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***Abstract*—An implicit discrete-time version of the twisting sliding-mode control algorithm is considered. The framework of variational inequalities is used to define the control input values. This provides the foundation for both the analysis of the controller and the numerical computations. The controller is shown to be well-defined and the discrete-time closed-loop system's fixed point is finite-time globally stable in the sense of Lyapunov. The analysis is led in the unperturbed case, and numerical simulations demonstrate the efficiency of the proposed controller when a disturbance acts on the system.**

***Index Terms*—sliding mode control, sampled-data system, discrete-time Lyapunov stability, implicit discretization, finite-time stability, robust control**

I. Introduction

Sliding-mode control (SMC) is widely appreciated for its robustness and ease of implementation. One of its drawbacks is the so-called chattering phenomenon, which consists of unwanted, high-frequency oscillations in the output (the sliding variable) and in the input (whose shape is a high-frequency bang-bang-like signal). Especially it has been recognized that the time-discretization of such set-valued controllers can have crucial consequences on the chattering [1], that is in this case a *numerical chattering*. Roughly speaking, *explicit* discretizations of the set-valued controller yield numerical chattering, even in the absence of perturbations. The regularization of the set-valued controller (usually a signum multifunction) at zero, is often considered as a universal remedy to chattering. However, it has severe drawbacks: it introduces additional control parameters (the regularization slopes), whose tuning is not so clear even in the case of a single attracting surface [2], and in the case of co-dimension ≥ 2 sliding surface, regularizations become even harder to analyze [3], [4]. Moreover, the precision is decreased and one has to choose a regularization with a steep slope, and a very small sampling period. In discrete-time, the choice of the sampling period as a function of this slope is not clear, even in the simplest cases [2]. Recently, an alternative to the explicit discretization method has been introduced, which is based on an *implicit* discretization of the set-valued controller [5], and takes the form in the simplest cases of a simple projection over the interval $[-1, 1]$. It has been successfully implemented and tested for first order SMC in [6], [2], [7], where it is demonstrated that the chattering at both the input and output signals, is drastically decreased compared with the explicit method. In particular it has been proved [8] that the explicit Euler method may yield instability, while the implicit one enjoys global stability. The twisting and super-twisting algorithms were tackled in [5], [6], with preliminary analytical and experimental results. The main properties of the implicit approach (which are impossible to obtain with the explicit one, or with a saturation) are: (i) a rigorous definition of the sliding-surface in discrete-time, (ii) global Lyapunov stability, (iii) finite-time convergence to the sliding-surface, (iv) convergence of the input to its continuous-time counterpart, (v) insensitivity of the control input with respect to the gains in the sliding-mode phase (in accordance with the continuous-time analysis using for instance Filippov's framework), (vi) robustness with respect to a large class of disturbances, and (vii) the possibility to choose large sampling periods without deteriorating too much the performance. Properties (v) (vi) (vii) have been validated experimentally in [6], [2], [7]. In parallel with discrete-time SMC analysis, a great deal of work has been dedicated in the past twenty years, to the analysis of the higher-order SMC (HOSMC), a class of SMC introduced by A. Levant [9], [10], [11], [12]. In particular, finding Lyapunov functions for the order-two twisting and super-twisting algorithms, has been a challenge [13], [14]. HOSMC is known to enable reduction of chattering, at the price however of tolerating only a smaller class of disturbances than the classical, first order SMC. But it is noteworthy that when discretized with an explicit method, HOSMC may also suffer from severe chattering effects, see [1] and experiments on the twisting algorithm in [6]. The goal of this article is to analyze in detail the implicit discretization of the twisting algorithm (in the disturbance-free case) and to show that, under a modification of the basic algorithm used in [6, Equation (5)], Lyapunov global stability and finite-time convergence results can be obtained, hence extending the results in [13] to the discrete-time case. As demonstrated experimentally in [6], the unmodified version of the implicit twisting scheme (called in this article the *regular implicit control*), results in a much better behavior (with important chattering reduction) than its explicitly discretized version. However, as shown in [15], the regular implicit control yields always, in theory, oscillations around the origin. The objective of this article is to prove that stronger stability properties can be obtained when the regular implicit discrete-time scheme is suitably modified, where the goal may also be to improve the precision of the closed-loop system.

This article is organized as follows: the discrete-time controller is introduced in Section II. The well-posedness (ex-

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istence and uniqueness of the controller) is shown in Section III, and the stability analysis of the closed-loop system is established in Section IV. The computation of the control law and simulations are presented in Section V, on a system with perturbations. Conclusions are given in Section VI. Some proofs are in the Appendix.

Notation and definitions: The tools used in this article are classical tools from Convex Analysis (see [16], [17]), and from Complementarity Theory ([18], [19]). Let $K \subseteq \mathbb{R}^p$ be a closed non-empty convex set. Its normal cone $\mathcal{N}_K$ is defined as follows: for any $x \in K$, $\mathcal{N}_K(x) = \{d \in \mathbb{R}^p \colon \langle d, y - x\rangle \leq 0, \forall y \in K\}$. The indicator function of $K \subseteq \mathbb{R}^p$ is defined as: $\delta_K(x) = 0$ if $x \in K$, $\delta_K(x) = +\infty$ if $x \notin K$. The support function σ_K is defined as: $\sigma_K(y) = \sup_{x\in K}\langle y, x\rangle$. It is the conjugate of the indicator function, that is $\sigma_K(y) = \delta_K^*(y) = \sup_{x\in\mathbb{R}^p}[\langle x, y\rangle - \delta_K(x)]$. Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$ be convex, proper (that is $f(x) < +\infty$ for at least one x, and $f(x) > -\infty$ for all x), and lower semi-continuous (lsc) function (that is, its epigraph epi $f := \{(x, \alpha) \in \mathbb{R}^p \times \mathbb{R} \colon f(x) \leq \alpha\}$ is a closed set). Its subdifferential ∂f is defined as: $\partial f(x) := \{g : f(z) \geq f(x) + \langle g, z - x\rangle,\ \forall z\}$. When K is closed convex non-empty, then δ_K is proper, convex and lsc, and one has $\partial\delta_K(x) = \mathcal{N}_K(x)$ for all $x \in K$. Using the definition, it is easy to see that $\mathcal{N}_K(-x) = -\mathcal{N}_{-K}(x)$.

Fact 1. *The subdifferentials of the indicator and the support function are inverses, that is* $y \in \partial\delta_K^*(x) = \partial\sigma_K(x) \Leftrightarrow x \in \mathcal{N}_K(y)$ *for all* $x, y \in \mathbb{R}^p$.

