On the Weak Solvability and the Optimal Control of a Frictional Contact Problem with Normal Compliance

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Abstract. In the present work we consider a frictional contact model with normal compliance. Firstly, we discuss the weak solvability of the model by means of two variational approaches. In a first approach the weak solution is a solution of a quasivariational inequality. In a second approach the weak solution is a solution of a mixed variational problem with solution-dependent set of Lagrange multipliers. Nextly, the paper focuses on the boundary optimal control of the model. Existence results, an optimality condition and some convergence results are presented.

Keywords: contact model, friction, normal compliance, weak solutions, optimal control, optimality condition, convergences.

1 Introduction

The present paper focuses on the weak solvability and the boundary optimal control of the following contact model.

Problem 1. Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^3$ such that

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega,$$

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_1,$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2,$$

$$-\sigma_\nu = p_\nu(u_\nu - g_a) \quad \text{on } \Gamma_3,$$

$$\left.\begin{array}{l} \|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_\nu - g_a), \\ \boldsymbol{\sigma}_\tau = -\,p_\tau(u_\nu - g_a)\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if} \quad \boldsymbol{u}_\tau \ne \boldsymbol{0} \end{array}\right\} \quad \text{on } \Gamma_3.$$

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Notice that Ω is a bounded domain of $\mathbb{R}^3$ with smooth enough boundary, partitioned in three measurable part Γ_1, Γ_2 and Γ_3; $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, " $\cdot$ " denotes the inner product of two vectors, $\|\cdot\|$ denotes the Euclidean norm, $\boldsymbol{\nu}$ is the unit outward normal vector.

Problem 1 is a contact problem with the normal compliance condition, associated to the Coulomb's law of dry friction. A normal compliance condition was firstly proposed in [17]. Then, such a contact condition was used in many models, see e.g. the papers [2, 7–9, 20].

In the normal compliance contact condition

$$-\sigma_\nu = p_\nu(u_\nu - g_a) \quad \text{on} \quad \Gamma_3,$$

p_ν is a nonnegative prescribed function which vanishes for negative argument and $g_a > 0$ denotes the gap (the distance between the body and the obstacle on the normal direction). When $u_\nu < g_a$ there is no contact and the normal pressure vanishes. When there is contact then $u_\nu - g_a$ is positive and represents a measure of the interpenetration of the asperities. Then, the normal compliance condition shows that the foundation exerts a pressure on the body which depends on the penetration. For details on the physical significance of the model we refer to [22].

The rest of the paper has the following structure. Section 2 is devoted to the weak solvability of the model by means of two variational approaches. In Section 3 we discuss an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on a part of the boundary.

There are several works concerning the optimal control of variational inequalities, see for instance [3, 4, 6, 10, 15, 16, 18, 23]. Nevertheless, only few works are devoted to the optimal control of contact problems, see [1, 5, 14]. The present paper adds a new contribution.

2 On the weak solvability of the model

In this section we shall indicate two variational approaches in the study of Problem 1. Let us make the following assumptions.

Assumption 1 *$\mathcal{F}: \Omega\times\mathbb{S}^3 \to \mathbb{S}^3$, $\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}) = (\mathcal{F}_{ijkl}(\boldsymbol{x})\varepsilon_{jk})$ for all $\boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^3$, a.e. $\boldsymbol{x} \in \Omega$. $\mathcal{F}_{ijkl} = \mathcal{F}_{jikl} = \mathcal{F}_{klij} \in L^\infty(\Omega)$, $1 \le i,j,k,l \le 3$. There exists $m_\mathcal{F} > 0$ such that $\mathcal{F}(\boldsymbol{x},\boldsymbol{\tau}) : \boldsymbol{\tau} \ge m_\mathcal{F}\|\boldsymbol{\tau}\|^2$ for all $\boldsymbol{\tau} \in \mathbb{S}^3$,* a.e. $\boldsymbol{x}$ in Ω.

