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WARDROP EQUILIBRIA AND RELAXATION OSCILLATIONS IN QUEUING SYSTEMS

VIVEK S. BORKAR[1] AND CORINNE TOUATI[2]

ABSTRACT We analyze delay based routing in parallel queues wherein packets are routed to the queue(s) with the least estimated delay. We argue that the average queue occupancy converges to a Wardrop equilibrium profile, but the instantaneous profile may show either convergence or relaxation oscillations depending on the traffic. The analysis uses concepts and results from dynamical systems theory applied to limiting differential equations for the averaging iterations.

Key words routing in queues; delay estimation; Wardrop equilibria; monotone dynamics; relaxation oscillations

1 Introduction

We revisit the problem of delay based routing in order to highlight an interesting dynamic phenomenon. The scheme we analyze is very simple: Keep a running average of delays experienced on each route to track the expected average delay on that route for the current traffic profile, and route packets to the route(s) with least estimated delay. Using ideas from monotone

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dynamical systems, we show that the time-averaged occupancy in different routes settles to a Wadrop-like equilibrium, but this can be quite misleading if one were to infer the same for instantaneous traffic pattern. For high traffic density, this shows similar behavior, but for low traffic, one finds oscilatory behavior. We expmain this as relaxation oscillations caused by two time scales in the dynamics.

The article is organized as follows. The next section describes the model and the delay estimation and routing schemes Section 3 provides the theoretical analysis of these. Section 4 reports representative numerical experiments along with pertinent observations. Section 5 concludes with some remarks.

2 The adaptation scheme

The adaptation scheme is in two parts: delay estimation and routing. We describe each below. We assume that there are M sources and N routes.

1. *Delay estimation:* This component of the adaptation scheme keeps track of the running average of the delays incurred in each route. That is, for the ith route, we iteratively compute, with $\gamma_i(0) = 0$,

$$\gamma_i(n+1) = \frac{1}{n+1}\sum_{m=0}^{n+1} \zeta_i(m) \quad (1)$$

$$= \gamma_i(n) + \frac{1}{n+1}(\zeta_i(n+1) - \gamma_i(n)), \quad (2)$$

   where $\zeta_i(n)$ is the nth sample of delay observed on route i. We make a few important observations about this iteration.

   (a) The delay is recorded once a time stamped ack is received. In particular, the delay measured at a time point does not correspond to the packet sent at that time point, but to one sent earlier.

   (b) Because the delays are random, the acks are not necessarily received in the same order as that in which the packets were sent.

   (c) The 'clock' $n = 0, 1, 2, \cdots$ above is event-driven, i.e., an update is made when an ack is received.

(d) Some packets can be lost, amounting to infinite delay.

As explained in section V of [4], the first two issues amount to an asymptotically negligible error in the analysis in the 'stochastic approximation algorithm' (2). We shall not delve into this issue here, referring the interested reader to *ibid.* The third issue also does not affect the analysis, as argued in [2]. The last issue is more serious and was circumvented in [4] by using truncated delays, i.e., taking the delay to be a prescribed $\Delta > 0$ if the ack is not received in Δ units of time. We do the same here. This implicitly assumes that the outliers are rare and their omission does not affect the delay statistics too much. Thus we assume:

(A1) The laws of the delays are supported in a finite interval $[0, \Delta]$ with a strictly positive density w.r.t. the Lebesgue measure for each of them, and also for their finite dimensional joint marginals.

2. *The routing scheme:*

We assume that each source directs a packet to the route with the least estimated delay, choosing one uniformly in case of non-uniqueness. Let $\chi_{ij}(n) := I\{n\text{th packet of } i \text{ sent to route } j\}$
and

$$\lambda_{ij}(0) := 0,\ \lambda_{ij}(n) := \frac{1}{n}\sum_{m=1}^{n}\chi_{ij}(m),\ n \geq 0.$$

Let $\tilde{\gamma}_j(n) :=$ the most recent value of $\gamma_j(\cdot)$ recorded at time n. Our transmission policy is then described by:

$$\chi_{ij}(n) = I\{\tilde{\gamma}_j(n) \leq \tilde{\gamma}_k(n)\ \forall\ k\}$$

with (conditional) probability

$$\frac{1}{\sum_\ell I\{\tilde{\gamma}_\ell(n) \leq \tilde{\gamma}_k(n)\ \forall\ k\}}.$$

