Differentials and Semidifferentials for Metric Spaces of Shapes and Geometries

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Abstract. The *Hadamard semidifferential* retains the advantages of the differential calculus such as the *chain rule* and semiconvex functions are Hadamard semidifferentiable. The *semidifferential calculus* extends to subsets of $\mathbb{R}^n$ without Euclidean smooth structure. This set-up is an ideal tool to study the semidifferentiability of objective functions with respect to families of sets which are non-linear non-convex complete metric spaces. *Shape derivatives* are differentials for spaces endowed with *Courant metrics*. *Topological derivatives* are shown to be *semidifferentials* on the group of Lebesgue measurable characteristic functions.

Keywords: Semidifferential, shape and topological derivatives

1 Introduction

In the past decades, direct constructions of complete *metric spaces of shapes and geometries* (cf., for instance, M. C. Delfour and J.-P. Zolésio [7]) and, additional new ones (cf., M. C. Delfour [6]) have been given without appealing to the classical notions of atlases or smooth manifolds encountered in classical Differential Geometry. Since, at best, such spaces are groups, the issue of making sense of tangent spaces and differentials naturally arises not only for "differentiable" functions but also for large classes of "non-differentiable" functions.

In that context, the geometrical definition of a differentiable function of J. Hadamard [11] is especially interesting since it implicitly involves the construction of trajectories (or paths) and tangent vectors to trajectories living in the space under investigation. His definition was relaxed by M. Fréchet [10] in 1937 by dropping the requirement that the differential be linear with respect to the direction or tangent vector while preserving two important properties of the differential calculus: the continuity of the function and the chain rule. A vast litterature on differentials on topological spaces followed (cf., for instance, the survey papers of V. I. Averbuh and O. G. Smoljanov [3] in 1988 and the 207-page paper of M. Z. Nashed [12] for a rather complete account until 1971). The

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definition of Fréchet can be further relaxed to the one of semidifferential which handles convex and semiconvex functions while preserving the two properties.

Since the semidifferential is not required to be linear, they have far reaching consequences for a function $f : A \to B$ between arbitrary sets A and B. De facto, this relaxes the requirement that the tangent spaces in each points of the sets A and B be linear spaces. It is sufficient to work with tangent cones to A and B such as Bouligand's tangent cone to make sense of semidifferentials. Shortcircuiting the requirement of a smooth manifold makes it possible to directly study the tangent spaces to non-convex metric spaces of shapes and geometries.

We show that the metric group of Lebesgue measurable characteristic functions has semi-tangents and that the notion of *topological derivative* of J. Sokołowski and A. Zòchowski [14] is in fact a semidifferential obtained by dilatation of a point creating a hole. By extending this construction via *dilatations*, we also show that the tangent space contains distributions creating topological perturbations along curves and surfaces that can break the connectivity of the set. In the same spirit *dilatations* of *d-rectifiable* and some H_d-rectifiable compact sets (H_d, d-dimensional Hausdorff measure) of Ambrosio et al [1] also generate semi-tangents. *Orthogonal dilatations* of closed subsets of the boundary of a set of positive reach can also be used via Steiner formula (see Federer [8]).

2 Hadamard Differential and Semidifferential

Hadamard Differential. In 1923 J. Hadamard [11] gave a *geometrical definition* by using an *auxiliary function* $t \mapsto x(t) : \mathbb{R} \to \mathbb{R}^{\mathrm{N}}$ such that

$$x(0) = a \quad \text{and} \quad x'(0) \stackrel{\text{def}}{=} \lim_{t \to 0} \frac{x(t) - a}{t} \text{ exists in } \mathbb{R}^{\mathrm{N}},$$

where $\mathbb{R}$ is the field of real numbers. It defines a path that induces a perturbation or a variation of the point a. We shall use the terminology *time* for the auxiliary variable t and *admissible trajectory* for the auxiliary function x. Note that x need not be continuous or differentiable at $t \neq 0$. The *vector* $x'(0)$ *is the tangent to the trajectory* x at the point $x(0) = a$. Scaling t by an arbitrary non-zero real number generates a whole line tangent to x at a

