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► **To cite this version:**

Maria Catalisano, Luca Chiantini, Anthony Geramita, Alessandro Oneto. Waring-like decompositions of polynomials, 1. *Linear Algebra and its Applications*, Elsevier, 2017, 533, pp.311 - 325. 10.1016/j.laa.2017.07.021 . hal-01590206

**HAL Id: hal-01590206**

**<https://hal.inria.fr/hal-01590206>**

Submitted on 19 Sep 2017

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# WARING-LIKE DECOMPOSITIONS OF POLYNOMIALS, 1

MARIA V. CATALISANO, LUCA CHIANTINI, ANTHONY V. GERAMITA\*, AND ALESSANDRO ONETO

*In memory of Tony Geramita (1942–2016)*

ABSTRACT. Let  $f$  be a homogeneous form of degree  $d$  in  $n$  variables. A Waring decomposition of  $f$  is a way to express  $f$  as a sum of  $d^{\text{th}}$  powers of linear forms. In this paper we consider the decompositions of a form as a sum of expressions, each of which is a fixed monomial evaluated at linear forms.

## 1. INTRODUCTION

Let  $f \in k[x_1, \dots, x_n] = R = \bigoplus_{i \geq 0} R_i$ , ( $k = \bar{k}$  and  $n \geq 2$ ). The necessities of a given problem involving  $f$  often make it useful to have different ways to *decompose*  $f$ . E.g. in many computations it is useful to express a polynomial as a sum of monomials ordered in a specific way. Different applications call for different kinds of decompositions. The following papers give some interesting uses of some non-standard decompositions (see, e.g. [22], [19], [18], [11]).

One particular kind of decomposition that has received a great deal of attention is the *Waring decomposition*. This decomposition asks us to write  $f \in R_d$  in an efficient way as

$$f = \sum_{i=1}^s L_i^d$$

where each  $L_i$  is a linear form. There is an extensive literature on this decomposition with many interesting applications (see [18],[4],[23],[24], [10]).

In this paper we want to propose an extension of the notion of Waring decomposition. To explain the idea we introduce the following notation.

Fix an integer  $d$ . Let  $\mathcal{P}_r = \mathbb{Z}[Z_1, \dots, Z_r]$  and let  $\mathcal{M}_{r,d}$  be the subset of all monomials such that

$$M = Z_1^{d_1} \dots Z_r^{d_r}$$

and

- i*)  $d_i > 0$ , for all  $i$ ,
- ii*)  $d_1 + \dots + d_r = d$ .

Of course, *i*) and *ii*) imply  $r \leq d$ .

**Definition 1.1.** Let  $M$  be as above and let  $f \in R_d$ .

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2010 *Mathematics Subject Classification.* 14Q20, 13P05, 14M99, 14Q15.

*Key words and phrases.* Waring problems, polynomials, secant varieties.

\**During the submission of this paper, Tony Geramita passed away. In memory of a friend, besides a colleague and a mentor, the three remaining authors dedicate this paper to him.*

*i)* An  $M$ -decomposition of  $f$  having length  $s$  is an expression of the form

$$f = \sum_{j=1}^s L_{1,j}^{d_1} \cdot L_{2,j}^{d_2} \cdots L_{r,j}^{d_r},$$

where the  $L_{i,j}$  are linear forms.

*ii)* The  $M$ -rank of  $f$  is the least integer  $s$  such that  $f$  has an  $M$ -decomposition of length  $s$ .

*Remark 1.2.* *i)* If  $M = Z_1^d$  then the  $M$ -rank of  $f$  is known as the *Waring rank* of  $f$ .

*ii)* Every  $f \in R_d$  has an  $M = Z_1^{d_1} \cdots Z_r^{d_r}$ -decomposition of finite length for any choice of  $M$ . This is immediate from the fact that every  $f \in R_d$  has finite Waring rank and

$$L^d = L^{d_1} \cdots L^{d_r}.$$

*iii)* Two other special cases have received a great deal of attention recently: when  $M = Z_1 \cdots Z_r$  ( $r \geq 2$ ) then the forms in  $R_r$  which have  $M$ -rank equal to 1 are called *split forms* or *completely decomposable forms*.  $M$ -decompositions for these  $M$  are considered in [1], [2], [12], [27] and [30].

When  $M = Z_1^{d-1} Z_2$ ,  $M$ -decompositions were first considered in [13] and then in [6] and [5]. They arose naturally from a consideration of the secant varieties of the tangential varieties to the Veronese varieties. The recent work [3] is a major contribution to this decomposition problem.

*iv)* For the case of binary forms (i.e.  $n = 2$ ) this problem has its roots in the very foundations of modern algebra. Since the  $M$ -rank of a binary form is invariant under the usual  $SL_2(k)$  action on  $k[x_1, x_2]$ , we see in the works of Cayley, Salmon, Sylvester ([8], [26], [28]) the search for the invariants which characterize forms of  $M$ -rank 1 for special choices of  $M$ .

A modern treatment of these classical investigations (as well as advances on them) can be found in the lovely papers of Chipalkatti (see [14], [15], [16], [17], [9]). The bibliographies in these papers give a quick entry into the classical literature on the subject.