That is, we route the packet to one of the routes with minimum estimated delay at that instant, with equal probability. Hence

$$\lambda_{ij}(n+1) = \lambda_{ij}(n) + \frac{1}{n+1}\left(\chi_{ij}(n+1) - \lambda_{ij}(n)\right). \qquad (3)$$

3 Convergence analysis

In order to analyze the coupled system of iterations (2)-(3), we need to consider it on a common clock $n = 0, 1, 2, \cdots$. This can be dictated by the temporal sequencing of events, i.e., transmission of packets and reception of acks, and not by an actual 'physical' clock that ticks at the multiples of a fixed duration. For this purpose, define 'local clocks' for (2) and (3) resp. by:

$\eta_j(n) :=$ the number of acks received on route j till time n,
$\kappa_{ij}(n) :=$ the number of transmissions by source i along route j till time n.

Also define

$$\begin{aligned}\mathcal{F}_n &:= \sigma(\gamma_{kl}(\eta_{kl}(m)), \lambda_{kl}(\kappa_{kl}(m)), m \leq n, 1 \leq k, l \leq N),\\ \mathcal{G}_n &:= \sigma(\gamma_{kl}(\eta_{kl}(m))), m \leq n, 1 \leq k, l \leq N)\end{aligned}$$

for $n \geq 0$, i.e., the sigma field generated by all transmissions and ack receptions (resp., all acks receptions) till time n. Let $\delta_i(n), \theta_{ij}(n)$ denote the indicators of resp. an ack being received on route i at time n, and a transmission of a packet from source i to route j at time n. Write $\hat{\gamma}_i(n) := \gamma_i(\eta_i(n))(= \tilde{\gamma}_i(n))$ and $\hat{\lambda}_{ij}(n) := \lambda_{ij}(\kappa_{ij}(n))$. By adding and subtracting appropriate conditional expectations, the above iterations can be written as

$$\begin{aligned}\hat{\gamma}_i(n+1) &= \hat{\gamma}_i(n) + \frac{\delta_i(n+1)}{\eta_i(n+1)}\left(\zeta_i(\eta_i(n+1)) - \hat{\gamma}_i(n)\right)\\ &= \hat{\gamma}_i(n) + \frac{\delta_i(n+1)}{\eta_i(n+1)}\Big(E\left[\zeta_i(\eta_i(n+1))|\mathcal{G}_n\right] - \hat{\gamma}_i(n)\\ &\quad + M_i(n+1)\Big)\end{aligned} \tag{4}$$

$$\begin{aligned}\hat{\lambda}_{ij}(n+1) &= \hat{\lambda}_{ij}(n) + \frac{\theta_{ij}(n+1)}{\kappa_{ij}(n+1)}(\chi_{ij}(\kappa_{ij}(n+1)) - \hat{\lambda}_{ij}(n))\\ &= \hat{\lambda}_{ij}(n) + \frac{\theta_{ij}(n+1)}{\kappa_{ij}(n+1)}\Big(E\left[\chi_{ij}(\kappa_{ij}(n+1))|\mathcal{F}_n\right] - \hat{\lambda}_{ij}(n)\\ &\quad + M'_{ij}(n+1)\Big),\end{aligned} \tag{5}$$

where $\{M_{ij}(n)\}, \{M'_{ij}(n)\}$ are suitably defined $\{\mathcal{G}_n\}$- (resp., $\{\mathcal{F}_n\}$-) martingale difference sequences. We shall now make a key modelling assumption regarding the conditional expectations in (4), (5).

(A2) $E[\zeta_i(n)|\mathcal{G}_n]$ is of the form $F_j(\hat{\gamma}_1(n), \cdots, \hat{\gamma}_N(n))$ for a continuously differentiable $F_i : \mathcal{R}^N \mapsto \mathcal{R}$ satisfying

$$\frac{\partial F_i}{\partial x_j} > 0 \ \ \forall \ i \neq j. \tag{6}$$

The motivation behind this assumption is as follows. The increase in delay in one route, in view of our routing policy, drives packets to other routes causing greater delays there, which leads to (6). Let $F := [F_1, \cdots, F_N]^T : \mathcal{R}^N \mapsto \mathcal{R}^N$.

Remark The advantage of this assumption is that it is a very qualitative requirement which is perfectly reasonable in the present circumstances as pointed out above. In particular, it does not require specific modelling assumptions on the traffic. Thus, for example, we do not make any specific statistical assumptions on the traffic and also allow for other ambient traffic that contributes to the delays but is otherwise not a part of our analysis.

