Impact of Informatics on Mathematics and Its Teaching

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On the Importance of Epistemological Analysis to Feed Didactical Research

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Abstract. In this article, we come back to the seminal role of epistemology in didactics of sciences and partiularly in mathematics. We defend that the epistemological research on the interactions between mathematics and informatics is necessary to feed didactical research on today's mathematics learning and teaching situations, impacted by the development of informatics. We develop some examples to support this idea and propose some perspectives to attack this issue.

Keywords: Epistemology, Didactics, Education, Mathematics, Informatics, Computer Science, Interactions

Introduction

The teaching of mathematics has been questioned for more than 30 years by the development of computer science (informatics in the following) due to its strong relation with mathematics [11, 14]. Today, we witness the generalization of the teaching of informatics (inside or beside mathematics), the introduction in mathematics curricula of contents shared with informatics (like algorithmics or combinatorics), and the generalization of computers as tools for teaching, especially in mathematics. Naturally, those changes raise many educational questions that have partly already been studied. Although, we want to focus on some of these questions with the point of view of epistemology and its relation to didactics. We will exemplify these questions and show how important is the epistemology of the informatics-mathematics relation in order to tackle these issues.

Indeed, mathematics and informatics have strong links and a common history. More precisely,

(1) they share common foundations, structured by logics, and a specific relation with proof [7],
(2) there is a certain continuity between them with many fields developing at their interface,
(3) computer assisted mathematics changes the way some mathematicians work, emphasizing the experimental dimension of mathematics[22], and

(4) mathematics and informatics, sometimes classified as formal sciences, have a very similar relation to other sciences through modelling and simulation (this aspect will not be discussed in details here).

In the first section, we will elaborate on the role of epistemology in didactics and precise how it can be implement in the case of the interaction informatics-mathematics. In the second section, we will give examples of approaches to cope with these didactical questions and illustrate them: links between proof and algorithms, language issues, differences between mathematical and algorithmic thinkings, experimental mathematics, role of the computer and the appearance of new objects.

1 The Need for Epistemological Analysis of Interactions Between Informatics and Mathematics to Feed Didactical Research

1.1 On the Links Between Epistemology and Didactics of Sciences

Didactics of sciences, in the French tradition, and particularly didactics of mathematics, have historically a strong and fundamental relation to epistemology. Didactics of Sciences (or of a specific scientific discipline) is often defined as the study of the conditions of transmission of the knowledge of sciences (or of the discipline). Since Johsua and Dupin [12]:

> If one had to try a definition, one can say that the didactics of a discipline is the science that study, for a specific field (here sciences and mathematics), the phenomena of teaching, the conditions of the transmission of the own "culture" of an institution [...] and the conditions of acquisition of knowledge by a learner. The entry in this problematic is the reflection on the knowledge.[1] [12, p. 2]

It is in this reflection on the knowledge where the epistemology comes up. In the founding article "Épistémologie et didactique", Michèle Artigue [1] expresses this in terms of:

> Epistemological needs in didactics, that is, the needs that can be expressed in terms of the understanding of the processes whereby the mathematical concepts take form and grow and more generally, the understanding of the characteristics of mathematical activity.[2] [1, p. 243]

Epistemological analysis permits to take a step back on a taught concept, by taking into account the conditions of its genesis and the conditions of its existence. In this way, in our specific case, fields at the frontier between mathematics and informatics and their objects must be questioned and analysed. Artigue specifies:

[1] Our translation.
[2] Our translation.

> Beyond the conceptual analysis, the epistemology interplays in a more general level because what aims the teaching of mathematics is not only the transmissions of mathematical knowledge, it is more globally the transmission of a culture. It is about making the students enter the mathematical game. But, what is this mathematical game? What are the general thinking processes that govern it? It is the epistemological analysis [...] which is first concerned by these questions.[3][1, p. 246]

Some changing in the practice of mathematics linked with the development of informatics [22, 15, 6, 13] seems to us falling under this general level. It is important, then, to support the didactical work with analyses of the contemporary epistemology of mathematics and informatics, in order to take the evolutions of the practices of the field into account. Enlightening phenomena of *didactical transposition* [4] specifically comes under this epistemological approach and feet particularly well with questioning the curricular choices facing the mutations of a discipline:

