Structural Delay-1 Input-and-State Observability

Federica Garin

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***Abstract*— This paper studies structured discrete-time LTI systems, where the state-space matrices have a fixed zero pattern, and all other entries are free parameters. The goal is to obtain generic results, true for almost all values of the free parameters. This paper focuses on input-and-state observability, i.e., the property that both initial state and unknown input can be reconstructed from the outputs. First, a simpler statement is presented of a known characterization of generic input-and-state observability. Then, a novel characterization is given of generic left-invertibility with delay one, where the input can be reconstructed up to a single time-step earlier than the most recent output measurement. All characterizations are in terms of properties of graphs associated with the zero pattern.**

I. INTRODUCTION

The study of generic properties of structured systems has been an active research area since the 80's (see [1] and references therein) and has received a wide recent attention with the rise of network systems studies. A structured system is a linear system whose state-space matrices have a fixed pattern of zeros, and the other entries are free parameters. For a given choice of parameters, properties such as observability and controllability have classical algebraic characterizations. The study of generic properties aims at results depending on the zero-pattern only, and holding for almost all values of the free parameters. These results overcome complexity and ill-posedness of the algebraic characterizations, and add robustness w.r.t. paramenter uncertainties.

In this paper we study input-and-state observability, i.e., the possibility to reconstruct both the initial state and an unknown input, from the measured outputs. The input may represent the contribution of an unmodeled part of the system, a fault, or a malicious external attack. The ability to reconstruct it in addition to the state estimation is relevant in fault detection and isolation, fault tolerant control, and cyber-physical security. This problem has been addressed in the framework of structured systems by Boukhobza et al. [2], where the authors obtain a characterization of generic input-and-state observability. Here, we present a corollary of their main result. This characterization concerns joint reconstruction of initial state and unknown input, without specifying the delay in the input reconstruction, but most iterative filters for input-and-state estimation (see e.g. [3], [4]) try to reconstruct the input with delay one, namely they try to reconstruct $u(t-1)$ after measuring $y(t)$. This motivates the study of delay-1 input-and-state observability, where $x(0)$ and the input sequence $u(0),\dots,u(n-1)$ can be reconstructed from the output sequence $y(0),\dots,y(n)$. The study of this property for structured systems has been initiated in [5], [6]. The current paper considers a more general setup, with no assumption on the system matrices, while [5], [6] assumed no direct feedthrough of the input towards the output, and imposed a particular structure on the matrices relating the input to the state and the state to the output. However, this paper focuses on generic results and LTI systems only, while [5] gives both generic and strongly structural results for LTI systems with scalar input, and [6] considers strong structural results for LTV systems; strongly structural refers to results being true for all non-zero values of the free parameters, as opposed to 'almost all' values. Our main result is to characterize generic delay-1 input-and-state observability for general LTI structured systems (Sect. IV).

F. Garin is with Inria, Univ. Grenoble Alpes, CNRS, GIPSA-lab, F-38000 Grenoble, France, federica.garin@inria.fr

II. INPUT-AND-STATE OBSERVABILITY

Consider the discrete-time LTI system

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the unknown input, and $y(t) \in \mathbb{R}^k$ is the output.

The system (A,B,C,D) is *strongly observable* if the initial state $x(0)$ can be uniquely determined from the output sequence $y(0),\dots,y(n)$, despite the presence of the unknown input u. The system (A,B,C,D) is *delay-ℓ left-invertible* if the input $u(0)$ can be uniquely determined from the initial state $x(0)$ and the output sequence $y(0),\dots,y(\ell)$. The system is *left-invertible* if it is delay-ℓ left-invertible for some ℓ, or equivalently if it is delay-n left-invertible. The notion of *input-and-state observability* studied in [2] (thereby called 'state and input observability') corresponds to strong observability together with left-invertibility, without enforcing any particular constraint on the delay of the input reconstruction. The notion of input-and-state observability studied in [5], [6] and needed for running input-and-state estimation filters as in [3], [4] rather enforces a delay 1; in this paper it will be studied under the name of *delay-1 input-and-state observability*, and it corresponds to strong observability together with delay-1 left-invertibility.

The algebraic characterizations of the above-defined properties are classical and well-known. Below, we recall only two results, on which our structural studies can be built. Concerning input-and-state observability (without any constraint on the delay for input reconstruction), the structural results by Boukhobza et al. [2] are built upon the following PBH-like characterization:

Proposition 1: (A,B,C,D) is strongly observable and left-invertible if and only if

$$\forall z \in \mathbb{C},\ \text{rank} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = n + p. \qquad \square$$

A characterization of delay-ℓ left-invertibility is given by Massey and Sain in [7, Thm. 4]. Here we use it with $\ell = 1$.

