Dynamic Programming to Reconstruction Problems for a Macroeconomic Model

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Abstract. Perturbed inverse reconstruction problems for controlled dynamic systems are under consideration. A sample history of the actual trajectory is known. This trajectory is generated by a control, which isn't known. Moreover, the deviation of the samples from the actual trajectory satisfies the known estimate of the sample error. The inverse problem with perturbed (inaccurate) sample of trajectory consists of reconstructing trajectories which are close to the actual trajectory in C. Controls generating the trajectories should be close in L_2 to the normal control generating the actual trajectory and have the least norm in L_2. A numerical method for solving this problem is suggested. The application of the suggested method is illustrated by the graphics.

Keywords: inverse problem, positive and negative discrepancy, optimal control problem, dynamic programming

1 Introduction

A model of macroeconomics is considered due to works by E.G. Al'brekht [1]. The model has the form of two nonlinear ordinary differential equations. The right-hand sides of the equations depend on control parameters. The rate of taxation, the refunding rate and the currency exchange course are included in control parameters because they determine economical conditions for production activity. A sample history of the actual trajectory of the model is known. A numerical method is suggested and verified to reconstruct the actual trajectory and the control generating it. It is based on the method of the dynamic programming. Results of numerical calculations of the solutions of the inverse problem are represented for statistic data obtained from a reports of companies sent to local statistic offices in Russia.

2 Macroeconomic Model

Consider a model of a macroeconomic system, where the symbol x_1 denotes the gross product, x_2 denotes production costs, G denotes profits.

Let dynamics of $x_1(t)$, $x_2(t)$ be of the form

$$\begin{aligned} \frac{dx_1}{dt} &= u_1(t)\frac{\partial G(x_1,x_2)}{\partial x_1}, \\ \frac{dx_2}{dt} &= u_2(t)\frac{\partial G(x_1,x_2)}{\partial x_2} \end{aligned} \tag{1}$$

on a time interval $t \in [0,T]$. Here $u_1(t)$, $u_2(t)$ are control parameters, satisfying the geometric restrictions

$$|u_1| \leq U_1, \qquad |u_2| \leq U_2, \tag{2}$$

where $U_1 > 0, U_2 > 0$ are constants.

3 Known Data

We have got the following statistic data in the form of a table of parameters x_1^*, x_2^*, G^* measured at given instants t_i, $t_i = 0, 1, \ldots, N$, $t_0 = 0$, $t_N = T$,

$$\begin{array}{ccc} x_1^*(t_0), & x_1^*(t_1), \ldots, & x_1^*(t_N), \\ x_2^*(t_0), & x_2^*(t_1), \ldots, & x_2^*(t_N), \\ G^*(t_0), & G^*(t_1), \ldots, & G^*(t_N), \end{array}$$

where $x_1^*(t_i)$, $x_2^*(t_i)$ are measurements of the actual trajectory $x_{1*}(\cdot)$, $x_{2*}(\cdot)$ of the system (1) on the interval $[0,T]$.

4 Hypothesis

Following the Albrekht's works, we assume that the mathematical model of the measured dynamics meets the following assertions

• the structure of the function $G(x_1,x_2)$ has the form of the polynomial

$$G(x_1,x_2) = x_1x_2(a_0 + a_1x_1 + a_2x_2). \tag{3}$$

• the given statistic data are measurements of the actual trajectory $x_*(\cdot) = (x_{1*}(\cdot), x_{2*}(\cdot))$ and profit function $G(x_*(\cdot))$ with errors, while estimate δ on admissible errors is known.

$$\begin{aligned} |x_{1*}(t_i) - x_1^*(t_i)| \leq \delta, \qquad |x_{2*}(t_i) - x_2^*(t_i)| \leq \delta, \\ |G(x_{1*}(t_i), x_{2*}(t_i)) - G^*(t_i)| \leq \delta, \quad i = 0, 1, \ldots, N. \end{aligned} \tag{4}$$