Assumption 2 *$\boldsymbol{f}_0 \in L^2(\Omega)^3$, $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.*

Assumption 3 *$p_e : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ $(e \in \{\nu,\tau\})$. There exists $L_e > 0$ such that $|p_e(\boldsymbol{x},r_1) - p_e(\boldsymbol{x},r_2)| \le L_e\,|r_1 - r_2|$ for all r_1, $r_2 \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Gamma_3$. The mapping $\boldsymbol{x} \mapsto p_e(\boldsymbol{x},r)$ is measurable on Γ_3, for any $r \in \mathbb{R}$ and $p_e(\boldsymbol{x},r) = 0$ for all $r \le 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.*

Assumption 4 *$g_a \in L^2(\Gamma_3)$, $g_a(\boldsymbol{x}) \ge 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.*

Assumption 5 $m_{\mathcal{F}} > c_0^2(L_\nu + L_\tau)$.

Assumption 5 is a smallness assumption which was introduced mainly for mathematical reasons. However, for some materials and frictional contact conditions we have appropriate constants $m_{\mathcal{F}}$, L_ν and L_τ which fulfill Assumption 5. Notice that " : " denotes the inner product of two tensors and $c_0 = c_0(\Omega,\, \Gamma_1,\, \Gamma_3) > 0$ is a "trace constant" such that:

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^3} \le c_0\, \|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V, \tag{1}$$

where

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \,\}.$$

In a first approach the weak solution is a solution of a quasivariational inequality having as unknown the displacement field.

Problem 2. Find a displacement field $\boldsymbol{u} \in V$ such that

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V. \tag{2}$$

Herein,

$$A : V \to V \quad (A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)_s^{3\times 3}},$$

$$j : V \times V \to \mathbb{R} \quad j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(u_\nu - g_a)|v_\nu|\, d\Gamma + \int_{\Gamma_3} p_\tau(u_\nu - g_a)\|\boldsymbol{v}_\tau\|\, d\Gamma,$$

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, d\Gamma. \tag{3}$$

We have the following existence and uniqueness result.

Theorem 1. *Under Assumptions 1-5, Problem 2 has a unique weak solution.*

For the proof we refer to Theorem 5.30 in [22].

The second approach is a mixed variational approach. The mixed variational formulations are related to modern numerical techniques in order to approximate the weak solutions of contact models. Referring to numerical techniques for approximating weak solutions of contact problems via saddle point technique, we send the reader to, e.g., [19, 24, 25]. The functional frame is the following one.

$$\begin{aligned} &V = \{\boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{v} = 0 \text{ a.e. on } \Gamma_1\};\\ &S = \{\boldsymbol{w}|_{\Gamma_3} \,|\, \boldsymbol{w} \in V\};\\ &D = S'. \end{aligned}$$

Notice that $\boldsymbol{w}|_{\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{w} \in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3;\mathbb{R}^3)$ where $H^{1/2}(\Gamma_3;\mathbb{R}^3)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^3$. We use the Sobolev-Slobodeckii norm

$$\|\boldsymbol{\zeta}\|_S = \Big(\int_{\Gamma_3} \int_{\Gamma_3} \frac{\|\boldsymbol{\zeta}(\boldsymbol{x}) - \boldsymbol{\zeta}(\boldsymbol{y})\|^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^3} \, ds_x \, ds_y \Big)^{1/2}.$$

For each $\boldsymbol{\zeta} \in S$, $\zeta_\nu = \boldsymbol{\zeta} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\zeta}_\tau = \boldsymbol{\zeta} - \zeta_\nu \boldsymbol{\nu}$ a.e. on Γ_3.

Let us consider $\boldsymbol{f} \in V$, see (3), and let us define two bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ as follows:

$$\begin{aligned} &a(\cdot,\cdot) : V \times V \to \mathbb{R}, \quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\, dx; \\ &b(\cdot,\cdot) : V \times D \to \mathbb{R} \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v}|_{\Gamma_3} \rangle. \end{aligned}$$