*v)* The  $M$ -rank of a form obviously depends on  $M$ . E.g. recall that the Waring rank of  $xyz$  in  $k[x, y, z]$  is 4 (see e.g. [10]) while if  $M = Z_1^2 Z_2$ , the  $M$ -rank of  $xyz$  is 2. To see this note that the  $M$ -rank is bigger than 1, but

$$4xyz = x((y+z)^2 - (y-z)^2).$$

There is a geometric way of considering the problem of finding the  $M$ -rank of a polynomial  $f \in R_d$ ,  $M \in \mathcal{M}_{r,d}$ ,  $M = Z_1^{d_1} \cdots Z_r^{d_r}$ .

Let  $\mathbb{P}(R_1)$  be the projective space based on the  $k$ -vector space  $R_1$ . We define the morphism

$$\varphi_M : (\mathbb{P}(R_1))^r \longrightarrow \mathbb{P}(R_d) \simeq \mathbb{P}^{\binom{d+n-1}{n-1}-1},$$

where  $(\mathbb{P}(R_1))^r$  denotes the cartesian product of  $r$  copies of  $\mathbb{P}(R_1)$ , by

$$\varphi_M([L_1], \dots, [L_r]) = [L_1^{d_1} L_2^{d_2} \cdots L_r^{d_r}],$$

and denote the image of  $\varphi_M$  by  $\mathbb{X}_M$ .

*Remark 1.3.* *i)* Notice that when we have  $M = Z_1^d$ , then  $\mathbb{X}_M$  is precisely the  $d^{\text{th}}$  Veronese embedding of  $\mathbb{P}(R_1) = \mathbb{P}^{n-1}$  in  $\mathbb{P}(R_d)$ .

*ii)* In general, the variety  $\mathbb{X}_M$  is a projection of an appropriate Segre-Veronese variety. However, we will not use that fact in this paper.

iii) The equations which define the Veronese variety are well-known (see e.g. [25]). It would be interesting to find equations for the variety  $\mathbb{X}_M$  when  $M \neq Z_1^d$ . (See, however, [7] for the variety of split forms).

If  $\mathbb{X} \subseteq \mathbb{P}^t$  is any projective variety, then

$$\sigma_s(\mathbb{X}) := \overline{\{P \in \mathbb{P}^t \mid P \in \langle P_1, \dots, P_s \rangle, P_i \in \mathbb{X}\}}$$

is the  $s^{\text{th}}$ -secant variety of  $\mathbb{X}$ .

$$\sigma_1(\mathbb{X}) = \mathbb{X},$$

$$\sigma_2(\mathbb{X}) = \text{secant line variety to } \mathbb{X}, \text{ etc.}$$

*Remark 1.4.* When  $\mathbb{X} = \mathbb{X}_M$  and  $M = Z_1^d$ , so  $\mathbb{X}$  is the  $d^{\text{th}}$  Veronese embedding of  $\mathbb{P}^n$  then, if  $F \in R_d$  has Waring rank  $s$  we have  $[F] \in \sigma_s(\mathbb{X})$ . In particular, if  $s$  is the least integer for which  $\sigma_s(\mathbb{X}) = \mathbb{P}(R_d)$  then the generic element  $[F] \in \mathbb{P}(R_d)$  has Waring rank  $s$ .

This is the fundamental connection between the algebraic problem of finding the Waring rank of a generic form and the geometric problem of finding the dimensions of secant varieties to Veronese varieties.

One can ask about the dimensions of the secant varieties of any projective variety. More precisely: given  $\mathbb{X} \subset \mathbb{P}^t$  what is  $\dim \sigma_s(\mathbb{X})$  for  $s \geq 2$ ?

There is a reasonable guess which gives an upper bound for  $\dim \sigma_s(\mathbb{X})$ . It is obtained by counting parameters and observing that  $\sigma_s(\mathbb{X}) \subseteq \mathbb{P}^t$ , namely

$$(1.1) \quad \dim \sigma_s(\mathbb{X}) \leq \min\{s \dim \mathbb{X} + s - 1, t\}.$$

If we have equality in (1.1) for some  $s$ , then we say that  $\sigma_s(\mathbb{X})$  has the *expected dimension*, while if (1.1) is a strict inequality for some  $s$ , we say that  $\sigma_s(\mathbb{X})$  is *defective* and the difference

$$\min\{s \dim \mathbb{X} + s - 1, t\} - \dim \sigma_s(\mathbb{X}),$$

is called the *s-defect* of  $\mathbb{X}$ .

If  $\mathbb{X} \subseteq \mathbb{P}^t$  is non-degenerate then  $\sigma_s(\mathbb{X}) = \mathbb{P}^t$  for some  $s$  and so the  $s$ -defect is eventually zero, for all  $s \gg 0$ .

The particular problem we consider in this paper is that of finding  $\dim \sigma_s(\mathbb{X}_M)$  for  $R = k[x_1, \dots, x_n]$  and any  $M \in \mathcal{M}_{r,d}$ .