• such smooth continuous interpolations $y(\cdot) = (y_1(\cdot), y_2(\cdot))$ of the data $x^*(t_i) = (x_1^*(t_i), x_2^*(t_i))$, $i = 0, 1, \ldots, N$ are defined, that

$$\left|\frac{d^2y_i(t)}{dt^2}\right| \leq K, \quad K > 0,\ t \in [0,T],\ i = 1, 2, . \tag{5}$$

$$\|y(\cdot) - x_*(\cdot)\|_c \to 0, \text{ as } \delta \to 0. \tag{6}$$

5 Reconstruction Problems

The inverse problems are identification problem and reconstruction problem for the model, which supposes reconstructing such trajectories $x^\delta(\cdot)$ of system (1) generated by measurable controls $u^\delta(\cdot)$, satisfying (2), that

$$\|x^\delta(\cdot) - x_*(\cdot)\|_C = \max_{t\in[0,T]} \|x^\delta(t) - x_*(t)\| \to 0, \text{ as } \delta \to 0;$$

$$\|u^\delta(\cdot) - u_*(\cdot)\|_{L_2}^2 = \int\limits_0^T \|u^\delta(t) - u_*(t)\|^2 dt \to 0, \text{ as } \delta \to 0;$$

where $x_*(\cdot) = (x_{1*}(\cdot), x_{2*}(\cdot))$ is the actual trajectory on $[0,T]$ generated by "normal" control $u_*(\cdot) = (u_{1*}(\cdot), u_{2*}(\cdot))$, which has the minimal norm in $L_2([0,T], R^2)$. The method suggested below is based on the dynamic programming [2] for auxiliary optimal control problems. It can be interpreted as a modification of Tikhonov method [3]. The other approach to solutions of the inverse problems with the help of optimal feedbacks [4] in auxiliary optimal control problems was suggested in works by Osipov, Kryazhimskii [5].

6 Identification Problem for the Function $G(x_1, x_2)$

At first we consider the identification problem for parameters a_0, a_1, a_2 of the polynomial

$$G(x_1, x_2) = x_1 x_2 (a_0 + a_1 x_1 + a_2 x_2)$$

to obtain the best correspondence with the given statistic materials.

In order to do this, we apply the least square method to the statistic data

$$\sum_{i=0}^{N} [G^*(t_i) - G(x_1^*(t_i), x_2^*(t_i))]^2 \longrightarrow \min_{(a_0,a_1,a_2)}.$$

7 Auxiliary Optimal Control Problems (AOCPs)

We introduce the following AOCPs to solve the reconstruction problem. Consider dynamics of the form

$$\begin{aligned} \tfrac{dx_1}{dt} &= u_1 \tfrac{\partial G(x_1,x_2)}{\partial x_1}, \\ \tfrac{dx_2}{dt} &= u_2 \tfrac{\partial G(x_2,x_2)}{\partial x_2}, \end{aligned} \tag{7}$$

$$t \in [0,T], \qquad u = (u_1, u_2) \in P,$$
$$P = \{|u_1| \le U_1, \qquad |u_2| \le U_2,\}. \tag{8}$$

The set of admissible controls is defined as

$$U_{[0,T]} = \{\forall u(\cdot) \colon [0,T] \to P \text{ — measurable}\}.$$

We introduce the α-regularized positive discrepancy functional

$$I^{+}_{0,x_1^0,x_2^0}(u(\cdot)) = \int\limits_0^T \frac{[(y_1(t)-x_1(t))^2+(y_2(t)-x_2(t))^2]}{2} + \alpha^2\frac{(u_1^2(t)+u_2^2(t))}{2}dt, \tag{9}$$

where α is a small parameter. The functions $y_1(\cdot), y_2(\cdot)$ are interpolations of statistic data.

