertex-disjoint paths between x and y. The *local edge-connectivity* $\lambda(x, y)$ of distinct vertices x and y is the maximum number of edge-disjoint paths between x and y. Consider the following classes:

- $\mathcal{C}_0^k$: graphs with maximum degree k,
- $\mathcal{C}_1^k$: graphs such that $\lambda(x, y) \leq k$ for all pairs of distinct vertices x and y,
- $\mathcal{C}_2^k$: graphs such that $\kappa(x, y) \leq k$ for all pairs of distinct vertices x and y, and
- $\mathcal{C}_3^k$: graphs such that $\kappa(x, y) \leq k$ for all edges xy.

In each successive class, the connectivity constraint is relaxed; that is, $\mathcal{C}_0^k \subseteq \mathcal{C}_1^k \subseteq \mathcal{C}_2^k \subseteq \mathcal{C}_3^k$. For each class, there is a bound on the chromatic number; we give details shortly. Note also that each of the four classes is closed under taking subgraphs.

A graph G is *k-connected* if it has at least 2 vertices and $\kappa(x, y) \geq k$ for all distinct $x, y \in V(G)$. The *connectivity* of a graph G is the maximum integer k such that G is k-connected. A graph contained in one of the above classes has connectivity at most k. So, for each class, it may be of interest to start by considering the graphs that have connectivity precisely k. For each class $\mathcal{C}_i^k$, we denote by $\widehat{\mathcal{C}}_i^k$ the subclass containing the k-connected members of $\mathcal{C}_i^k$. A Hasse diagram illustrating the partial ordering of these classes under set inclusion is given in Figure 1.

A graph in $\mathcal{C}_1^k$ is said to have *maximal local edge-connectivity* k. Our first main result is a Brooks-type theorem for graphs with maximal local edge-connectivity k. An *odd wheel* is a graph obtained from a cycle of odd length by adding a vertex that is adjacent to every vertex of the cycle.

Theorem 1.2. *Let G be a k-connected graph with maximal local edge-connectivity k, for $k \geq 3$. Then G is k-colourable if and only if G is not a complete graph or an odd wheel.*

Note that an odd wheel is not 4-connected, so the condition that G is not an odd wheel is only required when $k = 3$.

Although every graph with maximum degree k has maximal local edge-connectivity k, Theorem 1.2 is not, strictly speaking, a generalisation of Brooks' theorem, since it only concerns such graphs that are k-connected. However, for $k = 3$ we prove an extension of Brooks' theorem that characterises which graphs with maximal local edge-connectivity 3 are 3-colourable, with no requirement on 3-connectivity.

Let G_1 and G_2 be graphs and, for $i \in \{1, 2\}$, let (u_i, v_i) be an ordered pair of adjacent vertices of G_i. We say that the *Hajós join* of G_1 and G_2 with respect to (u_1, v_1) and (u_2, v_2) is the graph obtained by deleting the edges u_1v_1 and u_2v_2 from G_1 and G_2, respectively, identifying the vertices u_1 and u_2, and adding a new edge joining v_1 and v_2. A *block* of a graph G is a maximal connected subgraph of G for which every two vertices or edges are contained in a simple cycle.

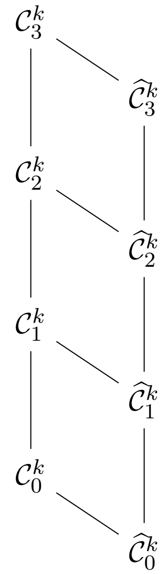


Figure 1: Hasse diagram of the graph classes defined by constraints on connectivity under $\subseteq$.

Theorem 1.3. *Let G be a graph with maximal local edge-connectivity* 3*. Then G is* 3*-colourable if and only if each block of G cannot be obtained from an odd wheel by performing a (possibly empty) sequence of Hajós joins with an odd wheel.*

For convenience, we call a graph that can be obtained from an odd wheel by performing a sequence of Hajós joins with odd wheels, a *wheel morass*. Suppose that G_1 and G_2 are wheel morasses. It can be shown, by a routine induction argument, that the Hajós join of G_1 and G_2 is itself a wheel morass.

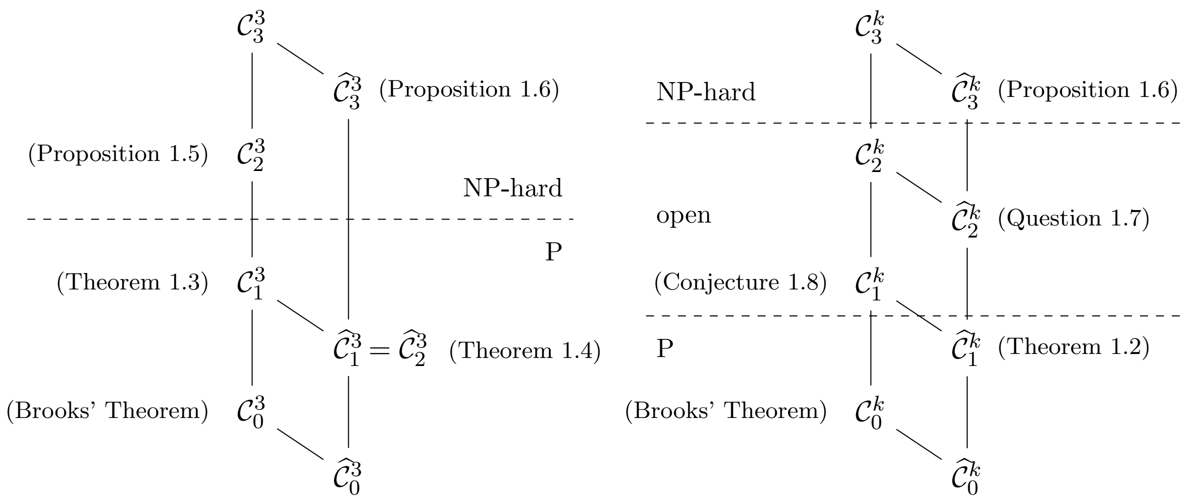
It follows from Theorems 1.2 and 1.3 that there is a polynomial-time algorithm that finds a k-colouring for a k-connected graph with maximal local edge-connectivity k, or determines that no such colouring exists; and there is a polynomial-time algorithm for finding an optimal colouring of any graph with maximal local edge-connectivity 3.

A graph in $\mathcal{C}_2^k$ is also said to have *maximal local connectivity* k. These graphs have been studied previously; primarily, the problem of determining bounds on the maximum number of possible edges in a graph with n vertices and maximal local connectivity k has received much attention (see [2, 14, 20, 23]). Note that for a k-connected graph G with maximal local connectivity k (that is, for G in $\widehat{\mathcal{C}}_2^k$), we have $\kappa(x, y) = k$ for all distinct $x, y \in V(G)$. When $k = 3$, it turns out that $\widehat{\mathcal{C}}_1^3 = \widehat{\mathcal{C}}_2^3$ (see Lemma 4.1). This leads to the following:

Theorem 1.4. *Let G be a* 3*-connected graph with maximal local connectivity* 3*. Then G is* 3*-colourable if and only if G is not an odd wheel. Moreover, there is a polynomial-time algorithm that finds an optimal colouring for G.*

However, we give an example in Section 4 to demonstrate that $\widehat{\mathcal{C}}_1^4 \neq \widehat{\mathcal{C}}_2^4$ (see Figure 5).

The class $\widehat{\mathcal{C}}_3^k$ is well known. A graph G is *minimally k-connected* if it is k-connected and the removal of any edge leads to a graph that is not k-connected. It is easy to check that a graph is in $\widehat{\mathcal{C}}_3^k$ if and only if it is minimally k-connected (see, for example, [2, Lemma 4.2]).



(a) $k = 3$ (b) $k \geq 4$

Figure 2: k-COLOURING complexity for graph classes defined by constraints on connectivity.

We now review known results regarding the bounds on the chromatic number of these classes. Mader proved that any graph with at least one edge contains a pair of adjacent vertices whose local connectivity is equal to the minimum of their degrees [21]. It follows that any graph in $\mathcal{C}_3^k$ has a vertex of degree at most k. This, in turn, implies that a graph in $\mathcal{C}_3^k$ is $(k+1)$-colourable. In particular, minimally k-connected graphs, and graphs with maximal local connectivity k, are all $(k+1)$-colourable.

Despite these results, it seems that, so far, the tractability of computing the chromatic number, or finding a k-colouring, for a graph in one of these classes has not been investigated. For fixed k, let k-COLOURING be the search problem that, given a graph G, finds a k-colouring for G, or determines that none exists. An overview of our findings in this paper is given in Figure 2, where we illustrate the complexity of k-COLOURING when restricted to the various classes defined by constraints on connectivity.

If $k = 1$, then $\mathcal{C}_3^k$ is the class of forests, so all the classes are trivial. For $k = 2$, since it is easy to determine if a graph is 2-colourable, and all graphs in $\mathcal{C}_3^k$ are 3-colourable, we may compute the chromatic number of any graph in $\mathcal{C}_3^k$ in polynomial time.

