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Abstract

An implicit discrete-time version of the twisting sliding-mode control algorithm is considered. The framework of variational inequalities is used to define the control input values. This provides the foundation for both the analysis of the controller and the numerical computations. The controller is shown to be well-defined and the discrete-time closed-loop system's fixed point is finite-time globally stable in the sense of Lyapunov.

Index Terms

sliding mode control, sampled-data system, discrete-time Lyapunov stability, implicit discretization, finite-time stability

I. Introduction

Sliding-mode control (SMC) is widely appreciated for its robustness and ease of implementation. One of its drawbacks is the so-called chattering phenomenon, which consists of unwanted, high-frequency oscillations in the output (the sliding variable) and in the input (whose shape is a high-frequency bang-bang-like signal). Especially it has been recognized that the time-dicretization of such set-valued controllers can have crucial consequences on the chattering, that is in this case a *numerical chattering*. Roughly speaking, *explicit* discretizations of the set-valued controller yield numerical chattering, as shown in [1]–[7] in various cases, even in the absence of perturbations.

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O. Huber is with the Wisconsin Institute for Discovery, UW-Madison, USA. V. Acary and B. Brogliato are with Univ. Grenoble-Alpes, INRIA Grenoble Rhône-Alpes, 655 avenue de l'Europe, Inovallée, 38334 Saint-Ismier, France. vincent.acary@inria.fr, bernard.brogliato@inria.fr

The regularization of the set-valued controller (usually a signum multifunction) at zero, is often considered as a universal remedy to chattering. However it has severe drawbacks: it introduces additional control parameters (the regularisation slopes), whose tuning is not so clear even in the case of a single attractive surface [8], and in the case of co-dimension ≥ 2 sliding surface regularizations become even harder to analyse [9]–[12]. Moreover the precision is decreased and one has to choose a regularization with a steep slope, and a very small sampling period. In discrete-time, the choice of the sampling period as a function of this slope is not clear, even in the simplest cases [8]. Recently, an alternative to the explicit discretization method has been introduced in [13], [14]. It is based on an *implicit* discretization of the set-valued controller, and takes the form in the simplest cases of a simple projection over the interval $[-1, 1]$. It has been successfully implemented and tested for first order SMC in [8], [15]–[17], see also [18], where it is demonstrated that the chattering at both the input and output signals, is drastically decreased compared with the explicit method. It has also been shown to apply to nonlinear systems with fixed finite-time convergence [19], to Lagrangian systems with parametric uncertainties and exogenous disturbances [20], and to linear time-invariant systems with both exogenous matched disturbances and parametric uncertainties [21]. In particular it has been proved [22], [23] that the explicit Euler method may yield instability, while the implicit one enjoys stability. The twisting and super-twisting algorithms were tackled in [14], [15] with preliminary analytical and experimental results. The main properties of the implicit approach (which are impossible to obtain with the explicit one) are: (i) a rigorous definition of the sliding-surface in discrete-time, (ii) global Lyapunov stability, (iii) finite-time convergence to the sliding-surface, (iv) convergence of the input to its continuous-time counterpart, (v) insensitivity of the control input with respect to the gains in the sliding-mode phase (in accordance with the continuous-time analysis using for instance Filippov's framework), (vi) robustness with respect to a large class of disturbances, and (vii) the possibility to choose large sampling periods without deteriorating too much the performance. Properties (v) (vi) (vii) have been validated experimentally in [8], [15]–[17]. In parallel with discrete-time SMC analysis, a great deal of work has been dedicated in the past twenty years, to the analysis of the higher-order SMC (HOSMC) introduced in [24], [25]. In particular, finding Lyapunov functions for the order-two twisting and super-twisting algorithms, has been a challenge tackled and solved in [26]–[28]. HOSMC is known to enable reduction of chattering (one reason being that algorihms like the super-twisting generate a continuous input signal [29]), at the price however of tolerating only a smaller class of disturbances than the classical, first order SMC. But it is noteworthy that when discretized with an explicit method, HOSMC may also suffer from severe chattering effects, see [7] and experiments on the twisting algorithm in [15]. The goal of this article is to analyse in detail the implicit discretisation of the twisting algorithm (in the disturbance-free case) and to show that, under a modification of the basic algorithm used in [15, Equation (5)], Lyapunov global stability and finite-time convergence results can be obtained, hence extending the results in [26] to the discrete-time case. As demonstrated experimentally in [15], the unmodified version of the implicit twisting scheme results in a much better behaviour (with important chattering reduction) than its explicitly discretized version. However as shown in [30], the unmodified scheme yields always, in theory, oscillations around the origin. The objective of this article is to prove that stronger stability properties can be obtained when the discrete-time scheme is modified in a suitable way, where the goal may also be to improve the precision of the closed-loop system.

This article is organized as follows: the discrete-time controller is introduced in Section II. In Section III, the uniqueness of solution to some affine variational inequalities is investigated. The well-posedness (existence and uniqueness of the controller) and stability analysis of the closed-loop system are established in Section IV. Conclusions are given in section V, and some mathematical details are provided in the Appendix.

Notations: $\mathcal{N}_K$ is the normal cone operator for the closed non empty convex set K, that is for any $x \in K$, $\mathcal{N}_K(x) = \{d \in \mathbb{R}^p : \langle d, y - x\rangle \leq 0, \forall y \in K\}$. The indicator function of K is $\delta_K(x) = 0$ if $x \in K$, $\delta_K(x) = +\infty$ if $x \notin K$. The support function δ^*_K is defined as $\delta^*_K(x) = \sup_{y\in K}\langle y, x\rangle$, and is the conjugate of the indicator function. The subdifferential of a convex, proper, lower semicontinuous function f is denoted by ∂f and is defined as $\partial f(x) \coloneqq \{g : f(z) \geq f(x) + \langle g, z - x\rangle \quad \forall z\}$. When K is closed convex non empty, one has $\partial\delta_K = \mathcal{N}_K$ for all $x \in K$. The set-valued signum function is $\mathrm{Sgn}(x) = -1$ if $x < 0$, $\mathrm{Sgn}(x) = 1$ if $x > 0$, $\mathrm{Sgn}(0) = [-1, 1]$. The unit ball for the maximum norm is $\mathcal{B}_\infty = \{x \in \mathbb{R}^n : ||x||_\infty := \max_{1\leq i\leq n} |x_i| \leq 1\}$.