We also consider the $\alpha-$regularized negative discrepancy functional

$$I^{-}_{0,x_1^0,x_2^0}(u(\cdot)) = \int\limits_0^T -\frac{[(y_1(t)-x_1(t))^2+(y_2(t)-x_2(t))^2]}{2} + \alpha^2\frac{(u_1^2(t)+u_2^2(t))}{2}dt. \tag{10}$$

8 Optimal Results in AOCPs

Let small parameters $\alpha > 0, \delta > 0$ be fixed and interpolations $y_1(\cdot), y_2(\cdot)$ of the statistic data be known. The aim of the AOCPs at an initial state $t = 0$, $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ is to minimize the cost functionals (10), (9) under the condition

$$x_1(T) = y_1(T), \quad x_2(T) = y_2(T). \tag{11}$$

The optimal results in the class $U_{[0,T]}$ are equal to

$$V^{\pm}(0, x_1^0, x_2^0) = \inf_{u(\cdot)\in U_{[0,T]}} I^{\pm}_{t_0,x_1^0,x_2^0}\big(u(\cdot)\big). \tag{12}$$

8.1 Hamiltonian

Let's consider the AOCP for the negative discrepancy functional (10). Let us denote

$$\omega_1(x) = \omega_1(x_1,x_2) = \frac{\partial G(x_1(t),x_2(t))}{\partial x_1} = a_0x_2 + 2a_1x_1x_2 + a_2x_2^2,$$
$$\omega_2(x) = \omega_2(x_1,x_2) = \frac{\partial G(x_1(t),x_2(t))}{\partial x_2} = a_0x_1 + a_1x_1^2 + 2a_2x_1x_2.$$

$$H^{\alpha}(t,x_1,x_2,s_1,s_2) = \min_{u\in P}\Big[s_1u_1\omega_1(x_1,x_2) + s_2u_2\omega_2(x_1,x_2) + \\ + \frac{\alpha^2(u_1^2+u_2^2)}{2} - \frac{(x_1-y_1(t))^2+(x_2-y_2(t))^2}{2}\Big] =$$

$$= \Big[s_1 u_1^0 \omega_1(x_1,x_2) + s_2 u_2^0 \omega_2(x_1,x_2) + \frac{\alpha^2({u_1^0}^2 + {u_2^0}^2)}{2} \\ - \frac{(x_1 - y_1(t))^2 + (x_2 - y_2(t))^2}{2}\Big]. \tag{13}$$

where for $i = 1, 2$,

$$u_i^0(x,s) = \begin{cases} -U_i, & \text{if} \quad -\frac{s_i\omega_i(x(t))}{\alpha^2} \le -U_i, \\ -\frac{s_1\omega_i(x(t))}{\alpha^2}, & \text{if} \quad -\frac{s_1\omega_i(x(t))}{\alpha^2} \in [-U_i, U_i], \\ U_i, & \text{if} \quad -\frac{s_1\omega_i(x(t))}{\alpha^2} \ge U_i. \end{cases}$$

So, for the simple case

$$u_i^0(x,s) \in [-U_i, U_i], \qquad i = 1, 2, \tag{14}$$

we get Hamiltonian of the form

$$H^\alpha(t, x_1, x_2, s_1, s_2) = -\frac{1}{2\alpha^2}(s_1^2 + s_2^2) - \frac{(x_1 - y_1(t))^2 + (x_2 - y_2(t))^2}{2}.$$

8.2 Characteristics

Necessary optimality conditions for the AOCPs has the following form [6, 7]: the characteristic system

$$\frac{dx_i}{dt} = \frac{\partial H^\alpha(t,x,s)}{\partial s_i}, \quad \frac{ds_i}{dt} = -\frac{\partial H^\alpha(t,x,s)}{\partial x_i}, \quad i = 1, 2, \quad t \in [0, T], \tag{15}$$

and the boundary conditions

$$x_i(T) = y_i(T), \quad s_i(T) = \xi_i, \left|\frac{\omega_i^2(x(T))\xi_i}{\alpha^2} - \dot{y}_i(T)\right| \le \delta, \quad i = 1, 2. \tag{16}$$

8.3 Characteristics for the Simple Case

Restrictions U_1, U_2 for admissible controls are usually unknown. To simplify the explanations we assume that U_1, U_2 are large enough to let interpolations $y(t)$ provide the simple case(14) with boundary conditions (16).