Proposition 2: The following are equivalent:

i) (A, B, C, D) is left-invertible with delay 1;

ii)

$$\text{rank} \begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} = p + \text{rank}\, D;$$

iii)

$$\text{rank} \begin{bmatrix} D & 0 & 0 \\ B & -I & 0 \\ 0 & C & D \end{bmatrix} = p + n + \text{rank}\, D. \qquad \square$$

Proof: The equivalence of i) and ii) is given in [7, Thm. 4]. We prove the equivalence of i) and iii) with a similar technique. First, we notice that the problem of reconstructing $u(0)$ from $x(0), y(0), y(1)$ is equivalent to the problem of reconstructing $u(0), x(1)$ from $x(0), y(0), y(1)$, since $x(0), u(0)$ fully determine $x(1)$. Re-writing the system dynamics (1), we obtain a linear system of equations:

$$\begin{bmatrix} D & 0 & 0 \\ B & -I & 0 \\ 0 & C & D \end{bmatrix} \begin{bmatrix} u(0) \\ x(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} y(0) - Cx(0) \\ -Ax(0) \\ y(1) \end{bmatrix}$$

The solution is unique for the first two blocks $u(0)$, $x(1)$ of the unknown vector if and only if

$$\text{rank} \begin{bmatrix} D & 0 & 0 \\ B & -I & 0 \\ 0 & C & D \end{bmatrix} = p + n + \text{rank} \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix}. \qquad \blacksquare$$

The goal of this paper is to find the structural counterparts of the algebraic characterizations in Propositions 1 and 2, i.e., to characterize when these properties are generically true, with conditions involving graphs describing the zero pattern.

III. Structural properties of network systems

A. Structured systems

A structured system is a family of systems sharing a same imposed zero-pattern for their matrices. We introduce zero-one valued matrices $\mathrm{A} \in \{0,1\}^{n\times n}$, $\mathrm{B} \in \{0,1\}^{n\times p}$, $\mathrm{C} \in \{0,1\}^{n\times k}$, $\mathrm{D} \in \{0,1\}^{k\times p}$ to describe the fixed zero positions. Namely, their zeros represent the direct interactions which cannot happen at all. On the other hand, the ones correspond to influences which are possible, without specifying the intensity of such an interaction. From a given pattern $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D})$, we construct a family of systems, where the ones are replaced by free real parameters. Denoting by a, b, c, d the number of ones in matrices $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ respectively, the space of parameters is $\mathbb{R}^m$ with $m = a + b + c + d$. We will use the notation $\alpha \in \mathbb{R}^a$ for the collection of parameters introduced in matrix A, and A_α for the matrix obtained replacing the ones with these parameters. Analogously we define $\beta \in \mathbb{R}^b$, $\gamma \in \mathbb{R}^c$, $\delta \in \mathbb{R}^d$, and B_β, C_γ, D_δ. Below is an example of structured system matrices.

Example 1:

$$\mathrm{A}_\alpha = \begin{bmatrix} 0 & 0 & 0 & \alpha_{14} & 0 & 0 \\ \alpha_{21} & 0 & 0 & 0 & \alpha_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{53} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{63} & 0 & 0 & 0 \end{bmatrix},\ \mathrm{B}_\beta = \begin{bmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathrm{C}_\gamma = \begin{bmatrix} \gamma_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 & 0 & 0 & \gamma_{26} \\ 0 & 0 & 0 & 0 & \gamma_{35} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{46} \end{bmatrix},\ \mathrm{D}_\delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta_{41} & 0 & \delta_{43} \end{bmatrix}.$$

B. Structural and generic properties

For a structured system, based on the structure $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D})$, one can try to find various kinds of properties.

First, one can show that there exists one choice of parameters $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^m$ such that $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma, \mathrm{D}_\delta)$ has a desired property.

Second, one can show that some property is true for almost all choices of parameters. 'Almost all' has a precise mathematical definition [1]: for all parameters, except those lying on a proper subvariety or $\mathbb{R}^m$. A probabilistic interpretation is that the property is true with probability one, if parameters are chosen at random according to any continuous distribution. If a property is true for almost all parameters, we will say that the property is *generically* true.

A third kind of results is to show that a property is true for all choices of parameters respecting the constraint that each individual parameter is non-zero. Such results are called *strongly structural*, and are beyond the scope of this paper.

Some properties have a peculiar quality: the conditions under which the property is true for one choice of parameters are the same conditions under which the property is generically true. In this paper, we will call such properties *structural*[1]. Relevant well-known examples of structural properties are controllability and observability [1].

In Sect. IV, we will study input-and-state observability and delay-1 input-and-state observability, discuss whether or not they are structural properties, and find necessary and sufficient conditions under which they are generically true for a structured system.

C. Graph representation of structured systems

An usual way to represent a structured system [1] is a directed graph $\mathcal{G}$, having vertex set $U \cup X \cup Y$ and edge set $E_\mathrm{A} \cup E_\mathrm{B} \cup E_\mathrm{C} \cup E_\mathrm{D}$, defined as follows:

- $U = \{u_1, \dots, u_p\}$ are the input vertices;
- $X = \{x_1, \dots, x_n\}$ are the state vertices;
- $Y = \{y_1, \dots, y_k\}$ are the output vertices;
- for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n\}$,
  $(x_i, x_j) \in E_\mathrm{A}$ if and only if $\mathrm{A}_{ji} = 1$;
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, n\}$,
  $(u_i, x_j) \in E_\mathrm{B}$ if and only if $\mathrm{B}_{ji} = 1$;
- for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$,
  $(x_i, y_j) \in E_\mathrm{C}$ if and only if $\mathrm{C}_{ji} = 1$;
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, k\}$,
  $(u_i, y_j) \in E_\mathrm{D}$ if and only if $\mathrm{D}_{ji} = 1$.