When $k = 3$, Theorem 1.4 implies that 3-COLOURING is polynomial-time solvable when restricted to $\widehat{\mathcal{C}}_2^3$. For the class $\mathcal{C}_1^3$, this problem remains polynomial-time solvable, by Theorem 1.3. One might hope to generalise these results in one of two other possible directions: to the class $\mathcal{C}_2^3$, or to $\widehat{\mathcal{C}}_3^3$. But any such attempt is likely to fail, due to the following results (see Sections 4 and 5 respectively):

Proposition 1.5. *For fixed $k \geq 3$, the problem of deciding if a $(k-1)$-connected graph with maximal local connectivity k is 3-colourable is NP-complete.*

Proposition 1.6. *For fixed $k \geq 3$, the problem of deciding if a minimally k-connected graph is k-colourable is NP-complete.*

Now consider when $k \geq 4$. It follows from Theorem 1.2 that k-COLOURING is polynomial-time

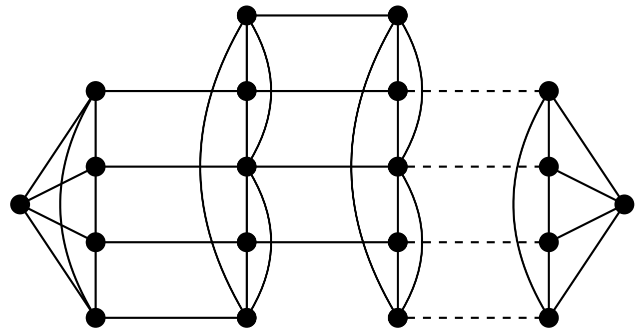


Figure 3: A 4-connected graph with maximal local edge-connectivity 4, and arbitrarily many vertices of degree more than 4.

solvable when restricted to $\widehat{\mathcal{C}}_1^k$. However, the complexity for the more general class $\widehat{\mathcal{C}}_2^k$ remains an interesting open problem:

Question 1.7. *For fixed $k \geq 4$, is there a polynomial-time algorithm that, given a k-connected graph G with maximal local connectivity k, finds a k-colouring of G, or determines that none exists?*

We also show that 3-COLOURABILITY is NP-complete for a graph in $\mathcal{C}_1^k$, when $k \geq 4$, so computing the chromatic number for a graph in this class, or in $\mathcal{C}_2^k$, is NP-hard, as is 3-COLOURING. However, the complexity of k-COLOURING (or k-COLOURABILITY) for these classes is unresolved. We make the following conjecture:

Conjecture 1.8. *For fixed $k \geq 4$, there is a polynomial-time algorithm that, given a graph G with maximal local edge-connectivity k, finds a k-colouring of G, or determines that none exists.*

It is worth noting that the class $\widehat{\mathcal{C}}_1^k$ is non-trivial. All k-connected k-regular graphs are members of the class, as are k-connected graphs with $n-1$ vertices of degree k and a single vertex of degree more than k. A member of the class can have arbitrarily many vertices of degree at least $k+1$. To see this for $k=3$, consider a graph $G'_{3,x}$, for $x \geq 3$, that is obtained from a grid graph $G_{3,x}$ (the Cartesian product of path graphs on 3 and x vertices) by adding two vertex-disjoint edges linking vertices of degree 2 at distance 2. The graph $G'_{3,x}$ is in $\widehat{\mathcal{C}}_1^3$, and has $x-2$ vertices of degree 4. A similar example can be constructed for any $k > 3$; for example, see Figure 3 for when $k=4$.

Finally, we consider a parameterization of k-COLOURING based on the number p_k of vertices of degree at least $k+1$. By Brooks' theorem, a graph G for which $p_k(G) = 0$ can be k-coloured in polynomial time, unless it is a complete graph or an odd cycle. We extend this to larger values of p_k, showing that, even when k is part of the input, finding a k-colouring for a graph is fixed-parameter tractable (FPT) when parameterized by p_k.

Theorem 1.9. *Let G be a graph with at most p vertices of degree more than k. There is a $\min\{k^p, p^p\} \cdot O(n+m)$-time algorithm for k-colouring G, or determining no such colouring exists.*

This paper is structured as follows. In the next section, we give preliminary definitions. In Section 3, we consider graphs with maximal local edge-connectivity k, and prove Theorems 1.2 and 1.3. We then consider the more general class of graphs with maximal local connectivity k, in Section 4, and prove Theorem 1.4 and Proposition 1.5. We present the proof of Proposition 1.6 in Section 5. Finally, in Section 6, we consider the problem of k-colouring a graph parameterized by the number of vertices of degree at least $k+1$, and prove Theorem 1.9.

2 Preliminaries

Our terminology and notation follows [3] unless otherwise specified. Throughout, we assume all graphs are simple. We say that paths are *internally disjoint* if they have no internal vertices in common. A *k-edge cut* is a k-element set $S \subseteq E(G)$ for which $G \backslash S$ is disconnected. A *k-vertex cut* is a k-element subset $Z \subseteq V(G)$ for which $G - Z$ is disconnected. We call the vertex of a 1-vertex cut a *cut-vertex*. For distinct non-adjacent vertices x and y, and $Z \subseteq V(G) \setminus \{x, y\}$, we say that *$Z$ separates x and y* when x and y belong to different components of $G - Z$. More generally, for disjoint, non-empty $X, Y, Z \subseteq V(G)$, we say that *Z separates X and Y* if, for each $x \in X$ and $y \in Y$, the vertices x and y are in different components of $G - Z$. We call a partition (X, Z, Y) of $V(G)$ a *k-separation* if $|Z| \leq k$ and Z separates X from Y. When G is k-connected and (X, Z, Y) is a k-separation of G, we have that $|Z| = k$. By Menger's theorem, if $\kappa(x, y) = k$ for non-adjacent vertices x and y, then there is a k-vertex cut that separates x and y. If $\kappa(x, y) = k \geq 2$ for adjacent vertices x and y, then there is a $(k-1)$-vertex cut in $G \backslash xy$ that separates x and y. We use these freely in the proof of Lemma 4.1.

We view a proper k-colouring of a graph G as a function $\phi : V(G) \to \{1, 2, \ldots, k\}$ where for every $uv \in E(G)$ we have $\phi(u) \neq \phi(v)$. For $X \subseteq V(G)$, we write $\phi(X)$ to denote the image of X under ϕ.

Given graphs G_1 and G_2, the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ is denoted $G_1 \cup G_2$.

A *diamond* is a graph obtained by removing an edge from K_4. We call the two degree-2 vertices of a diamond D the *pick* vertices of D.

3 Graphs with maximal local edge-connectivity k

In this section we prove Theorems 1.2 and 1.3.

Lovász provided a short proof of Brooks' theorem in [18]. The proof can easily be adapted to show that graphs with at most one vertex of degree more than k are often k-colourable. We make this precise in the next lemma; the proof is provided for completeness. A vertex is *dominating* if it is adjacent to every other vertex of the graph.

Lemma 3.1. *Let G be a 3-connected graph with at most one vertex of degree more than k, for $k \geq 3$, and no dominating vertices. Then G is k-colourable.*

Proof. Let h be a vertex of G with maximum degree. Since G has no dominating vertices and is connected, there is a vertex y at distance two from h. Let z_1 be a common neighbour of h and y. Since G is 3-connected, $G - \{h, y\}$ is connected. Let $z_1, z_2, \ldots, z_{n-2}$ be a search ordering of $G - \{h, y\}$ starting at z_1; that is, an ordering of $V(G - \{h, y\})$ where each vertex z_i, for $2 \leq i \leq n-2$, has a neighbour z_j with $j < i$. We colour G as follows. Assign h and y the colour 1, say. We can then (greedily) assign one of the k colours to each of $z_{n-2}, z_{n-3}, \ldots, z_2$ in turn, since at the time one of these vertices is considered, it has at most $k - 1$ neighbours that have already been assigned colours. Finally, we can colour z_1, since it has degree at most k, but at least two of its neighbours, h and y, are the same colour. □

Now we show that we can decompose a k-connected graph with maximal local edge-connectivity k into components each containing a single vertex of degree more than k.

Lemma 3.2. *Let G be a k-connected graph with maximal local edge-connectivity k, for $k \geq 3$, and at least two vertices of degree more than k. Then there exists a k-edge cut S such that one component of $G \backslash S$ contains precisely one vertex of degree more than k, and the edges of S are vertex disjoint.*

Proof. We say that a set of vertices $X_1 \subseteq V(G)$ is *good* if $|X_1| \leq n/2$ and $d(X_1) = k$, where $d(X_1)$ is the number of edges with one end in X_1 and the other end in $V(G) \setminus X_1$. If two good sets X_1 and X_2 have non-empty intersection, then $|X_1 \cup X_2| < n$, so $d(X_1 \cup X_2) \geq k$ by k-connectivity. As $d(X_1) + d(X_2) \geq d(X_1 \cup X_2) + d(X_1 \cap X_2)$ (see, for example, [3, Exercise 2.5.4(b)]), it follows that $d(X_1 \cap X_2) = k$. Thus, if a good set X_1 meets a good set X_2, then $X_1 \cap X_2$ is also good. This implies that if a vertex of degree more than k is in a good set, then there is unique minimal good set containing it. Since there is a k-edge cut between any two vertices, one of any two vertices is in a good set. Thus, all but at most one vertex of G is in a good set. Let X be a minimal good set containing at least one vertex of degree more than k. Suppose X contains distinct vertices x and y, each with degree more than k. Then there is k-edge cut separating them, so there is a good set containing exactly one of them. By taking the intersection of this good set with X, we obtain a good set that is a proper subset of X and contains at least one vertex of degree more than k; a contradiction. So X contains precisely one vertex of degree more than k. Now $d(X) = k$, since X is good, hence the k edges with one end in X and the other in $E(G) - X$ give an edge cut S.

It remains to show that the edges of S are vertex disjoint. Set $Y = V(G) \setminus X$, and let X_S (respectively, Y_S) be the set of vertices of X (respectively, Y) incident to an edge of S. Let $|X| = q$. Since every vertex in X has degree at least k, and X contains some vertex of degree more than k, we have that $\Sigma_{v \in X} d(v) \geq qk + 1$. If $q \leq k$, then, since each vertex in X has at most $q - 1$ neighbours in X, we have that $\Sigma_{v \in X} d(v) \leq q(q-1) + k \leq k(q-1) + k = qk$; a contradiction. So $X_S \neq X$ and, similarly, $Y_S \neq Y$. Now, since G is k-connected, there are k internally disjoint paths from any vertex in $X \setminus X_S$ to any vertex in $Y \setminus Y_S$. Each of these paths must contain a different edge of S. Thus S satisfies the requirements of the lemma. □

Next we show, loosely speaking, that if a graph G has a k-edge cut S where the edges in S have no vertices in common, then the problem of k-colouring G can essentially be reduced to finding k-colourings of the components of $G \backslash S$; the only bad case is when the vertices incident to S are coloured all the same colour in one component, and all different colours in the other.

Lemma 3.3. *Let G be a connected graph with a k-edge cut S, for $k \geq 3$, such that the edges of S are vertex-disjoint, and $G \backslash S$ consists of two components G_1 and G_2. Let V_i be the set of vertices in $V(G_i)$ incident to an edge of S, for $i \in \{1, 2\}$.*

(i) *Then G is k-colourable if and only if there exists a k-colouring ϕ_1 of G_1 and a k-colouring ϕ_2 of G_2 such that $\{|\phi_1(V_1)|, |\phi_2(V_2)|\} \neq \{1, k\}$.*

(ii) *Moreover, if ϕ_1 and ϕ_2 are k-colourings of G_1 and G_2, respectively, for which $\{|\phi_1(V_1)|, |\phi_2(V_2)|\} \neq \{1, k\}$, then there exists a permutation σ such that*

$$\phi(x) = \begin{cases} \phi_1(x) & \text{for } x \in V(G_1), \\ \sigma(\phi_2(x)) & \text{for } x \in V(G_2) \end{cases}$$

is a k-colouring of G.