The set-valued signum function is $\mathrm{Sgn}(x) = -1$ if $x < 0$, $\mathrm{Sgn}(x) = 1$ if $x > 0$, $\mathrm{Sgn}(0) = [-1, 1]$. Let $g(x) = |x|$, $x \in \mathbb{R}$, one has $\mathrm{Sgn}(x) = \partial g(x)$ for all $x \in \mathbb{R}$. Then $g(x) = \delta^*_{[-1,1]}(x)$, so that $y \in \mathrm{Sgn}(x) \Leftrightarrow x \in \partial\delta_{[-1,1]}(y) = \mathcal{N}_{[-1,1]}(y) = [0, +\infty)$ if $y = 1$, $(-\infty, 0]$ if $y = -1$, 0 if $y \in [-1, 1]$. Let the unit ball for the maximum norm be defined as $\mathcal{B}_\infty = \{x \in \mathbb{R}^n : ||x||_\infty := \max_{1\leq i\leq n}|x_i| \leq 1\}$.

Fact 2. *Let* $x \in \mathbb{R}^n$. *We have* $\delta^*_{\mathcal{B}_\infty} = |x_1| + |x_2| + \ldots + |x_n| = ||x||_2$, *and* $\partial\delta^*_{\mathcal{B}_\infty} = (\mathrm{Sgn}(x_1), \mathrm{Sgn}(x_2), \ldots, \mathrm{Sgn}(x_n))^T$.

Now let us give an explicit expression for the normal cone in the polyhedral case.

Fact 3. *[17, p. 67] Let K be a closed convex polyhedron defined as:*

$$K = \{x \in \mathbb{R}^n \mid Hx \leq b\}, \quad \text{with } H \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m.$$

The normal cone at a point $x \in K$ *is generated by the outward normals of the actives constraints:* $\mathcal{N}_K(x) = \left\{H^T_{\alpha\bullet} r, r \geq 0\right\}$, *with* $\alpha \in \{1, \ldots, m\}$ *the set of active constraints, that is for all* $i \in \alpha$, *we have* $H_{i\bullet}x = b_i$.

II. Discrete-time twisting controller

Let us quickly recall the basics of the twisting algorithm. We deal in this section with the unperturbed case. The case with a perturbation will be tackled in Section V. In the following, is its shown that a "naive" implicit discretization of the twisting algorithm, called the *regular implicit control* does not allow one to get the asymptotic stability. Thus, a modified algorithm is proposed. In the general setting, the sliding variable is supposed to be twice differentiable and has dynamics given by

$$\begin{aligned}\ddot\sigma &= a(x,t) + g_s(x,t)u\\ -u &\in a\,\mathrm{Sgn}(\sigma) + b\,\mathrm{Sgn}(\dot\sigma),\end{aligned}$$

with $a > b > 0$, and x is the state of the original plant. To simplify the analysis let us consider the case of a double integrator without perturbation, that is

$$\ddot\sigma \in -a\,\mathrm{Sgn}(\sigma) - b\,\mathrm{Sgn}(\dot\sigma).$$

Recasting this as a first order system, we get

$$\dot\Sigma = A\Sigma + B\lambda \quad \text{with} \quad A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, B = \begin{pmatrix}0 & 0\\ a & b\end{pmatrix}$$

$$\text{and} \quad \Sigma := \begin{pmatrix}\sigma\\ \dot\sigma\end{pmatrix}, -\lambda \in \mathrm{Sgn}\begin{pmatrix}\sigma\\ \dot\sigma\end{pmatrix} = \mathrm{Sgn}\,\Sigma.$$

The condition $a > b > 0$ ensures the finite-time global Lyapunov stability of the closed-loop fixed point, see [13]. Let us discretize the dynamics using the ZOH method. The discontinuous control input is implemented using the following discretization:

$$\Sigma_{k+1} = A^*\Sigma_k + B^*\lambda_{k+1} \quad \text{with} \quad \lambda_{k+1} = \begin{pmatrix}\lambda_{1,k+1}\\ \lambda_{2,k+1}\end{pmatrix} \quad (1)$$

$$\text{and} \quad A^* = \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}, B^* = h\begin{pmatrix}\frac{h}{2}\\ 1\end{pmatrix}\begin{pmatrix}a & b\end{pmatrix}. \qquad (2)$$

The piecewise constant control input value, to be calculated at time t_k and applied on $[t_k, t_{k+1})$ with $h := t_{k+1} - t_k > 0$ constant, is given by $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$. The discrete-time dynamics is thus given by

$$\begin{aligned}&(a)\ \ \sigma_{k+1} = \sigma_k + h\dot\sigma_k + \tfrac{h^2}{2}u_k\\ &(b)\ \ \dot\sigma_{k+1} = \dot\sigma_k + hu_k.\end{aligned} \qquad (3)$$

The precise definition of the relationship between λ_{k+1} and Σ_{k+1} is formalized later. First let us state a result which will be used later in Section IV for the stability analysis, and which concerns the set of points reaching the origin in one step.

Lemma 1. *Suppose that the piecewise constant control law is such that* u_k *can be freely chosen in* $[-u^M, u^M]$ *whenever* $\Sigma_{k+1} = 0$. *Then the origin of the closed-loop system* (3) *is only reachable from the line segment* S_0 *defined as*

$$S_0 := \left\{(\sigma_k, \dot\sigma_k) \in \mathbb{R}^2 \colon \sigma_k + \tfrac{h}{2}\dot\sigma_k = 0, |\dot\sigma_k| \leq hu^M\right\}.$$

The proof is in Appendix A. Therefore, a necessary condition for the global asymptotic stability of the closed-loop system is that S_0 is an attracting surface. If we were to perform a straightforward implicit discretization of the inclusion $-\lambda \in \mathrm{Sgn}\,\Sigma$ as done in [6], we would get

$$-\lambda_{k+1} \in \mathrm{Sgn}(\Sigma_{k+1}) \quad \Longleftrightarrow \quad -\Sigma_{k+1} \in \mathcal{N}_H(\lambda_{k+1}), \quad (4)$$

with $H := [-1, 1]^2$. Note that inclusion (4) together with (1) defines the regular implicit control. It has been shown in [15] that S_0 is not an attracting surface with a controller defined from the implicit discretization in (4), and that the set of points that can reach the origin is a set of measure zero. Whence, let us state a more general type of control law than (4), by setting

$$-\Sigma_{k+1} \in \mathcal{N}_K(\lambda_{k+1}), \qquad (5)$$

with K a polyhedral compact set. Then, we obtain from this inclusion and (1) the generalized equation (GE):

$$0 \in A^*\Sigma_k + B^*\lambda_{k+1} + \mathcal{N}_K(\lambda_{k+1}). \quad (6)$$

This type of inclusion is also known as an Affine Variational Inequality (AVI). Let us denote by $\mathrm{SOL}(M, q, K)$ the set of solutions to the AVI $0 \in Mx + q + \mathcal{N}_K(x)$. Therefore, we shall write $\lambda_{k+1} \in \mathrm{SOL}(B^*, A^*\Sigma_k, K)$ whenever λ_{k+1} is a solution to (6). The sliding variables and control input are then given as follows:

$$\begin{aligned} \Sigma_{k+1} &= A\Sigma_k + B^*\lambda_{k+1} \\ u_k &= a\lambda_{1,k+1} + b\lambda_{2,k+1}, \end{aligned} \quad (7)$$

whenever $\lambda_{k+1} \in \mathrm{SOL}(B^*, A^*\Sigma_k, K)$. Let us now study the existence and uniqueness of a solution to the AVI in (6).

