A simple approach for lower-bounding the distortion in any Hyperbolic embedding*

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Abstract

We answer open questions of [Verbeek and Suri, SOCG'14] on the relationships between *Gromov hyperbolicity* and the optimal stretch of graph embeddings in *Hyperbolic space*. Then, based on the relationships between hyperbolicity and *Cops and Robber* games, we turn necessary conditions for a graph to be Cop-win into sufficient conditions for a graph to have a large hyperbolicity (and so, no low-stretch embedding in Hyperbolic space). In doing so we derive lower-bounds on the hyperbolicity in various graph classes – such as Cayley graphs, distance-regular graphs and generalized polygons, to name a few. It partly fills in a gap in the literature on Gromov hyperbolicity, for which few lower-bound techniques are known.

Keywords: Gromov hyperbolicity; Hyperbolic space; Cop and Robber games.

1 Introduction

In a seminal paper [12], Kleinberg proved that every graph can be embedded in *Hyperbolic space* in such a way that, between any two vertices s and t, there exists an st-path in G where the Hyperbolic distance to t is monotonically decreasing. This fact has paved the way to an in-depth study of greedy routings in Hyperbolic space. In particular, Verbeek and Suri proved in [14] that for any embedding of $G = (V, E)$ in Hyperbolic space the multiplicative distortion of the distances is $\Omega(\delta(G)/\log\delta(G))$, with $\delta(G)$ being the *hyperbolicity* of the graph. Roughly, Gromov hyperbolicity is a parameter which gives bounds on the least distortion of the distances in a graph when its vertices are mapped to points in some "tree-like" metric space such as: (weighted) trees, Hyperbolic space, and more generally speaking spaces with negative curvature (formal definitions are postponed to Section 2). So far, tight upper-bounds on the hyperbolicity have been proved for various graph classes (*e.g.*, see [15]). For general graphs, it has been observed that this parameter is linearly upper-bounded by the diameter [3]. However, it is not clear when this bound is close to be optimal. More generally, there is a need for better understanding the cases where the hyperbolicity is large. As an example, lower-bounds on the hyperbolicity can be helpful in order to decide whether, given a graph, we should use greedy routing in Hyperbolic space or another routing method of the literature.

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Our contributions. *Quasi-cycles* were proved in [14] to be obstructions to low-stretch embeddings in Hyperbolic space. We first answer positively to an open question of the authors in [14] by proving that every large quasi-cycle has large hyperbolicity (Section 3.1). Then, we prove new lower-bounds for graph hyperbolicity based on the existence of regular graph powers (Section 3.2). This simple criterion is surprisingly powerful when applied to *distance-regular graphs* and their generalizations: where the number of walks of a given length between every two vertices only depends on their distance in the graph [4]. In Section 4 we prove lower-bounds on the hyperbolicity for all the graphs in these classes. It allows us to derive in one strike general results for some classes of graphs with applications in algebra, coding theory, design of interconnection networks, quantum theory and even finance — to name a few. In most cases, the lower-bound obtained is tight up to a constant-factor. From the computational point of view, our lower-bounds can be computed in $\mathcal{O}(|V||E|)$-time and $\mathcal{O}(|E|)$-space by counting the number of vertices in each layer of breadth-first-search trees. In contrast, the best-known combinatorial algorithms for computing graph hyperbolicity run in $\mathcal{O}(|V|^4)$-time and $\mathcal{O}(|V|^2)$-space [3][1]. We conclude this paper with some open questions.

2 Preliminaries

The graph terminology is from [2]. All graphs $G = (V, E)$ considered are finite, undirected, unweighted, simple (hence, with neither loops nor multiple edges) and connected. We write $dist_G(u, v)$ for the distance between every two vertices $u, v \in V$. Furthermore, for every $v \in V$ let $B_G(v, r) = \{u \in V \mid dist(u, v) \leq r\}$, and let $D_G(v, r) = B_G(v, r) \setminus B_G(v, r-1) = \{u \in V \mid dist_G(u, v) = r\}$. In particular, $N_G[v] = B_G(v, 1)$ and $N_G(v) = D_G(v, 1)$ are the closed and open neighbourhoods of v, respectively. The *diameter* of G is denoted by $diam(G) = \max_{u,v \in V} dist_G(u, v)$.