The paper is organized in the following way. In Section 2 we recall Terracini's Lemma, which is our main tool in finding  $\dim \sigma_s(\mathbb{X}_M)$ . Terracini's Lemma needs a description of the tangent space at a general point of  $\mathbb{X}_M$ . This tangent space corresponds to a vector space which is the graded piece of an ideal  $I$  in  $R$ . We use information about this ideal to find the dimensions we need.

In the third section we find the dimensions of all the secant varieties of  $\mathbb{X}_M$  for any  $M \in \mathcal{M}_{r,d}$ , for any  $r$  and any  $d$ , in case  $n = 2$ , i.e. for binary forms. We find that there are no defective secant varieties in this case.

In the fourth section we find the dimensions of the secant line varieties of  $\mathbb{X}_M$  for any  $n$  and for any  $M \in \mathcal{M}_{r,d}$  for any  $r$  and any  $d$ . In this family of cases we find exactly one defective secant line variety.

In the final section we use results of the previous sections and specialization in order to prove the non-defectiveness of new secant varieties.

## 2. PRELIMINARIES

Since the cases  $r = 1$  and  $d = 2$  were already treated and solved (see Remark 1.2), in this paper we assume  $r \geq 2$  and  $d \geq 3$ .

We begin by recalling the Lemma of Terracini [29].

**Lemma 2.1.** *Let  $\mathbb{X} \subseteq \mathbb{P}^t$  be a projective variety and let  $P \in \sigma_s(\mathbb{X})$  be a general point. If*

$$P \in \langle P_1, \dots, P_s \rangle$$

where  $P_1, \dots, P_s$  are general points of  $\mathbb{X}$ , then the tangent space to  $\sigma_s(\mathbb{X})$  at  $P$  is the linear span of tangent spaces to  $\mathbb{X}$  at  $P_1, \dots, P_s$ , i.e.,

$$T_P(\sigma_s(\mathbb{X})) = \langle T_{P_1}(\mathbb{X}), \dots, T_{P_s}(\mathbb{X}) \rangle.$$

To apply Terracini's Lemma to our situation we first need to calculate  $T_P(\mathbb{X}_M)$  for a general  $P$  in  $\mathbb{X}_M$ .

**Proposition 2.2.** *Let  $R = k[x_1, \dots, x_n]$  and let  $M \in \mathcal{M}_{r,d}$ ,*

$$M = Z_1^{d_1} \dots Z_r^{d_r}.$$

Let  $L_1, \dots, L_r$  be general linear forms in  $R_1$  so that  $P = [L_1^{d_1} \dots L_r^{d_r}]$  is a general point of  $\mathbb{X}_M = \varphi_M((\mathbb{P}^{n-1})^r) \subseteq \mathbb{P}^{\binom{d+n-1}{n-1}-1}$ .

If

$$F = L_1^{d_1} \dots L_r^{d_r} \text{ and } I_P = (F/L_1, \dots, F/L_r) = \bigoplus_{j \geq 0} (I_P)_j$$

then

$$T_P(\mathbb{X}_M) = \mathbb{P}((I_P)_d).$$

*Proof.* Since  $P = \varphi_M([L_1], \dots, [L_r])$ , the image of a line through the point  $([L_1], \dots, [L_r])$  in the direction  $([\tilde{L}_1], \dots, [\tilde{L}_r])$  is the curve on the variety  $\mathbb{X}_M$  whose points are parameterized by

$$[(L_1 + \lambda \tilde{L}_1)^{d_1} \dots (L_r + \lambda \tilde{L}_r)^{d_r}].$$

The tangent vector to this curve at  $P$  is given by the coefficient of  $\lambda$  in this last expression, that is,

$$(F/L_1) \tilde{L}_1 + \dots + (F/L_r) \tilde{L}_r.$$

These, for varying choices of the  $\tilde{L}_i$ , give that the tangent vectors at  $P$  are precisely the degree  $d$  piece of the ideal generated by the  $F/L_i$ . □

*Remark 2.3.* *i)* Note that if  $M = Z_1^{d_1} \dots Z_r^{d_r}$  and we are considering forms in  $k[x_1, \dots, x_n]$ , then  $\mathbb{X}_M = \varphi_M((\mathbb{P}^{n-1})^r)$ , Since the generic fibre of  $\varphi_M$  is finite, the dimension of  $T_P(\mathbb{X}_M)$  is  $r(n-1)$ .

