Characterization of ” low-pass filtering ” for some nonlinear ODE’s

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Abstract: The characterization of low-pass linear filters is nowadays a simple question relying on eigenvalue criteria, Bode diagram, Nyquist or Nichols plots. For nonlinear systems, the problem is more tricky. The present work tempts to develop a methodology for the characterization of “low-pass filtering” for nonlinear systems. Inspired by the DiPerna-Lions theory and relying on the theory of characteristics, the proposed method associates to the nonlinear ODE a linear transport PDE. A characterization of “low-pass filltering” is then deduced from developments on the PDE. Smooth ODE’s of the form $\dot{x} = F(x)$ are considered where the vector field $F(x)$ is perturbed by an additive, rapidly oscillating, noise m which may have a big magnitude $F(x + m)$. An intuitive observation is proved in this first contribution: if F has bounded derivatives, then the sensitivity of x to m decreases as the bound gets smaller or as m fluctuates faster.

Keywords: Big magnitude noise, Characterization of “low-pass filtering” for nonlinear systems, transport equation.

1. INTRODUCTION

The theory of linear filtering is nowadays widely taught, understood and utilized in concrete applications. It owes its popularity to its conceptual simplicity and to the associated graphical representations. The characterization of linear filters (low-pass, high-pass, band-pass etc.) relies on a simple eigenvalue criteria and elegant graphical representations of Bode and Nichols. Combined to the Fourier decomposition of signals, linear theory tools provide a design methodology to get rid off high-frequency noise. Remarkably, the linear theory do not necessitate specific conditions on the noise magnitude, all that is required is a sufficient separation between the noise spectrum and the informative signal spectrum. Therefore filtering out a large magnitude high-frequency noise is accomplished by an adequate design of the cut-off frequency and the order of a filter. The complexity of the filter design problem rises slightly in the context of feedback control since the “filter” (controller) should be dealing with conflicting requirements: noise filtering versus closed loop reactivity.

This harmonious picture breaks down when it comes to nonlinear systems, especially, nonlinear control and observer design. In the nonlinear observers literature, noise characterization (and in particular big magnitude and high-frequency noise) have not yet received a systematic treatment (see Besançon [2007], Fridman et.al. [2011] and Nijmeijer & Fossen [1999]). Moreover, for high-gain observers and for finite-time differentiators, noises with bounded magnitudes has been considered so far (Ahrens et al. [2009], Levant [2003]). Bounded noise means: as the noise magnitude gets smaller, the magnitude of the noise on the state of the observer or differentiator decreases. A natural question arises about the response of a nonlinear system (feedback, observer, differentiator...) when its state vector is affected by an additive noise which may have a big magnitude and high-frequency. This paper tempts to develop a methodology for a systematic analysis of this question. A big magnitude noise can be found in telecommunication systems, in mechanical systems, in power electronic devices, choppers etc. The problem is interesting since noises may have undesired effects on the system such as the excitation of hidden modes. For example, for a sliding mode controller of the form $u = -\text{sign}(x + \alpha\dot{x})$, if x is noise-free and $\dot{x}$ is estimated by an inadequate differentiator which induce a big magnitude noise, the proportional action x in the controller will be hidden by the noisy $\alpha\dot{x}$ and the controller performs far from design expectations.

The proposed methodology is greatly inspired from the DiPerna and Lions theory (DiPerna & Lions [1989]). In order to study existence and uniqueness questions related to Sobolev vector fields $F(x)$ where the Cauchy-Lipschitz theory fails, DiPerna and Lions proposed, using the theory of characteristics, to study first the solutions of the corresponding transport PDE then deduce theorems for the ODE. We note also that this theory was extended by Ambrosio [2004] to cover vector fields having bounded spatial variation.

Existence and uniqueness problems are not of our concern since sufficiently differentiable vector fields are studied in this first work. However, for the description of the “low-pass filter” character of some nonlinear systems, the theory of characteristics (Evans [1997]) is employed to write down the transport PDE corresponding to our nonlinear ODE. By considering distributional solutions for the transport PDE and imposing bounds on the derivatives of the vector field, we show that the nonlinear ODE filters out a high-frequency noise whose magnitude can be big. Transfering the problem to the study of a transport PDE offers a functional analytic framework (Brezis [2010]) in which an expression of the form $F(x + m) = F(x) + F'(\xi)m$,

$\xi \in (x, x+m)$ can be written down. The state x and the noise m are seen as elements in the Banach space of square summable functions and F a nonlinear mapping from this space into itself. Remarkably, if F is a continuous map, then F' corresponds to the derivative of F seen as a standard function. The additive noise on x is tranfered by this procedure to a multplicative one affecting $F(x)$ linearly. We note that a representation of big magnitude noises is provided in Riachy [2014] and Riachy et al. [2016], its filtering was done in Riachy [2014] by a least squares estimator and by mollification in Riachy et al. [2016]. The filtering through the nonlinear system was not invoked in these works.