II. DISCRETE-TIME TWISTING CONTROLLER

Let us quickly recall the basics of the twisting algorithm. In the general setting, the sliding variable is supposed to be twice differentiable and has dynamics given by

$$\ddot{\sigma} = a(x,t) + g_s(x,t)u$$
$$-u \in a\,\mathrm{Sgn}(\sigma) + b\,\mathrm{Sgn}(\dot{\sigma}),$$

with $a > b > 0$. To simplify the analysis let us consider the case of a double integrator, that is

$$\ddot{\sigma} \in -a\,\mathrm{Sgn}(\sigma) - b\,\mathrm{Sgn}(\dot{\sigma}).$$

Recasting this as a first order system, we get

$$\dot{\Sigma} = A\Sigma + B\lambda \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \tag{1}$$
$$\text{and} \quad \Sigma \coloneqq \begin{pmatrix} \sigma \\ \dot{\sigma} \end{pmatrix}, -\lambda \in \mathrm{Sgn}\begin{pmatrix} \sigma \\ \dot{\sigma} \end{pmatrix} = \mathrm{Sgn}\,\Sigma.$$

The condition $a > b > 0$ ensures the finite-time global Lyapunov stability of the closed-loop fixed point, see [26]. Let us discretize the dynamics using the ZOH method. The discontinuous control input is implemented using the following discretization:

$$\Sigma_{k+1} = A^*\Sigma_k + B^*\lambda_{k+1} \quad \text{with} \quad \lambda_{k+1} = \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \tag{2}$$
$$\text{and} \quad A^* = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, B^* = h \begin{pmatrix} \frac{h}{2} \\ 1 \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}. \tag{3}$$

The control input value (to be calculated at time t_k and applied on $[t_k, t_{k+1})$) is given by $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$. The discrete-time dynamics is thus given by

$$\begin{cases} \sigma_{k+1} = \sigma_k + h\dot{\sigma}_k + \frac{h^2}{2}(a\lambda_{1,k+1} + b\lambda_{2,k+1}) \\ \dot{\sigma}_{k+1} = \dot{\sigma}_k + h(a\lambda_{1,k+1} + b\lambda_{2,k+1}). \end{cases} \tag{4}$$

The precise definition of the relationship between λ_{k+1} and Σ_{k+1} is formalized later. Now let us compute the set of points reaching the origin in one step.

Lemma 1. *Suppose that the control law is such that u_k can be freely chosen in $[-u^M, u^M]$ whenever $\Sigma_{k+1} = 0$. Then the origin of the closed-loop system* (4) *is only reachable from the line segment S_0 defined as*

$$S_0 := \left\{(\sigma_k, \dot{\sigma}_k) \in \mathbb{R}^2 \colon \sigma_k + \tfrac{h}{2}\dot{\sigma}_k = 0, |\dot{\sigma}_k| \leq hu^M\right\}.$$

Proof. Let us first study the set of points Σ_k such that $\Sigma_{k+1} = (0, 0)$. Using the recurrence relations in (4), we get the system

$$\begin{cases} \sigma_k + h\dot{\sigma}_k + \frac{h^2}{2}u_k = 0 & \text{(5a)} \\ \dot{\sigma}_k + hu_k = 0. & \text{(5b)} \end{cases}$$

Inserting (5b) in (5a) yields $\sigma_k = \frac{h^2}{2}u_k$. Combining with (5b) we get $\sigma_k + \frac{h}{2}\dot{\sigma}_k = 0$. Hence, the origin can only be reached from this hyperplane. Since $|u_k| \leq u^M$, we get the constraint $|\dot{\sigma}_k| \leq hu^M$. □

Therefore, a necessary condition for the global asymptotic stability of the closed-loop system is that S_0 is an attractive surface. If we were to perform a straightforward implicit discretization of the inclusion $-\lambda \in \operatorname{Sgn}\Sigma$ as done in [15], we would get

$$-\lambda_{k+1} \in \operatorname{Sgn}(\Sigma_{k+1}) \quad \Longleftrightarrow \quad -\Sigma_{k+1} \in \mathcal{N}_H(\lambda_{k+1}), \tag{6}$$

with $H := [-1, 1]^2$. It has been shown in [30] that S_0 is not an attractive surface with a controller defined from the implicit discretization in (6), and that the set of points that can reach the origin is a set of measure zero. Whence, let us state a more general type of control law than (6) by setting

$$-\Sigma_{k+1} \in \mathcal{N}_K(\lambda_{k+1}),$$

with K a polyhedral compact set. Then, we obtain from that inclusion and (2) the generalized equation (GE):

$$0 \in A\Sigma_k + B^*\lambda_{k+1} + \mathcal{N}_K(\lambda_{k+1}). \tag{7}$$

The GE in (7) is equivalent to the AVI (see Definition 2): Find $\lambda \in K$ such that

$$\langle A\Sigma_k + B^*\lambda_{k+1}, y - \lambda_{k+1}\rangle \geq 0 \text{ for all } y \in K.$$

whenever λ_{k+1} a solution to (7). The sliding variables and control input are then given as follows:

$$\Sigma_{k+1} = A\Sigma_k + B^*\lambda_{k+1}$$

$$u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1},$$

whenever λ_{k+1} a solution to (7). Let us now study the existence and uniqueness of a solution to the Affine Variational Inequality (AVI) in (7).