Proof. First, we prove (ii), which implies that (i) holds in one direction. Let ϕ_1 and ϕ_2 be k-colourings of G_1 and G_2, respectively, for which $\{|\phi_1(V_1)|, |\phi_2(V_2)|\} \neq \{1, k\}$. We will construct an auxiliary graph H where the vertices are labelled by subsets of V_1 or V_2 in such a way that if we can k-colour H, then there exists a permutation σ such that ϕ, as defined in the statement of the lemma, is a k-colouring of G.

Let $(T_1, T_2, \ldots, T_{|\phi_1(V_1)|})$ be the partition of the vertices in V_1 into colour classes with respect to ϕ_1 and, likewise, let $(W_1, W_2, \ldots, W_{|\phi_2(V_2)|})$ be the partition of V_2 into colour classes with respect to ϕ_2. We construct a graph H consisting of $|\phi_1(V_1)|+|\phi_2(V_2)|$ vertices: for each $i \in \{1, 2, \ldots, |\phi_1(V_1)|\}$, we have a vertex $t_i \in V(H)$ labelled by T_i, and, for each $i \in \{1, 2, \ldots, |\phi_2(V_2)|\}$, we have a vertex $w_i \in V(H)$ labelled by W_i. Let $T = \{t_i : 1 \leq i \leq |\phi_1(V_1)|\}$ and let $W = \{w_i : 1 \leq i \leq |\phi_2(V_2)|\}$. Each $t \in T$ (respectively, $w \in W$) is adjacent to every vertex in $T - \{t\}$ (respectively, $W - \{w\}$). Finally, for each edge v_1v_2 in S, we add an edge between the vertex $t \in T$ labelled by the colour class containing v_1, and the vertex $w \in W$ labelled by the colour class containing v_2, omitting parallel edges. Thus there are at most k edges between vertices in T and vertices in W.

Now we show that H is k-colourable. Consider a vertex $t \in T$. If it has x neighbours in W, then it represents a colour class consisting of at least x vertices of V_1. So there are at most $k - x$ vertices in $T - \{t\}$, and hence t has degree at most $x + (k - x)$. It follows, by Brooks' theorem, that H is k-colourable unless it is a complete graph, as $k \geq 3$. Moreover, if $|V(H)| \leq k$, then H is k-colourable, so assume that $|V(H)| > k$. Then, without loss of generality, we may assume that $|T| > k/2$. Since there are at most k edges between vertices in T and vertices in W, and each vertex of T has the same number of neighbours in W, it follows that each vertex in T has a single neighbour in W. Since H is a complete graph, we have $|W| = 1$, and hence, recalling that $|V(H)| > k$, we have $|T| = k$. That is, $|\phi_1(V_1)| = k$ and $|\phi_2(V_2)| = 1$; a contradiction.

Now H is k-colourable. By permuting the colours of a k-colouring of H, we can obtain a k-colouring ψ such that $\psi|_{V_1} = \phi_1$. Then $\psi|_{V_2}$ induces a permutation σ of ϕ_2, in the obvious way, with the desired properties. This completes the proof of (ii).

Finally, we observe that when $\{|\phi_1(V_1)|, |\phi_2(V_2)|\} = \{1, k\}$ for every k-colouring ϕ_1 of G_1 and every k-colouring ϕ_2 of G_2, then G is not k-colourable. This completes the proof of (i). $\square$

Suppose that a graph G has a k-edge cut S that separates X from Y, where (X, Y) is a partition of $V(G)$. We fix the following notation for the remainder of this section. Let Y_S (respectively, X_S) be the subset of Y (respectively, X) consisting of vertices incident to an edge in S. Let G_X (respectively, G_Y) be the graph obtained from $G[X \cup Y_S]$ (respectively, $G[Y \cup X_S]$) by adding edges so that Y_S (respectively, X_S) is a clique.

Lemma 3.4. *Let G be a k-connected graph, for $k \geq 3$, with maximal local edge-connectivity k, and a k-edge cut S that separates X from Y, where (X, Y) partitions $V(G)$. Then G_X is k-connected and has maximal local edge-connectivity k.*

Proof. First we show that G_X has maximal local edge-connectivity k. The only vertices of degree more than k in G_X are in X. Suppose u and v are vertices in X of degree more than k. Clearly, for each uv-path in $G_X[X]$ there is a corresponding uv-path in $G[X]$. We show that there are at least as many edge-disjoint uv-paths that pass through an edge of S in G as there are in G_X; it follows that $\lambda_{G_X}(u, v) \leq \lambda_G(u, v) \leq k$. Since S is a k-edge cut in G_X, there are at most $\lfloor k/2 \rfloor$ edge-disjoint paths in G_X starting and ending at a vertex in X_S. Let y be a vertex in Y. Since G is k-connected, the Fan Lemma (see, for example, [3, Proposition 9.5]) implies that there are k paths from y to each member of X_S that meet only in y. Hence, there are $\lfloor k/2 \rfloor$ edge-disjoint paths in

$G[Y \cup X_S]$ starting and ending at a vertex in X_S. Thus, we deduce that G_X has maximal local edge-connectivity k.

We now show that G_X is k-connected, by demonstrating that $\kappa_{G_X}(u,v) \geq k$ for all distinct $u,v \in V(G_X)$. First, suppose that $u,v \in X$. Evidently, for each uv-path in $G[X]$ there is a corresponding uv-path in $G_X[X]$. Moreover, each uv-path in G that traverses an edge of S traverses two such edges xy and $x'y'$, say, where $x,x' \in X_S$ and $y,y' \in Y_S$. By replacing the $x'y'$-path in G with the edge $x'y'$ in G_X, we obtain a uv-path of G_X. We deduce that $\kappa_{G_X}(u,v) \geq \kappa_G(u,v) \geq k$ for any $u,v \in X$. Now suppose $u,v \in Y_S$. Then there are $k-1$ internally disjoint uv-paths in $G_X[Y_S]$. Pick $u',v' \in X_S$ such that uu' and vv' are in S. Since $G_X[X]$ is connected, there is at least one $u'v'$-path in $G_X[X]$, so there are k internally disjoint uv-paths in G_X. Finally, let $u \in X$ and $v \in Y_S$. Since G is k-connected, the Fan Lemma implies that there are k paths from u to each vertex of Y_S in G that meet only in u. Hence there are k such paths in G_X. Since Y_S is a clique in G_X, there are k internally disjoint uv-paths in G_X. Thus $\kappa_{G_X}(u,v) \geq k$ for all distinct $u,v \in V(G_X)$, as required. □

Proposition 3.5. *Let G be a k-connected graph, for $k \geq 3$, with maximal local edge-connectivity k and at least two vertices of degree more than k. Then G is k-colourable.*

Proof. The proof is by induction on the number of vertices of degree more than k. First we show that the proposition holds when G has precisely two vertices of degree more than k. Let x and y be distinct vertices of G with degree more than k. By Lemma 3.2, there is a k-edge cut S that separates X from Y, where $x \in X$, $y \in Y$, (X,Y) is a partition of $V(G)$, and X contains precisely one vertex of degree more than k. Consider the graph G_X; this graph is 3-connected by Lemma 3.4, and has no dominating vertices by definition. Hence, by Lemma 3.1, G_X is k-colourable. Moreover, in such a k-colouring, the vertices in Y_S are given k different colours, since they form a k-clique, and hence the vertices in X_S are not all the same colour. So $G_X[X] = G[X]$ is k-colourable in such a way that the vertices in X_S are not all the same colour. By symmetry, $G[Y]$ is k-colourable in such a way that the vertices in Y_S are not all the same colour. It follows, by Lemma 3.3, that G is k-colourable.

Now let G be a graph with p vertices of degree more than k, for $p > 2$. We assume that a k-connected graph with maximal local edge-connectivity k, and $p-1$ vertices of degree more than k is k-colourable. By Lemma 3.2, there is a k-edge cut S that separates X from Y, where X contains precisely one vertex x of degree more than k, and (X,Y) is a partition of $V(G)$. The graph G_Y is k-connected and has maximal local edge-connectivity k, by Lemma 3.4. Thus, by the induction assumption, G_Y is k-colourable. It follows that $G[Y]$ is k-colourable in such a way that the vertices in Y_S are not all the same colour. The graph G_X is 3-connected, by Lemma 3.4, so is k-colourable, by Lemma 3.1. So $G[X]$ is k-colourable in such a way that the vertices in X_S are not all the same colour. Thus, by Lemma 3.3, G is k-colourable. The proposition follows by induction. □

Proof of Theorem 1.2. Clearly if G is a complete graph, then G is K_{k+1} and is not k-colourable. If G is an odd wheel, then, since G is not 4-connected, we have $k = 3$, and G is not 3-colourable. This proves one direction. Now suppose G is not k-colourable and has p vertices of degree more than k. Then $p < 2$, by Proposition 3.5. If $p = 0$, then G is a complete graph, by Brooks' theorem (an odd cycle is not k-connected for any $k \geq 3$). If $p = 1$, then G has a dominating vertex v, by Lemma 3.1. Since $G - \{v\}$ is not $(k-1)$-colourable, and $G - \{v\}$ has maximum degree $k-1$, it follows, by Brooks' theorem, that $G - \{v\}$ is a complete graph or an odd cycle. Thus G is a complete graph or an odd wheel. □

Corollary 3.6. *Let G be a k-connected graph with maximal local edge-connectivity k. There is a polynomial-time algorithm that finds a k-colouring for G when G is k-colourable, or a $(k+1)$-colouring otherwise.*

Proof. Suppose G has at most one vertex of degree more than k. If G has no dominating vertices, then the proof of Lemma 3.1 leads to an algorithm for k-colouring G. Otherwise, when G has a dominating vertex v, the problem reduces to finding a $(k-1)$-colouring for $G - \{v\}$, where $G - \{v\}$ has maximum degree $k-1$. In either case, we have a linear-time algorithm for colouring G.