The paper is organized as follows. Section 2 provides a rough description of the problem while section 3 contains a description of the methodology adopted to formalize the problem. Sections 4 and 5 introduce the representation of the big-magnitude and rapidly oscillating noise and recall rudiments from calculus in a Banach space. Section 6 provides a description of a prototype problem for smooth vector fields and its solution.

2. PROBLEM DESCRIPTION

Consider a function $m(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ with the following characteristics:

- $m(t)$ is rapidly oscillating
- $m(t)$ may have a big magnitude
- its integral $\int_\Omega m(t)dt$ on some bounded interval $\Omega \subset \mathbb{R}^+$ is small.

A function satisfying the above characteristics *belongs* to the Sobolev space with negative exponent $\mathcal{H}^{-1}(\Omega)$. As an example, the function $m(t) = \frac{1}{\epsilon}\cos\left(\frac{t}{\epsilon^2}\right)$ has a big magnitude, oscillates rapidly and its integral $\int \frac{1}{\epsilon}\cos\left(\frac{t}{\epsilon^2}\right)dt = \epsilon \sin\left(\frac{t}{\epsilon^2}\right)$ is small as ϵ gets small.

Let $\mathbb{D}$ be an open subset of $\mathbb{R}^n$, $\mathbb{D} \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and consider the smooth nonlinear vector field:

$$F(x) : \mathbb{D} \mapsto \mathbb{R}^n.$$

Consider also a noise $\varpi(t) = [m_1(t), \cdots, m_n(t)]'$ where $m_1(t)$, $\cdots$, $m_n(t)$ are n rapidly oscillating functions as described previously in this section.

The objective is to characterize low-pass filtering capability of the nonlinear system associated to the vector field $F(x)$. We consider strong conditions in this first work and suppose that the first and second derivatives of F exist and are bounded. Less regular vector fields should be investigated in future publications. In particular, vector fields associated to homogeneous, finite-time convergent, ODE's (Levant [2003], Perruquetti et al. [2008]).

We will show that the flows $X^\star(t,x)$ and $X(t,x)$ (initiated at x for $t=0$) associated respectively to the vector fields:

$$\dot{X}^\star(t,x) = F(X^\star(t,x)) \tag{1}$$

and

$$\dot{X}(t,x) = F(X(t,x) + m(t)) \tag{2}$$

are close in the sense that some norm $\|X^\star - X\|$ gets smaller as $m(t)$ fluctuates faster.

Beyond this "prototype problem", the contribution of the present work is the description of a methodology which can be extended to more general vector fields. Our approach should be confronted in future publications to the nonlinear filtering theory based on stochastic differential equations driven by white noise (see Cacace et. al. [2016], Kallianpur & Karandikar [1985]).

3. THE METHODOLOGY

Our approach is inspired from the DiPerna and Lions theory (DiPerna & Lions [1989]) which we summarize in the following lines. Let us recall beforehand that the study of solutions of first order linear PDE's reduces, by the theory of characteristics, to the study of the existence and uniqueness of solutions for the corresponding nonlinear ODE's (Evans [1997]). This theory is widely used when the vector field associated to the ODE has Lipschitz regularity in space variable. Consider the vector field $F(x)$ and the associated flow $X(t,x)$ solving the equation:

$$\dot{X}(t,x) = F(X(t,x)).$$

The theory of characteristics is based on the observation that a smooth function $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is constant along a particular trajectory $X(t,x)$ if and only if it solves the transport equation

$$\partial_t u + F(x) \cdot \nabla_x u = 0$$
$$u(t,x) = \bar{u}(x)$$

In fact differentiate $u(t,x)$ with respect to time to find

$$\frac{du}{dt} = \partial_t u + F(x) \cdot \nabla_x u = 0. \tag{3}$$

For solving a transport PDE, a common practice is first to solve the associated ODE.

In order to study the existence and uniqueness of solutions for vector fields having Sobolev regularity, DiPerna and Lions (DiPerna & Lions [1989]) took the opposite direction. Using the method of characteristics they demonstrated that the existence and uniqueness of the ODE solutions can be deduced from the solutions of the PDE. They introduced the concept of renormalized solutions in addition to distributional solutions of the transport equation and showed that if a distributional solution is renormalized then solutions of the ODE exist. We note also that the DiPerna-Lions theory was extended to vector fields having bounded variations by Ambrosio (Ambrosio [2004]). In addition, Crippa & De Lellis [2008] and De Lellis [2008] proved differently the existence and uniqueness of solutions for Sobolev vector fields without using the transport PDE.