(A3) $\forall x, 1 + y \notin \text{spectrum}(DF(x)) \ \forall y \in \mathcal{R}$.

Lemma 1 The set $H := \{x : F(x) = x\}$ of fixed points of F is nonempty, each point in H is isolated, and there are no two points in H such that one is componentwise strictly larger than the other.

Proof Since delays are assumed to be bounded, F has a bounded range and the first claim is an easy consequence of the Brouwer fixed point theorem. In view of $(A3)$, $0 \notin \text{spectrum}(DF(x) - I)$. That is, $DF(x) - I$ is full rank at each point in F. The second claim then follows by the inverse function theorem. H is precisely the set of equilibria of the o.d.e.

$$\frac{d}{dt}x(t) = \frac{1}{N}(F(x(t)) - x(t)). \tag{7}$$

By (6), this is a cooperative o.d.e. [7], [8]. The third claim now follows Theorem 3.2, p. 38, [8]. □

Lemma 2 $\{\gamma_i(n)\}$ converge a.s. to a (possibly random) point in H.

Proof It suffices to show this for $\{\hat{\gamma}(n) := [\gamma_1(n), \cdots, \gamma_N(n)]^T, n \geq 0\}$. Iteration (4) is a stochastic approximation scheme taking values in $[0, \Delta]^N$. By the results of [2] (see also [3], Chapter 7), the limiting o.d.e. is (7). which is a *cooperative* o.d.e. in view of (6). By a well known theorem of Hirsch (Theorem 4.1, [7], pp. 435) it converges to H for a.e. initial conditions in $[0, \Delta]^N$. By **(A1)**, we may replace this by 'for a.s. all initial conditions'. It then follows by standard arguments as in [3], Chapter 2, that $\hat{\gamma}(n), n \geq 0$, converge a.s. to H. (See, e.g., the analysis of [5] for a related problem.) Point convergence follows from Corollary 4, p. 18, [3], combined with Lemma 1 above. □

By ($A3$), the equilibria of (7) are hyperbolic (i.e., the Jacobian matrix of the driving vector field, viz., $DF(x) - I$, does not have any eigenvalue on the imaginary axis for any $x \in H$). Let

$$H_s := \{x \in H : \text{ spectrum}(DF(x) - I) \subset \text{ the open left half plane}\},$$

i.e., the set of stable equilibria. Then under certain technical assumptions on the noise (see section 4.3, [3]), we can refine the conclusion of Lemma 2 to:

Lemma 2' $\{\gamma_i(n)\}$ converge a.s. to a (possibly random) point in H_s.

The key technical assumption required for this is that the noise in the stochastic approximation scheme, represented by the martingale difference sequence, be rich enough in the sense that have it have adequate variance along all directions in a cone transversal to stable manifolds of unstable equilibria (see *ibid.*). We have not verified this here, but this is not a serious omission as these conditions can be easily ensured by adding a small i.i.d. gaussian noise to the empirical delays prior to their averaging. In practice, this is not in general essential, the system noise is good enough.

Let $\gamma_j(\infty) := \lim_{n\uparrow\infty} \gamma_j(n)$. Letting $\mathcal{F}_\infty := \vee_{n\geq 0}\mathcal{F}_n$, we then have by Theorem 3.3.8, p. 56, [1],

$$\begin{aligned} E[\chi_{ij}(n)|\mathcal{F}_n] &\rightarrow P(\gamma_j(\infty) \leq \gamma_k(\infty)\ \forall k \mid \mathcal{F}_\infty) \\ &= I\{\gamma_j(\infty) \leq \gamma_k(\infty)\}\ \forall k. \end{aligned}$$

The limiting o.d.e. for (5) then is

$$\frac{d}{dt}y_{ij}(t) = I\{\gamma_j(\infty) \leq \gamma_k(\infty)\ \forall k\} - y_{ij}(t),\ 1 \leq i \leq M,\ 1 \leq j \leq N. \quad (8)$$

We now argue pathwise, in an almost sure sense. This is understood and not stated explicitly henceforth. As discussed in bullet (iv) of pp. 58-59 of [3], we have to view this as a *differential inclusion* in the following sense. Let $\psi := \{j : \gamma_j(\infty) = \min_k \gamma_k(\infty)\}$. Note that this is a *random* set. Then (8) is tantamount to the pair

$$\frac{d}{dt}y_{ij}(t) = -y_{ij}(t),\ 1 \leq i \leq M,\ j \notin \psi, \quad (9)$$

$$\frac{d}{dt}y_{ij}(t) \in [0,1] - y_{ij}(t),\ 1 \leq i \leq M,\ j \in \psi. \quad (10)$$