Definition 1 (4-points Condition, [10]). *Let $G = (V, E)$ be a connected graph. For every 4-tuple u, v, x, y of V, let $\delta(u, v, x, y)$ be defined as half of the difference between the two largest sums amongst: $S_1 = dist_G(u, v) + dist_G(x, y)$, $S_2 = dist_G(u, x) + dist_G(v, y)$, and $S_3 = dist_G(u, y) + dist_G(v, x)$. The graph hyperbolicity, denoted by $\delta(G)$, is equal to $\max_{u,v,x,y \in V} \delta(u, v, x, y)$. Moreover, we say that G is* δ-hyperbolic *for every $\delta \geq \delta(G)$.*

An $(s, s')^*$-dismantling ordering of $G = (V, E)$ is a total ordering $(v_1, v_2, \ldots, v_n)$ of V such that for every $i < n$, we have $B_G(v_i, s) \cap \{v_i, v_{i+1}, \ldots, v_n\} \subseteq B_G(v_j, s')$ for some $j > i$. We need the following characterization of Gromov hyperbolicity:

Lemma 2 ([5]). *Let $G = (V, E)$ be a graph. If G is δ-hyperbolic then it has a $(2r, r + 2\delta)^*$-dismantling ordering for every positive integer $r \geq 2\delta$. Conversely, if G has a $(s, s')^*$-dismantling ordering, for some $s' < s$, then it has hyperbolicity at most $16(s + s') \left\lceil \frac{s+s'}{s-s'} \right\rceil + 1/2$.*

Graphs with an $(1, 1)^*$-dismantling ordering are sometimes called Cop-win graphs. By Lemma 2, if G is δ-hyperbolic then its *graph power* $G^{4\delta}$ – obtained from G by adding an edge between every two distinct vertices that are at distance no more than 4δ in G – is Cop-win.

We will also need the following result to obtain stronger statements for bipartite graph classes.

[1]There are approximation algorithms running in $o(|V|^4)$-time – but they still require $\mathcal{O}(|V|^2)$-space [5]. Furthermore, the approximation factor in [5] is horrendous at 1569.

Lemma 3 ([7]). *Let $B = (V_0 \cup V_1, E)$ be a bipartite graph. For every $i \in \{0, 1\}$, let $G_i = (V_i, \{\{u, v\} \mid dist_B(u, v) = 2\})$. Then, $2\delta(G_i) \leq \delta(B) \leq 2\delta(G_i) + 2$ and the bounds are sharp.*

Notions local to a section are given in the appropriate sections.

3 New lower-bound techniques for graph hyperbolicity

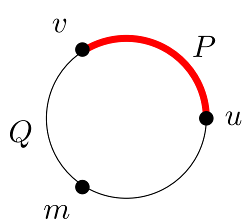
First we reinforce the relationship between the hyperbolicity of a graph and the minimum stretch of any of its Hyperbolic embeddings (Lemma 4). We then propose simple techniques in Section 3.2 in order to lower-bound the hyperbolicity of a given graph – based on combinatorial properties of its powers.

3.1 Hyperbolicity and Quasi-cycles

Given $G = (V, E)$, a cycle C of length n is an (α, β)-quasi-cycle if for every $u, v \in C$ such that $dist_C(u, v) \geq \beta n$ we have that $dist_G(u, v) \geq \alpha \cdot dist_C(u, v)$. Verbeek and Suri have proved in [14] that for every graph G with an (α, β)-quasi-cycle of length n, with $\alpha > 0$ and $\beta \leq 1/3$, any embedding of G in Hyperbolic space has multiplicative distortion $\Omega\left(\alpha n / \log n\right)$. They also proved that every graph G has an $(1/8, 1/3)$-quasi-cycle of length $\Omega(\delta(G))$. Answering an open question from [14], we now prove that the existence of a large quasi-cycle is a sufficient condition for having a large hyperbolicity.