We will be using the same strategy in the sequel. In order to show that solutions of (1) and (2) remain close to each other if both are initiated at the same x, we utilize the corresponding PDE. We consider smoothness and boundedness assumptions such that the ODE and the corresponding transport PDE both have classical solutions which are unique for a given initial condition. The proposed method can be applied in many situations, let us mention:

- nonlinear observer design: a nonlinear observer can be of the form $\dot{X} = F(X) + \Psi(Y)$ where Y is a noisy

measurement $Y = X + m$. This amounts to study the ODE $\dot{X} = F(X) + \Psi(X + m)$.
- feedback control design: consider the system $\dot{X} = F_1(X) + G(X)u$ with a feedback control $u = F(Y)$ and $Y = X + m$ a noisy measurement. The closed loop system is given by $\dot{X} = F_1(X) + G(X)F(X+m)$.

We note that the study of an ODE of the form $\dot{X} = F_1(X) + G(X)F(X + m)$, for example, can be done by straightforward adaptation of the forthcoming developments on (2).

4. REPRESENTATION OF THE BIG MAGNITUDE NOISE

Let $\Omega \subset \mathbb{R}^l$ be an open interval, take $w \in \mathcal{L}^2(\Omega)$, the space of square integrable functions equipped with the norm:

$$\|w\|^2_{\mathcal{L}^2(\Omega)} = \int_\Omega w^2 dz.$$

In the sequel l may take the value 1 for time dependent functions or n for state dependent vector fields. Thus, depending on the context, z can be t or the state vector x.

Consider the Sobolev space $\mathcal{H}^1(\Omega)$ of all functions $w \in \mathcal{L}^2(\Omega)$ whose distributional first order derivatives belong to $\mathcal{L}^2(\Omega)$. If there exist functions $\omega_i \in \mathcal{L}^2(\Omega)$, $i = 1 \cdots l$, such that:

$$\int_\Omega w(z)\partial_{z_i}\phi(z)dz = -\int_\Omega \omega_i(z)\phi(z)dz \tag{4}$$

for all $\phi \in \mathcal{C}^\infty_c(\Omega)$ where $\mathcal{C}^\infty_c(\Omega)$ is the space of infinitely differentiable and compactly supported functions, then the functions ω_i are said to be the distributional derivatives of w with respect to z. The space $\mathcal{H}^1(\Omega)$ is a Banach space under the norm:

$$\|w\|^2_{\mathcal{H}^1(\Omega)} \triangleq \|w\|^2_{\mathcal{L}^2(\Omega)} + \sum_{i=1}^{l} \|\partial_{z_i} w\|^2_{\mathcal{L}^2(\Omega)}. \tag{5}$$

With a little abuse of notation, $\partial_{z_i} w$, $i = 1 \cdots l$, will denote the distributional derivatives of w (4).

The space $\mathcal{H}^1(\Omega)$ is a Hilbert space equipped with the inner product:

$$\langle w_1, w_2\rangle_{\mathcal{H}^1(\Omega)} \triangleq \int_\Omega w_1 w_2 dz + \sum_{i=1}^{l}\int_\Omega (\partial_{z_i} w_1)(\partial_{z_i} w_2)dz. \tag{6}$$

A bounded linear functional on $\mathcal{H}^1(\Omega)$ is a bounded linear operator $L : \mathcal{H}^1(\Omega) \mapsto \mathbb{R}$. The set $\mathcal{H}^{-1}(\Omega)$ contains all such bounded linear functionals L, it is called the dual space of $\mathcal{H}^1(\Omega)$; $\mathcal{H}^{-1}(\Omega) = (\mathcal{H}^1(\Omega))'$.

By the Riesz representation theorem there exist, for each $L \in \mathcal{H}^{-1}(\Omega)$, $(l+1)$-functions, $f_0 \cdots f_l \in \mathcal{L}^2(\Omega)$, such that

$$\begin{aligned} L(w) &= \int_\Omega f_0 w dz + \sum_{i=1}^{l}\int_\Omega f_i(\partial_{z_i} w)dz,\ \forall w \in \mathcal{H}^1(\Omega), \\ &\triangleq \langle f_0, w\rangle + \sum_{i=1}^{l}\langle f_i, \partial_{z_i} w\rangle \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ (without subscripts) denotes the duality pairing of $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^{-1}(\Omega)$.

The functional $\|L\|_{\mathcal{H}^{-1}(\Omega)} : \mathcal{H}^{-1}(\Omega) \mapsto \mathbb{R}^+$ given by:

$$\|L\|_{\mathcal{H}^{-1}(\Omega)} \triangleq \sup_{w\in\mathcal{H}^1(\Omega), \|w\|_{\mathcal{H}^1(\Omega)}\le 1} |L(w)| \tag{7}$$

defines a norm on $\mathcal{H}^{-1}(\Omega)$ and turns it into a Banach space.