III. The Extended Implicit Twisting Controller

A. Definition of the control law

As eluded to above, the basic idea in (5) is to consider the variable λ_{k+1} to be defined as

$$-\Sigma_{k+1} \in \mathcal{N}_K(\lambda_{k+1}) \iff -\lambda_{k+1} \in \partial\delta^*_{-K}(\Sigma_{k+1}), \quad (8)$$

with K a bounded polytopic convex set. This can be seen as a generalization of the inclusion $-\lambda_{k+1} \in \mathrm{Sgn}(\Sigma_{k+1})$, which can be rewritten equivalently as $-\lambda_{k+1} \in \partial\delta^*_{[-1,1]^2}(\Sigma_{k+1})$ given that $\mathrm{Sgn}(\cdot) = \partial\delta^*_{[-1,1]^2}(\cdot)$ (see (4) and Fact 1). With the dynamics (7), the relation (8) is transformed into an AVI of the type (6).

Remark 1. The controller calculated from (6) is non-anticipative, since the solution of (6) depends only on Σ_k and h.

Lemma 1 provides us with an interesting insight on how to define the control input, such that the origin can be reached from a set of initial conditions with measure greater than zero in $\mathbb{R}^2$. Using this approach based on AVI is interesting since we want to be able to design a control law that steers a set with positive measure to the origin. To achieve this, we choose a set K, in which λ_{k+1} takes its values, that is not the box $[-1,1]^2$. We impose that a half-line containing a part of $-S_0$ belongs to the normal cone to K. Let us define K as a convex polytope:

$$K := \{x \in \mathbb{R}^2 \mid Ex \le b\} \text{ with } E \in \mathbb{R}^{4\times 2} \text{ and } b \in \mathbb{R}^4. \quad (9)$$

From Fact 3, we know that the normal cone is generated by the rows of E. Hence, a simple way to have a half-line containing a part of $-S_0$ as at least one line of the normal cones, is to include it as a constraint, that is at least one row of E has to be proportional to $(h/2, -1)$. To be more concrete, the square $[-1,1]^2$ admits the representation $\{x \in \mathbb{R}^2 \mid Hx \le b\}$ with $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. We propose to use instead the matrix

$$E(h) = \begin{pmatrix} 1 & 0 \\ -h/2 & 1 \\ -1 & 0 \\ h/2 & -1 \end{pmatrix}$$

in (9), where $E(0) = H$. The choice of the vector b depends on the constraints we want to impose on the control inputs. Let us discuss three possible choices:

$$b_1 = b, b_2(h) = \begin{pmatrix} 1 \\ 1-h/2 \\ 1 \\ 1-h/2 \end{pmatrix}, b_3(h) = \begin{pmatrix} 1 \\ 1+h/2 \\ 1 \\ 1+h/2 \end{pmatrix}. \quad (10)$$

which satisfy $b_1 = b_2(0) = b_3(0) = b$. With b_1, we obtain a parallelogram, which is not contained in the square $[-1,1]^2$. If the original control constraints were important to respect, then by using the vector b_2, the resulting set is a parallelogram contained in the unit square. Finally, another choice could be b_3, which gives a set containing the original square. Note that all those sets are symmetric with respect to the origin. Solving (6) with K defined by (9), gives us the value of the *extended implicit twisting control* input.

B. Existence and uniqueness of the control input

Let us now investigate the well-posedness of the extended implicit twisting controller. In the following, the sliding variables have the dynamics (3) and the control variables are defined by the inclusion (8), with the set K defined as in (9), and b_1 choosen as (10). The transformation from $\mathcal{B}_\infty = [-1,1]^2$ to K is given by

$$L = \begin{pmatrix} 1 & 0 \\ \frac{h}{2} & 1 \end{pmatrix}. \quad (11)$$

Recall that the control input is defined as $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$, with λ_{k+1} a solution to (6). The well-posedness of the controller is now investigated.

Lemma 2. *The system composed of the double integrator system and the implicit twisting controller as defined in* (8) *enjoys the existence and uniqueness of Σ_{k+1} and u_k. Moreover, if $\Sigma_{k+1} \neq 0$, then λ_{k+1} is also unique.*

The proof is in Appendix A.

Remark 2. In this section and the foregoing one, it has been proved that the controller exists and is unique and non anticipative. A way to *calculate* it on a real implementation, is indicated in Section V-A.

IV. Stability Analysis

Let us turn our attention to the stability analysis of the discrete-time closed-loop system (3) (8) (9) with b_1 in (10). It is clear that the origin is a fixed point, and we are now going to prove its global stability (which in passing proves that it is the unique fixed point).

Theorem 4. *Let $0 < h < 2$ and $a > \left(1 + \frac{h}{2}\right) b > 0$. The origin is the unique equilibrium of the discrete-time system* (3) *with the controller given by* (7) (8) *and is globally Lyapunov finite-time stable.*

Proof. The remaining part of this section is dedicated to the proof of this theorem. Before searching for a candidate Lyapunov function, let us provide some relations between the variables used in the twisting controller (3) and (8). First note that we can relate the support functions of the set K and of $[-1,1]^2$ as follows:

$$\begin{aligned}\delta^*_{-K}(x) &= \sup_{y\in -K}\langle y,x\rangle = \sup_{z\in-[-1,1]^2}\langle Lz,x\rangle\\ &= \sup_{z\in[-1,1]^2}\langle z, L^T x\rangle = \delta^*_{[-1,1]^2}(L^T x).\end{aligned}$$

Using the chain rule for proper lower semi-continuous convex functions [20, Theorem 23.9], we get: for all $\Sigma\in\mathbb{R}^2$,

$$\partial\delta^*_{-K}(\Sigma) = L\,\partial\delta^*_{[-1,1]^2}(L^T\Sigma).$$

Thus, using (11), the relation (8) can be rewritten at step k as:

$$-\lambda_{1,k}\in \operatorname{Sgn}(\sigma_k + \tfrac{h}{2}\dot\sigma_k) \tag{12}$$
$$-\lambda_{2,k}\in \operatorname{Sgn}(\dot\sigma_k) + \{-\tfrac{h}{2}\lambda_{1,k}\}. \tag{13}$$