For $j \notin \psi$, $y_{ij}(t) \to 0$, hence $\gamma_{ij}(n) \to 0$. That is, in the limit, the flows concentrate on routes with minimum average delay, which is Wardrop behaviour. We cannot say anything more specific regarding the dynamic behavior of (10), except that $\lim_{t\uparrow\infty}\sum_{j\in\psi} y_{ij}(t) = 1$, which is consistent with the above observation. Our observation can be alternatively summarized as:

Lemma 3 Almost surely, every limit point of $\{\gamma_{ij}(n),\ 1 \leq i \leq M,\ 1 \leq j \leq N,\ n \geq 0\}$ as $n \uparrow \infty$ is supported on routes with minimum average delay.

The delays for each of these routes are perforce equal and those on the rest are at least as much, in concordance with the definition of a Wardrop equilibrium. We now refine this result further using the last part of Lemma 1, under the following additional assumption:

$(A4)$ Average delay on a route is a strictly increasing continuous function of the flow therein.

Theorem 1 Almost surely, the flows concentrate on a Wardrop equilibrium and the corresponding average delay is uniquely specified.

Proof Only the uniqueness claim remains to be established. Suppose that there are two points x, y satisfying the Wardrop condition. Then some components of x (resp., y) are strictly positive and equal to (say) $a > 0$ (resp., $b > 0$) with the rest being zero. Suppose $a > b$. By the last claim of Lemma

1, there is at least one route (say $\hat{j}$) for which the corresponding component of x is zero and that of y is strictly positive. Let k be a route where the flow under x is strictly positive. Then moving an infinitesimal amount of flow from k to $\hat{j}$ will reduce its delay, a contradiction to the Wardrop property. The claim follows. $\square$

This observation, however, needs one serious qualification. What we have here is average behaviour. The pathwise behaviour, however, can vary drastically depending on the relative timescales. It is well known in game theory community that learning algorithms which lead to individual empirical frequencies of plays converging to a Nash or Wardrop equilibrium may in fact have a very poor actual performance depending on how the *joint* relative frequencies behave [10][3]. In the present instance, for example, suppose the delays are very high compared to the arrival rates. Then a desirable route will quickly get overcrowded because the feedback about delays has not yet registered with the users. Once it does and flags the route as undesirable, the entire flow may switch to another route, re-creating an identical situation there. This corresponds to considering (7) modified as

$$\frac{d}{dt}x(t) = \epsilon(F(x(t)) - x(t)),$$

where $\epsilon > 0$ is very small, indicating a significant separation of time scales. This is a singularly perturbed system which can show the so called relaxation oscillations (see, e.g., Chapter 12 of [9]). These capture precisely the phenomenon described above: The fast dynamics sees the slow one as quasi-static and may be analyzed (approximately) by freezing the latter at a constant value. The slow dynamics in turn sees the fast dynamics as quasi-equilibrated and may be analyzed (approximately) by replacing the latter by its equilibrium behavior. But these two effects may work at cross purposes. Thus, for example, the value of the slow dynamics frozen at one of its putative equilibria y_1 may affect the fast dynamics in a manner that pushes it quickly to a value that renders y_1 unstable for the slow dynamics and pushes

[3] Consider, e.g., the 'matching pennies' game wherein two players choose from $\{0, 1\}$ and get unit reward if their choices match, none otherwise. If both choose the sequence $1, 0, 1, 0, 1, 0, \cdots$, the reward is maximum, but if one of them switches to $0, 1, 0, 1, 0, 1, \cdots$, the reward is zero, although the empirical frequencies are identical in both, being given by $\{\frac{1}{2}, \frac{1}{2}\}$.

it slowly towards another value y_2 for which the opposite takes place. One can then expect to see rapid switches between y_1 and y_2 with a sojourn of a significant duration near each. This is the intuition behind relaxation oscillations. This suggests that the above phenomenon may be viewed as a special instance of relaxation oscillations.