Lemma 4. *For every $\alpha \leq 1, \beta \leq 1/3$, if $G = (V, E)$ has an (α, β)-quasi-cycle of length n then $\delta(G) = \Omega\left(\alpha^2 n\right)$.*

Proof. For simplicity, we will ignore the ceilings in the proof. Let C be an (α, β)-quasi-cycle of length n, which exists by the hypothesis. Let us pick $u, v \in C$ such that $dist_C(u, v) = n/3$. We can partition the cycle C into two uv-paths P, Q of respective length $n/3$ and $2n/3$. In this situation, since C is assumed to be an (α, β)-quasi-cycle and $\beta \leq 1/3$, we have $dist_G(u, v) \geq \alpha n/3$, and so, the respective lengths of P and Q are upper-bounded by $\frac{2}{\alpha} dist_G(u, v)$. Let $m \in Q$ be a middle-vertex, *i.e.*, chosen such that $dist_C(m, u) = |Q|/2$.



By the choice of m, $dist_C(m, P) = dist_C(m, u) = n/3$. Furthermore, since $\beta \leq 1/3$, it implies $dist_G(m, P) \geq \alpha n/3$. However, by Morse Lemma, almost shortest-paths stay close to each other in a hyperbolic graph. Precisely, the *Hausdorff distance* between P and Q is an $\mathcal{O}\left(\delta(G)/\alpha\right)$ (The Hausdorff distance between P and Q is defined as $\max\{dist_G(v, P) \mid v \in Q\} \cup \{dist_G(u, Q) \mid u \in P\}$) [13]. In particular, we have $\alpha n/3 \leq dist_G(m, Q) = \mathcal{O}\left(\delta(G)/\alpha\right)$. Altogether, $\delta(G) = \Omega\left(\alpha^2 n\right)$. □

For any tree T (0-hyperbolic graph) with maximum degree Δ, any embedding of T in *constant-dimensional* Hyperbolic space has distortion $\Omega(\log \Delta)$ [14]. Hence, our result negatively answers another open question of [14], which asked whether the best distortion in Hyperbolic embeddings can be upper-bounded by a function of quasi-cyclicity.

3.2 Hyperbolicity and Regular graph powers

Recall that for every $j \geq 1$, the j^{th} power of $G = (V, E)$ is the graph G^j that is obtained from G by adding an edge between every two distinct vertices u, v such that $dist_G(u, v) \leq j$. By Lemma 2, disproving that G^j is Cop-win, for some range of j, will give lower-bounds on $\delta(G)$. We have used this approach in [6] in order to prove that most underlying graphs of the data center interconnection networks have their hyperbolicity scaling with their diameter. Here we propose to do so using simple combinatorial properties of Cop-win graphs.

Proposition 5. *Let $G = (V, E)$ and $2 \leq r \leq diam(G)$ be such that G^{r-1} is regular. Then, $\delta(G) \geq \lceil r/2 \rceil /2$.*

Proof. Suppose for the sake of contradiction that $4\delta(G) < r$. In particular, since its hyperbolicity is always a half-integer, G is $\lfloor (r-1)/2 \rfloor /2$-hyperbolic, and so, by Lemma 2, it has a $(2\lceil (r-1)/2 \rceil, r-1)^*$-dismantling ordering. The latter ordering is also a $(r-1, r-1)^*$-dismantling ordering, hence G^{r-1} is Cop-win. However, since G^{r-1} is assumed to be regular, it must be a complete graph [1]. The latter contradicts that $r - 1 < diam(G)$. As a result, $4\delta(G) \geq r$, as desired. □