Lemma 2. *Whenever K is compact, the AVI* (7) *always has a solution.*

Proof. Since the mapping $L\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $L(z) = B^*z + A\Sigma_k$ is continuous and K is compact, we can apply Corollary 2.2.5, p. 148 in [31]. To make the presentation more self contained, let us prove this fact directly. Consider the mapping $\Lambda \mapsto \Pi_K(\Lambda - L(\Lambda))$, with Π_K the projector onto K. Since K is convex, Π_K is singled-valued, from K to itself, and also continuous. Then applying Brouwer fixed-point theorem yields that there exists Λ^0 in K such that $\Lambda^0 \in \Pi_K(\Lambda^0 - L(\Lambda^0))$. From [32, Prop. 6.17], $\Pi_K = (I + \mathcal{N}_K)^{-1}$. Therefore, it holds that $\Lambda^0 - L(\Lambda^0) \in (\Lambda^0 + \mathcal{N}_K\Lambda^0)$. Taking the inverse of the set-valued mapping yields $\Lambda^0 = (I + \mathcal{N}_K)^{-1}(\Lambda^0 - L(\Lambda^0))$, and the proof is complete. □

Remark 1. The controller calculated from (7) is non-anticipative, since the solution of (7) depends on Σ_k and h only.

To define the control law, the basic idea is to consider the variable λ_{k+1} to be defined as

$$-\lambda_{k+1} \in \partial\delta^*_{-K}(\Sigma_{k+1}), \tag{8}$$

with K a bounded polytopic convex set. This can be seen as a generalization of the inclusion $-\lambda_{k+1} \in \mathrm{Sgn}(\Sigma_{k+1})$, which can be rewritten equivalently as $-\lambda_{k+1} \in \partial\delta^*_{[-1,1]^2}(\Sigma_{k+1})$ given that $\mathrm{Sgn}(\cdot) = \partial\delta^*_{[-1,1]^2}(\cdot)$ (see (6) and Fact 2 in the Appendix). Lemma 1 provides us with an interesting insight on how to define the control input such that the origin can be reached from a set of initial conditions with measure greater than zero in $\mathbb{R}^2$. Using this approach based on AVI is interesting since we want to be able to design a control law that steers a set with positive measure to the origin. To achieve this, we choose a set K, in which λ_{k+1} takes its values, that is not the box $[-1,1]^2$. Rather we impose that a half-line containing a part of $-S_0$ belongs to the normal cone to K. Let us define K as a convex polytope:

$$K := \{x \in \mathbb{R}^2 \mid Ex \leq b\} \text{ with } E \in \mathbb{R}^{4\times 2} \text{ and } b \in \mathbb{R}^4. \tag{9}$$

From Fact 4 in the Appendix, we know that the normal cone is generated by the rows of E. Hence, a simple way to have a half-line containing a part of $-S_0$ as at least one line of the normal cones, is to include it as a constraint, that is at least one row of E has to be proportional to $(h/2, -1)$. To be more concrete, the square $[-1, 1]^2$ admits the representation $\{x \in \mathbb{R}^2 \mid Hx \leq b\}$ with $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. We propose to use instead the matrix

$$E = \begin{pmatrix} 1 & 0 \\ -h/2 & 1 \\ -1 & 0 \\ h/2 & -1 \end{pmatrix}$$

in (9), where $E = H$ when $h = 0$. The choice of the vector b depends on the constraints we want to impose on the control inputs. Let us discuss three possible choices:

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} 1 \\ 1 - h/2 \\ 1 \\ 1 - h/2 \end{pmatrix} \quad b_3 = \begin{pmatrix} 1 \\ 1 + h/2 \\ 1 \\ 1 + h/2 \end{pmatrix}. \tag{10}$$

which satisfy $b_1 = b_2 = b_3 = b$ when $h = 0$. With b_1, we obtain a parallelogram, which is not contained in the square $[-1, 1]^2$. If the original control constraints were important to respect, then by using the vector b_2, the resulting set is a parallelogram contained in the unit square. Finally, another choice could be b_3, which gives a set containing the original square. Note that all those sets are symmetric with respect to the origin.

The study of uniqueness of the control input is not as easy as with the classical SMC. This property would hold whenever the AVI (7) has a unique solution for all Σ_k. However, this does not hold since B^* is not a $\mathbf{P}$-matrix and not even positive-semidefinite. To see this, let us compute the symmetric part of B^*:

$$2B_s^* = B^* + B^{*T} = h \begin{pmatrix} ha & a + \frac{h}{2}b \\ a + \frac{h}{2}b & 2b \end{pmatrix},$$

whose determinant is equal to $-h\left(a-\frac{h}{2}b\right)^2<0$. Thus, the matrix B^* is indefinite. We could try to reformulate the AVI (7) into an LCP and see if the w-uniqueness (see Theorem 1 in the Appendix) property holds, but this does not work neither. In the following, we derive F-uniqueness results for some classes of AVIs that are an extension of those for LCPs, in the sense that the condition on the matrix is the same or close to the ones listed in Theorem 1.

III. F-UNIQUENESS OF AVIS

Since the control law involves the solution of an AVI, let us recall the condition on the matrix M of the $\mathrm{LCP}(q,M)$ for the w-uniqueness to hold:

Definition 1. [33, p. 155] Given a matrix $M\in\mathbb{R}^{n\times n}$, we say that any vector whose sign is reversed by M belongs to the nullspace of M if the following inequality holds:

$$[z_i(Mz)_i\leq 0 \text{ for all } i\in\{1,\dots,n\}]\quad\Longrightarrow\quad[Mz=0].$$

The results that we present here depend on the set K and the matrix M associated with the $\mathrm{AVI}(K,q,M)$, but not on the constant vector q. Let us start by considering the case where the set is $\mathcal{B}_\infty$, the unit ball for the maximum norm, before moving on to more generic sets.