Also, we define a variable set $\Lambda = \Lambda(\boldsymbol{\varphi})$,

$$\begin{aligned} \boldsymbol{\Lambda}(\boldsymbol{\varphi}) = \{ &\boldsymbol{\mu} \in D \,|\, \langle \boldsymbol{\mu}, \boldsymbol{v}|_{\Gamma_3} \rangle \\ &\leq \int_{\Gamma_3} (p_\nu(\boldsymbol{x}, \varphi_\nu(\boldsymbol{x}) - g_a)|v_\nu(\boldsymbol{x})| + p_\tau(\boldsymbol{x}, \varphi_\nu(\boldsymbol{x}) - g_a)\|\boldsymbol{v}_\tau(\boldsymbol{x})\|) d\Gamma \quad \boldsymbol{v} \in V \}. \end{aligned}$$

Notice that $\langle \cdot,\cdot \rangle$ denotes the duality pairing between D and S.

The second variational formulation of Problem 1 is the following one.

Problem 3. Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}(\boldsymbol{u}) \subset D$ such that

$$\begin{aligned} &a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V, \\ &b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \qquad \leq 0 \qquad\quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\Lambda}(\boldsymbol{u}). \end{aligned}$$

In this second approach, a weak solution is a pair consisting of the displacement field and a Lagrange multiplier related to the friction force.

Theorem 2. *Under Assumptions 1-4, Problem 3 has at least one solution.*

The proof of Theorem 2, based on the abstract results we have got in [11], can be found in the very recent paper [12].

Remark 1. Treating the model in the first approach we can prove the existence and the uniqueness of the weak solution. But, the approximation of the weak solution is based on a regularization/penalization technique. Treating the model in the second approach we are led to a generalized saddle point problem. Recall that, for weak formulations in Contact Mechanics via saddle point problems, efficient algorithms can be written in order to approximate the weak solution (see primal-dual active set strategies). But, there are a few open questions here:

- the study of the uniqueness of the weak solution of the mixed variational formulation Problem 3;
- a priori error estimates; algorithms.

3 Boundary optimal control

Let us discuss in this section a boundary optimal control problem related to our contact problem.

For a fixed function $\boldsymbol{f}_0 \in L^2(\Omega)^3$, we consider the following *state problem.*

(PS1) *Let $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$ (called control) be given. Find $\boldsymbol{u} \in V$ such that*

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, dx \qquad (4)$$
$$+ \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V.$$

According to Theorem 1, for every control $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$, the *state problem* (PS1) has a unique solution $\boldsymbol{u} \in V$, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f}_2)$. In addition, the following estimation takes place:

$$\|\boldsymbol{u}\|_V \leq \frac{1}{m_{\mathcal{F}}}(\|\boldsymbol{f}_0\|_{L^2(\Omega)^3} + c_0\|\boldsymbol{f}_2\|_{L^2(\Gamma_2)^3}),$$

where $m_{\mathcal{F}}$ is the constant in Assumption 1 and c_0 appears in (1).

Now, we would like to act a control on Γ_2 such that the resulting stress $\boldsymbol{\sigma}$ be as close as possible to a given target

$$\boldsymbol{\sigma}_d = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_d)$$

where $\boldsymbol{u}_d$ is a given function.

Let Q_∞ be the real Banach space

$$Q_\infty = \{\mathcal{F} = \mathcal{F}_{ijkl} \,|\, \mathcal{F}_{ijkl} = \mathcal{F}_{jikl} = \mathcal{F}_{klij} \in L^\infty(\Omega),\ 1 \leq i,j,k,l \leq 3\}$$

endowed with the norm $\|\mathcal{F}\|_\infty = \max_{1\leq i,j,k,l\leq 3} \|\mathcal{F}_{ijkl}\|_{L^\infty(\Omega)}$. According to [22], page 97,

$$\|\mathcal{F}\boldsymbol{\tau}\|_{L^2(\Omega)_s^{3\times 3}} \leq 3\,\|\mathcal{F}\|_\infty \|\boldsymbol{\tau}\|_{L^2(\Omega)_s^{3\times 3}} \quad \text{for all } \boldsymbol{\tau} \in L^2(\Omega)_s^{3\times 3}.$$

Therefore,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_d\|_{L^2(\Omega)_s^{3\times 3}} \leq 3 \max_{1\leq i,j,k,l\leq 3} \|\mathcal{F}_{ijkl}\|_{L^\infty(\Omega)}\, \|\boldsymbol{u} - \boldsymbol{u}_d\|_V.$$

Thus, $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_d$ will be close to one another if the difference between the functions $\boldsymbol{u}$ and $\boldsymbol{u}_d$ is small in the sense of $V-$norm.

