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Laurent Burlion, Luca Zaccarian, Henry de Plinval, Sophie Tarbouriech

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Detailed proof of the Lemma 1 used in the manuscript "Discontinuous model recovery anti-windup solution for image based visual servoing" [1]

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As mentioned in [1], the proof of technical Lemma 1 arises from brute force calculations of all possible cases. The details were omitted due to page restrictions. The proof of this result is reported below:

Lemma 1 *For any selection of* $(\tilde{x}, x) = (\tilde{p}, \tilde{v}, p, v) \in \mathbb{R}^4$, *it holds that*

$$\max_{\delta_p \in \Upsilon(\tilde{x}, x)} v\delta_p \leq k_p^g \frac{\lambda_m}{\lambda_0} \left(2|\tilde{x}||v| + |\tilde{x}|^2\right) + 3\bar{u}_p|\tilde{x}|. \tag{1}$$

To better follow the proof of Lemma 1, it is important to keep in mind the definition of M_I (defined in eq. (11),[1]) plus the following basic properties of the saturation function used in [1]: given $\bar{u}_p > 0$,

- **(P1)**: $\text{sat}_{\bar{u}_p}$ is 1-Lipschitz
- **(P2)**: $\forall x$, $|\text{sat}_{\bar{u}_p}(x)| \leq \bar{u}_p$
- **(P3)**: $\forall k_p^g > 0, \zeta > 0$, $\forall p \neq \tilde{p}$,
$$\frac{\text{sat}_{\bar{u}_p}(k_p^g \zeta p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta \tilde{p})}{p - \tilde{p}} \geq 0 \tag{2}$$
- **(P4)**: $\forall \chi > 0$, $\forall x, y$ then
$$xy \geq 0 \Longrightarrow y\text{sat}_{\bar{u}_p}(\chi x) \geq 0 \tag{3}$$
$$xy \leq 0 \Longrightarrow y\text{sat}_{\bar{u}_p}(\chi x) \leq 0 \tag{4}$$

Proof. Given any pair $(\tilde{x}, x) = (\tilde{p}, \tilde{v}, p, v)$, consider set $\Upsilon(\tilde{x}, x)$ defined in eq. (44),[1]. In particular, consider any selection of $\zeta \in M_I(pv)$, of $\zeta_{aw} \in M_I((p - \tilde{p})(v - \tilde{v}))$ and any $\mu \in [\lambda_0, \lambda_m]$. (From the definition of M_I and eq. (4),[1], it is clear that $\zeta, \zeta_{aw}, \mu > 0$).
Then denote:

$$\psi := v\left(\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta_{aw} \mu (p - \tilde{p}))\right) \tag{5}$$

where $k_p^g > 0$.
The proof of the lemma amounts to showing that

$$\psi \leq k_p^g \frac{\lambda_m}{\lambda_0} \left(2|\tilde{x}||v| + |\tilde{x}|^2\right) + 3\bar{u}_p|\tilde{x}| \tag{6}$$

This is done by way of a lengthy but simple study of five possible cases:

1. Suppose
$$(p - \tilde{p})(v - \tilde{v}) < 0 \quad \& \quad pv \geq 0 \tag{7}$$

In this case, $\zeta_{aw} = \frac{1}{\lambda_m}$ and (5) develops as follows:

$$\begin{aligned}
\psi &= v\left[\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_m}(p-\tilde{p})\right)\right] && (8)\\
&\leq v\left[\mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_m} p\right) - \mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_m} u(p-\tilde{p})\right)\right] \\
&\quad + v sat_{\bar{u}_p}(k_p^g \zeta \mu p) && (9)\\
&\leq k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| + v sat_{\bar{u}_p}(k_p^g \zeta \mu p) && (10)
\end{aligned}$$

where the penultimate inequality was obtained using **(P4)**, i.e $0 \leq v sat_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_m} p\right)$.
where the last inequality was obtained using **(P1)** and $0 < \frac{\mu}{\lambda_m} \leq 1 < \frac{\lambda_m}{\lambda_0}$.
Moreover, it is readily seen that $(p-\tilde{p})(v-\tilde{v}) < 0$ and **(P3)** imply:

$$\left(\mathrm{sat}_{\bar{u}_p}\left(k_p^g \zeta \mu p\right) - \mathrm{sat}_{\bar{u}_p}\left(k_p^g \zeta \mu \tilde{p}\right)\right)(v-\tilde{v}) \leq 0 \tag{11}$$

Thus,

$$\begin{aligned}
0 \leq v sat_{\bar{u}_p}\left(k_p^g \zeta \mu p\right) &\leq sat_{\bar{u}_p}\left(k_p^g \zeta \mu \tilde{p}\right)(v-\tilde{v}) \\
&\quad + sat_{\bar{u}_p}\left(k_p^g \zeta \mu p\right)\tilde{v} && (12)
\end{aligned}$$

Rewriting (7) as follows

$$0 \leq pv < \tilde{p}(v-\tilde{v}) + p\tilde{v} \tag{13}$$

and combining (10),(12),(13) one successively gets (using **(P1)** and **(P2)**):

$$\begin{aligned}
\psi &\leq k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| + k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{p}(v-\tilde{v})| + \bar{u}_p|\tilde{v}| && (14)\\
&\leq k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| + k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{p}v| + k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{p}\tilde{v}| + \bar{u}_p|\tilde{v}| && (15)\\
&\leq 2k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| + k_p^g \frac{\lambda_m}{2\lambda_0}|\tilde{x}|^2 + \bar{u}_p|\tilde{x}| && (16)
\end{aligned}$$

which means that (6) is satisfied in case (7).

