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# Optimal Fuzzy Controller Mapped from LQR under Power and Torque Constraints

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**Abstract.** Dealing with a LQR controller surface subject to power and torque constraints, is an issue of nonlinear problem that is difficult to implement. This paper employs a fuzzy controller surface to replace the LQR surface subject to power and torque constraints by using class stacking, least square and Sugeno-type fuzzy inference mode. Through this type of transformation, called “Optimal fuzzy controller mapped from LQR”, control of the system remains optimal.

**Keywords:** LQR, FLC, Fuzzy, Sugeno, Optimal, Class stacking.

## 1 Linear Quadratic Regulator (LQR)

The development of optimal control for a MIMO system [1][2] is mature nowadays. Approach of linear quadratic regulator (LQR) offers Kalman gain for state feedback to regulate a MIMO system in state-space form. But subject to power and torque limitation of motor drive, implementation of system becomes difficult using LQR approach. This section introduces some basis for LQR approach for further improvement.

### 1.1 LQR Approach

For a MIMO system in state-space form

$$\dot{X} = AX + Bu \quad (1)$$

where  $X \in R^n$  is the state vector. The control  $u \in R^m$  includes state-feedback torque in motor application, command for desired state output and disturbance that the system might encounter. Control  $u$  is thus expressed by

$$u = -KX + u_{com} + u_{dist} \quad (2)$$

where  $K$ : Kalman gain.  $u_{dist}$ : disturbance torque (colored noise).  $u_{com}$ : command for desired state output. Eq. (2) is a general form of input in LQG (Linear Quadratic

Gaussian) problems. When  $\mathbf{u}_{dist} = \mathbf{0}$  and  $\mathbf{u}_{com} \neq \mathbf{0}$ , the system becomes a tracking problem expressed by

$$\dot{\mathbf{X}} = (\mathbf{A} - \mathbf{BK})\mathbf{X} + \mathbf{B}\mathbf{u}_{com} \quad (3)$$

At steady state it satisfies

$$\mathbf{0} = (\mathbf{A} - \mathbf{BK})\mathbf{X} + \mathbf{B}\mathbf{u}_{com} \quad (4)$$

In case  $\mathbf{u}_{com} = \mathbf{0}$  in Eq. (4), the system becomes a regulator problem further called “linear quadratic regulator (LQR)”, if we select a gain  $\mathbf{K}$ , called Kalman gain such that  $\mathbf{u} = -\mathbf{K}\mathbf{X}$  to minimize the cost  $J$  in quadratic form as

$$J = \frac{1}{2} \mathbf{X}^T(T) \mathbf{S}(T) \mathbf{X} + \int_0^T \left[ \frac{1}{2} (\mathbf{X}^T \mathbf{Q} \mathbf{X} + \mathbf{u}^T \mathbf{R} \mathbf{u}) + \lambda^T (\mathbf{A} \mathbf{X} + \mathbf{B} \mathbf{u} - \dot{\mathbf{X}}) \right] dt \quad (5)$$

where  $\mathbf{Q}$ : dynamic state weighting,  $\mathbf{R}$ : control weighting  $\mathbf{S}$ : terminal state weighting and  $\lambda$ : Lagrange multiplier. After several algebra operations to minimize the cost, we have the following Hamiltonian matrix presenting as the transition matrix in Eq. (6):

$$\begin{bmatrix} \dot{\mathbf{X}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \lambda \end{bmatrix} \quad (6)$$

$$\lambda(t) = \mathbf{S}(t) \mathbf{X}(t) \quad (7)$$

$$\lambda(T) = \mathbf{S}(T) \mathbf{X}(T) \quad (8)$$

$$\mathbf{U}(t) = -\mathbf{R}^{-1} \mathbf{B}^T \lambda(t) = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{S}(t) \mathbf{X}(t) = -\mathbf{K}(t) \mathbf{X}(t) \quad (9)$$

The steady gain is

$$\mathbf{K} = \lim_{t \rightarrow \infty} \mathbf{K}(t) = \mathbf{R}^{-1} \mathbf{B}^T \lim_{t \rightarrow \infty} \mathbf{S}(t) = \mathbf{R}^{-1} \mathbf{B} \boldsymbol{\psi}_{21} (\boldsymbol{\psi}_{11})^{-1} \quad (10)$$

where  $\boldsymbol{\psi}_{21}$  and  $\boldsymbol{\psi}_{11}$  are the first-column block matrix of  $\boldsymbol{\Psi}$  which is the modal form of the Hamiltonian matrix in Eq. (6).

## 1.2 Constraints of LQR Control

While implementing the optimal control  $\mathbf{u}$  aforementioned, we always meet difficulty when subject to constraints of torque limit  $\mathbf{u}_l$  and power limit  $p_l$ . The torque constraint, caused by stall current in motor application, is expressed by

$$\mathbf{u} \leq \mathbf{u}_l, \quad \mathbf{u}_l = k_i \mathbf{i}_s \quad (11)$$

where  $k_i$ : current constant of motor.  $\mathbf{i}_s$ : stall current of motor. As usual the control  $\mathbf{u}$  is associated with states of position and rate. Mechanical power  $p_m$ , constrained by maximum power  $p_l$  of motor, is represented by

$$p_m = \mathbf{u}^T \dot{\boldsymbol{\theta}} \leq p_l \quad (12)$$

where  $\dot{\boldsymbol{\theta}}$  is rate-related state vector.