This gives rise to the following bounds:

$$\begin{aligned}&|\lambda_{1,k}|\le 1 \qquad |\lambda_{2,k}|\le 1+\tfrac{h}{2}\\ &|u_k|\le a + b(1+\tfrac{h}{2}).\end{aligned} \tag{14}$$

Now, let us provide additional relations between the controller and the sliding variables. The inclusion (12) can be inverted as:

$$\begin{aligned}&\sigma_k + \frac{h}{2}\dot\sigma_k \in \mathcal{N}_{[-1,1]}(-\lambda_{1,k}),\\ \iff\ &\forall\lambda_1'\in[-1,1],\ (\lambda_1'+\lambda_{1,k})\left(\sigma_k+\frac{h}{2}\dot\sigma_k\right)\le 0.\end{aligned} \tag{15}$$

Note that from (12), we have

$$\left|\sigma_k + \frac{h}{2}\dot\sigma_k\right| = -\lambda_{1,k}\left(\sigma_k+\frac{h}{2}\dot\sigma_k\right). \tag{16}$$

Also, whenever $h<2$, the relation (13) implies that

$$-\lambda_{2,k}\dot\sigma_k > 0 \text{ if } \dot\sigma_k\ne 0 \text{ and } -\lambda_{2,k}\dot\sigma_k = 0 \text{ if } \dot\sigma_k = 0, \tag{17}$$

since for $\dot\sigma_k\neq 0$, the first term in the right-hand side of (13) always dominates the second one. Now that we have those relations ready for use, let us propose the Lyapunov function candidate:

$$V_k := V(\sigma_k,\dot\sigma_k) = a\left|\sigma_k+\frac{h}{2}\dot\sigma_k\right| + \frac{1}{2}\dot\sigma_k^2 - \frac{h}{2}b\lambda_{2,k}\dot\sigma_k \tag{18}$$

Using (16) yields

$$\begin{aligned}V_k &= -a\lambda_{1,k}(\sigma_k+\tfrac{h}{2}\dot\sigma_k) + \tfrac{1}{2}\dot\sigma_k^2 - \tfrac{h}{2}b\lambda_{2,k}\dot\sigma_k\\ &= \begin{pmatrix}-a\lambda_{1,k} & \tfrac{1}{2}(\dot\sigma_k - ah\lambda_{1,k} - hb\lambda_{2,k})\end{pmatrix}\Sigma_k.\end{aligned} \tag{19}$$

Starting from (18) and using (17), it is easy to assess that if $h<2$, then $V_k(\cdot)$ is positive everywhere except at the origin where it vanishes, and that it is also radially unbounded. The remaining part is to prove that this function decreases between two iterates, that is $\Delta V_k := V_{k+1}-V_k < 0$ whenever $V_k\neq 0$. Recall that the dynamics of the system is in (3). First note that inserting (3)(b) in (19), we can write

$$V_{k+1} = -a\lambda_{1,k+1}\sigma_{k+1} + \tfrac{1}{2}\dot\sigma_k\dot\sigma_{k+1}.$$

Now we investigate the evolution of V_k:

$$\begin{aligned}\Delta V_k = &-a\lambda_{1,k+1}\left(\sigma_k + h\dot\sigma_k + \tfrac{h^2}{2}\overbrace{(a\lambda_{1,k+1}+b\lambda_{2,k+1})}^{\star}\right)\\ &+a\lambda_{1,k}\sigma_k + \tfrac{1}{2}(\dot\sigma_k\dot\sigma_{k+1} - \dot\sigma_k(\underbrace{\dot\sigma_k}_{\star} - ah\lambda_{1,k} - hb\lambda_{2,k})),\end{aligned}$$

where we used (19) to get the last term. Using (3)(b), we substitute the terms tagged with $\star$ to get

$$\begin{aligned}\Delta V_k &= a(\lambda_{1,k}-\lambda_{1,k+1})\sigma_k - \tfrac{ah}{2}\lambda_{1,k+1}(\dot\sigma_{k+1}+\dot\sigma_k)\\ &\quad+\tfrac{h}{2}(a\lambda_{1,k+1}+b\lambda_{2,k+1}+a\lambda_{1,k}+b\lambda_{2,k})\dot\sigma_k\\ &= a(\lambda_{1,k}-\lambda_{1,k+1})(\sigma_k+\tfrac{h}{2}\dot\sigma_k) + \tfrac{h}{2}(-a\lambda_{1,k+1}\dot\sigma_{k+1}\\ &\quad+(a\lambda_{1,k+1}+b\lambda_{2,k+1}+b\lambda_{2,k})\dot\sigma_k).\end{aligned}$$

Let us replace $\dot\sigma_{k+1}$ with its expression in (3)(b), to obtain:

$$\begin{aligned}&\Delta V_k = a(\lambda_{1,k}-\lambda_{1,k+1})(\sigma_k+\tfrac{h}{2}\dot\sigma_k)\\ &+\tfrac{h}{2}(-ah\lambda_{1,k+1}(a\lambda_{1,k+1}+b\lambda_{2,k+1}) + (b\lambda_{2,k+1}+b\lambda_{2,k})\dot\sigma_k).\end{aligned}$$

Using again relation (3)(b) to replace the term $\lambda_{2,k+1}\dot\sigma_k$, we get

$$\begin{aligned}&\Delta V_k = a(\lambda_{1,k}-\lambda_{1,k+1})(\sigma_k+\tfrac{h}{2}\dot\sigma_k)+\\ &\frac{h}{2}\Bigg(-ah\lambda_{1,k+1}(a\lambda_{1,k+1}+b\lambda_{2,k+1}) + b\lambda_{2,k+1}\dot\sigma_{k+1}\\ &\qquad -hb\lambda_{2,k+1}(a\lambda_{1,k+1}+b\lambda_{2,k+1})) + b\lambda_{2,k}\dot\sigma_k\Bigg).\end{aligned}$$

A final rearrangement in the second term yields

$$\begin{aligned}&\Delta V_k = a(\lambda_{1,k}-\lambda_{1,k+1})(\sigma_k+\tfrac{h}{2}\dot\sigma_k)\\ &-\tfrac{h^2}{2}(a\lambda_{1,k+1}+b\lambda_{2,k+1})^2 + \tfrac{bh}{2}(\lambda_{2,k+1}\dot\sigma_{k+1}+\lambda_{2,k}\dot\sigma_k).\end{aligned} \tag{20}$$