The characteristic system for the simple case has the form:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\frac{s_1(t)}{\alpha^2}\omega_1^2(x_1(t), x_2(t)), \\ \frac{dx_2(t)}{dt} &= -\frac{s_2(t)}{\alpha^2}\omega_2^2(x_1(t), x_2(t)), \\ \frac{ds_1(t)}{dt} &= x_1(t) - y_1(t) + \\ &+\frac{s_1^2(t)}{\alpha^2}F_1(x_1(t), x_2(t)) + \frac{s_2^2(t)}{\alpha^2}F_2(x_1(t), x_2(t)), \\ \frac{ds_2(t)}{dt} &= x_2(t) - y_2(t) + \\ &+\frac{s_1^2(t)}{\alpha^2}F_3(x_1(t), x_2(t)) + \frac{s_2^2(t)}{\alpha^2}F_4(x_1(t), x_2(t)), \end{aligned} \tag{17}$$

where

$$
\begin{aligned}
F_1(x_1,x_2) &= 2a_1x_2(a_0x_2 + 2a_1x_1x_2 + a_2x_2^2),\\
F_2(x_1,x_2) &= (a_0 + 2a_1x_1 + 2a_2x_2)(a_0x_1 + a_1x_1^2 + 2a_2x_1x_2),\\
F_3(x_1,x_2) &= (a_0 + 2a_1x_1 + 2a_2x_2)(a_0x_2 + 2a_1x_1x_2 + a_2x_2^2),\\
F_4(x_1,x_2) &= 2a_2x_1(a_0x_1 + a_1x_1^2 + 2a_2x_1x_2),
\end{aligned}
$$

boundary conditions

$$
\begin{gathered}
x_1(T) = y_1(T),\ x_2(T) = y_2(T),\\
\xi_1^- \le s_1(T) = \xi_1 \le \xi_1^+,\ \xi_2^- \le s_2(T) = \xi_2 \le \xi_2^+,
\end{gathered}
\tag{18}
$$

where

$$
\begin{gathered}
\xi_i^- = -\frac{\dot y_i(T)\alpha^2}{\omega_i(y(T))^2} - \frac{\delta\alpha^2}{\omega_i(y(T))^2},\\
\xi_i^+ = -\frac{\dot y_i(T)\alpha^2}{\omega_i(y(T))^2} + \frac{\delta\alpha^2}{\omega_i(y(T))^2},\\
i = 1,2.
\end{gathered}
\tag{19}
$$

9 Solutions of Inverse Problems

Let us pick such characteristics (15)–(19) $x_\delta^\alpha(\cdot)$ and the realizations of extremal feedbacks $u_\delta^\alpha[t] = u^\alpha(t, x_\delta^\alpha(t), s_\delta^\alpha(t))$, generating them, which satisfy the relations:

$$
\begin{gathered}
\|x(0,\xi) - y(0)\| \le \alpha + \delta,\\
I^{\pm}_{0,x_\delta^\alpha(0)}(u_\delta^\alpha[\cdot]) = \min_{\|x(0,\xi)-y(0)\|\le\alpha+\delta} I^{\pm}_{0,x(0)}(u^\alpha(\cdot)) = V^{\pm}(0, x_\delta^\alpha(0)),\\
u^\alpha(t) = u^\alpha(t, x(t,\xi), s(t,\xi)), \quad t \in [0,T].
\end{gathered}
\tag{20}
$$

We have got that these characteristics $x_\delta^\alpha(\cdot,\xi)$ and controls $u_\delta^\alpha[\cdot]$, generating them, provide solutions to the inverse problems [8–10].