To give an example of a target of interest, $\boldsymbol{u}_d$, we can consider $\boldsymbol{u}_d = 0$. In this situation, by acting a control on Γ_2, the tension $\boldsymbol{\sigma}$ is small in the sense of L^2- norm, even if the volume forces $\boldsymbol{f}_0$ does not vanish in Ω.

Let $\alpha, \beta > 0$ be two positive constants and let us define the following functional

$$L : L^2(\Gamma_2)^3 \times V \to \mathbb{R}, \quad L(\boldsymbol{f}_2, \boldsymbol{u}) = \frac{\alpha}{2}\|\boldsymbol{u} - \boldsymbol{u}_d\|_V^2 + \frac{\beta}{2}\|\boldsymbol{f}_2\|_{L^2(\Gamma_2)^3}^2.$$

Furthermore, we denote

$$\mathcal{V}_{ad} = \{[\boldsymbol{u}, \boldsymbol{f}_2] \mid [\boldsymbol{u}, \boldsymbol{f}_2] \in V \times L^2(\Gamma_2), \text{ such that (4) is verified}\}.$$

(POC1) *Find* $[\boldsymbol{u}^*, \boldsymbol{f}_2^*] \in \mathcal{V}_{ad}$ *such that* $L(\boldsymbol{f}_2^*, \boldsymbol{u}^*) = \min\limits_{[\boldsymbol{u}, \boldsymbol{f}_2] \in \mathcal{V}_{ad}} \Big\{ L(\boldsymbol{f}_2, \boldsymbol{u}) \Big\}$.

A solution of (POC1) is called *an optimal pair*. The second component of the optimal pair is called *an optimal control*.

Theorem 3. *Problem* $(POC1)$ *has at least one solution* $(\boldsymbol{u}^*, \boldsymbol{f}_2^*)$.

Let us fix $\rho > 0$ and $\boldsymbol{f}_0 \in L^2(\Omega)^3$.

We introduce the following regularized state problem.

(PS2) *Let* $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$ *(called regularized control) be given. Find* $\boldsymbol{u} \in V$ *such that*

$$\begin{aligned}(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j_\rho(\boldsymbol{u}, \boldsymbol{v}) - j_\rho(\boldsymbol{u}, \boldsymbol{u}) &\geq (\boldsymbol{f}_0, \boldsymbol{v} - \boldsymbol{u})_{L^2(\Omega)^3} \\ &+ (\boldsymbol{f}_2, \boldsymbol{v} - \boldsymbol{u})_{L^2(\Gamma_2)^3} \qquad \text{for all } \boldsymbol{v} \in V.\end{aligned} \tag{5}$$

Herein, $j_\rho : V \times V \to \mathbb{R}$ is defined as follows,

$$j_\rho(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x}))(\sqrt{(v_\nu(\boldsymbol{x}))^2 + \rho^2} - \rho)d\Gamma$$

$$+ \int_{\Gamma_3} p_\tau^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x}))(\sqrt{\|\boldsymbol{v}_\tau(\boldsymbol{x})\|^2 + \rho^2} - \rho)d\Gamma \quad \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in V,$$

where p_e^ρ, $e \in \{\nu, \tau\}$, satisfies the following assumptions.