Conversely, any functions $f_0 \cdots f_l \in \mathcal{L}^2(\Omega)$, determine an element $L_{f_0\cdots f_l}(w)$ of $\mathcal{H}^{-1}(\Omega)$ by:

$$L_{f_0\cdots f_l}(w) = \langle f_0, w\rangle + \langle f_1, \partial_{z_1} w\rangle + \cdots + \langle f_l, \partial_{z_l} w\rangle.$$

Consider the set:

$$\mathcal{H}^{-1}_\epsilon(\Omega) = \{f_0, \cdots, f_l \in \mathcal{L}^2(\Omega);\ \|L_{f_0\cdots f_l}\|_{\mathcal{H}^{-1}(\Omega)} < \epsilon\} \tag{8}$$

which, for a small $\epsilon > 0$, contains, among others, large magnitude but rapidly oscillating functions, such that their integral is less than ϵ. Indeed, take $w = 1$ then $|L_{f_0\cdots f_l}(1)| = |\int_\Omega f_0(z)dz| \le \|L_{f_0\cdots f_l}\|_{\mathcal{H}^{-1}(\Omega)} < \epsilon$. Note that

$$|\langle f_0, w\rangle| \le \|f_0\|_{\mathcal{L}^2(\mathbb{D})}\|w\|_{\mathcal{L}^2(\mathbb{D})} \le \|w\|_{\mathcal{L}^2(\mathbb{D})}\|L_{f_0}\|_{\mathcal{H}^{-1}(\mathbb{D})}. \tag{9}$$

As previously noted, depending on the context, Ω may correspond to a time interval or an open bounded domain in $\mathbb{R}^n$ where n is the dimension of the state vector.

5. CALCULUS IN A BANACH SPACE

Let $\mathcal{M}$ and $\mathcal{N}$ be two Banach spaces, let $\mathcal{D}$ be an open subset of $\mathcal{M}$ and consider F, a mapping from $\mathcal{D}$ into $\mathcal{N}$:

$$F : \mathcal{D} \mapsto \mathcal{N}. \tag{10}$$

Given $x, h \in \mathcal{D}$, if there exists a bounded linear map $DF : \mathcal{M} \mapsto \mathcal{N}$, satisfying:

$$\lim_{\|h\|\mapsto 0} \frac{\|F(x+h) - F(x) - DFh\|}{\|h\|} = 0, \tag{11}$$

then F is said to be differentiable at x and DF is called the Fréchet derivative of F at x. Moreover, if F is differentiable at x, then the mapping DF is uniquely defined. In addition, If F is bounded in a neighborhood of x then DF is also bounded.

Example 5.1. Let $\mathcal{D} = \mathcal{M} = \mathcal{N} = \mathcal{L}^2(\Omega)$, $\Omega \subset \mathbb{R}^+$. Consider a continuously differentiable function $\phi : \mathbb{R} \mapsto \mathbb{R}$ and let $F : \mathcal{M} \mapsto \mathcal{N}$ be the composition $F(x) = \phi \circ x$ with $x \in \mathcal{M}$. The quantity $[F(x)](t)$ corresponds to the evaluation of $F(x)$ at a given time instant $t \in \Omega$: $[F(x)](t) = \phi(x(t))$. By using the standard mean value theorem in $\mathbb{R}$ we obtain, for almost every t:

$$\begin{aligned} [F(x+h) - F(x)](t) &= \phi(x(t) + h(t)) - \phi(x(t)) \\ &= \phi'(x(t) + \theta(t)h(t))h(t), \end{aligned}$$

where $0 < \theta(t) < 1$.

Let:

$$DF = \phi' \circ x,$$

and verify, at almost every t, that:

$$\begin{aligned} [F(x+h) - F(x) - DFh](t) &= \phi'(x(t) + \theta(t)h(t))h(t) \\ &\quad - \phi'(x(t))h(t). \end{aligned}$$

Taking the norm in the previous equation we have

$$\|F(x+h) - F(x) - DFh\| \le \|\phi' \circ (x + \theta h) - \phi' \circ x\|\|h\|$$

By the continuity of ϕ' we show that DF is indeed the derivative of F at x: $DF(x) = \phi' \circ x$. $\square$