2. Suppose

$$(p-\tilde{p})(v-\tilde{v}) < 0 \quad \& \quad pv < 0 \tag{17}$$

In this case, $\zeta = \frac{1}{\lambda_m}$ and (5) develops as follows:

$$\begin{aligned}
\psi &\leq v\left(\mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_m} p\right) - \mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_m}(p-\tilde{p})\right)\right) \\
& && (18)\\
&\leq k_p^g \frac{\mu}{\lambda_m}|v||\tilde{p}| \leq k_p^g|\tilde{x}||v| \leq k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| && (19)
\end{aligned}$$

Where the second inequality was obtained using property **(P1)**. This means that (6) is satisfied in case (17).

3. Suppose

$$(p-\tilde{p})(v-\tilde{v}) > 0 \quad \& \quad pv \leq 0 \tag{20}$$

In this case, $\zeta_{aw} = \frac{1}{\lambda_0}$ and (5) develops as follows

$$\begin{aligned}
\psi &= v\left[\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu p) - \mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_0}(p-\tilde{p})\right)\right] && (21)\\
&\leq 0 - v\mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_0}(p-\tilde{p})\right) && (22)\\
&\leq -(v-\tilde{v})\mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_0}(p-\tilde{p})\right)\\
&\quad -\tilde{v}\mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_0}(p-\tilde{p})\right) && (23)\\
&\leq 0 + \bar{u}_p|\tilde{v}| \leq \bar{u}_p|\tilde{x}| && (24)
\end{aligned}$$

where the first inequality is obtained using **(P4)**
where the penultimate inequality is obtained using **(P2)**
From the last inequality, (6) is thus satisfied in case (20).

4. Suppose

$$(p-\tilde{p})(v-\tilde{v}) > 0 \quad \& \quad pv > 0 \tag{25}$$

In this case, $\zeta = \zeta_{aw} = \frac{1}{\lambda_0}$ and (5) develops as follows

$$\begin{aligned}
\psi &= v\left(\mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_0} p\right) - \mathrm{sat}_{\bar{u}_p}\left(k_p^g \frac{\mu}{\lambda_0}(p-\tilde{p})\right)\right) && (26)\\
&\leq k_p^g \mu \frac{|v\tilde{p}|}{\lambda_0} \leq k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| && (27)
\end{aligned}$$

which means that (6) is satisfied in case (25).

5. Suppose

$$(p-\tilde{p})(v-\tilde{v}) = 0 \tag{28}$$

In this case, (5) develops as follows:

$$\begin{aligned}
\psi &= v(\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu p) - \mathrm{sat}_{\bar{u}_p}(k_p^g \zeta_{aw}\mu(p-\tilde{p}))) && (29)\\
&= v\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu p) - \tilde{v}\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta_{aw}\mu(p-\tilde{p})) && (30)\\
&\leq v\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu p) + \bar{u}_p|\tilde{v}| && (31)
\end{aligned}$$

Moreover, it is readily seen that $(p-\tilde{p})(v-\tilde{v}) = 0$ implies that $p = \tilde{p}$ or $v = \tilde{v}$, which implies:

$$(\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu p) - \mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu\tilde{p}))(v-\tilde{v}) = 0 \tag{32}$$

Combining (31) and (32), one successively gets:

$$\begin{aligned}
\psi &\leq \tilde{v}\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu p) + \mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu\tilde{p})(v-\tilde{v}) + \bar{u}_p|\tilde{v}| && (33)\\
&\leq \bar{u}_p|\tilde{x}| + \mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu\tilde{p})v + |\mathrm{sat}_{\bar{u}_p}(k_p^g \zeta\mu\tilde{p})\tilde{v}| + \bar{u}_p|\tilde{v}| && (34)\\
&\leq k_p^g \frac{\lambda_m}{\lambda_0}|\tilde{x}||v| + 3\bar{u}_p|\tilde{x}| && (35)
\end{aligned}$$

where the last inequality is obtained using **(P1)** and **(P2)**. which means that (6) is satisfied in case (28). The proof is now completed.

References

[1] L. Burlion, L. Zaccarian, H. de Plinval and S. Tarbouriech *"Discontinuous model recovery anti-windup solution for image based visual servoing"*, submitted to Automatica, 2017.