## 2 Class Stacking

From Eq. (2), the LQR control is  $\mathbf{u} = -\mathbf{K}\mathbf{X} = [u_1 u_2 \cdots u_w]^T$  where  $\mathbf{K} = [k_{ji}] \in R^{w \times n}$  and

$u_j = -\sum_{i=1}^n k_{ji} x_i$  for  $j=1, \dots, w$ . Since each component  $u_j$  of  $\mathbf{u}$  will be solved one

by one by the same approach of class stacking,  $u_j$  will be replaced

by  $u = \sum_{i=1}^n k_i x_i$  for illustration without loss of generality. Obviously for  $j^{\text{th}}$  component

$u_j$  of  $\mathbf{u}$ ,  $k_i = -k_{ji}$ . The form of  $u = \sum_{i=1}^n k_i x_i$  is a hyper plane of  $n$ -D. All the states

$x_i$  for  $i=1, \dots, n$  are sorted out into two classes. One class, having  $m$  states, is position-related and piled up by

$$v_1 = \sum_{i=1}^m k_i x_i \quad (13)$$

and the other class, having  $n-m$  states, is rate-related and piled up by

$$v_2 = \sum_{i=m+1}^n k_i x_i \quad (14)$$

Then

$$u = v_1 + v_2 \quad (15)$$

The torque constraint is

$$v_1 + v_2 = u \leq u_l \quad (16)$$

The power constraint  $u v_2 \leq p_l$  from Eq. (12) can be rewritten by

$$(v_1 + v_2) v_2 \leq p_l \quad (17)$$

Eq. (13)~(15) thus reduce the system from  $n$ -D into 3-D with variables  $v_1$  and  $v_2$ .

Eq. (16) indicates a plane  $v_1 + v_2 = u$  with known LQR control  $u$  but upper bounded

by  $u_l$ . Eq. (17) indicates a nonlinear constraint. The process, called "class stacking",

thus make it feasible to plot a hyper plane of  $n$ -D in terms of 3-D. An example using computer-aided plot is given as follows. A LQR control  $u$ , with number of total states  $n=5$  and number of position-related states  $m=3$ , is expressed by

$$u = \sum_{i=1}^5 k_i x_i = v_1 + v_2, \quad v_1 = \sum_{i=1}^3 k_i x_i, \quad v_2 = \sum_{i=4}^5 k_i x_i \quad (18)$$

where  $|x_i| \leq 1$  (normalized) and  $k_i = i$  for  $i = 1, \dots, 5$ . Obviously maximum of  $v_1$  is 6 and maximum of  $v_2$  is 9, subject to constraints:  $v_1 + v_2 \leq 12$  (Upper bound) and  $(v_1 + v_2)v_2 \leq 75$  (Inequality constraint).

By using computer-aided plot [3], Fig. 1(a)-(d) shows a sequence of transformations from LQR surface to cases of constraints are obtained. Fig. 1(a) shows the interception of a LQR surface, a torque-constrained surface and a hyperboloid due to power constraint. Fig. 1(b) shows LQR surface bounded by all constraints.

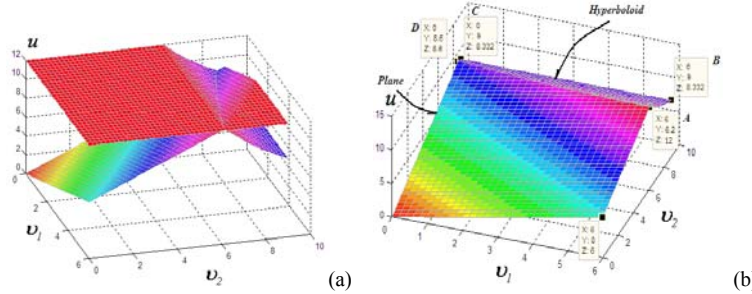


Fig. 1. Computer-aided plot (a) Interception of LQR plane, torque constraint plane and power constraint hyperboloid (b) LQR surface bounded by all constraints.

### 3 Fuzzy Logic Controller (FLC)

Although the approach of class stacking reduces the hyper plane from the LQR controller of  $n$ -D into a bounded and constrained plane in 3-D, the 3-D controller surface shown in Fig. 1.(b) is still nonlinear. Therefore we may employ a fuzzy-logic-controller (FLC) to implement this 3-D controller surface shown in Fig. 1.(b).

#### 3.1 Fuzzy inference system (FIS)

A fuzzy logic controller (FLC) [4][5][6] is configured by processing a sequence of fuzzification, inference and defuzzification as follows. The fuzzy inference system (FIS) to be selected is Sugeno model [7] given in Fig. 2 with  $v_1$  and  $v_2$  as input fuzzy variables and the control  $u$  as known output variable which is bounded, constrained and expressed by plane  $s1$ :  $u = v_1 + v_2$ , plane  $s2$ :  $u = 12$  (Upper bound) and hyperboloid:  $u v_2 \leq 75$  (Inequality constraint).