Proposition 1. *Consider an* $\mathrm{AVI}(\mathcal{B}_\infty,q,M)$ *in* $\mathbb{R}^n$ *with* $\mathcal{B}_\infty$ *the unit ball for the maximum norm. If every vector whose sign is reversed by* M *belongs to the nullspace of* M*, then the AVI enjoys the F-uniqueness property.*

Proof. We rewrite the $\mathrm{AVI}(\mathcal{B}_\infty,q,M)$ as the generalized equation $0\in Mz+q+\mathcal{N}_{\mathcal{B}_\infty}(z)$. Let $w:=Mz+q$ and recall from Facts 2 and 3 that $-w\in\mathcal{N}_{\mathcal{B}_\infty}(z)\iff -z\in\mathrm{Sgn}(w)$. Note that $\mathrm{Sgn}\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is a maximal monotone operator with the remarkable property that it is component-wise maximal monotone, since for all $i\in\{1,\dots,n\}$ it holds $\mathrm{Sgn}_i(w)=\mathrm{Sgn}(w_i)$. This observation is pivotal to the proof. Now suppose that the AVI has multiple solutions and let z^1,z^2 be any two of them with $\Delta z:=z^1-z^2$. Let $\Delta w:=w^1-w^2=M\Delta z$. Using the definition of the maximal monotonicity gives:

$$\langle z_i^1-z_i^2,w_i^1-w_i^2\rangle\leq 0\qquad\text{for all } i\in\{1,\dots,n\},$$

or more compactly written: $(\Delta z)_i (M\Delta z)_i \leq 0$. This implies that the sign of Δz is reversed by M. Hence, by the hypothesis on M, Δz belongs to its nullspace and therefore $\Delta w = 0$, completing the proof. $\square$

Let us now extend this to a bigger class of AVIs by considering convex compact polytopes other than $\mathcal{B}_\infty$. To do so, we need to recast the equivalence like (6) into a more general framework. Fact 2 states this relation in the more general setting of a relation between the subdifferentials of an indicator function $\delta_K(\,\cdot\,)$ and a support function $\delta_K^*(\,\cdot\,)$. Therefore, we shall substitute the equivalence (6) by:

$$-w \in \mathcal{N}_{\mathcal{B}_\infty}(z) = \partial\delta_{\mathcal{B}_\infty}(z) \quad \Longleftrightarrow \quad z \in \partial\delta^*_{\mathcal{B}_\infty}(-w).$$

The next proposition deals with AVIs where the set is polyhedral but not box-shaped.

Proposition 2 (F-uniqueness of an AVI)**.** *Consider an* $\mathrm{AVI}(K, q, M)$ *in* $\mathbb{R}^n$ *with* K *a non-empty convex polytope. If there exists a nondegenerate linear transformation, with a matrix representation* $L \in \mathbb{R}^{n\times n}$*, from* $\mathcal{B}_\infty$ *to* K*, and if every vector whose sign is reversed by* $\widetilde{M} := L^T M L$ *belongs to the nullspace of* $\widetilde{M}$*, then the AVI enjoys the F-uniqueness property.*

Proof. We try to show that the statement holds by transforming the $\mathrm{AVI}(K, q, M)$ into an $\mathrm{AVI}(\mathcal{B}_\infty, \widetilde{q}, \widetilde{M})$ and applying the previous result. The existence of the linear transformation $L \in \mathbb{R}^{n\times n}$ enables us to write that for every $z \in K$, there exists a unique $y \in \mathcal{B}_\infty$ such that $z = Ly$. The core part in the transformation between the two AVIs is how to relate the normal cones of the sets K and $\mathcal{B}_\infty$. Let us start by stating the relation between the indicator functions of the two sets: for all pairs z and y such that $z = Ly$, one has $\delta_K(z) = \delta_{\mathcal{B}_\infty}(y) = \delta_{\mathcal{B}_\infty}(L^{-1}z)$, since L is nonsingular. Using the chain rule for convex functions [34, Theorem 23.9], we get

$$\partial\delta_K(z) = L^{-T}\partial\delta_{\mathcal{B}_\infty}(L^{-1}z)$$

so that:

$$L^T\partial\delta_K(z) = \partial\delta_{\mathcal{B}_\infty}(L^{-1}z).$$

Rewriting this equality using normal cones, we have

$$L^T\mathcal{N}_K(z) = \mathcal{N}_{\mathcal{B}_\infty}(L^{-1}z). \tag{11}$$

Let us now transform the $\mathrm{AVI}(K, q, M)$

$$0 \in Mz + q + \mathcal{N}_K(z), \tag{12}$$

by noticing that z is solution of this AVI if and only if it is a solution of

$$0 \in L^T Mz + L^T q + L^T \mathcal{N}_K(z). \tag{13}$$

Using (11) and moving to the y variable, we get

$$0 \in L^T MLy + L^T q + \mathcal{N}_{\mathcal{B}_\infty}(y).$$

Letting $\widetilde{M} \coloneqq L^T ML$ and $\widetilde{q} \coloneqq L^T q$, we obtain

$$0 \in \widetilde{M}y + \widetilde{q} + \mathcal{N}_{\mathcal{B}_\infty}(y).$$

We can apply Proposition 1 on this AVI given the hypothesis on $\widetilde{M}$. Hence, we get that for any solution y of this AVI, $\widetilde{M}y + \widetilde{q}$ is unique, and if we have two distinct solutions y^1 and y^2, the difference $\Delta y \coloneqq y^1 - y^2$ is in $\ker \widetilde{M}$. This amounts to the F-uniqueness of the AVI (13) and the fact that if there exists two distinct solutions z^1 and z^2, the difference $\Delta z \coloneqq z^1 - z^2$ is in $\ker L^T M = \ker M$ since L is nonsingular. Therefore, $Mz + q$ is unique for all solutions z of the AVI (12) and the proof is complete.

□

Remark 2. It does not seem easy to characterize the matrices L and M such that every vector whose sign is reversed by $L^T ML$ belongs to the nullspace of $L^T ML$. Let us consider the case when M enjoys this property and is a rank-one matrix, that is $M = vu^T$ for some $u, v \in \mathbb{R}^n$. Now the matrix $L^T ML = L^T vu^T L = (Lv)(Lu)^T = \tilde{v}\tilde{u}^T$ is another rank-one matrix. By construction, the columns of $L^T ML$ are linearly dependent, therefore this matrix is $\mathbf{P}_0$ if for all i, $\tilde{v}_i \tilde{u}_i \geq 0$. The case when a diagonal element of $L^T ML$ is 0 requires special attention: the condition on the determinant implies that the column must be the zero vector. This prevents any element of $\tilde{u}$ from being zero.