Assumption 6 $p_e^\rho : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$. *The mapping* $\boldsymbol{x} \mapsto p_e^\rho(\boldsymbol{x}, r)$ *is measurable on* Γ_3 *for any* $r \in \mathbb{R}$, *and* $p_e^\rho(\boldsymbol{x}, r) = 0$ *for all* $r \leq 0$, *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Assumption 7 $p_e^\rho(\boldsymbol{x}, \cdot) \in C^1(\mathbb{R})$ *a.e. on* $\boldsymbol{x} \in \Gamma_3$. *There exists* $M_e > 0$ *such that* $|p_e^\rho(\boldsymbol{x}, r)| \leq M_e$ *for all* $r \in \mathbb{R}$, *a.e.* $\boldsymbol{x} \in \Gamma_3$. *In addition* $|\partial_2 p_e^\rho(\boldsymbol{x}, r)| \leq L_e$ *for all* $r \in \mathbb{R}$, *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Assumption 8 *There exists* $G_e : \mathbb{R}_+ \to \mathbb{R}_+$ $(e \in \{\nu, \tau\})$ *such that* $|p_e^\rho(\boldsymbol{x}, r) - p_e(\boldsymbol{x}, r)| \leq G_e(\rho)$ *for all* $r \in \mathbb{R}$, *a.e.* $\boldsymbol{x} \in \Gamma_3$ *and* $\lim_{\rho \to 0} G_e(\rho) = 0$.

Notice that the functional $j_\rho(\cdot, \cdot)$ has the following properties:

• for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $j_\rho(\boldsymbol{u}, \boldsymbol{v}) \geq 0$; $j_\rho(\boldsymbol{u}, 0_V) = 0$;

• for all $\boldsymbol{u} \in V$, $j_\rho(\boldsymbol{u}, \cdot) : V \to \mathbb{R}$ is a convex and Gâteaux differentiable functional;

• $j_\rho(\boldsymbol{\eta}_1, \boldsymbol{v}_2) - j_\rho(\boldsymbol{\eta}_1, \boldsymbol{v}_1) + j_\rho(\boldsymbol{\eta}_2, \boldsymbol{v}_1) - j_\rho(\boldsymbol{\eta}_2, \boldsymbol{v}_2) \leq c_0^2(L_\nu + L_\tau)\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V$ for all $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$.

• for all $\boldsymbol{u}, \boldsymbol{v} \in V$, there exists $\nabla_2 j_\rho(\boldsymbol{u}, \boldsymbol{v}) \in V$ such that

$$\lim_{h \to 0} \frac{j_\rho(\boldsymbol{u}, \boldsymbol{v} + h\boldsymbol{w}) - j_\rho(\boldsymbol{u}, \boldsymbol{v})}{h} = (\nabla_2 j_\rho(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w})_V \quad \text{ for all } \boldsymbol{w} \in V.$$

$$(\nabla_2 j_\rho(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})_V = \int_{\Gamma_3} p_\nu^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x})) \frac{v_\nu(\boldsymbol{x}) w_\nu(\boldsymbol{x})}{\sqrt{(v_\nu(\boldsymbol{x}))^2 + \rho^2}}\, d\Gamma + \int_{\Gamma_3} p_\tau^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x})) \frac{\boldsymbol{v}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x})}{\sqrt{\|\boldsymbol{v}_\tau(\boldsymbol{x})\|^2 + \rho^2}}\, d\Gamma.$$

The regularized state problem has a unique solution $\boldsymbol{u}^\rho \in V$ that depends Lipschitz continuously on $\boldsymbol{f}$. This is a straightforward consequence of an abstract result in the theory of the quasivariational inequalities, see e.g. Theorem 3.7, in [21].

For every $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$, the problem (PS2) has a unique solution $\boldsymbol{u} \in V$, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f}_2)$. In addition,

$$\|\boldsymbol{u}\|_V \le \frac{1}{m_{\mathcal{F}}}(\|\boldsymbol{f}_0\|_{L^2(\Omega)^3} + c_0\|\boldsymbol{f}_2\|_{L^2(\Gamma_2)^3}).$$

There exists an unique $\boldsymbol{z} \in V$ such that

$$(\boldsymbol{z},\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in V.$$

Furthermore, there exists an unique $\boldsymbol{y}(\boldsymbol{f}_2) \in V$ such that

$$(\boldsymbol{y}(\boldsymbol{f}_2),\boldsymbol{v})_V = \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V.$$

Let $\boldsymbol{u} \in V$ be the unique solution of (PS2).
Let us define

$$\partial_2 j_\rho(\boldsymbol{u},\boldsymbol{u}) = \{\boldsymbol{\zeta} \in V \mid j_\rho(\boldsymbol{u},\boldsymbol{v}) - j_\rho(\boldsymbol{u},\boldsymbol{u}) \ge (\boldsymbol{\zeta}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V\}.$$