#### 3.2 Fuzzification

Input fuzzy variables are  $U_1$  and  $U_2$ .  $U_1$  has five triangle membership functions, equally partitioned and spaced as shown in Fig. 2. Membership functions are defined by linguistic terms, i.e. S, SM, M, MB and B. Output variable is the control  $u$  that has five membership functions, i.e. S+, SM+, M+, MB+ and B as shown in Fig. 2.

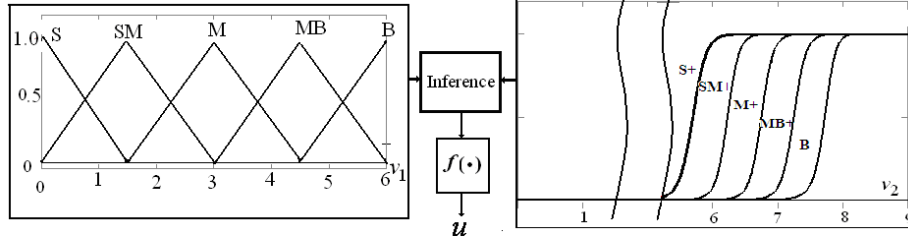


Fig. 2. Fuzzy inference system (FIS) using Sugeno model

### 3.3 Inference

Output variable is the control  $u$  that is composed of plane  $s1$  and a hyperboloid. The plane  $s1$  is  $u = U_1 + U_2$ . The hyperboloid  $ABCD$  in Fig. 1(b) is approximately replaced by another plane in Fig. 3.(a), named  $s2$  and obtained by least square with

$$s2: u = A [U_2 \ U_1 \ 1]^T \quad (19)$$

$$A = [-0.8025 \ 0.2566 \ 14.7253] \quad (20)$$

$A$  is obtained by

$$A^T = (Y^T Y)^{-1} (Y^T Z) \quad (21)$$

with

$$Y = \begin{bmatrix} U_{2A} & U_{1A} & 1 \\ U_{2B} & U_{1B} & 1 \\ U_{2C} & U_{1C} & 1 \\ U_{2D} & U_{1D} & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} u_A \\ u_B \\ u_C \\ u_D \end{bmatrix} \quad (22)$$

The selected  $A$  is able to minimize error  $\|Z - YA\|$ .

After finishing the definition of linear planes, we are able to develop the inference that includes ten rules. The former half governs the surface  $s2$  and the latter half governs the surface  $s1$ . Rules are written as follows.

1. If ( $U_1$  is S) and ( $U_2$  is B) then ( $u$  is  $s2$ );
2. If ( $U_1$  is SM) and ( $U_2$  is MB+) then ( $u$  is  $s2$ );
3. If ( $U_1$  is M) and ( $U_2$  is M+) then ( $u$  is  $s2$ );
4. If ( $U_1$  is MB) and ( $U_2$  is SM+) then ( $u$  is  $s2$ );
5. If ( $U_1$  is B) and ( $U_2$  is S+) then ( $u$  is  $s2$ );
6. If ( $U_1$  is S) and ( $U_2$  is not B) then ( $u$  is  $s1$ );
7. If ( $U_1$  is SM) and ( $U_2$  is not MB+) then ( $u$  is  $s1$ );
8. If ( $U_1$  is M) and ( $U_2$  is not M+) then ( $u$  is  $s1$ );
9. If ( $U_1$  is MB) and ( $U_2$  is not SM+) then ( $u$  is  $s1$ );
10. If ( $U_1$  is B) and ( $U_2$  is not S+) then ( $u$  is  $s1$ ).

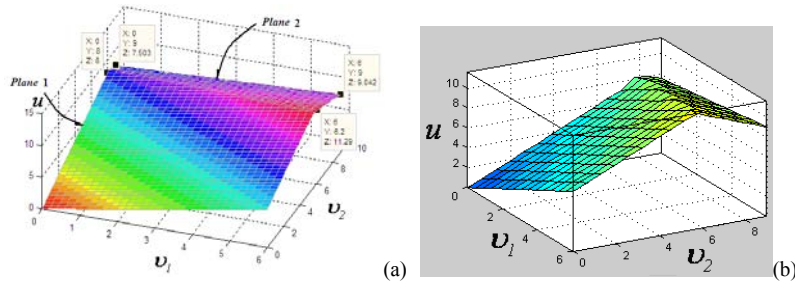


Fig. 3.(a) Hyperboloid is approximately replaced by plane  $s_2$  (b) Rule surface after fuzzy defuzzification.

### 3.4 . Defuzzification

A rule surface after defuzzification is generated as shown in Fig. 3(b) which is observed to be an approximation of Fig. 3.(a).

## 4 Conclusion

LQR control surface in hyperspace, subject to power and torque constraints, is difficult to implement. This paper employs class stacking to transform LQR surface in hyper space into a surface in 3-D and further applies a Sugeno-type fuzzy logic controller to plot the 3-D nonlinear surface. This approach makes LQR feasible and visible.

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