> The epistemological analysis also enables the didactician to measure the differences that exist between the academic knowledge [...] and the taught knowledge. Indeed, although School lives in the fiction that consists of seeing in the taught objects some copies, simplified but correct, of the objects of Science, the epistemological analysis, by allowing us to understand what leads the evolution of the scientific knowledge, helps us to be aware of the distance existing between the economies of the two systems.[4][1, p. 244-245]

Epistemology also gives keys to understand the students' errors and misconceptions, and can supports didactical arguments to design or organize the curricula. But more generally, by revealing the nature of the discipline and the specificity of the concepts, epistemology contributes to didactics by allowing to make enter *the knowledge* into the teaching relation between *the teacher* and *the student*, and thus permits a better understanding of the phenomena linked to the teaching and learning of these concepts.

1.2 The Specific Case of the Relation between Mathematics and Informatics

We claim that this approach can be extended to the mathematics-informatics interactions, and particularly the impacts of informatics on mathematics and its practice.

As mentioned above, epistemology of mathematics and informatics can enlighten didactical issues on two sides :

– **On concepts and objects.** Epistemology contributes to a better understanding of the nature of the objects and the way they have appeared and

[3] Our translation.
[4] Our translation.

developed. On this point, it is clear that the new objects developed at the interface between mathematics and informatics, and also the specific fields (such as combinatorics, discrete mathematics, algorithmics, operational research, cryptography, ...) must be precisely analyzed.
- **On the foundations and nature of the disciplines.** Informatics has impacted the way mathematics are dealt with, the practices of the mathematicians, and the changes are deep. Epistemology can document didactical research in order to understand and guide the evolutions of the mathematics curricula. Even deeper, in the interactions between mathematics and informatics, the questions of the foundations, the role and place of logic, algorithms, language and proof, must be taken into account.

In the second section, we will exemplify this and illustrate how epistemology of informatics-mathematics can improve didactical questions that arise from mathematics-informatics interactions from different perspectives.

In practice, didactics and epistemology interplays in two directions (often intertwined). Sometimes, a didactical phenomenon requires an epistemological perspective, but, epistemological inquiries often precede a didactical research, in the sense that it can reveal phenomena that were invisible, and can help the didactician to tackle a question with a sharp look.

Taking into account the context of this article – and our goal to illustrate the needs for interactions between epistemology and didactics of mathematics and informatics – we will introduce the examples from the epistemological questions to the didactical issues.

2 Approaches and Examples

In this section, we will illustrate what we have defended in the first section by presenting different epistemological lenses through which the interactions between mathematics and informatics in education can be viewed. The different perspectives we will develop are not supposed to cover all the questions of the area and aim at being developed in further works.

Our goal here is to show how the epistemological analysis, by enlightening the specificity of mathematics-informatics activity, the way concepts appear and develop, and phenomena like didactical transposition, contributes to what Artigue calls *epistemological vigilance in didactics* [1], permits to understand errors and obstacles of students and bring keys to organize the teaching and learning of mathematics, informatics and fields at their frontier.

2.1 Proof and Algorithm

Epistemological considerations In a previous work [20] we have underlined that algorithm and proof are linked in many different ways. First of all, it is important to point out that an algorithm is an effective method for solving a general problem. Based on various definitions encountered in the literature and on an epistemological investigation on the concept, we have selected the following definition of algorithm:

> An algorithm is a problem solving procedure, which can be applied on a set of instances of the problem and that produce, in a finite number of constructive, unambiguous and organised steps, the answer to the problem for any of those instances.[5]

The *Theory of Complexity* (see [9] for instance) defines a problem as a couple (I, Q) where I is a set of instances and Q a question that can concern any instance of I[6]. An algorithm `A` solves a problem $P = (I, Q)$ if for any instance of I, `A` give an right answer to Q after a finite number of steps. Within this point of view, proving an algorithm then consists of proving:

- Termination of the algorithm: for any instance of the problem, the algorithm gives an answer to the question after a finite number of steps;
- Correctness of the algorithm: for any instance of the problem, the algorithm gives a correct answer to the question.