Therefore,

$$\boldsymbol{z} + \boldsymbol{y}(\boldsymbol{f}_2) - A\boldsymbol{u} \in \partial_2 j_\rho(\boldsymbol{u},\boldsymbol{u}).$$

Since $j_\rho(\cdot,\cdot)$ is convex and Gâteaux differentiable in the second argument, we can write

$$\partial_2 j_\rho(\boldsymbol{u},\boldsymbol{u}) = \{\nabla_2 j_\rho(\boldsymbol{u},\boldsymbol{u})\}.$$

Thus, we are led to the following operatorial equation

$$A\boldsymbol{u} + \nabla_2\, j_\rho(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{z} + \boldsymbol{y}(\boldsymbol{f}_2).$$

Let us define the admissible set,

$$\mathcal{V}_{ad}^\rho = \{[\boldsymbol{u},\boldsymbol{f}_2] \mid [\boldsymbol{u},\boldsymbol{f}_2] \in V \times L^2(\Gamma_2)^3, \text{ such that (5) is verified}\}.$$

Using the functional L, we introduce the *regularized optimal control problem*,
(POC2) *Find* $[\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}_2] \in \mathcal{V}_{ad}^\rho$ *such that* $L(\bar{\boldsymbol{f}}_2, \bar{\boldsymbol{u}}) = \min_{[\boldsymbol{u},\boldsymbol{f}_2] \in \mathcal{V}_{ad}^\rho} \Big\{ L(\boldsymbol{f}_2, \boldsymbol{u}) \Big\}.$

Theorem 4. *The problem (POC2) has at least one solution* $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}_2)$.

A solution of (POC2) is called *a regularized optimal pair* and the second component $\bar{\boldsymbol{f}}_2$ is called *a regularized optimal control.*

The following result hold true.

Theorem 5. *(An optimality condition) Any regularized optimal control* $\bar{\boldsymbol{f}}_2$ *verifies*

$$\bar{\boldsymbol{f}}_2 = -\frac{1}{\beta}\boldsymbol{\gamma}(p(\bar{\boldsymbol{f}}_2)),$$

where $\boldsymbol{\gamma}$ *is the trace operator and* $\boldsymbol{p}(\bar{\boldsymbol{f}}_2)$ *is the unique solution of the variational equation*

$$\alpha(\boldsymbol{u}(\bar{\boldsymbol{f}}_2) - \boldsymbol{u}_d, \boldsymbol{w})_V = (\boldsymbol{p}(\bar{\boldsymbol{f}}_2), A\boldsymbol{w} + D_2^2 j_\rho(\boldsymbol{u}(\bar{\boldsymbol{f}}_2), \boldsymbol{u}(\bar{\boldsymbol{f}}_2))\boldsymbol{w})_V \quad \textit{for all } \boldsymbol{w} \in V,$$

$\boldsymbol{u}(\bar{\boldsymbol{f}}_2)$ *being the solution of (PS2) with* $\boldsymbol{f}_2 = \bar{\boldsymbol{f}}_2$.

Herein, for all $\boldsymbol{v} \in V$, writing $\boldsymbol{u}$ instead of $\boldsymbol{u}(\bar{\boldsymbol{f}}_2)$,