*ii)* It is easy to see that if  $F$  and the  $L_i$  are as above, then

$$\begin{aligned} I_P &= (F/L_1, \dots, F/L_r) \\ (2.1) \quad &= (L_1^{d_1-1} \dots L_r^{d_r-1}) \cdot (L_2 L_3 \dots L_r, L_1 L_3 \dots L_r, \dots, L_1 L_2 \dots L_{r-1}). \end{aligned}$$

**Corollary 2.4** (for binary forms). *Let  $R = k[x_1, x_2]$ ,  $M \in \mathcal{M}_{r,d}$  ( $r \geq 2$ )*

$$M = Z_1^{d_1} \cdots Z_r^{d_r}.$$

*If  $L_1, L_2, \dots, L_r$  are general linear forms in  $R_1$ ,  $P = [L_1^{d_1} \cdots L_r^{d_r}] \in \mathbb{X}_M$  and we set  $I'$  to be the principal ideal*

$$I' := (L_1^{d_1-1} \cdots L_r^{d_r-1})$$

*then we have*

$$T_P(\mathbb{X}_M) = \mathbb{P}(I'_d).$$

*Proof.* In view of equation (2.1) above we first consider the ideal

$$J := (L_2 L_3 \cdots L_r, L_1 L_3 \cdots L_r, \dots, L_1 L_2 \cdots L_{r-1}).$$

*Claim 2.5.*  $J = (x_1, x_2)^{r-1}$

*Proof.* (of the Claim) By induction on  $r$ . Obvious for  $r = 2$  so let  $r > 2$ . Since

$$J = (L_r \cdot (L_2 L_3 \cdots L_{r-1}), L_1 L_3 \cdots L_{r-1}, \dots, L_1 L_2 \cdots L_{r-2}), L_1 L_2 \cdots L_{r-1}),$$

we have, by the induction hypothesis, that

$$J = (L_r \cdot (x_1, x_2)^{r-2}, L_1 L_2 \cdots L_{r-1}).$$

Since  $R = k[x_1, x_2]$ ,  $L_1 L_2 \cdots L_{r-1}$  is a general form in  $R_{r-1}$ , hence not in the space  $L_r(x_1, x_2)^{r-2}$ . This last implies that  $\dim J_{r-1} = r$ . Since the ideal  $J$  begins in degree  $r - 1$  we are done with the proof of the claim.  $\square$

Now, using Claim 2.5, equation (2.1) and Proposition 2.2 we have that

$$I_P = (L_1^{d_1-1} \cdots L_r^{d_r-1})(x_1, x_2)^{r-1} \subseteq (L_1^{d_1-1} \cdots L_r^{d_r-1})$$

and since  $(x_1, x_2)^{r-1} = \bigoplus_{j \geq r-1} R_j$ , we have that

$$(I_P)_d = (L_1^{d_1-1} \cdots L_r^{d_r-1})_d$$

$\square$

**Corollary 2.6.** *Let  $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$ . Let  $P = \varphi_M([L_1], \dots, [L_r])$  be a general point of  $\mathbb{X}_M$ , where the  $L_i$  are general linear forms in  $k[x_1, \dots, x_n]$  and let  $I_P$  be as in Proposition 2.2. Then,*

$$I_P = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cap (\bigcap_{1 \leq i < j \leq r} (L_i, L_j)^{d_i+d_j-1}).$$

*Proof.* By Remark 2.3 we need to prove that the two ideals

$$I_P = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cdot (L_2 L_3 \cdots L_r, \dots, L_1 \cdots L_{r-1})$$

and

$$J = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cap (\bigcap_{1 \leq i < j \leq r} (L_i, L_j)^{d_i+d_j-1})$$

are equal.

Since each generator of  $I_P$  is in  $J$ , we have  $I_P \subseteq J$ .

Now let  $h = L_1^{d_1-1} \cdots L_r^{d_r-1}$ , and suppose that  $f = hg \in J$ . Since the  $L_i$  are general linear forms we have

$$h \in (L_i, L_j)^{d_i+d_j-2} \quad \text{for every } 1 \leq i < j \leq r,$$

but

$$h \notin (L_i, L_j)^{d_i+d_j-1} \quad \text{for every } 1 \leq i < j \leq r,$$

so  $g \in \text{rad}((L_i, L_j)^{d_i+d_j-1}) = (L_i, L_j)$  for all  $1 \leq i < j \leq r$ .

Since  $\bigcap_{1 \leq i < j \leq r} (L_i, L_j) = (L_2 \cdots L_r, \dots, L_1 \cdots L_{r-1})$  we are done. □

### 3. THE BINARY CASE

For binary forms there is a simple theorem covering all cases.

**Theorem 3.1.** *Let  $R = k[x, y] = \bigoplus_{j \geq 0} R_j$  and let  $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$  for any  $r$  and any  $d$  with  $r \leq d$ . Then  $\sigma_s(\mathbb{X}_M)$  has the expected dimension for every  $s$ , i.e.*

$$\dim \sigma_s(\mathbb{X}_M) = \min\{s \dim \mathbb{X}_M + (s-1), d\} = \min\{sr + s - 1, d\}$$

for every  $s$  and every  $M$ .

*Proof.* Since every form in  $R$  of degree  $d$  splits as a product of linear forms and the general form of degree  $d$  has no repeated factors, we conclude that for  $M = Z_1 \cdots Z_d$ ,  $\mathbb{X}_M = \mathbb{P}(R_d) = \mathbb{P}^d$ . This takes care of the case  $r = d$ . Now, for the rest of the proof, assume that  $r < d$ .