This construction is depicted in Fig. 1.

Another classical representation makes use of a bipartite graph $\mathcal{H}$, with left vertex set $U \cup X$, right vertex set $X' \cup Y$, and edge set $\bar{E}_\mathrm{A} \cup \bar{E}_\mathrm{B} \cup \bar{E}_\mathrm{C} \cup \bar{E}_\mathrm{D}$, defined as follows:

- $U = \{u_1, \dots, u_p\}$, $X = \{x_1, \dots, x_n\}$;
- $X' = \{x'_1, \dots, x'_n\}$, $Y = \{y_1, \dots, y_k\}$;

[1]The vocabulary might vary in different papers; sometimes 'structurally' is used in place of the above-defined 'generically'.

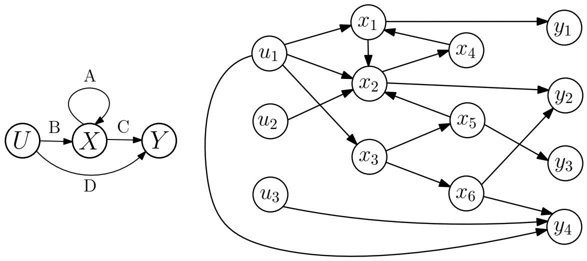

Fig. 1. Directed graph $\mathcal{G}$ associated with a structured system. Left: pictorial reminder of the construction of $\mathcal{G}$. Right: graph $\mathcal{G}$ for the system in Example 1.

- for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n\}$,
  $\{x_i, x'_j\} \in \bar{E}_{\mathrm{A}}$ if and only if $\mathrm{A}_{ji} = 1$;
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, n\}$,
  $\{u_i, x'_j\} \in \bar{E}_{\mathrm{B}}$ if and only if $\mathrm{B}_{ji} = 1$;
- for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$,
  $\{x_i, y_j\} \in \bar{E}_{\mathrm{C}}$ if and only if $\mathrm{C}_{ji} = 1$;
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, k\}$,
  $\{u_i, y_j\} \in \bar{E}_{\mathrm{D}}$ if and only if $\mathrm{D}_{ji} = 1$.

This construction is depicted in Fig. 2.

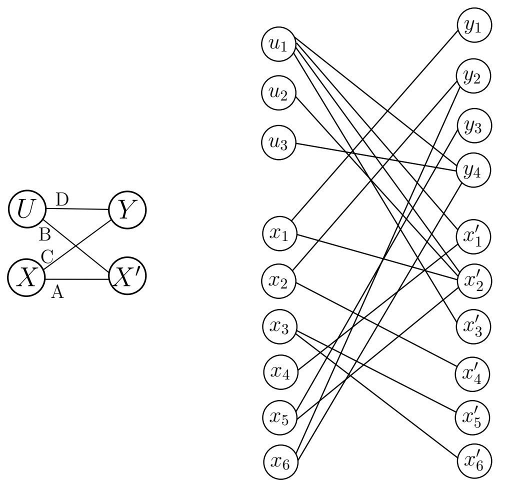

Fig. 2. Bipartite graph $\mathcal{H}$ associated with a structured system. Left: pictorial reminder of the construction of $\mathcal{H}$. Right: bipartite graph $\mathcal{H}$ for Example 1.

D. Graph vocabulary

Here we briefly remind some graph-theoretic definitions. More details can be found in [1], [2], [8].

A *matching* is a set of vertex-disjoint edges (any pair of edges does not have any common end-vertex). The size of a matching is its number of edges; a matching is a *maximum matching* if it has the maximum size among all matchings in the same graph. If a vertex is the end-point of an edge in the matching, it is said to be *saturated* by the matching.

In a directed graph $\mathcal{G}$, having fixed two disjoint subsets of vertices $U, Y \subset V$, a *linking* from U to Y is a set of vertex-disjoint paths from U to Y, namely such that each path has its starting vertex in U and ending vertex in Y, and moreover any vertex of the graph belongs to at most one path. The *size* of the linking, denoted with the cardinality symbol $\#$, is its number of paths, or, equivalently, the number of vertices in U belonging to some path in the linking. A linking is a *maximum linking* if it has maximum size. In analogy with the vocabulary for matchings, we will call *saturated* by the linking any vertex which belongs to a path in the linking. A set of vertices $S \subset V$ is called a vertex-separator of U and Y if the subgraph obtained removing S does not contain any path from U to Y. A vertex-separator is minimum if it has the smallest number of vertices. The well-known Menger Theorem states that the size of the maximum linking is equal to the size of the minimum vertex-separator. The set of *essential vertices* $V_{\text{ess}}(U, Y; \mathcal{G})$ is the set of vertices which are saturated by all maximum linkings from U to Y in $\mathcal{G}$. Equivalently, it is the union of all minimum vertex-separators of U and Y in $\mathcal{G}$. In this paper, we use the short notation V_{ess} to denote $V_{\text{ess}}(U, Y; \mathcal{G})$, where $\mathcal{G}$ is defined in Sect. III-C (Fig. 1), and U and Y are its input and output vertices.