Remark 3. Reasoning along the same lines, if we impose that M (or $L^T ML$) is a $\mathbf{P}$-matrix, then we get the uniqueness of the solution to the $\mathrm{AVI}(K, q, M)$. This result is already available in the first case, when the set K is box-shaped. To the best of our knowledge, the second proposition, when K is a polytope, has not been studied before.

Remark 4. Let us recall a result given for $\mathrm{VI}(K, F)$ regarding the F-uniqueness: the function $F(\cdot)$ has to be pseudo-monotone on K, that is for all x and y in K,

$$(x-y)^T F(y) \geq 0 \quad \Rightarrow \quad (x-y)^T F(x) \geq 0. \tag{14}$$

This condition is not necessarily fulfilled with an indefinite matrix that ensures the F-uniqueness of the AVI. Let us illustrate this fact with the AVI($[-1,1]^2, q, R$) and with the following rank-one matrix

$$R = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \text{ whose symmetric part is } R_s = \begin{pmatrix} 2 & 7/2 \\ 7/2 & 6 \end{pmatrix}.$$

The matrix R is indefinite since $\operatorname{tr} R_s = 8$ and $\det R_s = -1/4$. The condition (14) fails with $y = 0$ and x the eigenvector associated with the negative eigenvalue. To the best of our knowledge, the most general result for F-uniqueness of $\mathrm{AVI}(K, q, M)$, requires the matrix R to be PSD-plus (*i.e.*, R is positive-semidefinite and $x^T R x = 0$ implies $Rx = 0$ [31, p.151]). This class of matrices is further characterized in [35]. Hence, the positive-semidefiniteness of R is a necessary condition for the results from the literature to apply. Note that the F-uniqueness of the AVI with the matrix R can be established using Proposition 1.

IV. Analysis of the implicit twisting controller

Let us now investigate two properties of the implicit twisting controller: its well-posedness and finite-time Lyapunov stability. In the following, the sliding variables have the dynamics (4) and the discontinuous control variables are defined by the inclusion (8) with the set K defined as in (9) with the choice b_1 in (10). The transformation from $\mathcal{B}_\infty = [-1,1]^2$ to K is given by

$$L = \begin{pmatrix} 1 & 0 \\ \frac{h}{2} & 1 \end{pmatrix}. \tag{15}$$

A. Existence and uniqueness of the control input

Recall that the control input is defined as $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$. The well-posedness of the controller is now investigated.

Lemma 3. *The system composed of the double integrator system and the implicit twisting controller as defined in* (8) *enjoys the uniqueness of* Σ_{k+1} *and* u_k. *Moreover, if* $\Sigma_{k+1} \neq 0$, *then* λ_{k+1} *is also unique.*

Proof. The uniqueness of Σ_{k+1} is a direct application of Proposition 1: as noted in Remark 2, since B^* as in (3) is a rank-one matrix, $L^T B^* L$ is the rank-one matrix given by

$$L^T B^* L = h \begin{pmatrix} h \\ 1 \end{pmatrix} \begin{pmatrix} a + \frac{h}{2}b & b \end{pmatrix}.$$

This matrix is singular with positive diagonal elements. By construction its columns are linearly dependent. Hence, we have by Proposition 2 the uniqueness of Σ_{k+1}. As shown in the proof of this proposition, this implies that the difference between any two solutions lies the kernel of the linear mapping, here B^*. Note that $\ker B^* = \operatorname{span}(b, -a)^T$ and that u_k is defined as $(a, b)\lambda$. It is easy to see that any element of $\ker B^*$ is orthogonal to $(a, b)^T$, which ensures the uniqueness of u_k. For the last part of the statement, the uniqueness of Σ_{k+1} implies that if λ^1_{k+1} and λ^2_{k+1} are two distinct solutions, then their opposites are both in the set $\partial\delta^*_{-K}(\Sigma_{k+1})$ (see (8)). This implies that λ^1_{k+1} and λ^2_{k+1} are such that $\langle -\lambda^1_{k+1}, \Sigma_{k+1}\rangle = \langle -\lambda^2_{k+1}, \Sigma_{k+1}\rangle = \delta^*_{-K}(\Sigma_{k+1})$. Whence, $\Delta\lambda_{k+1} \coloneqq \lambda^1_{k+1} - \lambda^2_{k+1}$ is orthogonal to Σ_{k+1}. But we know that $\Delta\lambda \in \ker M = \operatorname{span}\left(\begin{smallmatrix} -b \\ a \end{smallmatrix}\right)$, which means that $\Sigma_{k+1} = s\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, for some $s \in \mathbb{R}$. We are interested in the case $\Sigma_{k+1} \neq 0$ which means that $s \neq 0$. Given that $a > b > 0$, we get that:

$$\partial\delta^*_{-K}(\Sigma_{k+1}) = \operatorname*{arg\,sup}_{y \in -K} \langle y, \Sigma_{k+1}\rangle = \begin{cases} \left\{ \begin{pmatrix} 1 \\ 1+\frac{h}{2} \end{pmatrix} \right\} & \text{if } s > 0 \\ \left\{ \begin{pmatrix} -1 \\ -1-\frac{h}{2} \end{pmatrix} \right\} & \text{if } s < 0. \end{cases}$$

The two sets in the right-hand side are both singletons: the solution λ_{k+1} of the AVI is therefore unique whenever $\Sigma_{k+1} \neq 0$. $\square$

B. Stability analysis

Let us now turn our attention to the stability analysis of the discrete-time closed-loop system (4) (8) (9) with b_1 in (10). It is clear that the origin is a fixed point and we are now going to prove its global stability (which in passing proves it is the unique fixed point) The conditions on the gain read:

$$a > \left(1 + \frac{h}{2}\right) b > 0, \tag{16}$$

which come from the forthcoming Lyapunov analysis. Before searching for a candidate Lyapunov function, let us provide some relations between the variables used in the twisting controller (4) and (8). First note that we can relate the support functions of the set K and $[-1,1]^2$:

$$\delta^*_{-K}(x) = \sup_{y \in -K} \langle y, x \rangle = \sup_{z \in -[-1,1]^2} \langle Lz, x \rangle$$
$$= \sup_{z \in [-1,1]^2} \langle z, L^T x \rangle = \delta^*_{[-1,1]^2}(L^T x).$$