Let us analyze the last equality term by term: using (15) with the choice $\lambda_1' = -\lambda_{k+1}$, the first term is nonpositive. The second term is clearly nonpositive and the third one too, using the relation (17). Thus, $\Delta V_k\le 0$. Let us show now that the variation ΔV_k is negative as long as the origin is not reached. The second term in (20) is zero if and only if

$$a\lambda_{1,k+1}+b\lambda_{2,k+1} = 0. \tag{21}$$

Using (14) and the second gain condition in Theorem 4, it follows that (21) has a solution if and only if $|\lambda_{1,k+1}|<1$. Thus, from (12) at step $k+1$, (21) holds if and only if

$$\sigma_{k+1} + \frac{h}{2}\dot\sigma_{k+1} = 0. \tag{22}$$

Using (21) in the dynamics (3)(b), we get that in such a case $\dot\sigma_{k+1} = \dot\sigma_k$. Going back to the analysis of (20), using (17), the last term in the right-hand side is zero if and only if $\dot\sigma_{k+1} = \dot\sigma_k = 0$, which combined with (22) implies that $\sigma_{k+1}=0$, and by (12) that $\sigma_k = 0$. Whence, ΔV_k can be zero only when the system has already reached the origin. Otherwise, $\Delta V_k<0$.

Let us now prove the finite-time stability. Remember that the three terms in the right-hand side of (20) are all nonpositive. Hence, we can find an upper bound of ΔV_k by considering only one of those terms. Let us take a closer look at the second one:

$$-\tfrac{h^2}{2}(a\lambda_{1,k+1}+b\lambda_{2,k+1})^2 = -\tfrac{h^2}{2}u_k^2.$$

We have shown that $u_k = 0 \Leftrightarrow |\lambda_{1,k+1}|<1 \Leftrightarrow \sigma_{k+1} + \frac{h}{2}\dot\sigma_{k+1} = 0$. Thus $u_k\neq 0 \Rightarrow |\lambda_{1,k+1}| = 1$ and $\sigma_{k+1} +$

$\frac{h}{2}\dot{\sigma}_{k+1} \neq 0$ (using (22)). Suppose that $\sigma_k + \frac{h}{2}\dot{\sigma}_k \neq 0$. Then $0 \neq u_k \in \pm(a - \frac{h}{2}b) + b\,\mathrm{Sgn}(\dot{\sigma}_{k+1})$. The second gain condition in the theorem statement gives us $a - b\frac{h}{2} - b > 0$. Therefore, we get that $u_k^2 \geq (a - (1 + \frac{h}{2})b)^2$, so that

$$\Delta V_k \leq -\frac{h^2}{2}u_k^2 \leq -\frac{h^2}{2}\left(a - (1 + \tfrac{h}{2})b\right)^2 < 0. \qquad (23)$$

Inequality (23) holds everywhere outside the line $\sigma_k + \frac{h}{2}\dot{\sigma}_k = 0$. From Lemma 1, we know that if the state of the system belongs to S_0, the origin is reached at the next time instant. If we prove that this segment is reachable in finite-time from any initial conditions, then the origin is globally finite-time reachable. Hence, given that (23) holds everywhere except on the line (22), we just need to bound ΔV_k away from 0 if Σ_{k+1} belongs to the line (22) minus S_0. Thus, suppose that Σ_{k+1} belongs to the line (22), and that $|\dot{\sigma}_{k+1}| > h(a + b)$. In the third term in the right-hand side of (20), and using (12) (13) we get:

$$\begin{aligned}\lambda_{2,k+1}\dot{\sigma}_{k+1} &= -|\dot{\sigma}_{k+1}| - \tfrac{h}{2}\lambda_{1,k+1}\dot{\sigma}_{k+1} \\ &\leq -(1 - \tfrac{h}{2})|\dot{\sigma}_{k+1}| \leq -h(1 - \tfrac{h}{2})(a + b).\end{aligned}$$

Hence, disregarding all other terms (which are all non positive) but $\lambda_{2,k+1}\dot{\sigma}_{k+1}$ in the right-hand side of (20), we obtain the upperbound:

$$\Delta V_k \leq -\tfrac{bh^2}{2}(1 - \tfrac{h}{2})(a + b), \qquad (24)$$

the right-hand side of (24) being negative for all $0 < h < 2$. For all $k \geq 0$, ΔV_k is smaller than the maximum of the right-hand side of (23) and (24). Both quantities being negative constants, the finite-time convergence to S_0 holds, hence the finite-time convergence to the origin. □

Remark 3. The discrete-time Lyapunov function $V_k(\cdot)$ is close to its continuous-time counterpart used in [13], which is given by $a|\sigma| + \dot{\sigma}^2/2$.

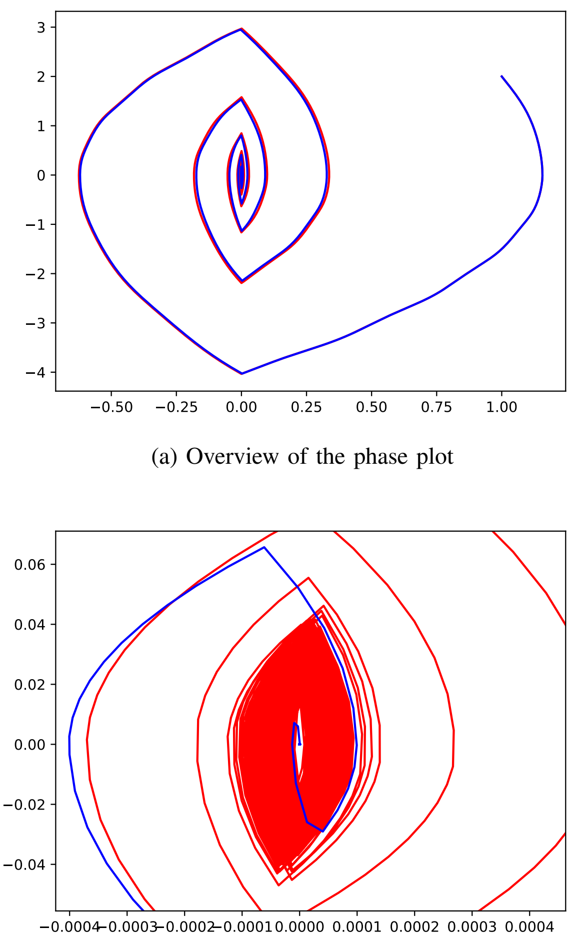
Corollary 1. *Let $0 < h < 2$ and $a > \left(1 + \frac{h}{2}\right)b > 0$. The origin is the unique equilibrium of the closed-loop system, consisting of a double-integrator with the piecewise constant controller obtained from* (7)-(8)*, and it is globally Lyapunov finite-time stable.*