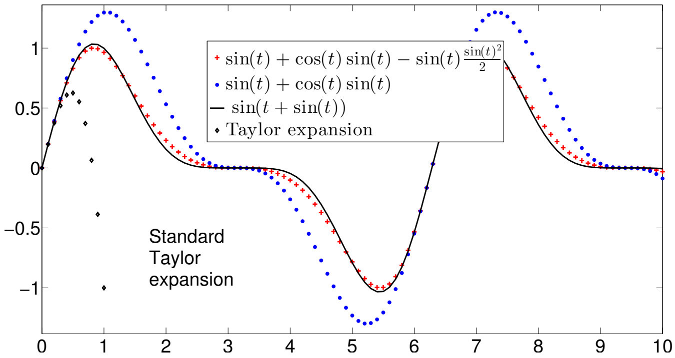

Fig. 1. Approximation of $\sin(t+\sin(t))$

Beyond the first derivative, it is possible to define a Taylor expansion in a Banach space, we illustrate this definition through an example. This example highlights the difference between the Taylor expansion in an $\mathcal{L}^p$ space and $\mathbb{R}$. Let Ω be a bounded open interval of $\mathbb{R}^+$ and consider the mapping $F=\sin(t+m)$, $F:\mathcal{L}^2(\Omega)\mapsto\mathcal{L}^2(\Omega)$ where t and m are seen as elements of $\mathcal{L}^2(\Omega)$. A Taylor expansion of F writes:

$$\sin(t+m)=\sin(t)+\cos(t)m-\sin(t)\frac{m^2}{2}-\cos(t)\frac{m^3}{6}+\cdots. \tag{12}$$

Figure 1 shows successive approximations for $m=\sin(t)$. On the other hand, the standard Taylor expansion in $\mathbb{R}$ gives $\sin(t+\sin(t))=0+2t+0-9\frac{t^3}{6}$.

Moreover, the function $\sin(t+m)$ where m corresponds to high frequency noise is plotted on figure 2, the truncation error is also plotted on figures 3, 4 and 5 for three different truncation orders.

Theorem 5.2. (Mean value theorem in Banach space). Let F be a real-valued mapping defined on an open set $\mathcal{D}$ in a Banach space. Let $a,b\in\mathcal{D}$. Assume that the interval

$$[a,b]=\{a+\theta(b-a);\ 0\le\theta\le 1\}$$

lies in $\mathcal{D}$. If F is continuous on $[a,b]$ and differentiable on the open interval (a,b) then for some $\xi\in(a,b)$

$$F(b)-F(a)=DF(\xi)(b-a).$$

□

Therefore, consider $x,m\in\mathcal{D}$ such that x and m satisfy the conditions of the previous Theorem, in particular, the interval $[x,x+m]=\{x+\theta m;\ 0\le\theta\le 1\}$ lies in $\mathcal{D}$. Then, $\exists\,\xi\in[x,x+m]$ such that

$$F(x+m)=F(x)+DF(\xi)m.$$

6. NONLINEAR LOW-PASS FILTERING: A PROTOTYPE PROBLEM AND ITS SOLUTION

Consider $\mathbb{D}$, an open subset of $\mathbb{R}^n$, and the nonlinear system:

$$\dot{X}=F(X) \tag{13}$$

where $X=[x_1,\cdots,x_n]'\in\mathbb{D}$ and $F(X):\mathbb{D}\mapsto\mathbb{R}^n$ a vector field satisfying the following assumption.

Assumption 6.1. $F(X)=[F_1(X),\cdots,F_n(X)]'$ is differentiable with respect $x_1,\ \cdots,x_n$ with bounded derivatives.