Combining Proposition 5 with Lemma 3, we obtain the following result for bipartite graphs:

Corollary 6. *Let $B = (V_0 \cup V_1, E)$ be a bipartite graph and for every $i \in \{0, 1\}$, let $G_i = (V_i, \{\{u, v\} \mid dist_B(u, v) = 2\})$. If G_i^{r-1} is a regular graph for some $2 \leq r \leq diam(G_i)$ then $\delta(B) \geq r/2$.*

Additional results. Our approach in this note can be combined with *algebraic* properties of Cop-win graphs to obtain even more lower-bounds on graph hyperbolicity. This was the approach taken in [6]. We can generalize the results obtained there, and prove for instance (under mild assumptions on the value of n) that every n-vertex transitive graph G is $\Omega(n)$-hyperbolic. We do not detail this part to keep the focus on *combinatorial arguments*, that are in our opinion simpler and computationally less expensive to be verified.

4 Application to some graph classes

Applying Proposition 5, we lower-bound the hyperbolicity in various graph classes. In most cases, the lower-bound is linear in the diameter of the graph. This is optimal up to a constant-factor since for every graph $G = (V, E)$, we have $\delta(G) \leq \lfloor diam(G)/2 \rfloor$ [3]. More precisely, a regular graph $G = (V, E)$ is *distance-regular* if for every $j, k \geq 0$, and for every $u, v \in V$, the number of vertices that are both at distance j from u and distance k from v in G only depends on $i = dist_G(u, v)$, j and k [4]. Distance-regular graphs have been generalized in many ways. We consider three of them in what follows. First, a graph $G = (V, E)$ is *distance degree regular* if if and only if all its powers are regular graphs [11]. We deduce the following from Proposition 5:

Corollary 7. *For every distance degree regular graph G that is not a clique, $\delta(G) \geq \lceil diam(G)/2 \rceil /2$.*

We stress that distance degree regular graphs are a common generalization of many interesting graph classes such as: Moore graphs, distance mean regular graphs, Cayley graphs and more generally vertex-transitive graphs. Details on these classes are omitted due to lack of space. Second, a graph $G = (V, E)$ is *distance-regularized* if for every $v \in V$ and every $r \geq 1$, any vertex of $D_G(v, r)$ has equal number of neighbours in $D_G(v, r-1)$, resp. in $D_G(v, r+1)$. The latter is a common generalization of distance-regular graphs and generalized polygons. Furthermore, as proved in [9], every distance-regularized graph is either (i) distance-regular; or (ii) is a bipartite graph $B = (V_0 \cup V_1, E)$ such that, for every $i \in \{0, 1\}$ the graph $G_i = (V_i, \{\{u, v\} \mid dist_B(u, v) = 2\})$ is distance-regular (such bipartite graphs are called *distance biregular*). Summarizing, we deduce the following from Corollaries 3 and 7:

Corollary 8. *For every distance-regularized graph G that is not a clique, $\delta(G) \geq \lfloor diam(G)/2 \rfloor /2$.*

Finally, a graph is called *(ℓ, m)-walk regular* if for every $0 \leq i \leq \ell$ and for every $0 \leq j \leq m$, the number of uv-walks of length i for any two vertices u and v at distance j is a constant $d_{i,j}$ independent of u and v [8]. We prove the m first powers of these graphs are regular, that gives a non trivial lower-bound on their hyperbolicity. Proofs are omitted due to lack of space.

Corollary 9. *Let $\ell \geq m \geq 2$ be integers. For every (ℓ, m)-walk regular graph $G = (V, E)$ that is not a clique, $\delta(G) \geq \lceil m/2 \rceil /2$.*

Open problems. So far, there are few reported lower-bounds on graph hyperbolicity. We ask whether our general approach in this note could be applied to the weaker notions of *r^{th}-order regular* graphs and *r^{th}-distance balanced* graphs in order to obtain new lower-bound techniques.

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