When G has at least two vertices x and y of degree more than k, we use the approach taken in the proof of Proposition 3.5. We can find a k-edge cut S that separates x and y in $O(km)$ time, by an application of the Ford-Fulkerson algorithm. Without loss of generality, x is contained in a component of $G\backslash S$ with at most $n/2$ vertices. It follows, by the proof of Lemma 3.2, that with $O(n)$ applications of the Ford-Fulkerson algorithm we can obtain an edge cut S' such that x is the only vertex of degree more than k in one component X of $G\backslash S'$. Thus we can find the desired k-edge cut S' in $O(knm) = O(nm)$ time. Let $Y = V(G) \setminus X$, and let G_X and G_Y be as defined just prior to Lemma 3.4. As G_X is 3-connected by Lemma 3.4, and has no dominating vertices by definition, we can find a k-colouring ϕ_X for G_X in linear time by Lemma 3.1. To find a k-colouring ϕ_Y for G_Y, if one exists, we repeat this process recursively. Then, by Lemma 3.3, we can extend ϕ_Y to a k-colouring of G by finding a permutation for ϕ_X, which can be done in constant time. When G has p vertices of degree more than k, this process takes $O(pnm)$ time. Since $p \leq n$, the algorithm runs in $O(n^2m)$ time. □

An extension of Brooks' theorem when $k = 3$

We now work towards proving Theorem 1.3. Recall that a wheel morass is either an odd wheel, or a graph that can be obtained from odd wheels by applying the Hajós join. We restate the theorem here in terms of wheel morasses:

Theorem 3.7. *Let G be a graph with maximal local edge-connectivity 3. Then G is 3-colourable if and only if each block of G is not a wheel morass.*

Let us now establish some properties of wheel morasses. A graph G is *k-critical* if $\chi(G) = k$ and every proper subgraph H of G has $\chi(H) < k$.

Proposition 3.8. *Let G be a wheel morass. Then*

(i) *G is 4-critical, and*

(ii) *for every two distinct vertices x and y, we have $\lambda(x, y) \geq 3$.*

Proof. (i) It is well known that the Hajós join of two k-critical graphs is k-critical (see, for example, [3, Exercise 14.2.9]). Since the odd wheels are 4-critical, we immediately get, by induction, that every wheel morass is 4-critical.

(ii) We prove this by induction on the number of Hajós joins. The result can easily be checked for odd wheels.

Assume now that G is the Hajós join of G_1 and G_2 with respect to (u_1, v_1) and (u_2, v_2). Let x and y be two vertices in G. If $x \in V(G_1)$ and $y \in V(G_1)$, then, by the induction hypothesis, there are three edge-disjoint xy-paths in G_1. If one them contains v_1u_1, then replace it by the concatenation of v_1v_2 and a v_2u_2-path in $G_2\backslash u_2v_2$ (such a path exists since $\lambda_{G_2}(u_2, v_2) \geq 3$ by the induction hypothesis). This results in three edge-disjoint xy-paths, so $\lambda_G(x,y) \geq 3$. Likewise, if $x \in V(G_2)$ and $y \in V(G_2)$, then $\lambda_G(x,y) \geq 3$.

Assume now that $x \in V(G_1)$ and $y \in V(G_2)$. Let us prove the following:

Claim 3.8.1. *In $G_1 \backslash u_1v_1$, there are three edge-disjoint paths P_1, P_2 and P_3 such that P_1 and P_2 are xu_1-paths and P_3 is an xv_1-path.*

Proof. By the induction hypothesis, there are three edge-disjoint xu_1-paths R_1, R_2, R_3 in G_1. If $v_1 \in V(R_1) \cup V(R_2) \cup V(R_3)$, then we may assume, without loss of generality, that $v_1 \in V(R_3)$ and $u_1v_1 \notin E(R_1) \cup E(R_2)$. Hence R_1, R_2 and the xv_1-subpath of R_3 are the desired paths. Now we may assume that $v_1 \notin V(R_1) \cup V(R_2) \cup V(R_3)$. Let Q be a shortest path from $z_1 \in V(R_1) \cup V(R_2) \cup V(R_3)$ to v_1 in $G \backslash u_1v_1$ (such a path exists by our connectivity assumption). Without loss of generality, $z_1 \in V(R_3)$. Hence the desired paths are R_1, R_2 and the concatenation of the xz_1-subpath of R_3 and Q. This proves Claim 3.8.1. □

By Claim 3.8.1 and symmetry, there are three edge-disjoint paths Q_1, Q_2 and Q_3 in $G_2 \backslash u_2v_2$ such that Q_1 and Q_2 are u_2y-paths and Q_3 is a v_2y-path. The paths obtained by concatenating P_1 and Q_1; P_2 and Q_2; and P_3, v_1v_2 and Q_3 are three edge-disjoint xy-paths in G, so $\lambda_G(x,y) \geq 3$. □

Proof of Theorem 1.3. If a block of G is a wheel morass, then this block has chromatic number 4 by Proposition 3.8(i), and thus $\chi(G) \geq 4$.

Conversely, assume that no block of G is a wheel morass. We will show that G is 3-colourable by induction on the number of vertices. We may assume that G is 2-connected (since if each block is 3-colourable, then it is straightforward to piece these 3-colourings together to obtain a 3-colouring of G). Moreover, if G is 3-connected, then the result follows from Theorem 1.2 since G is not an odd wheel. Henceforth, we assume that G is not 3-connected.

Let $(A, \{x,y\}, B)$ be a 2-separation of $V(G)$. Let H_A (respectively, H_B) be the graph obtained from $G_A = G[A \cup \{x,y\}]$ (respectively, $G_B = G[B \cup \{x,y\}]$) by adding an edge xy if it does not exist. Observe that since G is 2-connected, there is at least one xy-path in G_B, so H_A (and, similarly, H_B) has maximal local edge-connectivity 3.

Assume first that neither H_A nor H_B are wheel morasses. By the induction hypothesis, both H_A and H_B are 3-colourable. Thus, by piecing together a 3-colouring of H_A and a 3-colouring of H_B in both of which x is coloured 1 and y is coloured 2, we obtain a 3-colouring of G.

Henceforth, we may assume that H_A or H_B is a wheel morass. Without loss of generality, we assume that H_A is a wheel morass. Observe first that $xy \notin E(G)$. Indeed, if $xy \in E(G)$, then $\lambda_{H_A}(x,y) \leq 2$, since there is an xy-path in $G_B \backslash xy$, as G is 2-connected. Hence, by Proposition 3.8(ii), H_A is not a wheel morass; a contradiction.

Furthermore, Proposition 3.8(ii) implies that there are three edge-disjoint xy-paths in H_A, two of which are in G_A. Now, since $\lambda_G(x,y) \leq 3$, it follows that $\lambda_{G_B}(x,y) \leq 1$. But G_B is connected, since G is 2-connected, so there exists an edge $x'y'$ such that $G_B \backslash x'y'$ has two components: one, G_x, containing both x and x'; and the other, G_y, containing y and y'. We now distinguish two cases depending on whether or not $x = x'$ or $y = y'$.

- Assume first that $x \neq x'$ and $y \neq y'$. Let H_x (respectively, H_y) be the graph obtained from G_x (respectively, G_y) by adding the edge xx' (respectively, yy'), if it does not exist. Observe that the concatenation of an xy-path in G_A, a yy'-path in G_y, and $y'x'$ is a non-trivial xx'-path in G whose internal vertices are not in $V(G_x)$. Hence $\lambda_{G_x}(x,x') \leq 2$, so H_x has maximal local edge-connectivity 3. Moreover, G_x is not a wheel morass, by Proposition 3.8(ii), and hence G_x is 3-colourable, by the induction hypothesis. Let J be the graph obtained from $G - (V(G_x) \setminus \{x\})$ by adding the edge xy'. Since there is an xx'-path in G_x, the graph J has maximal local edge-connectivity 3. Hence, by the induction hypothesis, either J is 3-colourable or J is a wheel morass. In both cases, $G - (V(G_x) \setminus \{x\})$ is 3-colourable, by Proposition 3.8(i).

Suppose that $xx' \in E(G)$. Then, in every 3-colouring of G_x, the vertices x and x' have different colours. Consequently, one can find a 3-colouring c_1 of G_x and a 3-colouring c_2 of $G - (V(G_x) \setminus \{x\})$ such that $c_1(x) = c_2(x)$ and $c_1(x') \neq c_2(y')$. The union of these two colourings is a 3-colouring of G. Similarly, the result holds if $yy' \in E(G)$.

Henceforth, we may assume that xx' and yy' are not edges of G. If both H_x and H_y are wheel morasses, then G is also a wheel morass, obtained by taking the Hajós join of H_A and H_x with respect to (x, y) and (x, x'), and then the Hajós join of the resulting graph and H_y with respect to (y, x') and (y, y'). Hence, we may assume that one of H_x and H_y, say H_x, is not a wheel morass. Thus, by the induction hypothesis, H_x admits a 3-colouring c_1, which is a 3-colouring of G_x such that $c_1(x) \neq c_1(x')$. Since $G - (V(G_x) \setminus \{x\})$ is 3-colourable, one can find a 3-colouring c_2 of $G - (V(G_x) \setminus \{x\})$ such that $c_1(x) = c_2(x)$ and $c_1(x') \neq c_2(y')$. The union of c_1 and c_2 is a 3-colouring of G.

- Assume now that $x = x'$ or $y = y'$. Without loss of generality, $x = x'$. Let H_y be the graph obtained from G_y by adding the edge yy', if it does not exist. The graph H_y has maximal local edge-connectivity 3. If H_y is a wheel morass, then G is the Hajós join of H_A and H_y with respect to (y, x) and (y, y'), so G is also a wheel morass; a contradiction. If H_y is not a wheel morass, then by the induction hypothesis H_y admits a 3-colouring c_2, which is 3-colouring of G_y such that $c_2(y) \neq c_2(y')$. Now H_A is a wheel morass, so it is 4-critical by Proposition 3.8(i). Thus G_A admits a 3-colouring c_1 such that $c_1(x) = c_1(y)$. Without loss of generality, we may assume that $c_1(y) = c_2(y)$. Then the union of c_1 and c_2 is a 3-colouring of G. □

Corollary 3.9. *Let G be a graph with maximal local edge-connectivity 3. Then there is a polynomial-time algorithm that finds an optimal colouring for G.*

4 Graphs with maximal local connectivity k

We now consider the more general class of graphs with maximal local (vertex) connectivity k. First, we show that for a 3-connected graph, the notions of maximal local edge-connectivity 3 and maximal local connectivity 3 are equivalent.