Proof. The ZOH discretization being exact, we know by the previous Theorem that there exists $k_0 \in \mathbb{N}$ such that $\Sigma_{k_0} = 0$. Then for all $k > k_0$, $\Sigma_k = 0$, with the control input $u_k = 0$, as we can easily infer from (3). On each sampling interval $[t_k, t_{k+1}]$, $k > k_0$, the continuous-time system has the dynamics

$$\dot{\Sigma} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Sigma + \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} 0 = 0.$$

This concludes the proof. □

Remark 4. From (3) it follows that once the origin has been attained, then the (unperturbed) discrete-time system stays at the origin for all future times. Therefore, there is no numerical chattering with the implicit controller, contrarily to its explicit counterpart [1]. Let us illustrate this point numerically on the double integrator: on Fig. 1 the phase plot of the explicitly discretized control law, $u_k = -a\,\mathrm{sgn}(\sigma_k) - b\,\mathrm{sgn}(\dot{\sigma}_k)$, and the enhanced implicit twisting given by (7)-(8). This is confirmed experimentally in [6] with the regular implicit twisting controller calculated from (4). The enhanced implicit twisting controller is expected to provide even better results. The regular implicit twisting algorithm has also been analyzed in [5, Propositions 7 and 8], including a perturbation. It is shown therein that during the discrete-time sliding mode, the disturbance is attenuated with a factor h^2 on σ_k and with a factor h on $\dot{\sigma}_k$.



(a) Overview of the phase plot

(b) Zoom around the origin

Fig. 1: The blue trajectory is with the enhanced implicit control law, the red one is obtained with an explicit discretization. The timestep is $h = 10$ms.

V. Enhanced Implicit Twisting with perturbations

A. Computation of the control input

Let us now tackle the case with perturbations, with the same setup as before: we consider the ZOH discretization of a double integrator. Let us denote by $\widetilde{\Sigma}_k$ and $\widetilde{\Sigma}_{k+1}$ the sliding variables for the *real plant*, *i.e.*, with perturbations, whereas the sliding variables Σ_k and Σ_{k+1} are the variables of the *nominal system*, *i.e.*, without perturbation. The latter obey the same system (1), whereas $\widetilde{\Sigma}_{k+1}$ is given by:

$$\widetilde{\Sigma}_{k+1} = A^*\widetilde{\Sigma}_k + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u_k + d \quad \text{with}$$

$$u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1} \quad \text{and} \quad -\lambda_{k+1} \in \partial\delta^*_{-K}(\Sigma_{k+1}),$$

where $\mathbb{R}^2 \ni d = \mathcal{O}(h)$ captures the contribution from the perturbations. At each timestep, the value Σ_k of the nominal system is set to the value $\widetilde{\Sigma}_k$, measured (or estimated) on the real system. The inclusion $-\lambda_{k+1} \in \partial\delta^*_{-K}(\Sigma_{k+1})$ combined with the one step relation $\Sigma_{k+1} = A^*\widetilde{\Sigma}_k + B^*\lambda_{k+1}$ gives the AVI

$$0 \in A^*\widetilde{\Sigma}_k + B^*\lambda_{k+1} + \mathcal{N}_K(\lambda_{k+1}). \tag{25}$$

Again, a solution to this AVI is computed with data known at time t_k. Therefore, the control law is causal. Note that from these equations, it is easy to see that the discrepancy on one time step between Σ_{k+1} and $\widetilde{\Sigma}_{k+1}$, is d.

Let us now detail how a solution to the AVI (25) can be computed. Since we are dealing with an AVI of dimension 2, it is possible to apply a simple algorithm. Note that with our choice of K, each λ_i can take values in 3 sets: λ_1 in $\{1\}$, $\{-1\}$ and $[-1,1]$, and λ_2 in $\{1+\frac{h}{2}\lambda_1\}$, $\{-1+\frac{h}{2}\lambda_1\}$ and $[-1+\frac{h}{2}\lambda_1, 1+\frac{h}{2}\lambda_1]$. Each corresponds to the arguments of the Sgn multifunction to be either nonnegative, nonpositive and 0. This gives 9 possible cases to consider. Therefore, it is tractable to find a solution by enumerating the candidates. Thanks to the result of existence and uniqueness of the control input in Lemma 2, this approach is well defined. Let us first describe the enumeration procedure, before giving some heuristic to find a solution in a faster way.

First, whenever λ is one of the extreme point of K, then checking that it is a solution consists of computing Σ_{k+1} based on that value of λ, and check that the set $(\mathrm{Sgn}(\sigma_{k+1}), \mathrm{Sgn}(\dot\sigma_{k+1}) + \frac{h}{2}\,\mathrm{Sgn}(\sigma_{k+1}))^T$ contains that value.

When the candidate λ belongs to the boundary of set K, the procedure consists of computing u_k as a solution to either $\sigma_{k+1} = 0$ or $\dot\sigma_{k+1} = 0$, and checking that there exists a λ that can give that value u_k. Whenever we consider a candidate λ with $\lambda_1 = 0$ (resp. λ_2), we compute the value of u_k solution to $\sigma_{k+1} = 0$ (resp. $\dot\sigma = 0$). Then, we compute the value of λ_1 (resp. λ_2) giving that control input value u_k:

- $\lambda = (1, \lambda_2)$, then $\lambda_2 \in [-1+\frac{h}{2}, 1+\frac{h}{2}]$.
- $\lambda = (-1, \lambda_2)$, then $\lambda_2 \in [-1-\frac{h}{2}, 1-\frac{h}{2}]$.

For the case $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_2 - \frac{h}{2}\lambda_1 = \pm 1$, then we combine this with the equation $a\lambda_1 + b\lambda_2 = u_k$. It is easy to check that this 2×2 system always has a unique solution, our candidate λ. To check that it is in the constraint set K, it is sufficient that $\lambda_1 \in [-1, 1]$.

Finally, the last case is when $\Sigma_{k+1} = 0$ for which $\lambda \in K$. For this last case, the method consists in computing u_k as the solution of $\dot\sigma_{k+1}$. If $|u_k| < a + b(1+\frac{h}{2})$, then the value of σ_{k+1} is checked to be 0. If the latter condition holds, then the control input value is u_k. Indeed, one can check that whenever $|u_k| \le a + b(1+\frac{h}{2})$, the vector λ such that $\lambda_1 = u_k/(a+b(1+\frac{h}{2})$ and $\lambda_2 = (1+\frac{h}{2})\lambda_1$ belongs to K, and provides the required control input value.

A simple and efficient heuristic consists in trying first the value of $\lambda = (\mathrm{sgn}(\sigma_k), \mathrm{sgn}(\sigma_{k+1}) + \frac{h}{2}\,\mathrm{sgn}(\sigma_{k+1}))^T$, since the value of the control is likely not to change, especially away from the origin. Note that we used the single-valued signum function. If this candidate fails, then the remaining 8 combinations are tried.