Figs. ..., ... show typical oscillation behavior when traffic is low (few sources) whereas Figs — show the convergent behavior under heavy traffic. Note that the time average of the oscillations converges as predicted. There is, however, a subtle point involved regarding the nature of these oscillations. The route choices of the packets are seen to spend longer and longer times at any route. A trajectory spending longer and longer time in the neighborhood of an equilibrium before moving relatively quickly to another is a signature of a heteroclinic cycle [6], wherein the trajectory connects in succession two or more hyperbolic equilibria, the unstable manifold of one merging with the stable manifold of the next. This typically occurs in systems with symmetries, for which there is no compelling reason here. The catch is the time scaling inherent in the passage from the actual iteration to its o.d.e. limit. Treating the iterations above as stochastic approximation that approximate appropriate limiting o.d.e.s as stated calls for a time scaling $n \mapsto t(n) = \sum_{m=0}^{n} \frac{1}{m+1}$, because the iteration has $\frac{1}{n+1}$ as the 'step size' which doubles up as the discrete time step when we view it as an approximation to the limiting o.d.e. Since $t(n) = O(\log n)$, the time scaling is logarithmic. If we correct for it by plotting route choices against a logarithmic time axis, we get a periodic pattern. In ther words, when viewed on the correct time scale, the oscillations are periodic, which is a signature of relaxation oscillations caused by two time scales.

4 Numerical Analysis

For illustration purposes, consider a system with M sources and 2 routes. Each source i emits packets according to a Poison distribution of parameter $\lambda_i = 17/M$. In this way, the system has a total arrival flow following a Poison distribution of parameter 17. Each of the two route is modeled as a FIFO queue of service rate 10.

Thus, the Wardrop equilibria are the vectors $((\lambda_i^1)_{1\leq i\leq M}, (\lambda_i^2)_{1\leq i\leq M})$ such

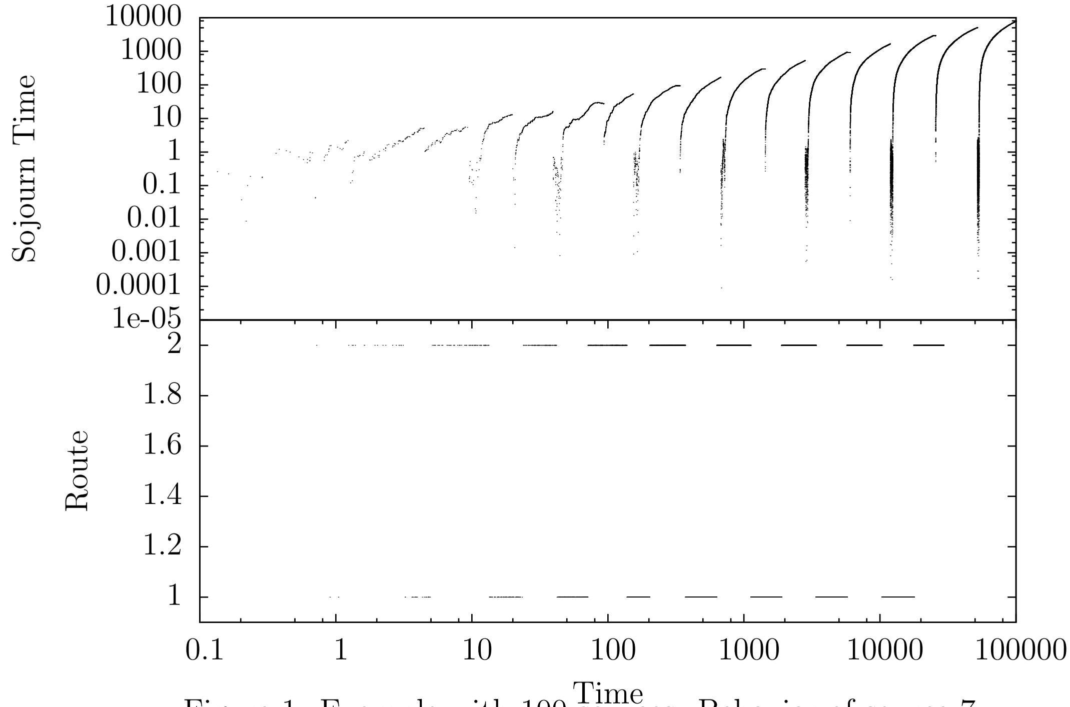

Figure 1: Example with 100 sources. Behavior of source 7.

that:

$$\forall i, 0 \leq \lambda_i^1 \leq 17/M, \quad \lambda_i^1 + \lambda_i^2 = 17/M \text{ and } \sum_{i=1}^{M} \lambda_i^1 = 8.5.$$

In other words, at the Wardrop equilibria, a Poison flow of rate 8.5 is sent on each route, which thus follows a M/M/1 formulation. Then, the average sojourn time for the packets is $\frac{1}{10-8.5} = 2/3$.

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