By Corollary 2.4 we know that if  $P = [L_1^{d_1} \cdots L_r^{d_r}]$  is a general point of  $\mathbb{X}_M$  (where  $L_1, \dots, L_r$  are general in  $R_1$ ) then

$$T_P(\mathbb{X}_M) = \mathbb{P}((I'_P)_d) \quad \text{where} \quad I'_P = (L_1^{d_1-1} \cdots L_r^{d_r-1}).$$

So, by Terracini's Lemma, if  $P_1, \dots, P_s$  are a set of  $s$  general points of  $\mathbb{X}_M$  then

$$\dim(\sigma_s(\mathbb{X}_M)) = \dim_k(I'_{P_1} + \cdots + I'_{P_s})_d - 1$$

where if  $P_j = [L_{j1}^{d_1} \cdots L_{jr}^{d_r}]$  then  $I'_{P_j} = (L_{j1}^{d_1-1} \cdots L_{jr}^{d_r-1})$ .

However, [21, Cor.2.3] states that for the special points  $Q_1 = [H_1^d], \dots, Q_s = [H_s^d] \in \mathbb{X}_M$  (where the  $H_j$  are general in  $R_1$ ) and for the ideal  $J = (H_1^{d-r}, \dots, H_s^{d-r})$  we have

$$\dim_k(J_d) = \min\{d+1, s(r+1)\} = \min\{d, sr + (s-1)\} + 1.$$

Since we know that  $\dim \sigma_s(\mathbb{X}_M) \leq \min\{d, sr + (s-1)\}$ , it follows (by semicontinuity) that

$$\dim(I'_{P_1} + \cdots + I'_{P_s})_d = \min\{d, sr + (s-1)\} + 1.$$

and so  $\sigma_s(\mathbb{X}_M)$  always has the expected dimension. □

### 4. THE SECANT LINE VARIETIES TO $\mathbb{X}_M$

In this section we will find the dimensions of the secant line varieties of  $\mathbb{X}_M$  for every  $M \in \mathcal{M}_{r,d}$  and for every polynomial ring  $R = k[x_1, \dots, x_n]$ .

When  $r = 1$  this is one part of the complete solution to Waring's Problem solved by Alexander and Hirschowitz in [4], so we will assume that  $r \geq 2$ . In the previous section we solved this problem for  $n = 2$ , so we may now assume that  $n \geq 3$ . Recall that we are also assuming  $d \geq 3$ .

The main result of this paper is the following theorem.

**Theorem 4.1.** *Let  $R = k[x_1, \dots, x_n]$ , let  $M \in \mathcal{M}_{r,d}$ ,  $n \geq 3$ ,  $r \geq 2$ ,  $d \geq 3$ ,*

$$M = Z_1^{d_1} \dots Z_r^{d_r}.$$

*Then  $\sigma_2(\mathbb{X}_M)$  is not defective, except for  $M = Z_1^2 Z_2$  and  $n = 3$ . For this last case  $\mathbb{X}_M$  has 2-defect equal to 1.*

*Proof.* We always have  $d \geq r$ . The case  $d = r$  is covered in [27, Theorem 4.4] so we may as well assume that  $d > r$  also.

By Terracini's Lemma we need to find the vector space dimension of

$$(I_{P_1} + I_{P_2})_d,$$

where  $P_1 = [L_1^{d_1} \dots L_r^{d_r}]$  and  $P_2 = [N_1^{d_1} \dots N_r^{d_r}]$  are points of  $\mathbb{X}_M$  and the

$$\{L_i, 1 \leq i \leq r\}, \{N_i, 1 \leq i \leq r\}$$

are general sets of linear forms in  $R$ .

By Corollary 2.6 we obtain

$$\begin{aligned} I_{P_1} &= (L_1^{d_1-1} \dots L_r^{d_r-1}) \cap (\cap_{1 \leq i < j \leq r} (L_i, L_j)^{d_i+d_j-1}), \\ I_{P_2} &= (N_1^{d_1-1} \dots N_r^{d_r-1}) \cap (\cap_{1 \leq i < j \leq r} (N_i, N_j)^{d_i+d_j-1}). \end{aligned}$$

By the exact sequence

$$(4.1) \quad 0 \longrightarrow (I_{P_1} \cap I_{P_2})_d \longrightarrow (I_{P_1} \oplus I_{P_2})_d \longrightarrow (I_{P_1} + I_{P_2})_d \longrightarrow 0,$$

and the fact that we know that  $\dim(I_{P_i})_d = r(n-1) + 1$ ,  $(i = 1, 2)$ , (see Remark 2.3), it is enough to find  $\dim(I_{P_1} \cap I_{P_2})_d$ .

Recall that the expected dimension of  $\sigma_2(\mathbb{X}_M)$  is

$$\text{exp. dim } \sigma_2(\mathbb{X}_M) = \min\{2r(n-1) + 1, \binom{d+n-1}{n-1} - 1\}.$$

Note that, if  $\dim(I_{P_1} \cap I_{P_2})_d = 0$ , then by (4.1) the dimension of  $\sigma_2(\mathbb{X}_M)$  is as expected.

Let  $V$  be the subscheme of  $\mathbb{P}^{n-1}$  defined by  $I_{P_1} \cap I_{P_2}$  and let  $f$  be a form of degree  $d$  in  $I_{P_1} \cap I_{P_2}$ . Clearly

$$f = L_1^{d_1-1} \dots L_r^{d_r-1} \cdot N_1^{d_1-1} \dots N_r^{d_r-1} \cdot g$$

where  $g$  is a form of degree  $d - 2(d-r) = 2r - d$ .