Hence, algorithmic problem solving includes a important dimension of proof and many tools and theories have been developed for proving algorithms and studying their properties. On an other hand, any constructive mathematical proof involves an implicit algorithm that can be made explicit. In particular, it is the case of any proof by induction, where an underlying process of construction can be detailed.

Actually, the theoretical result called Curry-Howard correspondence, shows that any proof can be seen as an algorithm and any algorithm as a proof. All those points of view make algorithm not only a tool for mathematics but also a object about which mathematics can be developed (the field called algorithmics indeed).

Didactical issues Based on these considerations about the relation between algorithm and proof, the concept of algorithm has to be questioned as a mean for a mediation in the teaching of mathematical proof, particularly in a context of presence of algorithmics in a mathematical curriculum, as we do have in France. In [19, 20] we studied how the concept of algorithm has been transposed in the French curricula and textbooks for high school and how proof could be handled and taught through algorithmics.

By analysing the algorithms that are proposed at his level, through the lens of proof and problem solving – and using the Instance-Question description of

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[5] We are aware that every definition has epistemological consequences. This definition emphasizes the notions of problem, input and output, and is compatible with theoretical models such as *Turing Machine*, or recursive functions. This choice is relevant in a perspective of didactics of mathematics' interaction with informatics.

[6] For example, the problem of testing primality can be expressed as the set of instances $\mathbb{N}^*$ of all positive integers and the question "Is it prime?" (or equivalently "Does the integer has a divisor different from 1 and itself?".

problems – we have been able to enlighten some didactical phenomena. Although there is a strong potential to deal with proof [18], French mathematical curricula are principally focused on dealing with algorithm as a tool, with regular confusions between algorithms and programs and between correctness of the algorithm – as the general method implemented – and "good" programming of the method in a specific language (respect of the syntax, interface management, . . .). Thus, they do not permit to deal with proof and generate, for instance, algorithms that solve a unique instance of a problem (figure 1), that simulate random phenomena, or consist of programming an interface for an already existing algorithm in the machine (figure 2). Such algorithms can not (or have no interest to) be proven in the meaning presented above, that is producing a correct answer for any instance of the problem.

To face such phenomena, developing rich activities involving algorithm and proof for mathematics at high school is necessary. In this direction, epistemology can give interesting perspectives for designing and experimenting such activities [20].

```
begin
      give r the value 1
      for i from 1 to 10 do
            give r the value r*i
      return r
end
```

Fig. 1. Factorial of 10, from a resource for grade 10 (our translation). *From the point of view of the proof of an algorithm, it is not possible to differentiate this algorithm from any other also producing the output* 3628800*.*

84 ALGO

1. The expression of a function being already entered in a calculator, design an algorithm which computes an approximation of the integral of this function between any a and b.

2. Program this algorithm on your calculator.

HELP: Use the functions `intégrFonction` or $\int dx$ of the calculator.

Fig. 2. Exercise from a textbook for grade 11 (our translation). *The aim of the exercise is to use the primitive function already present in the students' calculators to write a program that computes the integral of a function on a given interval. There is no issue about the way the primitives are found, the validity of the method or its precision.*

2.2 Language in Mathematics and Informatics

Variables in mathematics and informatics Different kinds of variables used in mathematics and informatics can be distinguished. Didactics of mathematics has already deeply analysed and documented the obstacles met when introducing variables in elementary algebra, and the different status that can have a letter in mathematics (parameter, generic element, unknown, ...).

In informatics, a variable stands for a place in the memory, and its content can change. The operation of giving a new value to a variable is called *assignment*, and can be represented with symbols such that "`=`", "`:=`" or "$\leftarrow$". This operation of assignment (not symmetric) must not get mixed up with the equality "=" in mathematics (symmetric). Even so, in the way they are used in informatics, variables can also have different status.