For the system in Example 1, the size of the maximum matching in $\mathcal{H}$ is $p + n$, the size of the maximum linking from U to Y in $\mathcal{G}$ is p, and $V_{\text{ess}} = U \cup \{x_2\} \cup \{y_4\}$.

E. Other graphs associated with $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D})$

The graphs $\mathcal{G}$ and $\mathcal{H}$ defined in Sect. III-C are equivalent descriptions of a structured system, containing all the information about $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$. In this subsection, we introduce some more graph constructions, that are not a standard description of the system, but that will be needed for the statement of the main results in Sect. IV.

The directed graph $\tilde{\mathcal{G}}$ (Fig. 3) is the subgraph of $\mathcal{G}$ obtained by removing all vertices belonging to V_{ess}, the set of essential vertices defined in Sect. III-D.

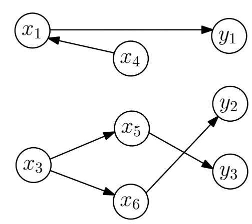

Fig. 3. The directed graph $\tilde{\mathcal{G}}$ is a subgraph of $\mathcal{G}$, obtained by removing all vertices belonging to V_{ess}. The figure shows $\tilde{\mathcal{G}}$ for Example 1.

The bipartite graph $\mathcal{D}$ (Fig. 4) is the subgraph of $\mathcal{H}$ having left vertex set U, right vertex set Y, and edge set $\bar{E}_{\mathrm{D}}$.

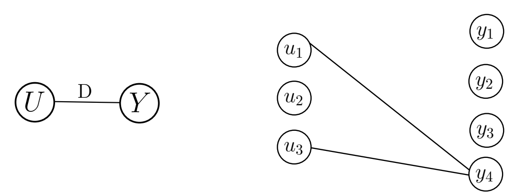

Fig. 4. Bipartite graph $\mathcal{D}$ associated with the output matrix D. Left: pictorial reminder of the construction. Right: bipartite graph $\mathcal{D}$ for Example 1.

The directed graph $\mathcal{K}$ (Fig. 5) has vertex set $U_0 \cup U_1 \cup X \cup Y_0 \cup Y_1$ and edge set $F_{\mathrm{D}}^0 \cup F_{\mathrm{D}}^1 \cup F_{\mathrm{B}} \cup F_{\mathrm{C}}$, where:

- $U_0 = \{u_1^0, \dots, u_p^0\}$, $U_1 = \{u_1^1, \dots, u_p^1\}$;

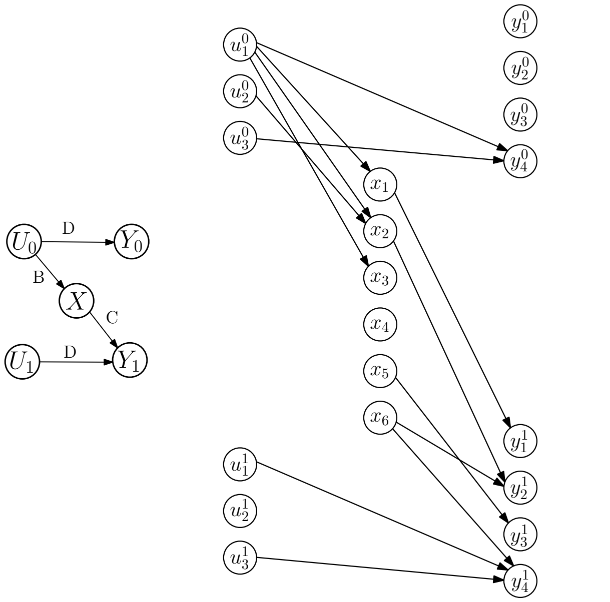

Fig. 5. Directed graph $\mathcal{K}$ used in Sect. IV-B. Left: pictorial reminder of the construction. Right: directed graph $\mathcal{K}$ for Example 1.

- $X = \{x_1, \dots, x_n\}$;
- $Y_0 = \{y_1^0, \dots, y_k^0\}$, $Y_1 = \{y_1^1, \dots, y_k^1\}$;
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, k\}$,
$$(u_i^0, y_j^0) \in F_{\mathrm{D}}^0 \text{ if and only if } \mathrm{D}_{ji} = 1$$
and
$$(u_i^1, y_j^1) \in F_{\mathrm{D}}^1 \text{ if and only if } \mathrm{D}_{ji} = 1;$$
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, n\}$,
$$(u_i^0, x_j) \in F_{\mathrm{B}} \text{ if and only if } \mathrm{B}_{ji} = 1;$$
- for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$,
$$(x_i, y_j^1) \in F_{\mathrm{C}} \text{ if and only if } \mathrm{C}_{ji} = 1.$$

The directed graph $\tilde{\mathcal{K}}$ is the subgraph of $\mathcal{K}$ having vertex set $U_0 \cup X \cup Y_1$ and edge set $F_{\mathrm{B}} \cup F_{\mathrm{C}}$.