□

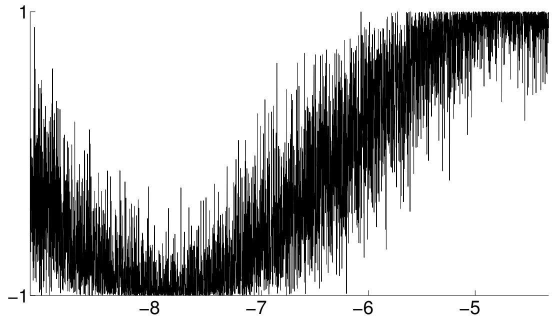

Fig. 2. $\sin(t+m)$

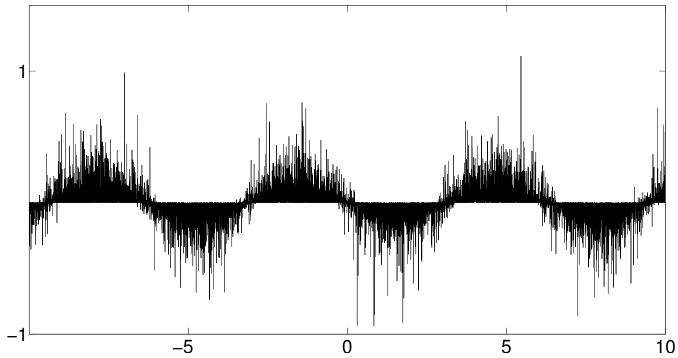

Fig. 3. $\sin(t+m)-[\sin(t)+\cos(t)m]$

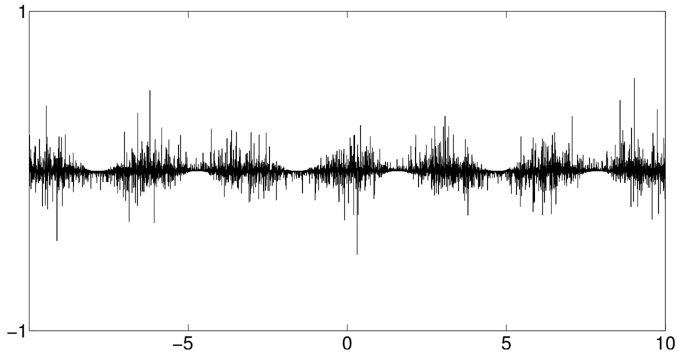

Fig. 4. $\sin(t+m)-\left[\sin(t)+\cos(t)m-\sin(t)\frac{m^2}{2}\right]$

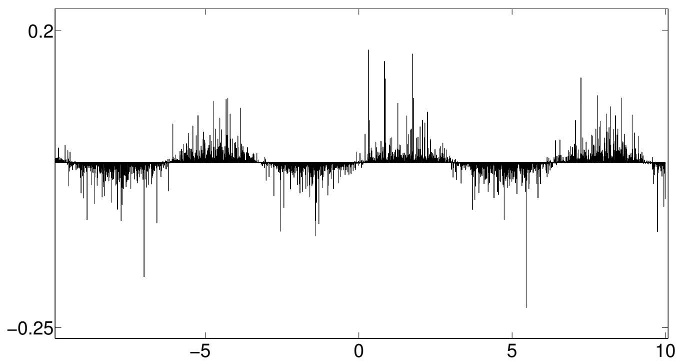

Fig. 5. $\sin(t+m)-\left[\sin(t)+\cos(t)m-\sin(t)\frac{m^2}{2}-\cos(t)\frac{m^3}{6}\right]$

For later use, we set

$$G(X) = \operatorname{div}_X F(X) = \frac{\partial F_1(X)}{\partial x_1} + \cdots + \frac{\partial F_n(X)}{\partial x_n}. \quad (14)$$

Assumption 6.2. The gradient of the scalar function $G(X)$ is bounded. □

Due to the boundedness of the derivatives (Assumption 6.1), the Cauchy-Lipschitz Theorem ensures the existence of a solution of (13). This solution is unique for a given initial condition $X(0) = x$.

Therefore a flow $X^\star(t,x)$ exists satisfying the differential equation:

$$\dot{X}^\star(t,x) = F(X^\star(t,x)). \quad (15)$$

Let $m_1(t), \cdots, m_n(t) \in \mathcal{H}_\epsilon^{-1}(\Omega)$ where $\mathcal{H}_\epsilon^{-1}(\Omega)$ is given by (8), $m = [m_1, \ \cdots, \ m_n]'$ and consider the differential equation:

$$\dot{X} = F(X + m) \triangleq F(t, X). \quad (16)$$

Once again, the Cauchy-Lipschitz Theorem ensures the existence and uniqueness of the solutions of (16) since $\mathcal{H}_\epsilon^{-1}(\Omega) \subset \mathcal{L}^2(\Omega)$ then $F(t,\cdot)$ is summable. It is then possible to associate a flow $X(t,x)$ initiated, for $t = 0$, at the same initial condition $X(0,x) = x$ and satisfying:

$$\dot{X}(t,x) = F(X(t,x) + m(t)). \quad (17)$$

A characterization of "low-pass filtering" can be given by an inequality of the form:

$$\|X^\star(t,x) - X(t,x)\|_{\mathcal{L}^2(\Omega)} \leq C\|m(t)\|_{\mathcal{H}^{-1}(\Omega)} \quad (18)$$

where the constant $C > 0$ depends on bounds on F and its derivatives.

To the vector field F, we associate the transport equation

$$\partial_t u + F(x) \cdot \partial_x u = 0, \quad (19)$$
$$u(0,x) = \bar{u}(x). \quad (20)$$

Assumption 6.3. $\bar{u}(x)$ is continuous and $|\bar{u}(x)| < 1, \forall x \in \mathbb{D}$. The set $\{x; \bar{u}(x) = c, \ c \text{ is a constant}\}$ has a zero $(n+1)$-Lebesgue measure. □

The existence and uniqueness of a function $u(t,x)$ satisfying (19) with the initial condition (20) is ensured by the theory of characteristics and Assumptions 6.1 and 6.2.