B. Numerical simulations

Let us illustrate the above developments on a disturbed double integrator. The control input value is supposed to be piecewise constant: $u(t) = u_k$ for $t \in [t_k, t_{k+1})$, $k \ge 0$. Three control laws are compared, with G and β such that $a = G$ and $b = G\beta$:

- *explicit control*: $u_k = -G(\mathrm{sgn}(\sigma_k) + \beta\,\mathrm{sgn}(\dot\sigma_k))$
- *regular implicit control*: $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$, and $\lambda_{k+1} \in \mathrm{SOL}(A^*\widetilde{\Sigma}_k, B^*, [-1,1]^2)$.
- *enhanced implicit control*: $u_k = a\lambda_{1,k+1} + b\lambda_{2,k+1}$, and $\lambda_{k+1} \in \mathrm{SOL}(A^*\widetilde{\Sigma}_k, B^*, K)$.

The plant dynamics with a perturbation, $\ddot\sigma(t) = u(t) + \alpha\sin(\omega t)$, is integrated using the Lsodar method from ODEPACK, for a high fidelity integration process mimicking the continuous-time plant. In the following, $\omega = 50$ and $\alpha = 1$. Since B^* is not symmetric positive definite, the AVI cannot be solved with a projection as for first-order SMC [7], [2]. However, the algorithm from [21] can always find a solution. Thanks to the linear nature of the problem, the maximum number of iterations is nine [22]. Furthermore, each iteration requires solving a linear system with only two variables. Hot-start strategy is also very easy to set up, since in the reaching phase the values of λ do not frequently change. The implementation of the proposed implicit control is available in the open-source software SICONOS and its module SICONOS/CONTROL [23]. Otherwise, it is possible to perform the simulation with the algorithm described in Section V-A.

Let us first show the control input values in Fig. 2, for the three control laws. The explicit control law behaves as a bang-bang signal, while the regular implicit twisting has a control magnitude about twice as big as the enhanced implicit control law. Let us now analyze Figures 3 and 4. The enhanced implicit control law has the best performance when taking into account the control input effort, the chattering and the precision (on both σ and $\dot\sigma$). The control effort is defined as $\|\{u_k\}\|_1$, the chattering is the total variation (TV) of that sequence: $TV(u) := \sum_k |u_{k+1} - u_k|$. The chattering of the sliding variables is also defined as the total variation of the sequences $\{\sigma_k\}$ and $\{\dot\sigma_k\}$. The following trends can be observed: the enhanced implicit control law always outperforms the other two. It provides the best precision, in the same range as the regular implicit control law (magenta crosses), but each time with less chattering or less control effort. As expected, the closed-loop system with the explicitly discretized control law delivers the worst performances.

The continuous time twisting algorithm enjoys the invariance of the control input with respect to the gain in the sliding phase, given that the magnitude of the control input is large enough to overcome the perturbation. This is illustrated in Fig. 4, which is obtained as follows: for each gain value, simulations for 49 initial conditions in a ball of radius $2e^{-2}$ around $(1, 2)$ are performed. The worst case performance is then used to create this picture. This methodology is needed since the controller obtained by a regular implicit discretization is very sensitive to the initial value, as documented in [15, Section 2.3.4]. From Fig. 4 it is clear that the enhanced implicit controller is not sensitive with respect to an increase of the control gain, hence confirming a property of the implicit first-order discrete-time SMC [2], [7]. On the other hand, both the explicitly discretized controller and the regular implicit controller are sensitive to an increase of the gain, linearly for the errors and the control effort.

Finally, let us investigate the influence of the sampling period on the closed-loop performance, depicted in Fig. 5. The average error on σ is quadratic for all control laws. Moreover the results about the control chattering show that the enhanced implicit control law achieves a much better performance than the other two controllers: its chattering level remains constant, whereas it is inversely proportional to the sampling period in the other two cases. This is also coherent with results obtained in [7], [2] for the first order SMC.

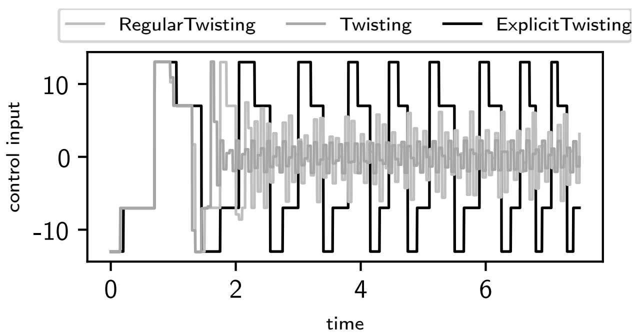


Fig. 2: Control inputs for the three control laws

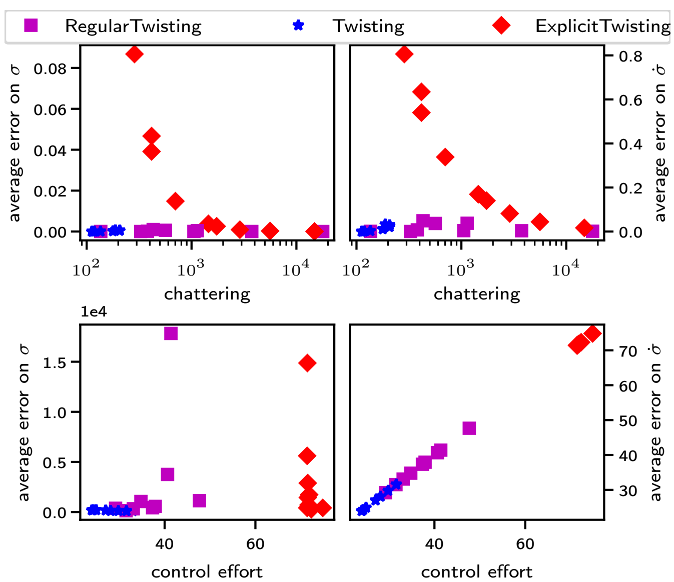


Fig. 3: Comparisons between the 3 controllers with different metrics. Each point represents a different sampling period with values between 50 ms and 1 ms.

VI. Conclusion

In this article, an enhanced implicit discrete-time twisting sliding-mode controller is studied. Fundamental properties are investigated: existence and uniqueness of the controller at each step, and the global finite-time Lyapunov stability of the closed-loop system. Affine variational inequalities are at the core of the analysis and computational algorithms. Numerical comparisons illustrate the superior performances of the enhanced implicit twisting controller and illustrate some properties shared by the continuous time and discrete-time twisting algorithms. Future work should tackle the analysis of the case with perturbations.