9.1 Assumptions

$A1$ Such constants $\alpha_0 > 0,\ \delta_0 > 0$ exist that state characteristics $x_1(t,\xi)$ and $x_2(t,\xi)$ of the form (15)–(19) for all $t \in [0,T]$ belong to the compact set Φ:

$$
\begin{gathered}
\Phi \supset \Phi(\delta,\alpha) \quad \forall\ \delta,\ \alpha: \quad 0 < \delta \le \delta_0, \quad 0 < \alpha \le \alpha_0,\\
\Phi(\delta,\alpha) = \Big\{(t,x)\colon t \in [0,T],\ x = x(t,\xi),\\
x(T,\xi) = y(T), \quad \left|\frac{\omega_i^2(x(T))\xi_i}{\alpha^2} - \dot y_i(T)\right| \le \delta, \quad i = 1,2\Big\}.
\end{gathered}
\tag{21}
$$

$A2$ For $(x_1,x_2) \in \Phi$ such constants $\underline{\omega}_i > 0,\ \bar\omega_i > 0,\ i = 1,2$ exist, that

$$
0 < \underline{\omega}_1^2 \le \omega_1^2(x_1(t),x_2(t)) \le \bar\omega_1^2, \quad 0 < \underline{\omega}_2^2 \le \omega_2^2(x_1(t),x_2(t)) \le \bar\omega_2^2, \quad t \in [0,T].
$$

9.2 Note

In the example below, one can choose such $\alpha_0 > 0$, $\delta_0 > 0$, $r > 0$, that

$$\Phi = \Phi^r = \{(t,x)\colon t \in [0,T],\ \|x - y(t)\| \le r\},$$
$$\min_{0\le t\le T} y_i(t) > 3r > 0,\ i = 1,2,$$

and assumptions $A1$–$A2$ are true.

9.3 Main Result

Let us consider AOCPs for the system (7), (8) at initial states

$$x(0) \in \{x:\ \|x - y(0)\| \le \delta + \alpha\}$$

where the aim is to reach the target set $\{T, x = y(T)\}$ and minimize the functional (10).

The following assertions are proven [9, 10].

Lemma 1. *Let $x_\delta^\alpha(t)$ be a solution of the AOCP* (7), (8), (10). *Let $u_\delta^\alpha(t)$ be a control generating $x_\delta^\alpha(t)$. If conditions* $\mathbf{A1}$–$\mathbf{A2}$ *are true in the problem, then such constant $c > 0$ exists that the following estimate takes place:*

$$I_{0,x_\delta^\alpha(0)}(u_\delta^\alpha(\cdot)) \le I_{0,x_*(0)}(u_*(\cdot)) + \zeta(\alpha,\delta), \quad \zeta(\alpha,\delta) = c\delta(\delta^2 + \alpha^2 U_*^2),$$

where $U_* = \max\{U_1, U_2\}$.

We introduce the functions

$$\phi(\alpha,\delta,h) = TMh\left(\frac{TMh}{2} + 2\delta + \alpha + \zeta(\alpha,\delta)\right), \qquad \rho(h) = nU_*T(K+M)h,$$

where $K,\ M$ are constant parameters.

Let us denote numerical approximations of the solution $x_\delta^\alpha(\cdot)$, $u_\delta^\alpha(\cdot)$ of AOCP (7), (8), (10) as $x_h(\cdot)$, $u_h(\cdot)$.

Theorem 1. *Let conditions* $\mathbf{A1}$ – $\mathbf{A2}$ *be true in AOCP* (7), (8), (10). *Then there exists such constants $M > 0$, $K > 0$ and parameters $h = h(\delta) > 0$, $\alpha = \alpha(\delta) > 0$, $\delta > 0$, satisfying the conditions $\lim_{\delta\to 0} h(\delta) = 0$, $\lim_{\delta\to 0} \alpha(\delta) = 0$,*

$$\lim_{\delta\to 0} \frac{2}{\alpha^2}\left(\phi(\alpha,\delta,h) + \rho(h) + \frac{T}{2}(Mh + \alpha + 2\delta + \zeta(\alpha,\delta))^2\right) = 0, \tag{22}$$

that the following relations are true

$$\lim_{\delta\to 0} \|x_{h(\delta)}(\cdot) - x_*(\cdot)\|_C = 0, \quad \lim_{\delta\to 0} \|u_{h(\delta)}(\cdot) - u_*(\cdot)\|_{L_2} = 0.$$