Using the chain rule for proper lower semi continuous convex functions [34, Theorem 23.9], we get: for all $\Sigma \in \mathbb{R}^2$,

$$\partial \delta^*_{-K}(\Sigma) = L \partial \delta^*_{[-1,1]^2}(L^T \Sigma).$$

Thus, using (15) the relation (8) can be rewritten at step k as:

$$-\lambda_{1,k} \in \operatorname{Sgn}(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k) \tag{17}$$

$$-\lambda_{2,k} \in \operatorname{Sgn}(\dot{\sigma}_k) + \tfrac{h}{2}\operatorname{Sgn}(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k). \tag{18}$$

This gives rise to the following bounds:

$$|\lambda_{1,k}| \leq 1 \qquad |\lambda_{2,k}| \leq 1 + \tfrac{h}{2} \tag{19}$$
$$|u_k| \leq a + b(1 + \tfrac{h}{2}).$$

Now let us provide additional relations between the controller and the sliding variables. The inclusion (17) can be inverted as:

$$\sigma_k + \frac{h}{2}\dot{\sigma}_k \in \mathcal{N}_{[-1,1]}(-\lambda_{1,k}),$$
$$\text{or:} \quad \forall \lambda'_1 \in [-1,1], \quad (\lambda'_1 + \lambda_{1,k})\left(\sigma_k + \frac{h}{2}\dot{\sigma}_k\right) \leq 0. \tag{20}$$

Note that from (17), we have

$$\left|\sigma_k + \frac{h}{2}\dot{\sigma}_k\right| = -\lambda_{1,k}\left(\sigma_k + \frac{h}{2}\dot{\sigma}_k\right).$$

Also, whenever $h < 2$, the relation (18) implies that

$$-\lambda_{2,k}\dot{\sigma}_k > 0 \text{ if } \dot{\sigma}_k \neq 0 \text{ and } -\lambda_{2,k}\dot{\sigma}_k = 0 \text{ if } \dot{\sigma}_k = 0, \tag{21}$$

since for $\dot{\sigma}_k \neq 0$, the first term in the right-hand side of (18) always dominates the second one. Now that we have those relations ready for use, let us propose the Lyapunov function candidate:

$$V_k := V(\sigma_k, \dot{\sigma}_k) = a\left|\sigma_k + \frac{h}{2}\dot{\sigma}_k\right| + \frac{1}{2}\dot{\sigma}_k^2 - \frac{h}{2}b\lambda_{2,k}\dot{\sigma}_k. \tag{22}$$

Introducing the variable λ_k yields

$$\begin{aligned} V_k &= -a\lambda_{1,k}(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k) + \tfrac{1}{2}\dot{\sigma}_k^2 - \tfrac{h}{2}b\lambda_{2,k}\dot{\sigma}_k \\ &= \begin{pmatrix} -a\lambda_{1,k} & \tfrac{1}{2}(\dot{\sigma}_k - ah\lambda_{1,k} - hb\lambda_{2,k}) \end{pmatrix} \Sigma_k. \end{aligned} \tag{23}$$

Starting from (22) and using (21), it is easy to assess that if $h < 2$, then $V(\cdot)_k$ is positive everywhere except at the origin where it vanishes, and that it is also radially unbounded. The remaining part is to prove that this function decreases between two iterates, that is $\Delta V_k \coloneqq V_{k+1} - V_k < 0$ whenever $V_k \neq 0$. Let us first recall the dynamics of the system:

$$\begin{cases} \sigma_{k+1} = \sigma_k + h\dot{\sigma}_k + \frac{h^2}{2}(a\lambda_{1,k+1} + b\lambda_{2,k+1}) & \text{(24a)} \\ \dot{\sigma}_{k+1} = \dot{\sigma}_k + h(a\lambda_{1,k+1} + b\lambda_{2,k+1}). & \text{(24b)} \end{cases}$$

First note that inserting (24b) in (23), we can write

$$V_{k+1} = -a\lambda_{1,k+1}\sigma_{k+1} + \tfrac{1}{2}\dot{\sigma}_k\dot{\sigma}_{k+1}.$$

Now we investigate the evolution of V_k:

$$\begin{aligned} \Delta V_k = -a\lambda_{1,k+1}\Big(\sigma_k + h\dot{\sigma}_k + \tfrac{h^2}{2}\overbrace{(a\lambda_{1,k+1} + b\lambda_{2,k+1})}^{\star}\Big) \\ +a\lambda_{1,k}\sigma_k + \tfrac{1}{2}(\dot{\sigma}_k\dot{\sigma}_{k+1} - \dot{\sigma}_k(\underbrace{\dot{\sigma}_k}_{\star} - ah\lambda_{1,k} - hb\lambda_{2,k})), \end{aligned}$$

where we used (23) to get the last term. Using (24b), we substitute the terms tagged with $\star$ to get

$$\begin{aligned} \Delta V_k &= a(\lambda_{1,k} - \lambda_{1,k+1})\sigma_k - \tfrac{ah}{2}\lambda_{1,k+1}(\dot{\sigma}_{k+1} + \dot{\sigma}_k) \\ &\quad + \tfrac{h}{2}(a\lambda_{1,k+1} + b\lambda_{2,k+1} + a\lambda_{1,k} + b\lambda_{2,k})\dot{\sigma}_k \\ &= a(\lambda_{1,k} - \lambda_{1,k+1})(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k) + \tfrac{h}{2}(-a\lambda_{1,k+1}\dot{\sigma}_{k+1} \\ &\quad + (a\lambda_{1,k+1} + b\lambda_{2,k+1} + b\lambda_{2,k})\dot{\sigma}_k). \end{aligned}$$

Let us replace $\dot{\sigma}_{k+1}$ with its expression in (24b), to obtain:

$$\begin{aligned} \Delta V_k = a(\lambda_{1,k} - \lambda_{1,k+1})(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k) \\ + \tfrac{h}{2}(-ah\lambda_{1,k+1}(a\lambda_{1,k+1} + b\lambda_{2,k+1}) + (b\lambda_{2,k+1} + b\lambda_{2,k})\dot{\sigma}_k). \end{aligned}$$