Elementary algebra and algorithmics or programming are often introduced simultaneously in curricula (grade 7-10 in the new French curricula for instance), and the associated notions of variables are developed in parallel and sometimes even used as a way to give meaning one to the other. There are didactical issues at this level to build the different notions of variables and the different uses of letters in mathematics, and the place that informatics is taking in school has to be taken into account in the teaching of elementary algebra.

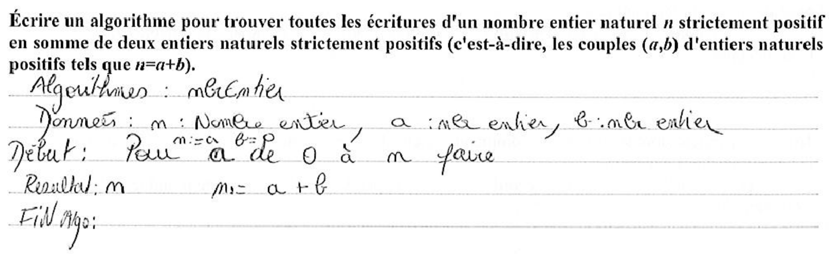
Actually, elementary algebra builds on elementary arithmetic[7] with continuities (for instance, algebraic thinking can be seen as a generalization of arithmetic thinking) but also discontinuities. For instance:

- Solving problems in arithmetic consists of starting from the known and determinate unknown values one by one until getting the solution of the problem whereas, in algebra, one describes the relations between the known and the unknowns, and then solves the equation(s) to get the value of the unknown;
- The meaning of the equal sign changes, as it announces the result of a computation in arithmetic (not symmetric) and gets different other meanings in elementary algebra (a universal equality in the case of the identity $(a+b)^2 = a^2+2ab+b^2$, an equality that can be true in the case of the equation $2x + 4 = 11$, an assignment in the instruction "Evaluate the expression $2(u - 1)(v + 3)$ where $u = 12$ and $v = 7$", ...).

The notion of algorithm includes an idea of generalization too, since its objective is to describe a general method for solving many instances of a same problem. This can be considered as shared with algebra. But, to a certain extend, there is also a continuity with arithmetic as an algorithm describes solving processes going from the known to the unknown step by step, which is not common with elementary algebra. Moreover, in the notion of assignment we can find common points with the equality used in elementary arithmetic.

These different status of symbols in algorithmics, arithmetic and algebra generate language difficulties that can persist at the beginning of the University, as in the example of figure 3.

[7] We denote by elementary arithmetic (or, simply, arithmetic), the use of the four arithmetic operations to solve problems as it is taught at primary school.



Écrire un algorithme pour trouver toutes les écritures d'un nombre entier naturel *n* strictement positif en somme de deux entiers naturels strictement positifs (c'est-à-dire, les couples (*a*,*b*) d'entiers naturels positifs tels que *n*=*a*+*b*).

Algorithmes : nbrEntier

Données : n : Nombre entier, a : nbr entier, b : nbr entier

n:=a b:=0

Début : Pour a de 0 à n faire

Resultat : n   n:= a+b

FIN Algo :

Translation :
Write an algorithm to find all the decompositions of a positive integer n as the sum of two positive integers (that is, the couples (a, b) such that $n = a + b$).
Algorithm: nbrInteger
Data: n:integer, a:integer, b:integer
n:=a b:=0
Begin: For a from 0 to n do
Result: n   *n:=a+b*
End Algo:

Fig. 3. Algorithm written by an undergraduate student of mathematics (first year). *The instruction* n:=a+b*, assignment of the value a+b to the variable n, is incorrect. But, an interpretation in terms of algebra of this same line – the result is n such that n=a+b, or equivalently with b=n-a – permits to possibly understand what tried to do the student. We can suppose that he mixed up variables in mathematics used with the sign = (and algebraic equivalences) and variables in informatics used with the assignment sign :=.*

The study of the way algebra arise and develop in history has been very useful to enlighten what happen in the scholar context of teaching and learning algebra. With the place taken by algorithmics and informatics today in school, it is necessary to take a new look on the development of algebra since the notion of algorithm (that, in its original meaning, comes from the mathematician al-Khwarizmi, whose writings are considered as seminal for the development of early algebra) developed in the same movement, in link with the idea of describing general solving processes (for more details, see [3]). This epistemological work have meaning only if it is done while keeping an eye on the contemporary epistemology of the concept algorithm too and the way it has grown and evolved, since the beginning of the XXth century, and with the development of informatics.