The bipartite graph $\mathcal{N}$ (Fig. 6) has left vertex set $U_0 \cup X \cup U_1$ and right vertex set Y_0, X', Y_1, with U_0, U_1, Y_0, Y_1, X same as in the definition of $\mathcal{K}$, and $X' = \{x'_1, \dots, x'_n\}$. The edge set of $\mathcal{N}$ is $\bar{F}_{\mathrm{D}}^0 \cup \bar{F}_{\mathrm{D}}^1 \cup \bar{F}_{\mathrm{B}} \cup \bar{F}_{\mathrm{C}} \cup \bar{F}_I$, where:

- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, k\}$,
$$\{(u_i^0, y_j^0\} \in \bar{F}_{\mathrm{D}}^0 \text{ if and only if } \mathrm{D}_{ji} = 1$$
and
$$\{u_i^1, y_j^1\} \in \bar{F}_{\mathrm{D}}^1 \text{ if and only if } \mathrm{D}_{ji} = 1;$$
- for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, n\}$,
$$\{u_i^0, x'_j\} \in \bar{F}_{\mathrm{B}} \text{ if and only if } \mathrm{B}_{ji} = 1;$$
- for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$,
$$\{x_i, y_j^1\} \in \bar{F}_{\mathrm{C}} \text{ if and only if } \mathrm{C}_{ji} = 1;$$
- $\bar{F}_I = \big\{\{x_i, x'_i\}, i = 1, \dots, n\big\}$.

IV. Main results

A. Input-and-state observability

From [2], we know that input-and-state observability is a structural property, and we can obtain the following characterization.

Theorem 1: The system $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma, \mathrm{D}_\delta)$ is generically strongly observable and left-invertible if and only if the following two conditions hold:

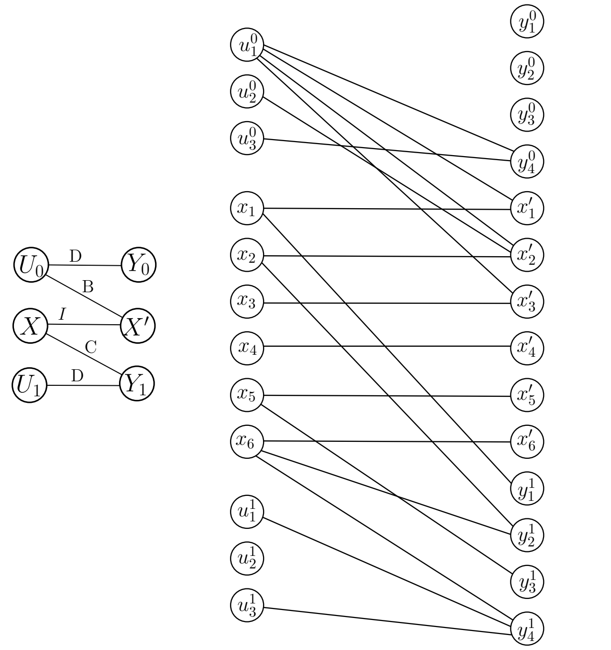

Fig. 6. Bipartite graph $\mathcal{N}$ used in Sect. V. Left: pictorial reminder of the construction. Right: bipartite graph $\mathcal{N}$ for Example 1.

(a) The bipartite graph $\mathcal{H}$ contains a matching of size $p+n$;
(b) The directed graph $\tilde{\mathcal{G}}$ is output-connected, i.e., from each of its vertices not in Y, there exists a path to some vertex in Y. □

Remark 1: Condition (a) in Thm. 1 implies that there exists a linking of size p from U to Y in $\mathcal{G}$. This also implies that $U \subseteq V_{\text{ess}}$, so that there is no input vertex in $\tilde{\mathcal{G}}$. Hence, condition (b) is equivalent to the following:

(b') For all $x \in X \setminus V_{\text{ess}}$, there exists a path in $\mathcal{G}$ from x to Y, such that no vertex of the path belongs to V_{ess}. □