10 Numerical experiments

Results of application of the suggested numerical method via AOCP with the functional $I^{-}_{(0,x(0))}(u(\cdot))$ are exposed on the figures 1–6 below.

Note that the results obtained via AOCP with the functional $I^{+}_{(0,x(0))}(u(\cdot))$ are not so satisfying (see figures 7,8). This is because of the properties of characteristics in the considered AOCPs.

We used the data on the industry of the Ural Region in Russia for the period 1970–1985 (10000 Rubles = 1) due to paper [1]:

t	Year	Gross regional product x_1^*	Costs x_2^*	Profit G^*
0	1970	37.88	21.69	6.17
1	1971	40.63	23.70	6.31
2	1972	43.25	25.45	6.68
3	1973	46.00	27.30	6.98
4	1974	49.33	29.44	7.04
5	1975	53.04	32.16	7.27
6	1976	57.03	35.01	7.62
7	1977	59.85	36.92	8.00
8	1978	62.72	38.69	8.27
9	1979	63.45	38.76	8.42
10	1980	65.74	39.96	8.61
11	1981	65.90	39.75	8.21
12	1982	69.22	41.31	9.65
13	1983	64.52	37.86	9.28
14	1984	71.03	42.04	10.26
15	1985	74.69	45.05	10.76

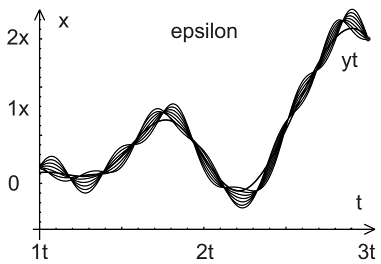

Fig. 1. Trajectory bundle obtained with $\alpha^2 = 10^{-4},\ t \in [1, 1.5]$

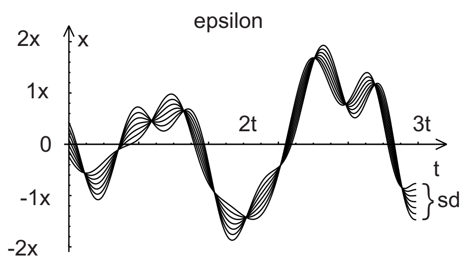

Fig. 2. Controls bundle obtained with $\alpha^2 = 10^{-4}, t \in [1, 1.5]$.

Fig. 3. Discrepancy $x_1(t) - y_1(t)$ with $\alpha^2 = 10^{-4}, \quad t \in [0, 1.5]$

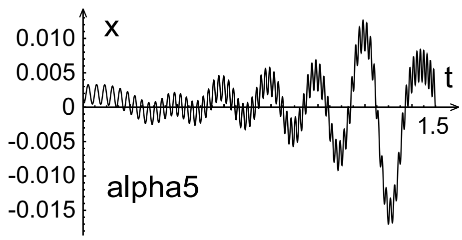

Fig. 4. Discrepancy $x_1(t) - y_1(t)$ with $\alpha^2 = 10^{-5}, \quad t \in [0, 1.5]$