Lemma 4.1. *Let G be a 3-connected graph with maximal local connectivity 3. Then G has maximal local edge-connectivity 3.*

Proof. Consider two vertices x and y with four edge-disjoint paths between them. We will show that there is a pair of vertices with four internally disjoint paths between them, contradicting that G has maximal local connectivity 3. First we assume that x and y are not adjacent. Let (X, S, Y) be a 3-separation with $x \in X$ and $y \in Y$ such that X is inclusion-wise minimal. Let $S = \{v_1, v_2, v_3\}$; note that 3-connectivity implies that every vertex in S has a neighbour both in X and Y. Each of

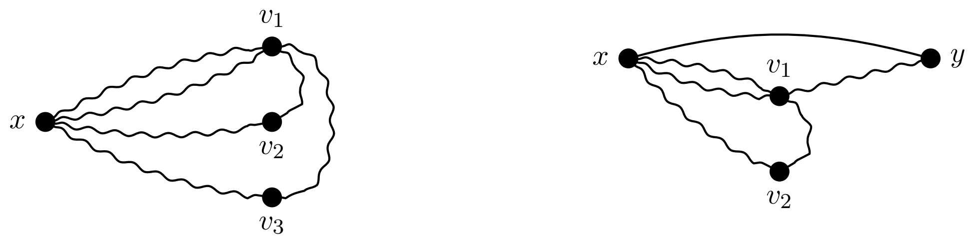


Figure 4: The four internally disjoint xv_1-paths obtained in the proof of Lemma 4.1, when x and y are non-adjacent (left) or adjacent (right). Wiggly lines represent internally disjoint paths.

the four paths has, when going from x to y, a last vertex in $X \cup S$. This vertex has to be in S, so we can assume, without loss of generality, that v_1 is the last such vertex of at least two of the four edge-disjoint paths. This means that v_1 has at least two neighbours in Y.

We will show that there are four internally vertex-disjoint paths in $G[X \cup S]$: two xv_1-paths, an xv_2-path and an xv_3-path. Let G' be the graph obtained from $G[X \cup S]$ by introducing a new vertex v_1' that is adjacent to every neighbour of v_1 in $X \cup S$. If G' contains four paths connecting x and $S' := \{v_1, v_1', v_2, v_3\}$ that meet only in x, then the required four paths exist in $G[X \cup S]$. If there are no four such paths in G', then a max-flow min-cut argument (with x having infinite capacity and every other vertex having unit capacity) shows that there is a set S^* of at most three vertices, with $x \notin S^*$, that separate x and S'. It is not possible that $S^* \subset S'$: then every vertex in the non-empty set $S' \setminus S^*$ remains reachable from x (using that every vertex of S' has a neighbour in X). Therefore, S^* has at least one vertex in X and hence the set of vertices reachable from x in $G' - S^*$ is a proper subset of X. It follows that S^* implies the existence of a 3-separation contradicting the minimality of X.

Next we prove that there are internally disjoint v_1v_2- and v_1v_3-paths in $G[S \cup Y]$. Recall that v_1 has two neighbours in Y. Suppose, towards a contradiction, that given any v_1v_2-path and v_1v_3-path in $G[S \cup Y]$, these paths are not internally disjoint. Then, in $G[S \cup Y]$, there is a cut-vertex w that separates v_1 and $\{v_2, v_3\}$. Since v_1 has two neighbours in Y, there is a vertex $q \in Y$ that is adjacent to v_1 and distinct from w. As w is a cut-vertex in $G[S \cup Y]$, every qv_2- or qv_3-path passes through w. Hence $\{w, v_1\}$ separates q from x in G, contradicting 3-connectivity.

Now there are internally disjoint xv_1-, xv_1-, xv_2-, and xv_3-paths in X and internally disjoint v_1v_2- and v_1v_3-paths in Y. Thus, as shown Figure 4, there are four internally disjoint xv_1-paths, contradicting the fact that the local connectivity $\kappa(x, v_1)$ is at most 3.

A similar argument applies when x and y are adjacent. In this case, $G\backslash xy$ has a 2-vertex cut. Let (X, S, Y) be a 2-separation of $G\backslash xy$ with $x \in X$ and $y \in Y$ such that X is inclusion-wise minimal, and let $S = \{v_1, v_2\}$. Since $G\backslash xy$ is 2-connected, v_1 and v_2 each have a neighbour in X and a neighbour in Y. Each of the three xy-paths in $G\backslash xy$ has a last vertex in S, so we may assume, without loss of generality, that v_1 is the last vertex of at least two of the three, and hence v_1 has at least two neighbours in Y. Let G' be the graph obtained from $G[X \cup S]$ by introducing a new vertex v_1' that is adjacent to every neighbour of v_1 in $X \cup S$, and let $S' = \{v_1, v_1', v_2\}$. If G' does not contain three paths from x to S' that meet only in x, then, by a max-flow min-cut argument as in the case where x and y are not adjacent, we deduce there is a set S^* of at most two vertices that separate x and S'. Since $S^* \not\subset S'$, this contradicts the minimality of X.

It remains to prove that there are internally disjoint v_1y- and v_1v_2-paths in $G[Y \cup S]$. Suppose

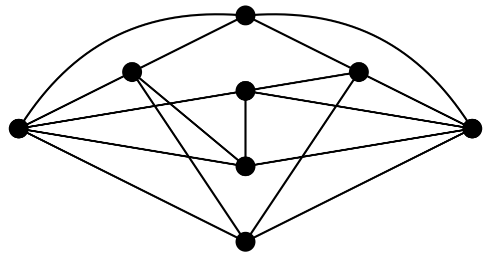


Figure 5: A 4-connected graph with maximal local connectivity 4, but maximal local edge-connectivity 5.

not. Then, in $G[Y \cup S]$, there is a cut-vertex w that separates v_1 and $\{v_2, y\}$. Since v_1 has at least two neighbours in Y, one of these neighbours q is distinct from w. As every qv_2- or qy-path in $G[Y \cup S]$ passes through w, it follows that $\{w, v_1\}$ separates q from x in G, contradicting 3-connectivity. This completes the proof of Lemma 4.1. □

At this juncture, we observe that the proof of Lemma 4.1 relies on properties specific to 3-connected graphs with local connectivity 3. For $k \geq 4$, a k-connected graph with maximal local connectivity k may not have maximal local edge-connectivity k; an example is given in Figure 5.

Theorem 1.4 now follows immediately from Theorem 1.2, Corollary 3.6, and Lemma 4.1. One might hope to generalise this result to all graphs with maximal local connectivity 3, for a result analogous to Theorem 1.3. But this hope will not be realised, unless P=NP, since deciding if a 2-connected graph with maximal local connectivity 3 is 3-colourable is NP-complete. We prove this using a reduction from the unrestricted version of 3-COLOURABILITY. Given an instance of this problem, we replace each vertex of degree at least four with a gadget that ensures that the resulting graph has maximal local connectivity 3. Shortly, we describe this gadget; first, we require some definitions.

We call the graph obtained from two copies of a diamond, by identifying a pick vertex from each, a *serial diamond pair* and denote it D_2. We call the two degree-2 vertices of D_2 the *ends*. A tree is *cubic* if all vertices have either degree one or degree three. A degree-1 vertex is a *leaf*; and an edge that is incident to a leaf is a *pendant* edge, whereas an edge that is incident to two degree-3 vertices is an *internal* edge.

For $l \geq 4$, let T be a cubic tree with l leaves. For each pendant edge xy, we remove xy, take a copy of a diamond D and identify, firstly, the vertex x with one pick vertex of D, and, secondly, y with the other pick vertex of D. For each internal edge xy, we remove xy, take a copy of D_2 and identify, firstly, the vertex x with one end of D_2, and, secondly, y with the other end of D_2. A degree-2 vertex in the resulting graph T' corresponds to a leaf of T; we call such a vertex an *outlet*. We also call T' a *hub gadget* with l outlets. Observe that for any integer $l \geq 4$, there exists a hub gadget with exactly l outlets. When T' is used to replace a vertex h, we say T' is the *hub gadget of* h. An example of a hub gadget with four outlets is shown in Figure 6.

Proposition 4.2. *The problem of deciding if a* 2*-connected graph with maximal local connectivity* 3 *is* 3*-colourable is NP-complete.*

Proof. Let G be an instance of 3-COLOURABILITY. We may assume that G is 2-connected. For each $v \in V(G)$ such that $d(v) \geq 4$, we replace v with a hub gadget with outlets $p_1, p_2, \ldots, p_{d(v)}$,

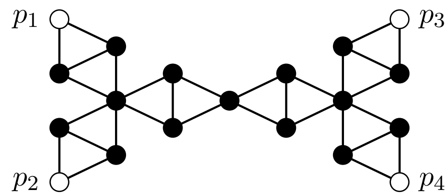


Figure 6: A hub gadget with four outlets p_1, p_2, p_3 and p_4.

such that each neighbour n_i of v in G is adjacent to p_i, for $i \in \{1, 2, \ldots, d(v)\}$. Thus each outlet has degree three in the resulting graph G'.

It is clear that G' is 2-connected. Now we show that G' has maximal local connectivity 3. Clearly $\kappa(x, y) \leq 3$ if $d(x) \leq 3$ or $d(y) \leq 3$. Suppose $d(x), d(y) \geq 4$. Then x and y belong to a hub gadget and are not outlets. So x belongs to either two or three diamonds, each with a pick vertex distinct from x. Let P be the set of these pick vertices. When $y \notin P$, an xy-path must pass through some $p \in P$, so $\kappa(x, y) \leq 3$ as required. Otherwise, x and y are pick vertices of a diamond D, and there are two internally vertex disjoint xy-paths in D. But D is contained in a serial diamond pair D_2, and all other xy-paths must pass through the end of D_2 distinct from x and y. So $\kappa(x, y) \leq 3$, as required.

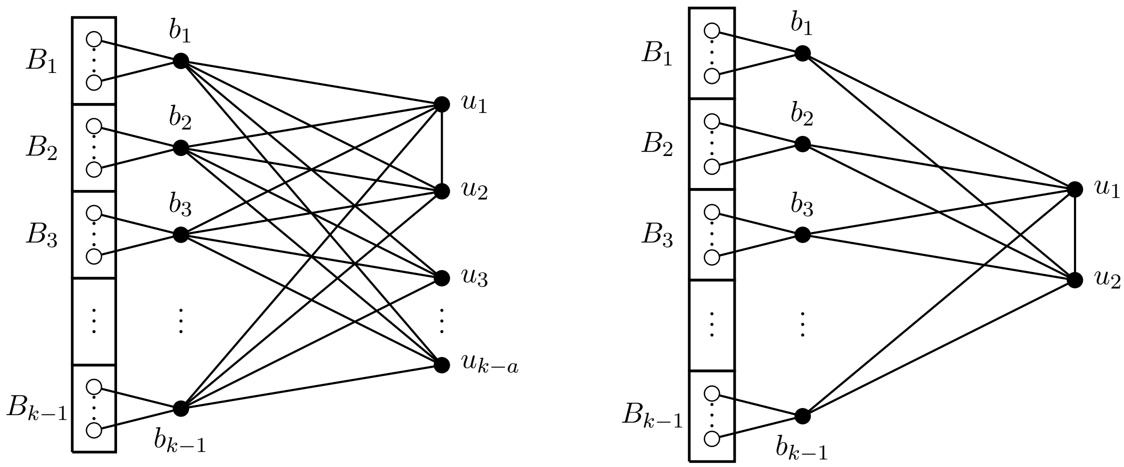
Suppose G is 3-colourable and let ϕ be a 3-colouring of G. We show that G' is 3-colourable. Start by colouring each vertex v in $V(G) \cap V(G')$ the colour $\phi(v)$. For each hub gadget H of G' corresponding to a vertex h of G, colour every pick vertex of a diamond in H the colour $\phi(h)$. Clearly, each outlet is given a different colour to its neighbours in $V(G)$ since ϕ is a 3-colouring of G. The remaining two vertices of each diamond contained in H have two neighbours the same colour $\phi(h)$, so can be coloured using the other two available colours. Thus G' is 3-colourable.