If  $2r - d < 0$ , of course there are no forms of this degree, hence  $(I_{P_1} \cap I_{P_2})_d = 0$  and we are done. So assume  $2r - d \geq 0$ .

The form  $g$  vanishes on the residual scheme  $W$  of  $V$  with respect to the  $2r$  multiple hyperplanes  $\{L_i = 0\}$  and  $\{N_i = 0\}$  ( $1 \leq i \leq r$ ). It is easy to see that  $W$  is defined by the ideal

$$(\cap_{1 \leq i < j \leq r} (L_i, L_j)) \cap (\cap_{1 \leq i < j \leq r} (N_i, N_j)).$$

Now,  $g$  cannot be divisible by all the  $L_i$  and  $N_j$  because otherwise it would have degree at least  $2r$ .

Without loss of generality assume that  $g$  is not divisible by  $L_1$ , and let  $H$  be the hyperplane defined by  $L_1$ . The form  $g$  cuts out on  $H$  a hypersurface  $\mathcal{S}$  of  $H$  having degree  $2r - d$ , and containing the  $r - 1$  hyperplanes of  $H$  cut out by  $L_2, \dots, L_r$ . Hence, in order for  $g$  to exist,  $2r - d$  has to be at least  $r - 1$ , that is,  $d \leq r + 1$ . But we are assuming  $d \geq r + 1$ , so we get that  $d = r + 1$ .

It follows that  $\mathcal{S}$  has degree  $r - 1$ , contains the  $r - 1$  hyperplanes of  $H$  cut out by  $L_2, \dots, L_r$ , and contains the trace on  $H$  of the schemes defined by the ideals  $(N_i, N_j)$ . Since the  $N_i$  are generic



with respect to  $H$  and to the  $L_i$ 's, the only possibility for  $g$  to exist is that the schemes  $Y_{i,j}$  defined by the ideals  $(N_i, N_j)$  do not intersect  $H$ . Since  $H \simeq \mathbb{P}^{n-2} \subseteq \mathbb{P}^{n-1}$  and  $Y_{i,j} \simeq \mathbb{P}^{n-3} \subseteq \mathbb{P}^{n-1}$ , then  $H \cap Y_{i,j} = \emptyset$  only for  $n \leq 3$ .

Therefore, we are left with the following cases:

Case 1:  $n = 3$ ,  $r \geq 3$ ,  $d = r + 1$ ;

Case 2:  $n = 3$ ,  $r = 2$ ,  $d = 3$ .

In Case 1, the form  $g$  has degree  $2r - d = r - 1$  and it should vanish at the scheme  $W$  which is a union of  $2\binom{r}{2}$  simple points in  $\mathbb{P}^2$ . The dimension of the space of homogeneous polynomials of degree  $r - 1$  is  $\binom{r+1}{2}$ , which is smaller than  $2\binom{r}{2}$ , for  $r \geq 3$ . Then, such a  $g$  cannot exist.

In Case 2,  $M = Z_1^2 Z_2$ , the form  $g$  has degree  $2r - d = 1$  and the scheme  $W$  is the union of 2 points of  $\mathbb{P}^2$ . Hence  $g$  can exist and describes the line through the two points. It follows that  $\dim(I_{P_1} \cap I_{P_2})_3 = 1$ . So from (2.1) we get

$$\dim(I_{P_1} + I_{P_2})_3 = 9,$$

that is,  $\dim \sigma_2(\mathbb{X}_M) = 8$ . But  $\text{exp. dim } \sigma_2(\mathbb{X}_M) = 9$ , and so  $\mathbb{X}_M$  has 2-defect = 1 and we are done.  $\square$

*Remark 4.2.* The exceptional case noted above was observed in [13] in connection with the study of the secant varieties of the tangential varieties to Veronese varieties.

*Example.* We claim that the hypersurface in  $\mathbb{P}^9$  containing all those cubic forms of  $k[x, y, z]$  which can be written  $L_1^2 L_2 + N_1^2 N_2$  (with the  $L_i, N_i$  linear forms) is precisely the hypersurface in  $\mathbb{P}^9$  containing all singular cubics. It is well-known that the (closure) of the set of cubic plane curves with a double point is a hypersurface in  $\mathbb{P}^9$ . It will be enough to show that every nodal cubic can be written in the desired form (since cuspidal cubics can, after a change of variables, always be written in the form  $y^3 + x^2 z$ ).

First recall (see [20]) that every nodal cubic can, after a change of variables, be written in the form

$$xyz - x^3 - y^3.$$

With a further change of variables given by

$$x = -X - Y; \quad y = X - Y; \quad z = -Z,$$

we get

$$X^2(6Y + Z) + Y^2(2Y - Z).$$

## 5. INDUCTIVE RESULTS

We hope that our results above could provide a starting point for further investigations on  $\mathcal{M}$ -decompositions, when the number  $s$  of summands increase. The main obstruction to extend plainly our arguments resides in the fact that when  $r$  is close to  $n$  and  $s > 2$ , then the shape of polynomials lying in the intersection of a tangent space  $T_P$  with the *span* of two or more tangent spaces to  $\mathbb{X}_M$  is not easy to control.