Proof of Thm. 1: This result is an equivalent rephrasing of [2, Coroll. 7], where the two conditions (a) and (b) are described differently. The equivalence is immediate for condition (a), while condition (b) requires a careful look at the definitions and properties given in [2] and [9]. Due to space limitation, we refer to [2] for notation and definitions. [2, Coroll. 7] is reported in the thesis [9] as Corollary 2.5, followed by two equivalent reformulations of condition (b), one of which is: $U_0 \cup X \setminus X_1 \subseteq V_{\text{ess}}$. U_0 is a suitably-defined subset of U, and under (a) it is surely a subset of V_{ess}, as discussed in Remark 1. $X \setminus X_1$, also called Δ_0 in [2], is the union of the following three disjoint sets, as discussed in [2, page 1207]: 1) the state vertices which cannot be linked to Y, 2) the state vertices belonging to V_{ess}, and 3) the state vertices $x \notin V_{\text{ess}}$ such that all paths from x to Y have some vertex belonging to V_{ess}. Clearly, asking that $X \setminus X_1 \subseteq V_{\text{ess}}$ means asking that its first and third subsets are empty, i.e., all state vertex $x \notin V_{\text{ess}}$ has some path to Y such that no vertex of the path belongs to V_{ess}. This is (b') in Remark 1. ■

The statement of Thm. 1 is simpler than the one of [2, Coroll. 7]. However, there is no claim of added computational simplicity: the two formulations are equivalent, and require the same algorithm to be verified.

B. *Delay*-1 *left invertibility*

Left-invertibility is a structural property, characterized as follows.

Proposition 3 ([10, Thm. 2]): The system $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma, \mathrm{D}_\delta)$ is generically left-invertible if and only if the directed graph $\mathcal{G}$ contains a linking of size p from U to Y. $\square$

In case $\mathrm{D} = 0$, also delay-1 left-invertibility is a strucural property. Indeed, the algebraic characterization in Prop. 2 i) simplifies to $\mathrm{rank}(\mathrm{C}_\gamma \mathrm{B}_\beta) = p$, and having full rank is a structural property [8, Sect. 2.1.3]. The generic rank of a product of matrices is characterized in [11]: $\mathrm{rank}(\mathrm{C}_\gamma \mathrm{B}_\beta)$ equals the size of the maximum linking from U_0 to Y_1 in the directed graph $\tilde{\mathcal{K}}$. Hence, we obtain:

Proposition 4: If $\mathrm{D} = 0$, the system $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma)$ is generically delay-1 left-invertible if and only if the directed graph $\tilde{\mathcal{K}}$ contains a linking from U_0 to Y_1 of size p. $\square$

For the general case with no assumptions on D, delay-1 left-invertibility is not a structural property. Our main result is the following characterization of generic delay-1 left-invertibility. The result should be interpreted as follows: if the given condition is true, then the system is delay-1 left-invertible for almost all parameters; if the condition fails, then the system might be delay-1 left-invertible for some parameters, but at most for 'few' parameter vectors, lying in a proper subvariety of $\mathbb{R}^m$.

Theorem 2: The system $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma, \mathrm{D}_\delta)$ is generically left-invertible with delay 1 if and only if the directed graph $\mathcal{K}$ contains a linking of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$, where r is the size of the maximum matching in $\mathcal{D}$. $\square$

The proof of this result is given in Section V.

C. *Delay-1 input-and-state observability*

From Theorems 1 and 2, we immediately obtain the following characterization of generic delay-1 input-and-state observability, which is the main result of this paper.

Corollary 1: The system $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma, \mathrm{D}_\delta)$ is generically strongly observable and left-invertible with delay 1 if and only if the following three conditions hold:

(a) The bipartite graph $\mathcal{H}$ contains a matching of size $p+n$;
(b) The directed graph $\tilde{\mathcal{G}}$ is output-connected;
(c) The directed graph $\mathcal{K}$ contains a linking of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$, where r is the size of the maximum matching in $\mathcal{D}$. $\square$

Applying this result to Example 1, it is easy to see that such system is generically delay-1 input-and-state observable.

V. Proof of Thm. 2

Throughout this section, r denotes the size of the maximum matching in $\mathcal{D}$.

A. *Proof of sufficiency*

Using item ii) of Prop. 2, we will prove that the existence of the linking described in Thm. 1 is sufficient to ensure that the system is generically delay-1 left-observable.

The following lemma ensures that, given a linking as in Thm. 2, one can construct another linking, with all the same properties, and with the additional property that it uses the 'same' edges from F_{D}^0 and F_{D}^1, where 'same' means 'with the same index', i.e., (u_i^0, y_j^0) and (u_i^1, y_j^1) with same i and j.

Lemma 1: If there exists a linking L in $\mathcal{K}$ of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$, then we can construct another such linking $\tilde{L}$ with the additional property that $(u_i^0, y_j^0) \in \tilde{L}$ if and only if $(u_i^1, y_j^1) \in \tilde{L}$.

Proof: The proof is constructive. First, notice that L can be partitioned in the following three sets:

- L_0, containing paths from U_0 to Y_0;
- L_X, containing paths from U_0 to Y_1 (each such path contains a vertex from X);
- L_1, containing paths from U_1 to Y_1.

Each path in L_0 and L_1 has length one, i.e., it is a single edge, while each path in L_X has length two. Moreover, $L_0 \cup L_X$ saturates U_0, and L_1 saturates r vertices of U_1.