Now suppose that G' is 3-colourable. Each pick vertex of a diamond must have the same colour in a 3-colouring of G', so all outlets of a hub gadget have the same colour. Let H be the hub gadget of h, where $h \in V(G)$. We colour h with the colour of all the outlets of H in the 3-colouring of G'. For each vertex $v \in V(G) \cap V(G')$, we colour v with the same colour as in the 3-colouring of G', thus obtaining a 3-colouring of G. □

A similar approach can be used to show that 3-COLOURABILITY remains NP-complete for $(k-1)$-connected graphs with maximal local edge-connectivity k, for any $k \geq 4$. To prove this, we first require the following lemma:

Lemma 4.3. *Let $k \geq 3$ and $j \geq 1$. Then k-COLOURABILITY remains NP-complete when restricted to j-connected graphs.*

Proof. We show that k-COLOURABILITY restricted to j-connected graphs is reducible to k-COLOURABILITY restricted to $(j+1)$-connected graphs, for any fixed $j \geq 1$. Let G_0 be a j-connected graph; we construct a $(j+1)$-connected graph G' such that G_0 is k-colourable if and only if G' is. Let S_0 be a j-vertex cut in G_0, let $s \in S_0$, and let G_1 be the graph obtained from G_0 by introducing a single vertex s' with the same neighbourhood as s. Now if S' is a j'-vertex cut in G_1, for $j' \leq j$, then S', or $S' \setminus \{s'\}$, is a j'-vertex cut, or $(j'-1)$-vertex cut, in G_0. Since S_0 is not a j-vertex cut in G_1, it follows that G_1 has strictly fewer j-vertex cuts than G_0. Repeat this process for each j-vertex cut S_i in G_i (there are polynomially many), and let G' be the resulting graph. Then G'



(a) $G_{l,k}$, with $l \leq (k-2)(k-1)$

(b) $H_{l,k}$, with $l > (k-2)(k-1)$

Figure 7: Gadgets and intermediate gadgets used in the proof of Proposition 4.4.

has no vertex cuts of size at most j, so G' is $(j+1)$-connected. Moreover, it is straightforward to verify that G' is k-colourable if and only if G_0 is k-colourable. $\square$

We perform a reduction from k-COLOURABILITY restricted to $(k-1)$-connected graphs (which is NP-complete by Lemma 4.3). Let G be a $(k-1)$-connected graph. For each vertex v with $d(v) \geq k+1$, we will "replace" it with a gadget in such a way that the resulting graph G' remains $(k-1)$-connected, G' is 3-colourable if and only if G is 3-colourable, and no vertex of G' has degree greater than k.

We will describe, momentarily, a gadget $G_{l,k}$ used to replace a vertex v of degree l, where $l > k$, with vertices $x_1, x_2, \dots, x_l \in V(G_{l,k})$ called the *outlets* of $G_{l,k}$. Let $G_{l,k}$ be a gadget, and let G be a graph with a vertex v of degree $l > k$. We say that we *attach $G_{l,k}$ to G at v* when we perform the following operation: relabel the vertices of G such that $V(G) \cap V(G_{l,k}) = N_G(v) = \{x_1, x_2, \dots, x_l\}$, and construct the graph $(G \cup G_{l,k}) - \{v\}$.

We now give a recursive description of $G_{l,k}$. First, suppose that $l \leq (k-2)(k-1)$. Let $a = \lceil l/(k-1) \rceil$, and let $(B_1, B_2, \dots, B_{k-1})$ be a partition of $\{x_1, x_2, \dots, x_l\}$ into $k-1$ cells of size $a-1$ or a. We construct $G_{l,k}$ starting from a copy of the complete bipartite graph $K_{k-1,k-a}$ where the vertices of the $(k-1)$-vertex partite set are labelled $b_1, b_2, \dots, b_{k-1}$, and the remaining vertices are labelled $u_1, u_2, \dots, u_{k-a}$. Since $k \geq 4$ and $2 \leq a \leq k-2$, we have $k-a \geq 2$. Add an edge u_1u_2, and for each $i \in \{1, 2, \dots, k-1\}$ and $w \in B_i$, add an edge wb_i. We call the resulting graph $G_{l,k}$ and it is illustrated in Figure 7(a).

Now suppose $l > (k-2)(k-1)$. Let $(B_1, B_2, \dots, B_{k-1})$ be a partition of $\{x_1, x_2, \dots, x_l\}$ such that $|B_i| = k-2$ for $i \in \{1, 2, \dots, k-2\}$, and $|B_{k-1}| > k-2$. Take a copy of $K_{k-1,1,1}$, labelling the vertices of the $(k-1)$-vertex partite set as $b_1, b_2, \dots, b_{k-1}$, and the other two vertices u_1 and u_2. For each $i \in \{1, 2, \dots, k-1\}$, and for each $w \in B_i$, we introduce an edge wb_i. Label the resulting graph $H_{l,k}$; we call $H_{l,k}$ an *intermediate gadget* (see Figure 7(b)). Let $l_1 = d_{H_{l,k}}(b_{k-1})$. Since $l_1 = l - (k-2)^2 + 2$, we have $k+1 \leq l_1 \leq l-2$. The graph $G_{l,k}$ is obtained by attaching $G_{l_1,k}$ to $H_{l,k}$ at b_{k-1}. An example of such a gadget, for $l = 10, k = 4$, is given in Figure 8, and the intermediate gadgets involved in its construction are given in Figure 9.

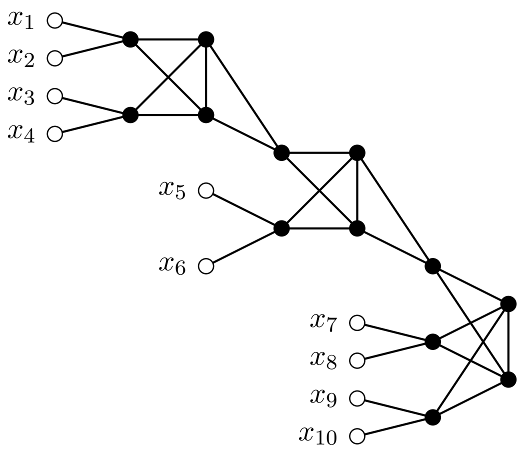
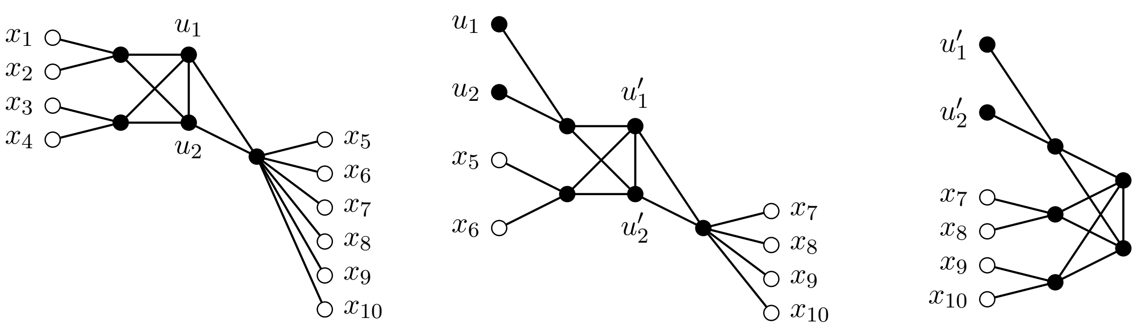


Figure 8: An example of a gadget, $G_{10,4}$.



(a) $H_{10,4}$ (b) $H_{8,4}$ (c) $G_{6,4}$

Figure 9: The intermediate gadgets used in the construction of $G_{10,4}$.

Proposition 4.4. *For any fixed $k \geq 4$, the problem of deciding if a $(k-1)$-connected graph with maximal local edge-connectivity k is 3-colourable is NP-complete.*

Proof. Let G be a $(k-1)$-connected graph, and let G' be the graph obtained by attaching a gadget $G_{d(v),k}$ to G at v for each vertex v of degree at least $k+1$. It is not difficult to verify that G' can be constructed in polynomial time and that every vertex of G' has degree at most k, so G' has maximal local edge-connectivity k. Moreover, for all distinct $i, j \in \{1, 2, \ldots, k-1\}$, the vertices $\{b_i, b_j, u_1, u_2\}$ induce a diamond in G', so the pick vertices $\{b_1, b_2, \ldots, b_{k-1}\}$ of these diamonds must have the same colour in a 3-colouring of G'. Now, given a 3-colouring of G', we can 3-colour G, where a vertex $v \in V(G)$ replaced by a gadget $G_{l,k}$ in G' is given the colour shared by the vertices $\{b_1, b_2, \ldots, b_{k-1}\}$ of $G_{l,k}$. It is also straightforward to verify that if G is 3-colourable, then G' is 3-colourable.

It remains to show that G' is $(k-1)$-connected. We may assume, by induction, that G' is obtained from G by attaching one gadget $G_{l,k}$. Moreover, when $l > (k-2)(k-1)$, we can view the attachment of a gadget $G_{l,k}$ as a sequence of attachments of intermediate gadgets $H_{l_0,k}, H_{l_1,k}, H_{l_2,k}, \ldots, H_{l_{s-1},k}, G_{l_s,k}$ where $l = l_0$ and $l_i = l_{i-1} - (k-2)^2 + 2$ for $i \in \{1, 2, \ldots, s\}$, and $l_s \leq (k-2)(k-1)$. We need only to show that the attachment of a gadget $G_{l,k}$, or of an intermediate gadget $H_{l,k}$, preserves $(k-1)$-connectivity.