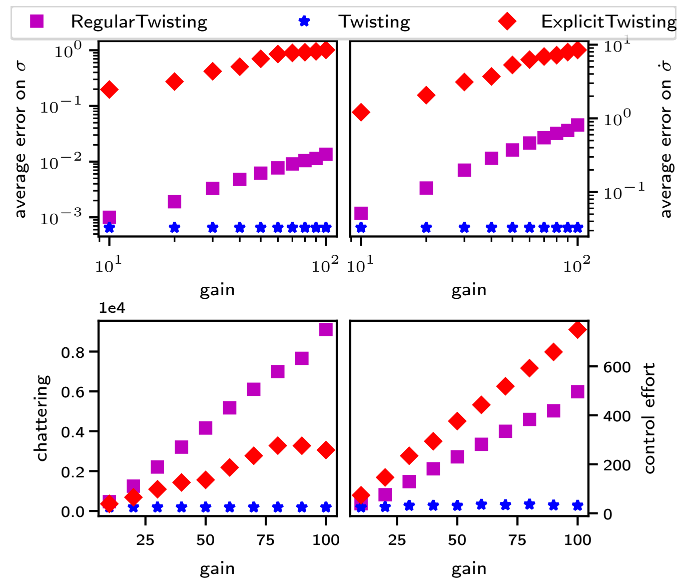


Fig. 4: Evolution of the performance of the controllers with respect to the gain.

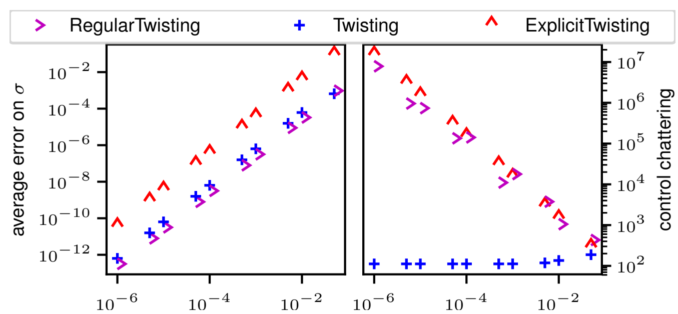


Fig. 5: Evolution of the performance of the controllers with respect to the sampling period (in seconds).

Appendix A
Proof of Lemmae 1 and 2

Proof of Lemma 1. Let us first study the set of points Σ_k such that $\Sigma_{k+1} = (0, 0)$. Using the recurrence relations in (3), we get the system

$$\begin{cases} \sigma_k + h\dot{\sigma}_k + \frac{h^2}{2}u_k = 0 & \text{(26a)} \\ \dot{\sigma}_k + hu_k = 0. & \text{(26b)} \end{cases}$$

Inserting (26b) in (26a) yields $\sigma_k = \frac{h^2}{2}u_k$. Combining with (26b) we get $\sigma_k + \frac{h}{2}\dot{\sigma}_k = 0$. Hence, the origin can only be reached from this hyperplane. Since $|u_k| \leq u^M$, we get the constraint $|\dot{\sigma}_k| \leq hu^M$. □

Proof of Lemma 2. The existence of a solution to the AVI (6) is ensures by the continuity of the functional L defined as $\lambda \mapsto A^*\Sigma_k + B^*\lambda$, and the compactness of the convex set K. The outline of the proof (that can be found in [19, Corollary 2.2.5, p. 148]), is as follows: consider the mapping $\Lambda \mapsto \Pi_K(\Lambda - L(\Lambda))$, with Π_K the projector onto K. Since K is convex, it is single valued from $K \to K$. By Brouwer's fixed-point theorem, there exists a fixed point Λ^0. From [16, Prop. 6.17], $\Pi_K^{-1} = (I + \mathcal{N}_K)$, in the sense of set-valued mappings. Hence, $\Pi_K^{-1}(\Lambda^0) = \Lambda^0 + \mathcal{N}_K(\Lambda^0) = \{\Lambda^0 - L(\Lambda^0)\}$. Rearranging terms gives that $-L(\Lambda^0) \in \mathcal{N}_K(\Lambda^0)$, whence Λ^0 is a solution to the AVI.

For the uniqueness property, note that since B^*, from (2), is a rank-one matrix, $L^T B^* L$ is also a rank-one matrix given by the outer product $L^T B^* L = h(h,1)^T \ (a + \frac{h}{2}b, b)$. This matrix is singular with positive diagonal elements. By construction its columns are linearly dependent. Hence, we have by [15, Proposition 2.3.4 and Remark 2.3.5], the uniqueness of Σ_{k+1}. As shown in the proof of that proposition, this implies that the difference between any two solutions lies the kernel of the linear mapping, here B^*. Note that $\ker B^* = \operatorname{span}(b, -a)^T$ and that u_k is defined as $(a,b)\lambda$. It is easy to see that any element of $\ker B^*$ is orthogonal to $(a,b)^T$, which ensures the uniqueness of u_k. For the last part of the statement, the uniqueness of Σ_{k+1} implies that if λ^1_{k+1} and λ^2_{k+1} are two distinct solutions, then their opposites are both in the set $\partial\delta^*_{-K}(\Sigma_{k+1})$ (see (8)). This implies that λ^1_{k+1} and λ^2_{k+1} are such that $\langle -\lambda^1_{k+1}, \Sigma_{k+1}\rangle = \langle -\lambda^2_{k+1}, \Sigma_{k+1}\rangle = \delta^*_{-K}(\Sigma_{k+1})$. Whence, $\Delta\lambda_{k+1} := \lambda^1_{k+1} - \lambda^2_{k+1}$ is orthogonal to Σ_{k+1}. But we know that $\Delta\lambda \in \ker M = \operatorname{span}\left(\begin{smallmatrix}-b\\ a\end{smallmatrix}\right)$, which means that $\Sigma_{k+1} = s\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right)$, for some $s \in \mathbb{R}$. We are interested in the case $\Sigma_{k+1} \neq 0$ which means that $s \neq 0$. Given that $a > b > 0$, we get that:

$$\partial\delta^*_{-K}(\Sigma_{k+1}) = \operatorname*{arg\,sup}_{y \in -K} \langle y, \Sigma_{k+1}\rangle = \begin{cases} \left\{\begin{pmatrix} 1 \\ 1+\frac{h}{2}\end{pmatrix}\right\} & \text{if } s > 0 \\ \left\{\begin{pmatrix} -1 \\ -1-\frac{h}{2}\end{pmatrix}\right\} & \text{if } s < 0. \end{cases}$$

The two sets in the right-hand side are both singletons: the solution λ_{k+1} of the AVI is therefore unique whenever $\Sigma_{k+1} \neq 0$. □

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