For didactical issues, it is important to go on developing an epistemological framework articulating the trio algebra-arithmetic-algorithmics, both from an historical and contemporary perspective. This is essential to think the ways this trio can interplay in mathematics teaching and learning, in a synergy that could permit to face the language difficulties that algebra and algorithmics raise up.

Logic, language and proof Language aspects, in relation with logics, are also important in proof teaching and learning issues [8]. These proof issues are already involved in algebra and algorithmics questions, but the development of *proof assistants*, as tools for accompanying mathematicians in their tasks changes the way mathematical activity can be done. It questions the epistemology of mathematics under the influence of logics and informatics.

This raises didactical questions about learning proof and the place of formal proof in high school mathematics, as some tools for teaching and learning proof based on proof assistants are now being developed[8]. Then, many issues about the language of formal proofs appear and didactical and epistemological studies must be developed to bring a better understanding of the potentials and limits of such tools.

Of course, these questions are not far from the questions that raise the choices of programming languages in the teaching of mathematics and informatics and the effects of their nature and structure.

2.3 Algorithmic Thinking and Mathematical Thinking

Two thinkings to be articulated These thinkings have much in common but have fundamental differences [13, 19]. Knuth [13] insists on the fact that algorithmic thinking (considered in his paper as the thinking of informatics or computer science) has two important differences from mathematical thinking:

- the notion of assignment (that we already developed in the previous section);
- the notion of complexity, which he considers as absent from mathematical thinking.

[8] For example the project *Edukera*, based on the COQ proof assistant. See http://edukera.com/.

Nevertheless, algorithmic thinking can be present in mathematics, especially with the use of computers in mathematics and the way mathematical thinking and algorithmic thinking interplay in fields at the interface between mathematics and informatics.

Actually, these thinkings may be in contradiction with each other. For instance, when asking what is a (good) answer, with an emphasis on complexity and efficiency in informatics: a mathematical simplification of a formula is not always easier to compute than the original formula [21, 17], and can not be considered as an algorithmic answer. For example, if the cardinal of a set $\#S_n$ depending on a integer n is given by a formula $f(n)$, from the point of view of informatics and complexity it is sometimes more efficient to enumerate the set S_n than to evaluate the formula $f(n)$.

This issue also has to be analysed from an epistemological point of view in order to feed didactical research on teaching and learning mathematics and informatics, especially in order to develop their interactions in school.

An example: binary search algorithm and bisection method In France, in resources for algorithmics in high school mathematics, *binary search algorithm* (BSA) and *bissection method* (BM) (both often called *Recherche dichotomique*) are often proposed together. Binary search is often introduced first, to introduce the divide-and-conquer algorithm for researching a element in a sorted list (it is the best algorithm in terms of complexity). It is then used to construct and justify the binary search method that find a root of a continuous function f on an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different signs.

An epistemological analysis of the relation between BSA and BM reveals the weakness of this introduction of BM. Indeed, even if they are both based on the divide-and-conquer paradigm, BSA and BM are not directly related.

BSA is an algorithm on discrete objects (sorted list of a finite set of elements), whose complexity can be evaluated. More precisely, the problem of researching an element in a sorted list can be described as a couple (Instances, Question):

Problem P_{SESL} (searching an element in a sorted list):

- Instances : Any sorted list S of elements and any element e[9]
- Question : Does e appears in S?[10]

The BSA solves this problem and it can be proven to be the best algorithm in terms of worst-case complexity[11].

On the contrary, the BM can not be considered as the best algorithm for finding the roots in the given problem. Indeed, as it is a numerical method, it

[9] We suppose that the elements are of a given type and do not enter in more technical details.

[10] It is supposed here that any elements of the list can be compared to e.