Loosely speaking, we start by proving that the gadget, or intermediate gadget, itself is sufficiently connected. Let $l > k \geq 4$. If $l \leq (k-2)(k-1)$, then set $J_{l,k} = G_{l,k}$, otherwise set $J_{l,k} = H_{l,k}$. Let K_l be a copy of the complete graph with vertex set $N_G(v) = \{x_1, x_2, \ldots, x_l\}$. We will prove that $J'_{l,k} = J_{l,k} \cup K_l$ is $(k-1)$-connected. Let (X, Z, Y) be a j-separation of $J'_{l,k}$, for some $j < k-1$, such that Z is a minimal vertex cut. Set $U = \{u_1, u_2\}$ if $J_{l,k} = H_{l,k}$ and $U = \{u_1, \ldots u_{k-a}\}$ if $J_{l,k} = G_{l,k}$. Suppose $u_1 \in X$. Then $N(u_1) = \{b_1, b_2, \ldots, b_{k-1}\} \cup \{u_2\}$ is contained in $X \cup Z$. If $|U| \geq 3$ and there exists $u \in U \setminus \{u_1, u_2\}$ such that $u \in Y$, then $N(u) = \{b_1, b_2, \ldots, b_{k-1}\} \subseteq Z$, contradicting $|Z| < k-1$. So $U \cup \{b_1, \ldots, b_{k-1}\} \subseteq X \cup Z$ and thus $Y \subseteq \{x_1, x_2, \ldots, x_l\}$. For $i = 1, \ldots, k-1$, either $b_i \in Z$ or $B_i \subseteq Z$, so $|Z| \geq k-1$, a contradiction. Now we may assume that $u_1 \in Z$ and, by symmetry, $u_2 \in Z$. Since Z is minimal, u_1 has a neighbour in X and a neighbour in Y. Suppose $b_i \in X$ and $b_j \in Y$ for $i, j \in \{1, 2, \ldots, k-1\}$. In order for Z to separate b_i and b_j, we require $U \cup B_i \subseteq Z$ or $U \cup B_j \subseteq Z$, so $|Z| \geq k-1$, a contradiction. Thus $J'_{l,k}$ is $(k-1)$-connected.

We now return to proving that G' is $(k-1)$-connected. Suppose, towards a contradiction, that G' is not $(k-1)$-connected. Let (X, Z, Y) be a j-separation of G' for some $j < k-1$, such that Z is a minimal vertex cut. We denote by $G'_{l,k}$ the subgraph of $G_{l,k}$ obtained by deleting the vertices in common with G, namely $N_G(v)$. Note that $V(G')$ is the disjoint union of $V(G-v)$ and $V(G'_{l,k})$.

First, suppose that $Z \subseteq V(G-v)$. Since $G'_{l,k}$ is connected, and $Z \subseteq V(G-v)$, we deduce that, without loss of generality, $V(G'_{l,k}) \subseteq Y$. It follows that Z is a j-vertex cut that separates X from $(Y \setminus V(G'_{l,k})) \cup \{v\}$ in G; a contradiction.

Now suppose that $Z \nsubseteq V(G-v)$. Moreover, suppose that $X \cap V(G-v)$ and $Y \cap V(G-v)$ are both non-empty. Then $Z \cup V(G'_{l,k})$ separates $X \cap V(G-v)$ from $Y \cap V(G-v)$ in G', so $(Z \setminus V(G'_{l,k})) \cup \{v\}$ is a vertex cut of G and since $Z \nsubseteq V(G)$, we have $|(Z \setminus V(G'_{l,k})) \cup \{v\}| \leq j < k-1$, a contradiction to the fact that G is $(k-1)$-connected. Thus, either $X \subseteq V(G'_{l,k})$ or $Y \subseteq V(G'_{l,k})$. Assume without loss of generality that $X \subseteq V(G'_{l,k})$ and $V(G-v) \subseteq Y \cup Z$. Since $|N_G(v)| \geq k$, $Y \cap N_G(v) \neq \emptyset$. Hence $Z \cap (V(G'_{l,k}) \cup N_G(v))$ separates $Y \cap (V(G'_{l,k}) \cup N_G(v))$ from X in $J'_{l,k}$, a contradiction. $\square$

Proposition 1.5 now follows from Proposition 4.2 and Proposition 4.4. Note that Proposition 4.4 rules out (unless P=NP) the possibility of a polynomial-time algorithm that computes the chromatic number (or finds an optimal colouring) for a graph with maximal local edge-connectivity k, for $k \geq 4$. However, there may exist a polynomial-time algorithm that, given such a graph, finds a k-colouring or determines that none exists. Thus, a result in the style of Theorem 1.3 that characterises when graphs with maximal local edge-connectivity 4 are 4-colourable remains a possibility.

5 Minimally k-connected graphs

In this section we prove that deciding if a minimally k-connected graph is k-colourable is NP-complete. To do this, we perform a reduction from the following problem, where k is a fixed integer at least three. A hypergraph is *k-uniform* if each hyperedge is of size k.

k-UNIFORM HYPERGRAPH k-COLOURABILITY
Instance: A k-uniform hypergraph H.
Question: Is there a k-colouring of H for which no edge is monochromatic?

The problem of deciding if a hypergraph is 2-colourable is well known to be NP-complete [17], and the search problem of finding a k-colouring for a k-uniform hypergraph, for $k \geq 3$, is shown in [7] to be NP-hard, even when restricted to such hypergraphs that are $(k-1)$-colourable. However, to the best of our knowledge, no proof that k-UNIFORM HYPERGRAPH k-COLOURABILITY is NP-complete has been published, so we provide one here for completeness.

Proposition 5.1. *The problem k-UNIFORM HYPERGRAPH k-COLOURABILITY is NP-complete for fixed $k \geq 3$.*

Proof. Let $V_1, V_2, \dots, V_k$ be k disjoint sets each consisting of k distinct vertices, and let H_0 be the k-uniform hypergraph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ whose hyperedges consist of all k-element subsets of $V(H_0)$ not in $\{V_1, V_2, \dots, V_k\}$. Then, in a k-colouring of H_0, for each subset X of $V(H_0)$ of size at least k, either X is not monochromatic or X is one of $V_1, V_2, \dots, V_k$. It follows that a k-colouring of H_0 is unique, up to a permutation of the colours: each V_i, for $i \in \{1, 2, \dots, k\}$, is monochromatic, and for distinct $i, j \in \{1, 2, \dots, n\}$, the colours given to $v_i \in V_i$ and $v_j \in V_j$ are distinct.

We perform a reduction from k-COLOURABILITY. Let G be a graph. We construct a k-uniform hypergraph H as follows. Start with the hypergraph on the vertex set $V(H_0) \cup V(G)$, where $V(H_0)$ and $V(G)$ are disjoint, and containing all the hyperedges of H_0. For each edge uv of G and each $i \in \{1, 2, \dots, k\}$, introduce a hyperedge consisting of u, v, and $k-2$ vertices of V_i. Each such hyperedge enforces that in a k-colouring of H, the vertices u and v do not both have colour i. Thus, if H is k-colourable, then G is k-colourable. Now suppose that ϕ is a k-colouring of G. Then, by assigning a vertex $v \in V(G)$ the colour $\phi(v)$ in H, and colouring each vertex $v \in V(H_0)$ the colour i if $v \in V_i$, we obtain a k-colouring of H. This completes the proof. □

Proof of Proposition 1.6. We perform a reduction from k-UNIFORM HYPERGRAPH k-COLOURABILITY. Let H be a k-uniform hypergraph with vertex set $\{v_1, v_2, \dots, v_h\}$. We will construct a minimally k-connected graph G with $\{v_1, v_2, \dots, v_h\} \subseteq V(G)$. For each hyperedge $e = u_1 u_2 \cdots u_k$, where $u_i \in \{v_1, v_2, \dots, v_h\}$ for $i \in \{1, 2, \dots, k\}$, let P_e be the graph on $2k$ vertices that is the union of the complete graph K_k on the vertices $\{k_1, k_2, \dots, k_k\}$, and k vertex-disjoint edges $\{u_1 k_1, u_2 k_2, \dots, u_k k_k\}$. For $k = 3$, this graph is given in Figure 10. For each $l \in \{1, 2, \dots, h\}$,

Figure 10: Gadgets used in the proof of Proposition 1.6.

let Q_l be the graph on $3k-1$ vertices obtained from the complete bipartite graph $K_{k,k-1}$ with k-element partite set $\{b_1, b_2, \ldots, b_k\}$ by adding k vertex-disjoint edges $b_i v_j$ where $j \equiv i + l \pmod{h}$ for each $i \in \{1, 2, \ldots, k\}$. For $k = 3$, this graph is given in Figure 10. Finally, we obtain G from the union of the i graphs Q_i, for each $i \in \{1, 2, \ldots, h\}$, and the $|E(H)|$ graphs P_e, for each $e \in E(H)$. Note that a vertex v_i, for $i \in \{1, 2, \ldots, h\}$, is common to $Q_{i-k}, Q_{i-k+1}, \ldots, Q_{i-1}$ (with indices interpreted modulo h) and P_e for any hyperedge e containing v_i.

Suppose we have a k-colouring for G. Then, since each vertex of a K_k subgraph is coloured a different colour, each set of vertices $\{u_1, u_2, \ldots, u_k\}$ corresponding to a hyperedge e is not monochromatic. So the vertex colouring of the graph G gives us a colouring of the hypergraph H where no hyperedge is monochromatic.

Now suppose we have a colouring ϕ of H where no hyperedge is monochromatic. Starting from the colouring on $\{v_1, v_2, \ldots, v_h\}$ given by ϕ, we can extend this to a colouring of G as follows. Consider a Q_l subgraph. For each $v \in \{v_{l+1}, v_{l+2}, \ldots, v_{l+k}\}$, if $\phi(v) \neq 1$, we assign the vertex adjacent to v in Q_l colour 1; otherwise, it is assigned colour 2. The remaining $k-1$ vertices of Q_l can then be assigned colour 3. Now consider a P_e subgraph. Let $U = \{u_1, u_2, \ldots, u_k\}$, let C be the set of colours $\phi(U)$, and let σ be a permutation of C with no fixed points; such a permutation exists since no hyperedge of H is monochromatic, so $|C| > 1$. For each $c \in C$, pick a vertex $u \in U$ with $\phi(u) = c$, and colour the vertex adjacent to u in P_e the colour $\sigma(\phi(u))$. Now each of the remaining $k - |C|$ uncoloured vertices can be assigned one of the $k - |C|$ unused colours arbitrarily. So G is k-colourable if and only if H is k-colourable.