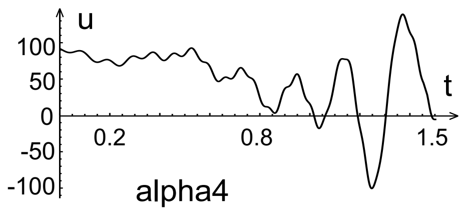

Fig. 5. Control $u_1(t)$ with $\alpha^2 = 10^{-4}$, $t \in [0, 1.5]$

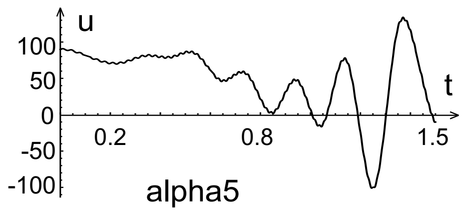

Fig. 6. Control $u_1(t)$ with $\alpha^2 = 10^{-5}$, $t \in [0, 1.5]$

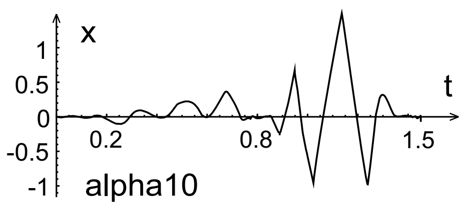

Fig. 7. Discrepancy $x_1(t) - y_1(t)$ for functional $I^+(\cdot)$ with $\alpha^2 = 10^{-10}$, $t \in [0, 1.5]$

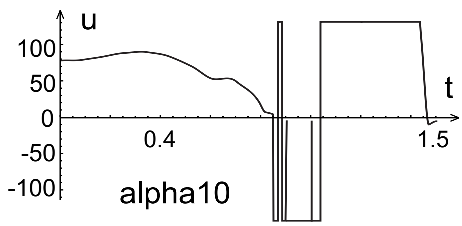

Fig. 8. Control $u_1(t)$ for functional $I^+(\cdot)$ with $\alpha^2 = 10^{-10}$, $t \in [0, 1.5]$

11 Perspectives

The suggested numerical method can be applied in the following directions.

– Identification and reconstruction of dynamic models of production activity for single firms, various branches of industry or industry and economics of a region.
– Investigating properties of the examined object.
– A short-term and long-term prediction and analysis of scenarios of the process development in the future.
– Analysis of the production plan and construction of feedback controls realizing the plan.

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References

1. **Al'brekht E.G.** *Methods of construction and identification of mathematical models for macroeconomic processes* // electronic journal "Investigated in Russia" (http:://zhurnal.ape.relarn.ru/articles/2002/005.pdf) [in Russian]
2. **Bellman R.** *Dynamic Programming.* Princeton: Princeton University Press., 1957. 340 p.
3. **Tihonov A.N.** *Solution of incorrectly formulated problems and the regularization method* //Soviet Math. 1963. Vol.4. P.1035–1038.
4. **Krasovskii N.N., Subbotin A.I.** *Game–Theoretical Control Problems.* NY: Springer-Verlag. 1987.
5. **Osipov Yu.S., Kryazhimskii A.V.** *Inverse Problems for Ordinary Differential Equations: Dynamical Solutions.* London: Gordon and Breach, 1995.
6. **Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F.** *The Mathematical Theory of Optimal Processes.* Moscow.: Nauka, 1961. 392 p. Wiley, New York, 1962.
7. **Clarke F.** *Optimization and nonsmooth analysis.* New York: John Wiley and Sons, Inc., 1983. 308 p.
8. **Subbotina N.N., Kolpakova E.A, Tokmantsev T.B., Shagalova L.G.** *The method of characteristics for Hamilton–Jacobi–Bellman equation.* Ekaterinburg: RIO UrO RAN, 2013. 144 p. [in Russian]
9. **Krupennikov E.A.** *Validation of a solution method for the problem of reconstructing the dynamics of a mecroeconomic system* // Proceedings of the Institute of Mathematics and Mechanics, Ekaterinburg: RIO UrO RAN. 2015. Vol.21. No.2. P.100–112. [in Russian]
10. **Subbotina N.N., Tokmantsev T.B.** *A Study of the Stability of Solutions to Inverse Problems of Dynamics of Control Systems under Perturbations of Initial Data//* Proceedings of the Steklov Institute of Mathematics, 2015, Vol. 291, Suppl. 1, pp. S1–S17.