The first construction turns L into $\bar{L}$ (which is decomposed in $\bar{L}_0$, $\bar{L}_X$, and $\bar{L}_1$, similarly to above), so that $\bar{L}$ is a linking saturating U_0 and r vertices of U_1, with the additional property that $\bar{L}_0$ and $\bar{L}_1$ saturate the 'same' y's, in the sense that y_j^0 is saturated by $\bar{L}_0$ if and only if y_j^1 is saturated by $\bar{L}_1$. We will use the notation J, $\bar{J}$ to denote the set of indices j such that y_j^0 is saturated by L_0 and $\bar{L}_0$, respectively, and K, $\bar{K}$ will be the set of indices k such that y_k^1 is saturated by L_1 and $\bar{L}_1$. The aim of the construction is to obtain $\bar{K} \subseteq \bar{J}$.

For each $k \in K \setminus J$, let (u_i^1, y_k^1) be the edge in L_1 that saturates y_k^1. By definition of $K \setminus J$, y_k^0 is not saturated in L_0, and the goal is to make it saturated in $\bar{L}_0$. There are two cases:

- If u_i^0 is saturated in L_0, then we remove the edge saturating it, say (u_i^0, y_h^0), and we replace it by (u_i^0, y_k^0).
- If u_i^0 is not saturated in L_0 (and hence u_i^0 is saturated in L_X), then we add a new edge (u_i^0, y_k^0), and we remove from L_X the path that was saturating u_i^0.

Every step of this construction preserves the property of having a linking such that $\bar{L}_0 \cup \bar{L}_X$ saturates U_0 and $\#\bar{L}_1 = r$. Moreover, having considered all $k \in K \setminus J$, we obtain $\bar{K} \setminus \bar{J} = \emptyset$, i.e., $\bar{K} \subseteq \bar{J}$.

Now we prove that, with no further construction, we actually have $\bar{K} = \bar{J}$. Indeed, we know that $\#\bar{K} = \#\bar{L}_1 = r$ and that $\#\bar{J} = \#\bar{L}_0 \leq r$; the latter inequality is true, because the linking $\bar{L}_0$ has a natural corresponence with a matching in $\mathcal{D}$, and hence it cannot have a size larger than the size r of a maximum matching in $\mathcal{D}$. Having $\bar{K} \subseteq \bar{J}$ and $\#\bar{J} \leq \#\bar{K}$ leads to the conclusion that $\bar{K} = \bar{J}$.

Now we are ready for the final step of the construction. We define $\tilde{L}$ where $\tilde{L}_0 = \bar{L}_0$ and $\tilde{L}_X = \bar{L}_X$ are left unchanged, while $\tilde{L}_1$ saturates the same set of y's as $\bar{L}_1$ (the ones with index in $\bar{K}$), but it does it using the 'same' edges as $\tilde{L}_0$: $(u_i^1, y_j^1) \in \tilde{L}_1$ if and only if $(u_i^0, y_j^0) \in \tilde{L}_0$. $\blacksquare$

Using the linking constructed in Lemma 1, we can exhibit one choice of parameters $\alpha, \beta, \gamma, \delta$ such that $\mathrm{rank} \begin{bmatrix} \mathrm{D}_\delta & 0 \\ \mathrm{C}_\gamma \mathrm{B}_\beta & \mathrm{D}_\delta \end{bmatrix} = p + r$, as follows.

Lemma 2: If the graph $\mathcal{K}$ contains a linking L of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$, then there exists a vector of parameters $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^m$ such that $\mathrm{rank} \begin{bmatrix} \mathrm{D}_\delta & 0 \\ \mathrm{C}_\gamma \mathrm{B}_\beta & \mathrm{D}_\delta \end{bmatrix} = p + r$. $\square$

Proof: We construct $\tilde{L}$ as in Lemma 1. Then we set parameters to be 1 in correspondence of edges of $\mathcal{K}$ that belong to $\tilde{L}$, i.e., $\delta_{ji}=1$ if $(u_i^0, y_j^0) \in \tilde{L}_0$ (which also means $(u_i^1, y_j^1) \in \tilde{L}_1$), $\beta_{ki}=1$ and $\gamma_{jk}=1$ if $(u_i^0, x_k, y_j^1) \in \tilde{L}_X$. All other parameters are set to zero. Now consider $\left[\begin{smallmatrix} \mathrm{D}_\delta & 0 \\ \mathrm{C}_\gamma \mathrm{B}_\beta & \mathrm{D}_\delta \end{smallmatrix}\right]$ and label its $2p$ columns and its $2k$ rows using vertices in $U_0 \cup U_1$, and in $Y_0 \cup Y_1$, respectively. With the above-described choice of parameters, an entry of the matrix corresponding to a pair of vertices u, y is equal to one if there is a path in $\tilde{L}$ from u to y, and to zero else. Since $\tilde{L}$ is a linking saturating U_0 and r vertices of U_1, the matrix has $p-r$ columns of the right block $\left[\begin{smallmatrix} 0 \\ \mathrm{D}_\delta \end{smallmatrix}\right]$ being all-zero (corresponding to the unsaturated vertices in U_1), and the remaining $p+r$ columns have one '1' per column, all in different rows (corresponding to the saturated vertices of $Y_0 \cup Y_1$). This proves that the matrix has rank $p+r$. ■