For every edge xy of G, at least one of x or y has degree k, so $G \backslash xy$ is at most $(k-1)$-connected. Moreover, it is not difficult to see there are at least k internally disjoint paths between any pair of vertices, so G is k-connected. Hence G is minimally k-connected, as required. $\square$

6 Graphs with a bounded number of vertices of degree more than k

In this section we prove Theorem 1.9. The proof of this result relies on a generalisation of Brooks' theorem established independently by Borodin [4] and by Erdős, Rubin and Taylor [8].

A *list assignment* for a graph G is a function L that associates to every vertex $v \in V(G)$ a set $L(v)$ of integers that are called the *colours associated with* v. A *degree-list-assignment* of a graph G is a list assignment L such that $|L(v)| \geq d_G(v)$ for every $v \in V(G)$. An *L-colouring* of G is a function c from $V(G)$ such that, for all $v \in V(G)$, we have $c(v) \in L(v)$, and, for all edges uv, we have $c(u) \neq c(v)$. A graph G is *L-colourable* if it admits at least one L-colouring. A graph G is *degree-choosable* if G is L-colourable for any degree-list-assignment L. A graph is a *Gallai tree* if it is connected and each of its blocks is either a complete graph or an odd cycle.

Theorem 6.1 (Borodin [4], Erdős, Rubin and Taylor [8])**.** *Let G be a connected graph. Then G is degree-choosable if and only if G is not a Gallai tree. Moreover, if G is not a Gallai tree, then there is an $O(m)$-time algorithm that, given a degree-list-assignment L, finds an L-colouring.*

We need now to study L-colourings of Gallai trees. Let G be a Gallai tree together with a list assignment L. Suppose that G has a cut-vertex and consider a leaf block B attaching at v. We say that L is *B-uniform* if the list $L(u)$ is the same for all $u \in V(B) \setminus \{v\}$ and satisfies $|L(u)| = d(u)$. When L is B-uniform, we define the list assignment $L_{\overline{B}}$ of $G - V(B - \{v\})$ as follows: for all $w \neq v$, $L_{\overline{B}}(w) = L(w)$ and $L_{\overline{B}}(v) = L(v) \setminus L(u)$ for some, and thus any, $u \in V(B) \setminus \{v\}$.

Lemma 6.2. *If a Gallai tree G has a cut-vertex v, a leaf block B attaching at v, and a list assignment L such that L is B-uniform, then G is L-colourable if and only if $G - V(B - \{v\})$ is $L_{\overline{B}}$-colourable.*

Proof. We deal only with the case when B is an odd cycle (the case when B is a complete graph is similar). Up to a relabelling of the colours, we may assume that every vertex of $B - \{v\}$ is assigned the list $\{1, 2\}$.

Suppose that $G - V(B - \{v\})$ is $L_{\overline{B}}$-colourable. In the colouring of $G - V(B - \{v\})$, the colours $\{1, 2\}$ are not used for v (from the definition of $L_{\overline{B}}$), so they can be used to colour $B - \{v\}$, showing that G is L-colourable.

Suppose conversely that G is L-colourable. We note that $B - \{v\}$ is a path of odd length and, in any L-colouring, its ends must receive colours 1 and 2, because of the parity. It follows that v is not coloured with 1 or 2. Therefore, the restriction of the colouring to $G - V(B - \{v\})$ is an $L_{\overline{B}}$-colouring, showing that $G - V(B - \{v\})$ is $L_{\overline{B}}$-colourable. □

When G is a graph together with a degree-list-assignment L, we say a vertex has a *long* list $L(v)$ when $|L(v)| > d(v)$.

Lemma 6.3. *There is an $O(m)$-time algorithm whose input is a connected graph G together with a degree-list-assignment L such that at least one vertex has a long list, and whose output is an L-colouring of G.*

Proof. In time $O(m)$, a vertex v whose list is long can be identified. The algorithm then runs a search of the graph (a depth-first search, for instance) starting at v. This gives a linear ordering of the vertices starting at v: $v = v_1 < v_2 < \cdots < v_n$, such that, for every $i \in \{2, 3, \dots, n\}$, the vertex v_i has at least one neighbour v_j with $j < i$. The greedy colouring algorithm starting at v_n then yields an L-colouring of G. □

Proposition 6.4. *There is an $O(m)$-time algorithm whose input is a Gallai tree G together with a degree-list-assignment L, and whose output is an L-colouring of G or a certificate that no such colouring exists.*

Proof. The algorithm first checks whether one of the lists is long, and if so runs the algorithm from Lemma 6.3. Otherwise the classical $O(m)$-time algorithm of Tarjan [24] finds the block decomposition of G.

Loop step: If G is not a clique or an odd cycle, then it has a cut-vertex v and a leaf block B attaching at v. The algorithm checks whether L is B-uniform (which is easy in time $O(|V(B)|)$), and if so, as in the proof of Lemma 6.2, colours the vertices of $B - \{v\}$, removes them, updates the list $L(v)$, and repeats the loop step again.

If B is not uniform, then the algorithm identifies in $B - \{v\}$ two adjacent vertices u, u' with different lists. So, up to swapping u and u', there is a colour c in $L(u)$ that is not present in $L(u')$. Then the algorithm gives colour c to u, removes u from G, and removes colour c from the lists of all neighbours of u. The resulting graph is a connected graph together with a degree-list-assignment, and the list of u' is long. Therefore, we may complete the colouring by Lemma 6.3.

Hence, we may assume that the algorithm repeats the loop step until the removal of leaf blocks finally leads to a clique or an odd cycle. Then, if all lists of vertices are equal, obviously no colouring exists, and the sequence of calls to Lemma 6.2 certifies that G has no L-colouring. Otherwise, a colouring can be found by Lemma 6.3. □

Proof of Theorem 1.9. Let X be the set of p vertices of degree more than k in G. We guess what could be the colouring on those vertices. There are at most k^p possibilities.

For each, we check whether it can be extended to a k-colouring of the whole graph. To do so we consider $H = G - X$, and for every vertex $v \in V(G) \setminus X$, we use the list assignment $L(v)$ given by the list of colours in $\{1, 2, \ldots, k\}$ that are not used on a neighbour of v in X. Clearly, $|L(v)| \geq k - |N(v) \cap X| \geq d_H(x)$, so we have a degree-list-assignment.

Next we find the connected components of H in $O(n + m)$. Then for each component C, we check if C is a Gallai tree or not. If not, then we use the $O(m)$ algorithm of Theorem 6.1 to L-colour C. If it is a Gallai tree, then we rely on Proposition 6.4.

The running time of the algorithm described above is $k^p O(n + m)$. If $k > p$, then we may assume, without loss of generality, that only the first p colours are used on the p vertices of degree more than k. Therefore, we have to try only p^p possibilities for colouring these vertices. Thus, we obtain an algorithm that runs in time $\min\{k^p, p^p\} \cdot O(n + m)$. □

Theorem 1.9 immediately implies a fixed-parameter tractability result.

Corollary 6.5. *The problem* k-COLOURABILITY*, when parameterized by the number of vertices of degree more than k, is FPT.*

Let us consider now the problem restricted to the case when G is planar. Then only the $k = 3$ case makes sense: for $k \leq 2$, the problem is polynomial-time solvable, while for $k \geq 4$ the colouring always exists by the Four Colour Theorem. For $k = 3$, Theorem 1.9 gives an algorithm with running time $3^p \cdot O(n + m)$. On general graphs, this is essentially best possible, in the following sense. The *Exponential-Time Hypothesis (ETH)*, formulated by Impagliazzo, Paturi, and Zane [10], implies that n-variable 3-SAT cannot be solved in time $2^{o(n)}$. It is known that ETH further implies that 3-COLOURABILITY on an n-vertex graph cannot be solved in time $2^{o(n)}$ [15]. It follows that in the algorithm given by Theorem 1.9 for 3-COLOURABILITY, the exponential dependence on p cannot be improved to $2^{o(p)}$: as p is at most the number of vertices, such an algorithm could be used to solve 3-COLOURABILITY in time $2^{o(n)}$ on any graph.

Corollary 6.6. *Assuming ETH, there is no $2^{o(p)} \cdot n^{O(1)}$ time algorithm for* 3-COLOURABILITY*, where p is the number of vertices with degree more than 3.*

However, on planar graphs we can do substantially better. There are several examples in the parameterized-algorithms literature [5, 6, 12, 13, 16, 22] where significantly better algorithms are known when the problem is restricted to planar graphs, and, in particular, a square root appears in the running time. In most cases, the square root comes from the use of the Excluded Grid Theorem for planar graphs, stating that if a planar graph has treewidth w, then it contains an $\Omega(w) \times \Omega(w)$ grid minor. Often this result is invoked not on the input graph itself, but on some other graph derived from it in a nontrivial way. This is also the case with this problem.

Theorem 6.7. *Let G be a planar graph with at most p vertices of degree more than 3. There is a $2^{O(\sqrt{p})}(n+m)$-time algorithm for 3-colouring G, or determining no such colouring exists.*

Proof. Let X be the set of p vertices of degree more than 3 in G. If a component C of $G-X$ is not a Gallai tree, then, by Theorem 6.1, we can extend a colouring of $G \setminus C$ to a colouring of G in linear time (similar to the proof of Theorem 1.9). Thus, we may assume that each component of $G-X$ is a Gallai tree. It is well known that the treewidth of a graph is the maximum treewidth of one of its blocks (see, for example, [19]). Since a planar Gallai tree has no cliques of size more than 4, and an odd cycle has treewidth 2, a planar Gallai tree has treewidth at most 3. Therefore, the deletion of X from G reduces the treewidth of the resulting graph to a constant. Let w be the treewidth of G. Since G is planar, it contains a $\Omega(w) \times \Omega(w)$ grid minor, so $\Omega(w^2)$ vertices need to be deleted in order to reduce the treewidth of G to a constant. This implies that $p = |X| = \Omega(w^2)$, or in other words, G has treewidth $O(\sqrt{p})$. Therefore, after computing a constant-factor approximation of the tree decomposition (using, for example, the algorithm of Bodlaender et al. [1] or Kammer and Tholey [11]), we can use a standard 3-colouring on the tree decomposition to solve the problem in time $2^{O(\sqrt{p})} \cdot n$. □

It is known that, assuming ETH, 3-COLOURABILITY cannot be solved in time $2^{o(\sqrt{n})}$ on planar graphs [15]. This implies that the $2^{O(\sqrt{p})}$ factor in Theorem 6.7 is best possible: assuming ETH, it cannot be replaced by $2^{o(\sqrt{p})}$.

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