In this final section, in which as above we assume  $n \geq 3$ , we show how by Theorem 3.1 and Theorem 4.1 we can improve our knowledge on the defectivity of  $\mathcal{M}$ -decompositions. We will need some preliminary algebraic remarks.

**Proposition 5.1.** *Let  $I, J$  be ideals in the polynomial ring  $R = k[x_1, \dots, x_n]$ , both generated by elements of degree  $d-1$ , for some  $d \geq 3$ . Assume that  $I_d \cap J_d = (0)$ . Then after adding one variable  $y$  to  $R$ , we still get  $IR[y]_d \cap JR[y]_d = (0)$ .*

*Proof.* By our assumptions on the generators of  $I$ , the elements of  $IR[y]_d$  are of the form  $Ay + B$ , with  $A \in I_{d-1}$  and  $B \in I_d$ . Similarly the elements of  $JR[y]_d$  are of the form  $Cy + D$ , with  $C \in J_{d-1}$  and  $D \in J_d$ . Since  $y$  is independent mod  $R$ , the equality  $Ay + B = Cy + D$  yields  $A = C$ ,  $B = D$ . Hence  $A = B = 0$  by assumption.  $\square$

Next, fix  $d \geq 3$  and go back to the ideals  $I_P$  defined in the previous sections, where  $P$  is a splitting form of fixed type:

$$M = Z_1^{d_1} \cdots Z_r^{d_r}, \quad d = d_1 + \cdots + d_r.$$

**Proposition 5.2.** *Assume that for a general choice of splitting forms  $P_1, \dots, P_s \in k[x_1, \dots, x_t]$  of type  $M$  in  $t < n$  variables the ideal  $I = I_{P_1} + \cdots + I_{P_s} \subset k[x_1, \dots, x_t]$  has the degree  $d$  piece of (maximal) dimension  $s(r+1)$ . Then for a general choice of splitting forms  $Q_1, \dots, Q_s \in R$  of type  $M$ , the ideal  $J = I_{Q_1} + \cdots + I_{Q_s} \subset R$  has a degree  $d$  piece of (maximal) dimension  $sr(n-1) + s$ .*

*Proof.* We make induction on  $s$ . Notice that it is enough to prove that the degree  $d$  piece of the extended ideal  $IR$  has dimension  $sr(n-1) + s$ , because then, by semicontinuity, the same will hold for a general choice of the  $Q_i$ 's.

For the case  $s = 1$ , take  $P_1 = [L_1^{d_1} \cdots L_r^{d_r}]$ , where the  $L_i$ 's are general linear forms in  $k[x_1, \dots, x_t]$ . The vector space  $(IR)_d$  is spanned by the  $(t-1)r+1$  independent forms in the  $t$  variables  $x_1, \dots, x_t$  which are a basis for  $I_d$ , plus the  $r(n-t)$  independent forms  $x_i P_1 / L_j$ , for all  $i = t+1, \dots, n$  and  $j = 1, \dots, r$ . It follows that  $\dim(IR)_d = (t-1)r+1 + r(n-t) = r(n-1) + 1$ .

For higher  $s$ , notice that our assumption implies that  $I' = I_{P_1} + \cdots + I_{P_{s-1}}$  has a null intersection with  $I_{P_s}$  in degree  $d$ , inside  $k[x_1, \dots, x_t]$ . Thus also the extensions of  $I'$  and  $I_{P_s}$  to  $R$  have a null intersection (apply  $n-t$  times the previous Proposition 5.1). Since, by the inductive hypothesis,  $\dim(I'R)_d = (s-1)r(n-1) + (s-1)$ , we have that

$$\dim(IR)_d = (s-1)r(n-1) + (s-1) + r(n-1) + 1 = sr(n-1) + s,$$

and the result follows.  $\square$

Now we can apply the previous proposition to extend Theorem 3.1 to many variables.

**Proposition 5.3.** *Let  $R = k[x_1, \dots, x_n]$  and let  $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$ ,  $r \leq d$ . Assume  $s(r+1) \leq d+1$ . Then, for every  $M$ ,  $\sigma_s(\mathbb{X}_M)$  has the expected dimension*

$$\dim \sigma_s(\mathbb{X}_M) = s \dim \mathbb{X}_M + (s-1) = sr(n-1) + s - 1.$$

*Proof.* The result is true in two variables, by Theorem 3.1. In more than two variables, specialize the  $s$  points  $P_i$ 's to forms in two variables: the result remains true by Proposition 5.2.  $\square$

The condition  $s(r+1) \leq d+1$  is fundamental in the previous argument, because we need that, after the specialization, each  $I_{P_i}$  has a null intersection with the span of the remaining  $I_{P_j}$  (in  $k[x_1, x_2]$ ). We can improve the previous result with the trick of separating the variables.