$$\begin{aligned}(D_2^2 j_\rho(\boldsymbol{u}, \boldsymbol{u})\boldsymbol{v}, \boldsymbol{w})_V = &\int_{\Gamma_3} \partial_2 p_\nu^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x}))\frac{u_\nu(\boldsymbol{x}) v_\nu(\boldsymbol{x}) w_\nu(\boldsymbol{x})}{\sqrt{u_\nu(\boldsymbol{x})^2 + \rho^2}}\, d\Gamma \\ &+ \int_{\Gamma_3} \partial_2 p_\tau^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x}))\frac{\boldsymbol{u}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x}) v_\nu(\boldsymbol{x})}{\sqrt{\|u_\tau(\boldsymbol{x})\|^2 + \rho^2}}\, d\Gamma \\ &+ \int_{\Gamma_3} + p_\nu^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x}))\frac{v_\nu(\boldsymbol{x}) w_\nu(\boldsymbol{x})\rho^2}{(u_\nu(\boldsymbol{x})^2 + \rho^2)^{3/2}}\, d\Gamma \\ + \int_{\Gamma_3} p_\tau^\rho(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g_a(\boldsymbol{x}))&\frac{\boldsymbol{v}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x})(\|\boldsymbol{u}_\tau\|^2 + \rho^2) - (\boldsymbol{u}_\tau \cdot \boldsymbol{w}_\tau)(\boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau)}{(\|\boldsymbol{u}_\tau(\boldsymbol{x})\|^2 + \rho^2)^{3/2}}\, d\Gamma.\end{aligned}$$

The main tool in the proof of Theorem 5 is a Lions's Theorem, which we recall here for the convenience of the reader.

Theorem 6. *Let* $\mathcal{B}$ *be a Banach space,* X *and* Y *two reflexive Banach spaces. Let also be given two* $\mathcal{C}^1$ *functions* $F : \mathcal{B} \times X \to Y$, $L : \mathcal{B} \times X \to \mathbb{R}$. *We suppose that, for all* $\beta \in \mathcal{B}$,

i) There exists a unique $\tilde{u}(\beta)$ *such that* $F(\beta, \tilde{u}(\beta)) = 0$,

ii) $\partial_2 F(\beta, \tilde{u}(\beta))$ *is an isomorphism from* X *onto* Y.

Then, $J(\beta) = L(\beta, \tilde{u}(\beta))$ *is differentiable and, for every* $\zeta \in \mathcal{B}$,

$$\frac{dJ}{d\beta}(\beta)\zeta = \partial_1 L(\beta, \tilde{u}(\beta))\zeta - \langle p(\beta), \partial_1 F(\beta, \tilde{u}(\beta))\zeta\rangle_{Y', Y},$$

where $p(\beta) \in Y'$ *is the adjoint state, unique solution of*

$$\Big[\partial_2 F(\beta, \tilde{u}(\beta))\Big]^* p(\beta) = \partial_2 L(\beta, \tilde{u}(\beta)) \quad \textit{in } X'.$$

For the proof of Theorem 6 we refer to, e.g., [1].

Let us indicate in the last part of this section two convergence results. The first one involves the unique solution of the regularized state problem (PS2) and the unique solution of the state problem (PS1).

Theorem 7. *Let* $\rho > 0$, $\boldsymbol{f}_0 \in L^2(\Omega)^3$ *and* $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$ *be given. If* $\boldsymbol{u}^\rho$, $\boldsymbol{u} \in V$ *are the solutions of problems* $(PS2)$ *and* $(PS1)$, *respectively, then,*

$$\boldsymbol{u}^\rho \to \boldsymbol{u} \text{ in } V \text{ as } \rho \to 0.$$

Next, we have a convergence result involving the solutions of the problems (POC2) and (POC1).

Theorem 8. *Let* $[\bar{\boldsymbol{u}}^\rho, \bar{\boldsymbol{f}}_2{}^\rho]$ *be a solution of the problem (POC2). Then, there exists a solution of the problem (POC1),* $[\boldsymbol{u}^*, \boldsymbol{f}_2^*]$ *, such that*

$$\bar{\boldsymbol{u}}^\rho \to \boldsymbol{u}^* \text{ in } V \text{ as } \rho \to 0,$$
$$\bar{\boldsymbol{f}}_2{}^\rho \rightharpoonup \boldsymbol{f}_2^* \text{ in } L^2(\Gamma_2)^3 \text{ as } \rho \to 0.$$

Theorems 3-8 are new results; their proofs will be published in [13].

Let us mention here some open questions:

- $\bar{\boldsymbol{f}}_2{}^\rho \to \boldsymbol{f}_2^*$ in $L^2(\Gamma_2)^3$ as $\rho \to 0$;
- an optimality condition for (PS1);
- to study the boundary optimal control of the model by means of the mixed variational formulation.

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