Using again relation (24b) to replace the term $\lambda_{2,k+1}\dot{\sigma}_k$, we get

$$\Delta V_k = a(\lambda_{1,k} - \lambda_{1,k+1})(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k) + \frac{h}{2}\Bigg(-ah\lambda_{1,k+1}(a\lambda_{1,k+1} + b\lambda_{2,k+1}) + b\lambda_{2,k+1}\dot{\sigma}_{k+1} - hb\lambda_{2,k+1}(a\lambda_{1,k+1} + b\lambda_{2,k+1})) + b\lambda_{2,k}\dot{\sigma}_k \Bigg).$$

A final rearrangement in the second term yields

$$\Delta V_k = a(\lambda_{1,k} - \lambda_{1,k+1})(\sigma_k + \tfrac{h}{2}\dot{\sigma}_k) - \tfrac{h^2}{2}(a\lambda_{1,k+1} + b\lambda_{2,k+1})^2 + \tfrac{bh}{2}(\lambda_{2,k+1}\dot{\sigma}_{k+1} + \lambda_{2,k}\dot{\sigma}_k). \tag{25}$$

Let us analyze the last equality term by term: using (20) with the choice $\lambda_1' = -\lambda_{k+1}$, the first term is nonpositive. The second term is clearly nonpositive and the third one too, using the relation (21). Thus $\Delta V_k \leq 0$. Let us show now that the variation ΔV_k is negative as long as the origin is not reached. The second term in (25) is zero only if

$$a\lambda_{1,k+1} + b\lambda_{2,k+1} = 0. \tag{26}$$

Using (19) and the condition (16), it follows that (26) has a solution if and only if $|\lambda_{1,k+1}| < 1$. Thus, from (17) at step $k+1$, (26) holds if and only if

$$\sigma_{k+1} + \frac{h}{2}\dot{\sigma}_{k+1} = 0. \tag{27}$$

Using (26) in the dynamics (24b), we get that in such a case $\dot{\sigma}_{k+1} = \dot{\sigma}_k$. Going back to the analysis of (25), using (21), the last term in the right-hand side is zero if and only if $\dot{\sigma}_{k+1} = \dot{\sigma}_k = 0$, which combined with (27) implies that $\sigma_{k+1} = 0$, and by (17) that $\sigma_k = 0$. Whence, ΔV_k can be zero only when the system has already reached the origin. Otherwise $\Delta V_k < 0$.

Let us now prove the finite-time stability. Remember that the three terms in the right-hand side of (25) are all nonpositive. Hence, we can find an upper bound of ΔV_k by considering only one of those terms. Let us take a closer look at the second one:

$$-\tfrac{h^2}{2}(a\lambda_{1,k+1} + b\lambda_{2,k+1})^2 = -\tfrac{h^2}{2}u_k^2.$$

We have shown that $u_k = 0 \Leftrightarrow |\lambda_{1,k+1}| < 1 \Leftrightarrow \sigma_{k+1} + \frac{h}{2}\dot{\sigma}_{k+1} = 0$. Thus $u_k \neq 0 \Rightarrow |\lambda_{1,k+1}| = 1$ and $\sigma_{k+1} + \frac{h}{2}\dot{\sigma}_{k+1} \neq 0$ (using (27)). Now we have $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$ so that $u_k =$

$-\left(a+b\frac{h}{2}\right)\mathrm{Sgn}(\sigma_{k+1}+\frac{h}{2}\dot\sigma_{k+1})-b\mathrm{Sgn}(\dot\sigma_{k+1})$, where we used the fact that $\mathrm{Sgn}(\sigma_{k+1}+\frac{h}{2}\dot\sigma_{k+1})=\pm 1$. Therefore $u_k \geq -\left(a+b\frac{h}{2}\right)\mathrm{Sgn}(\sigma_{k+1}+\frac{h}{2}\dot\sigma_{k+1})-b$, and:

$$
\begin{aligned}
u_k^2 \;&= \left(a+b\tfrac{h}{2}\right)^2+b^2+\left(a+b\tfrac{h}{2}\right)b\mathrm{Sgn}(\dot\sigma_{k+1})\\
&\times\mathrm{Sgn}\left(\sigma_{k+1}+\tfrac{h}{2}\dot\sigma_{k+1}\right)\\
&\geq a^2+b^2\tfrac{h^2}{4}+abh+b^2-ab-b^2\tfrac{h}{2}\\
&= a^2+abh+b^2-ab-b^2\tfrac{h^2}{4}\\
&= \left(a+b\tfrac{h}{2}\right)\left(a-b\tfrac{h}{2}\right)+ab(h-1)+b^2\\
&> \left(a+b\tfrac{h}{2}\right)b+ab(h-1)=b^2\tfrac{h}{2}+abh>0
\end{aligned}
$$

where we used (16) that implies that $a-b\frac{h}{2}>b$. Therefore we infer that

$$\Delta V_k \leq -\frac{h^2}{2}u_k^2 \leq -b\frac{h^3}{2}\left(\frac{b}{2}+a\right)<0. \tag{28}$$

Inequality (28) holds everywhere outside the line $\sigma_k+\frac{h}{2}\dot\sigma_k=0$. From Lemma 1, we know that if the state of the system belongs to S_0, the origin is reached at the next time instant. So if we prove that this segment is reachable in finite-time from any initial conditions, then the origin is globally finite-time reachable. Hence, given that (28) holds everywhere except on the line (27), we just need to bound ΔV_k away from 0 if Σ_{k+1} belongs to the line (27) minus S_0. Thus, suppose that Σ_{k+1} belongs to the line (27) and that $|\dot\sigma_{k+1}|>h(a+b)$. In the third term in the right-hand side of (25), and using (17) (18) we get:

$$
\begin{aligned}
\lambda_{2,k+1}\dot\sigma_{k+1} &= -|\dot\sigma_{k+1}|-\tfrac{h}{2}\lambda_{1,k+1}\dot\sigma_{k+1}\\
&\leq -(1-\tfrac{h}{2})|\dot\sigma_{k+1}| \leq -h(1-\tfrac{h}{2})(a+b).
\end{aligned}
$$

Hence, disregarding all other terms (which are all non positive) but $\lambda_{2,k+1}\dot\sigma_{k+1}$ in the right-hand side of (25), we obtain the upperbound:

$$\Delta V_k \leq -\tfrac{bh^2}{2}(1-\tfrac{h}{2})(a+b), \tag{29}$$

the right-hand side of (29) being negative for all $h<2$. For all $k\geq 0$, ΔV_k is smaller than the maximum of the right-hand side of (28) and (29). Both quantities being negative constants, the finite-time convergence to S_0 holds, hence the finite-time convergence to the origin. We summarize the developments in the following proposition.