**Proposition 5.4.** *For a fixed  $j \in \{2, \dots, n-2\}$ , let  $R_1 = k[x_1, \dots, x_j]$  and  $R_2 = k[x_{j+1}, \dots, x_n]$ . Let  $I \subset R_1$ ,  $J \subset R_2$  be ideals generated by elements of degree  $d-1$ , for some  $d \geq 3$ . Then  $(IR)_d \cap (JR)_d = (0)$ .*

*Proof.* Notice that the elements of  $(IR)_d$  have degree at least  $d-1$  in  $x_1, \dots, x_j$ , while the elements of  $(JR)_d$  have degree at most 1 in  $x_1, \dots, x_j$ . Since  $d \geq 3$ , the two things cannot match, and the claim follows.  $\square$

Now, by dividing the points  $P_1, \dots, P_s$  in groups and specializing each group to forms in two distinct variables, we get

**Proposition 5.5.** *Let  $R = k[x_1, \dots, x_n]$  and let  $M = Z_1^{d_1} \dots Z_r^{d_r} \in \mathcal{M}_{r,d}$ ,  $r \leq d$ , and  $d \geq 3$ . Set  $m = \lfloor \frac{n}{2} \rfloor$  and  $s' = \lfloor \frac{d+1}{r+1} \rfloor$ . Then, for every  $s \leq s'm$  and for every  $M$ ,  $\sigma_s(\mathbb{X}_M)$  has the expected dimension*

$$\dim \sigma_s(\mathbb{X}_M) = s \dim \mathbb{X}_M + (s-1) = sr(n-1) + s - 1.$$

*Proof.* It is enough to prove the statement for  $s = s'm$ . For each  $j = 1, \dots, m$  take  $s'$  forms  $P_{1j}, \dots, P_{s'j} \in \mathbb{X}_M$  in the variables  $x_{2j-1}, x_{2j}$ . The ideal

$$I_j = (I_{P_{1j}} + \dots + I_{P_{s'j}})$$

in  $k[x_{2j-1}, x_{2j}]$  has the expected dimension  $s'r + s'$  in degree  $d$ . By Proposition 5.2 with  $t = 2$ , the extension of  $I_j$  to  $R$  has the expected dimension  $s'r(n-1) + s'$  in degree  $d$ .

Then the sum  $I = I_1 + \dots + I_m$  has the expected dimension  $m(s'r(n-1) + s') = sr(n-1) + s$  in degree  $d$ , because by Proposition 5.4 every  $I_j$  has null intersection with the sum of the remaining  $I_k$ 's,  $k \neq j$ .  $\square$

We can use the idea of separating the variables also in connection with Theorem 4.1, by specializing pairs of points  $P_i$ 's to forms in three distinct variables. The statement below covers some cases which do not fit with the numerical assumptions of Proposition 5.5.

**Proposition 5.6.** *Let  $R = k[x_1, \dots, x_n]$ ,  $r \geq 2$ ,  $d \geq 3$ . Let  $M \in \mathcal{M}_{r,d}$  be different from  $Z_1^2 Z_2$ . Then  $\sigma_s(\mathbb{X}_M)$  is not defective for  $s \leq 2 \lfloor \frac{n}{3} \rfloor$ .*

*Proof.* The case  $s = 2$  is provided by Theorem 4.1. For higher  $s$ , it suffices to prove the statement for  $s = 2 \lfloor \frac{n}{3} \rfloor$ . For all  $i = 1, \dots, \lfloor \frac{n}{3} \rfloor$  choose general points  $P_{2i-1}, P_{2i} \in \mathbb{X}_M$  which involve only the variables  $x_{3i-2}, x_{3i-1}, x_{3i}$ . By Theorem 4.1 and its proof, we know that the ideals  $I_{P_{2i+1}}, I_{P_{2i+2}}$  have null intersection in degree  $d$ , considered as ideals in  $k[x_{3i+1}, x_{3i+2}, x_{3i+3}]$ . Thus their sum  $I_i = I_{P_{2i-1}} + I_{P_{2i}}$  has the expected dimension in degree  $d$ . The same remains true when we extend the ideals to  $R$ , by Proposition 5.2 with  $t = 3$ . Now it is enough to apply Proposition 5.4 to show that the sum of the  $I_i$ 's has the expected dimension in degree  $d$ .  $\square$

The results proved in Proposition 5.5 or Proposition 5.6 by separating the variables are clearly not optimal, in the sense that they cannot cover the full range of all possible  $M$ -ranks. In order to get a complete classification of defective varieties  $\mathbb{X}_M$  for  $s > 2$ , new ideas should be introduced.

## ACKNOWLEDGEMENTS

The first author wishes to thank Queen's University, in the person of the third author, for their kind hospitality during the preparation of this work. The first and third authors enjoyed support from NSERC (Canada). The first and second authors were also supported by GNSAGA of INDAM and by MIUR funds (PRIN 2010-11 prot. 2010S47ARA-004 - Geometria delle Varietà Algebriche) (Italy). The fourth author was supported by Jubileumsdonationen, K & A Wallenbergs Stiftelse (Sweden) during visits to the first and the third author.

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