[11] Here, the worst-case complexity of an algorithm for searching an element in a sorted list is given by the function c that associate to an positive integer n be the maximum number $c(n)$ of comparisons that will make the algorithm for searching any element e in a n-elements list.

does not deal with a discrete situation, as we are dealing with real functions. The notion of complexity does not fit with the situation, and generally numerical methods are compared in terms of convergence speed. More over, even if we give a precision for the root approximation, the problem does not translate into the problem P_{SESL} above (as the function f is not increasing on $[a, b]$).

Actually, the origin and the meaning of BM comes from the proof *by dichotomy* of the intermediate value theorem, which builds a sequence that converges to a root of the function.

This short example shows how algorithmic thinking has to be taken into account in the didactical analysis of mathematical situations that involves informatics, and how epistemological vigilance can enlighten a didactical question. It seems clear that it is necessary to develop a joint epistemological reflection on mathematics and informatics aspects of such situations, in a didactical perspective.

2.4 Experimental Mathematics, Role of the Computer and New Objects

Experimental mathematics and role of the computer Informatics allowed to develop or renew experimental aspects of mathematics [2, 22]. From an epistemological point of view, to a certain extend, the nature of mathematics can be considered as unchanged with computer-assisted mathematics, but works like [5, 6] show that there are changes, due to the use of computer and influence of informatics, that must be considered in the practice of the mathematics. Hamming [10] illustrate well this point:

> It is like the statement that, regarded solely as a form of transportation, modern automobiles and aeroplanes are no different than walking. [...] A jet plane is around two orders of magnitude faster than unaided human transportation, while modern computers are around six orders of magnitude faster than hand computation. [10, p. 1]

In a didactical perspective, it is clear that these practical changes in the mathematical activity have to be taken into account [15, 16] and it is important to study the way they are reflected and transposed into the teaching of mathematics – the way they change the "economy of the system" in the meaning of Artigue [1].

New objects The development of informatics brought new objects in mathematics, mainly of discrete type. They were sometimes present but not considered seriously before informatics [13]. Introducing computer in the classroom inevitably leads to mathematical questions about such objects (explicitly presented or not) like representations of numbers in a machine when programming or discrete lines when using a dynamic geometry software or plotting curves. This directly questions the consistency of the mathematics curricula and the necessity of questioning those curricula regarding new fields arising in mathematics.

As an example, in the French curricula of mathematics for high school has been introduced recently algorithmics[12]. The national curricula states that "algorithmics has a natural place in all the branches of mathematics and the problems posed must be in relation with the other parts of the [mathematical] curricula [...] but also with other disciplines and every day life.".

An in-depth analyse of the place and role stood by algorithms in mathematics permits to see that algorithms have not the same role and the same importance in every branches of mathematics. Indeed, two important mathematical domains for developing algorithmics – arithmetic and discrete mathematics (graph theory, combinatorics, ...) – are absent from these curricula. As a consequence, in French textbooks and resources for high school in mathematics, one can observe that their are very few propositions of algorithmic activities in some branches, and they are sometimes very poor from an algorithmic point of view. Actually, most of the algorithms found in the French high school mathematics are numerical methods in the chapters of mathematical analysis and simulations for statistic and probabilities activities.

This absence of mathematical objects that could be source of rich algorithmic problems can explain the difficulties noticed by mathematics teachers to keep algorithmics alive in French high school.

Conclusion and Perspectives

In conclusion, we have defended the importance of considering the epistemology of informatics in the didactics of mathematics. It seems important to take into account the way concepts in informatics and mathematics arose, the links informatics had and have with mathematics and also the specificities that distinguished it from mathematics, the role of logics and language and the place of proof. Through our examples we can distinguish two big lines that must be studied:

1. the relations between proof, language, algorithm, programming and logic in mathematics and informatics, and
2. the new fields and questions appearing at the mathematics-informatics interface, the discrete mathematics and the representation of objects in mathematics and informatics.

The examples presented in this article open perspectives in this direction. They show that specific concepts at the informatics-mathematics interface must be analysed, and point out the need for general frameworks in order to analyse mathematics-informatics interactions. In this goal, language issues must be emphasized.

To conclude, we underline the need for a cooperation between research in history, epistemology and didactics of informatics and mathematics to tackle these issues.

[12] Since 2009.

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