By straightforward manipulations, equation (19) can be rewritten as follows where we replace u by $u^\star$ to denote the noise-free solution:

$$\partial_t u^\star + \operatorname{div}_x(F(x)u^\star) - G(x)u^\star = 0 \quad (21)$$
$$u^\star(0,x) = \bar{u}(x). \quad (22)$$

The function $G(x) = \operatorname{div}_x(F(x))$ is given by (14).

The transport equation associated to the perturbed vector field is given by:

$$\partial_t u + \operatorname{div}_x(F(x+m)u) - G(x+m)u = 0 \quad (23)$$
$$u(0,x) = \bar{u}(x). \quad (24)$$

Let $\Omega = [0,+\infty)$ and let $\mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ be the space of infinitely differentiable functions which are compactly supported in $\mathbb{R}\times\mathbb{R}^n$. Suppose that $u(t,x) = 0$ for $t < 0$. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ then the following formulas are obtained by integration by parts:

$$\int_0^{+\infty} \int_{\mathbb{R}^n} u^\star \left(\partial_t\psi + \nabla_x\psi \cdot F + \psi G(x)\right) dxdt = -\int_{\mathbb{R}^n} \bar{u}(x)\psi(0,x)dx \quad (25)$$

and

$$\int_0^{+\infty} \int_{\mathbb{R}^n} u \left(\partial_t\psi + \nabla_x\psi \cdot F(x+m) + \psi G(x+m)\right) dxdt = -\int_{\mathbb{R}^n} \bar{u}(x)\psi(0,x)dx \quad (26)$$

A function $u^\star$ (resp. u) $\in \mathcal{L}^\infty_{\text{loc}}(\Omega \times \mathbb{R}^n)$ is said to be a distributional solution of (21) (resp. (23)) if (25) (resp. (26)) holds $\forall \ \psi(t,x) \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$.

Note that equations (25) and (26) have the same right-hand-side.

Consider $x_1, \cdots, x_n$ as square summable functions, $x_1, \cdots, x_n, m \in \mathcal{L}^2(\mathbb{D})$ and

$$F_i : \mathcal{L}^2(\mathbb{D}) \mapsto \mathcal{L}^2(\mathbb{D}), \ i = 1, \cdots, n.$$

By assumption 6.1, F_i can be seen as continuously differentiable mappings associating to an element of $\mathbb{R}^n$, an element of $\mathbb{R}^n$. Therefore, following example 5.1, the Fréchet derivative denoted $D_{x_j}F_i, \ i,j = 1\cdots n$, corresponds to the differential of F_i seen as a function in $\mathbb{R}$. Introduce the notation:

$$\nabla_x F_i(x) = [D_{x_1}F_i, \cdots, D_{x_n}F_i], \ i = 1, \cdots, n,$$

and notice that the sum of two square integrable functions is square integrable, then all the sets:

$$\{x_i + \theta m_j, \ 0 \leq \theta \leq 1\}, \ i,j = 1\cdots n,$$

are subsets of $\mathcal{L}^2(\mathbb{D})$. Therefore, there exist n^2 functions in $\mathcal{L}^2(\mathbb{D})$ $\xi_{i,j}$, $i,j = 1\cdots n$, such that the following is verified:

$$F_1(x+m) = F_1(x) + \nabla_x F_1(\xi_{1,1}, \cdots, \xi_{1,n}) \cdot m$$
$$\vdots \qquad \vdots$$
$$F_m(x+m) = F_m(x) + \nabla_x F_m(\xi_{n,1}, \cdots, \xi_{n,n}) \cdot m.$$

By a similar procedure, there exist $\zeta_1, \cdots, \zeta_n$ such that

$$G(x+m) = G(x) + \nabla_x G(\zeta_1, \cdots, \zeta_n) \cdot m.$$

Let $\nabla_x F(\xi) = [\nabla_x F_1 \cdots \nabla_x F_m]'$ where the letter ξ in (ξ) quotes the dependence of $\nabla_x F$ on the n^2 $\mathcal{L}^2(\mathbb{D})$ functions $\xi_{1,1}, \cdots \xi_{n,n}$. Using the previous expressions in (26) we get:

$$\int_0^\infty \int_{\mathbb{R}^n} (u^\star - u)\left(\partial_t\psi + F \cdot \nabla_x\psi + \psi G(x)\right) dxdt = \int_0^\infty \int_{\mathbb{R}^n} u(t,x)[\nabla_x\psi \cdot \nabla_x F(\xi) \cdot m + \psi \nabla_x G(\xi_{n+1}) \cdot m]dxdt. \quad (27)$$