Proposition 3. *Let $h<2$. The origin is the unique equilibrium of the discrete-time system* (24) *with the controller given by* (8) *and is globally Lyapunov finite-time stable.*

Remark 5. The discrete-time Lyapunov function V_k is close to its continuous-time counterpart used in [26], which is given by $a|\sigma| + \dot{\sigma}^2/2$.

Corollary 1. *Let* $h < 2$*. The origin is the unique equilibrium of the continuous-time system* (1) *with the piecewise constant controller obtained from* (8)*, and it is globally Lyapunov finite-time stable.*

Proof. The ZOH discretization being exact, we know by the previous proposition that there exists $k_0 \in \mathbb{N}$ such that $\Sigma_{k_0} = 0$. Then for all $k > k_0$, $\Sigma_k = 0$, with the control input $u_k = 0$, as we can easily infer from (24). On each sampling interval $[t_k, t_{k+1}]$, $k > k_0$, the continuous-time system has the dynamics

$$\dot{\Sigma} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Sigma + \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} 0 = 0.$$

This concludes the proof. □

Remark 6. From (4) it follows that once the origin has been attained, then the discrete-time system stays at the origin for all future times. Therefore there is no numerical chattering with the implicit controller, contrarily to its explicit counterpart [7]. This is confirmed experimentally in [15] with the non modified implicit twisting controller calculated from (6). The modified version of the implicit controller is expected to provide event better results.

V. Conclusion

In this article, an implicit discrete-time twisting sliding-mode controller is studied. Fundamental properties are investigated: existence and uniqueness of the controller at each step, and the global finite-time Lyapunov stability of the closed-loop system. Affine variational inequalities are at the core of the analysis and computation algorithms. Future work should tackle the case with a perturbation.

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APPENDIX

Definition 2. [31] Given a set $K \subset \mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$, the variational inequality $VI(K,F)$ is defined as: Find a vector $x \in K$ such that $\langle y - x, F(x)\rangle \geq 0$ for all $y \in K$. When $F(\,\cdot\,)$ is the affine function $F(x) = Mx + q$, this is an affine variational inequality (AVI) denoted as AVI(K, q, M).

Definition 3. [31] [F-uniqueness] A $\mathrm{VI}(K,F)$ is said to be *F-unique* if $F(\mathrm{SOL}(K,F))$ is at most a singleton. Then we say that the *F-uniqueness* of the VI holds.

Definition 4. Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The linear complementarity problem LCP(q, M) with unknown λ is defined as: Find λ such that $\lambda \geq 0$, $M\lambda + q \geq 0$, $\lambda^T(M\lambda + q) = 0$. This is written compactly as: $0 \leq \lambda \perp M\lambda + q \geq 0$.

Definition 5. A matrix $M \in \mathbb{R}^{n\times n}$ is a P-matrix if all its principal minors are positive. It is a P_0-matrix if its principal minors are non negative.

Theorem 1. *[33] Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:*

- *If the $\mathrm{LCP}(q, M)$ is solvable and $\tilde{z}$, $\hat{z}$ are any two solutions, then $M\tilde{z} = M\hat{z}$ (if this is the case, then we say that the w-*uniqueness* of the LCP holds).*
- *Every vector whose sign is reversed by M belongs to the nullspace of M, that is*

$$[z_i(Mz)_i \leq 0 \, \textit{for all } i \in \{1, \dots, n\}] \quad \Rightarrow \quad [Mz = 0].$$

- *M is a $\mathbf{P_0}$-matrix and for each $\mathcal{I} \subseteq \{1, \dots, n\}$ with $\det M_{\mathcal{I}\mathcal{I}} = 0$, the columns of $M_{\bullet\mathcal{I}}$ are linearly dependent.*

The notation $M_{\bullet\mathcal{I}}$ is for the matrix made of all columns of M indexed in the set $\mathcal{I}$. The notation $M_{\mathcal{I}\mathcal{I}}$ is for the principal submatrix of M obtained by deleting rows and columns which are not indexed in the set $\mathcal{I}$. In the same way $M_{i\bullet}$ is the ith row of M.

Fact 2. *The subdifferentials of the indicator and the support function are inverses, that is: $y \in \partial\delta^*_K(x) \quad \Longleftrightarrow \quad x \in \mathcal{N}_K(y)$.*

Fact 3. *Let $x \in \mathbb{R}^n$. We have $\delta^*_{\mathcal{B}_\infty} = |x_1| + |x_2| + \ldots + [x_n| = ||x||_1$, and $\partial\delta^*_{\mathcal{B}_\infty} = (\mathrm{Sgn}(x_1), \mathrm{Sgn}(x_2), \dots, \mathrm{Sgn}(x_n))^T$*

Fact 4. *[36, p. 67] Let K be a closed convex polyhedron defined as:*

$$K = \{x \in \mathbb{R}^n \mid Hx \leq b\}, \quad \textit{with } H \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m.$$

The normal cone at a point $x \in K$ is generated by the outward normals of the actives constraints:

$$\mathcal{N}_K(x) = \left\{H^T_{\alpha\bullet} r, r \geq 0\right\},$$

with $\alpha \in \{1, \dots, m\}$ the set of active constraints, that is for all $i \in \alpha$, we have $H_{i\bullet}x = b_i$.