Lemma 3: If the graph $\mathcal{K}$ contains a linking of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$, then, for almost all choices of parameters, $\operatorname{rank} \mathrm{D}_\delta = r$ and $\operatorname{rank} \left[\begin{smallmatrix} \mathrm{D}_\delta & 0 \\ \mathrm{C}_\gamma \mathrm{B}_\beta & \mathrm{D}_\delta \end{smallmatrix}\right] = p+r$. □

Proof: Having full column rank is a structural property. Hence, since Lemma 2 exhibits one particular choice of parameters such that $\operatorname{rank} \left[\begin{smallmatrix} \mathrm{D}_\delta & 0 \\ \mathrm{C}_\gamma \mathrm{B}_\beta & \mathrm{D}_\delta \end{smallmatrix}\right] = p+r$, then the same is true generically. Moreover, $\operatorname{rank} \mathrm{D}_\delta$ is generically equal to r [8, Sect. 2.1.3]. ■

By Prop. 2 ii), this proves the sufficiency part of Thm. 2.

B. *Proof of necessity*

For the necessity part, we will use item iii) in Prop. 2.

Lemma 4: For any vector of parameters $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^m$, the rank of the matrix $N = \left[\begin{smallmatrix} \mathrm{D}_\delta & 0 & 0 \\ \mathrm{B}_\beta & -I & 0 \\ 0 & \mathrm{C}_\gamma & Dd \end{smallmatrix}\right]$ is smaller than or equal to the size of the maximum matching in the bipartite graph $\mathcal{N}$. □

Proof: The bipartite graph $\mathcal{N}$ is the so-called Coates graph of the matrix N: it represents its zero pattern, since associating columns of N with the left vertex set $U_0 \cup X \cup U_1$ and rows with the right vertex set $Y_0 \cup X' \cup Y_1$, the edges correspond to the positions not fixed to zero. A classical result [8, Sect. 2.3.1] shows that the rank of a matrix, for any choice of the entries not fixed to zero, is upper bounded by the size of the maximum matching in the Coates graph. ■

Lemma 5: The existence of a linking of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$ in $\mathcal{K}$ is a necessary condition for $\operatorname{rank} N = p+n+r$, where $N = \left[\begin{smallmatrix} \mathrm{D}_\delta & 0 & 0 \\ \mathrm{B}_\beta & -I & 0 \\ 0 & \mathrm{C}_\gamma & Dd \end{smallmatrix}\right]$. □

Proof: From Lemma 4, $\operatorname{rank} N = p+n+r$ implies the existence of a matching M of size $p+n+r$ in $\mathcal{N}$. But this also implies the existence of a linking L of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$ in $\mathcal{K}$. Indeed, L can be constructed from M, as follows. First, notice that M saturates U_0, X and r vertices of U_1 (since it cannot saturate more than r vertices of U_1, otherwise there would exist a matching of size larger than r in a subgraph of $\mathcal{N}$ isomorphic to $\mathcal{D}$). For each $u_i^0 \in U_0$, if u_i^0 is saturated in M by an edge $\{u_i^0, y_j^0\}$ with $y_j^0 \in Y_0$, then add the edge (u_i^0, y_j^0) to L; else, u_i^0 is saturated by an edge $\{u_i^0, x_k'\}$ with $x_k' \in X'$. In this case, look at the corresponding vertex $x_k \in X$ (with the same index k); x_k is saturated in L by an edge $\{x_k, y_h^1\}$ with $y_h^1 \in Y_1$; add to L the path saturating u_i^0 using the two above edges: (u_i^0, x_k, y_h^1). Finally add to L the r edges corresponding to the edges saturating vertices of U_1 in M. ■

Lemma 6: For each vector of parameters such that $\operatorname{rank} \mathrm{D}_\delta = r$, the existence of a linking of size $p+r$ from $U_0 \cup U_1$ to $Y_0 \cup Y_1$ in $\mathcal{K}$ is a necessary condition for delay-1 left-invertibility of $(\mathrm{A}_\alpha, \mathrm{B}_\beta, \mathrm{C}_\gamma, \mathrm{D}_\delta)$. □

Proof: This follows from Prop. 2 and Lemma 5. ■

Since $\operatorname{rank} \mathrm{D}_\delta$ is generically equal to r [8, Sect. 2.3.1], this ends the proof of the necessity part of Thm. 2.

VI. Conclusion

In this paper we have studied structured systems. Based on graphs describing the system structure, we have given necessary and sufficient conditions for generic input-and-state observability. We have particularly addressed delay-1 input-and-state observability, where the input is to be reconstructed with only one time-step of delay. Open problems include alternative equivalent characterizations, results for LTV systems, strongly structural results (holding for all non-zero parameters, as opposed to generically), and the study of delay-ℓ left-invertibility for $\ell > 1$.

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