By the maximum principle, it is well known that $\inf u(t,x) \geq \inf \bar{u}(x)$ and $\sup u(t,x) \leq \sup \bar{u}(x)$. Hence, by assumption 6.3, $|u(t,x)| < 1, \ \forall t, \ \forall x$. With:

$$\nabla_x\psi = [\partial_{x_1}\psi, \cdots, \partial_{x_n}\psi],$$

we have:

$$\nabla_x\psi \cdot \nabla_x F(\xi) \cdot m = [\partial_{x_1}\psi, \cdots, \partial_{x_n}\psi] \times \begin{bmatrix} \partial_{x_1}F_1 & \cdots & \partial_{x_n}F_1 \\ \vdots & & \vdots \\ \partial_{x_1}F_n & \cdots & \partial_{x_n}F_n \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$

$$= [\partial_{x_1}\psi, \cdots, \partial_{x_n}\psi] \times \begin{bmatrix} \partial_{x_1}F_1 m_1 + \cdots + \partial_{x_n}F_1 m_n \\ \vdots \\ \partial_{x_1}F_n m_1 + \cdots + \partial_{x_n}F_n m_n \end{bmatrix}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (\partial_{x_i}\psi)\left(\partial_{x_j}F_i\right) m_j.$$

The symbol $\partial_{x_j}F_i$ is used instead of $D_{x_j}F_i$ because the Fréchet derivative correspond to partial derivative of F and F is seen as a real valued function on $\mathbb{R}$.

Since the test functions $(\partial_{x_i}\psi)$, $i = 1\cdots n$, are compactly supported on $\mathbb{R}^n$, we have after n integrations by parts:

$$\int_{\mathbb{R}^n} (\partial_{x_i}\psi)\left(\partial_{x_j}F_i\right) m_j dx$$
$$= (-1)^n \int_{\mathbb{R}^n} \partial_x \left((\partial_{x_i}\psi)\left(\partial_{x_j}F_i\right)\right)\left(\int_{\mathbb{R}^n} m_j dx\right) dx$$

where $\partial_x(\cdot) \triangleq \partial_{x_1}\cdots\partial_{x_n}(\cdot)$. Since we assumed the boundedness of the derivatives of F (Assumption 6.1) and following the inequality given by (9) we ensure the existence of a positive constant c_i satisfying the following bound:

$$\left|\int_{\mathbb{R}^n} \partial_x \left((\partial_{x_i}\psi)\left(\partial_{x_j}F_i\right)\right)\left(\int_{\mathbb{R}^n} m_j dx\right) dx\right| \leq c_i\epsilon$$

with $\left\|\int_{\mathbb{R}^n} m_j dx\right\|_{\mathcal{H}^{-1}(\mathbb{D})} < \epsilon$. Applying the same procedure to other entries in the right-hand-side of (27), we find that there exists a positive constant C such that:

$$\left\|\int_0^\infty \int_{\mathbb{R}^n} (u^\star - u)\left(\partial_t\psi + F\cdot\nabla_x\psi + \psi G(x)\right) dxdt\right\|_{\mathcal{L}^2(\mathbb{D})} \leq C\epsilon.$$

Since the above inequality is verified $\forall\psi \in \mathcal{C}_c^\infty(\Omega\times\mathbb{R}^n)$ and by assumption 6.3, we conclude that there exist positive constants $\bar{C}$ and $\bar{\epsilon}$ such that

$$\|u^\star - u\|_{\mathcal{L}^2(\mathbb{D})} \leq \bar{C}\bar{\epsilon}.$$

Finally, since both the transport PDE and the ODE admit unique solutions for given initial conditions $\bar{u}(x)$ and x respectively (Recall that uniqueness means that two solutions of the ODE issued from different initial conditions do not intersect each other.), there exist positive constants $\tilde{C}$ and $\tilde{\epsilon}$ such that:

$$\|X^\star(t,x) - X(t,x)\|_{\mathcal{L}^2(\mathbb{D})} \leq \tilde{C}\tilde{\epsilon}.$$

We show by this bound that the solution of the perturbed ODE $X(t,x)$ remain close, in the $\mathcal{L}^2$-norm, to the solution of the noise-free one $X^\star(t,x)$. The error between $X(t)$ and $X(t) + m(t)$ can be big due to big magnitude of $m(t)$, however, the solutions are close to each other. This proves that our nonlinear ODE is a low-pass filter.

7. CONCLUSION

In this paper, a characterization of "low-pass filtering" for nonlinear systems is proposed. Under regularity conditions on the vector field associated to the ODE, it is shown that a nonlinear system acts as a low-pass filter if the vector field has bounded derivatives. The extension of the method to less regular vector fields is an interesting future research direction.

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