Definition 1. A function $f : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^K$ *is Hadamard differentiable at* $a \in \mathbb{R}^{\mathrm{N}}$ if

(i) for all *admissible trajectories* x at a the limit
$$(f \circ x)'(0) \stackrel{\text{def}}{=} \lim_{t \to 0} \frac{f(x(t)) - f(a)}{t} \text{ exists in } \mathbb{R}^K$$

(ii) and there exists a *linear function* $Df(a) : \mathbb{R}^N \to \mathbb{R}^K$ such that for all admissible trajectories x at a
$$(f \circ x)'(0) = Df(a)\,(x'(0)).$$

$Df(a)$ is the *differential of* f *at* a. □

The definition of *Hadamard differentiability* is equivalent to the one of Fréchet differentiability in finite dimension. In Banach and Fréchet spaces, a *Hadamard differentiable function* at a point a is *continuous at* a and the *chain rule is applicable*. In 1937, Fréchet [10] *insisted* on the fact that the *definition of Hadamard is more general than his* since it extends to functions $f : X \to \mathbb{R}^K$ defined on *topological vector spaces* X that are not *normed vector spaces*. Furthermore, in *Banach spaces of functions*, we can consider the set of *tangent vectors (functions)* $x'(0)$ as *weak limits* ... and even as *distributions*.

In his 1937 paper, Fréchet [10] observed that, in *function spaces*, the Hadamard differentiability is *not only* a notion more general than the one he introduced in 1911 but that the *linearity in part (ii) is not necessary* to preserve the continuity of the function and the chain rule. He gives the following example:

$$f(x_1, x_2) \stackrel{\text{def}}{=} x\sqrt{\frac{x_1^2}{x_1^2 + x_2^2}} \quad \text{for } (x_1, x_2) \neq (0,0) \quad \text{and} \quad f(0,0) \stackrel{\text{def}}{=} 0.$$

([10, p. 239]). Indeed, it is readily checked that for any trajectory $x : \mathbb{R}^2 \to \mathbb{R}$ such that $x(0) = (0,0)$ and $x'(0)$ exists

$$(f \circ x)'(0) \stackrel{\text{def}}{=} \lim_{t \to 0} \frac{f(x(t)) - f(0,0)}{t} = f(x'(0)).$$

Hadamard always insisted on the linearity and this new notion was criticized by P. Lévy. Yet, his example shows that such *nondifferentiable functions* exist.

Hadamard Semidifferential. By relaxing the linearity, we can deal with some families of non-differentiable functions. Unfortunately, some *convex continuous functions* and, in particular, the *norm* $\|x\|$ in $a = 0$, are *not differentiable* in this relaxed sense. To get around this, we need the notion of *semidifferential.*

For instance, in the case of the Euclidean norm $x \mapsto f(x) = \|x\| : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ at $x = 0$, consider a *semi-trajectory* $x : [0, +\infty) \to \mathbb{R}^N$ through the origin $x(0) = 0$ for which the right-hand limit $x'(0^+)$ exists. We get at $a = 0$

$$(f \circ x)'(0^+) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{f(x(t)) - f(0)}{t} = \lim_{t \searrow 0} \left\| \frac{x(t) - x(0)}{t} \right\| = \|x'(0^+)\|,$$

where the notation $t \searrow 0$ means that t goes to 0 by strictly positive values. We have a similar result for *convex and semiconvex* continuous functions. When $(f \circ x)'(0^+)$ is not a linear function of the semi-tangent $x'(0^+)$, we say that the function is *semidifferentiable.*

From Linear to Non-convex Spaces. The *hypothesis of linearity of the differential* is also a severe restriction to define a differential for a function $f : A \subset \mathbb{R}^{\mathrm{N}} \to B \subset \mathbb{R}^K$ since it requires that the *tangent space to* A at a and the *tangent space to* B at $f(a)$ be *linear subspaces*. This necessitates that the sets A and B be *sufficiently smooth* in the sense that, at each point of A and of B, the tangent spaces be linear subspaces of $\mathbb{R}^{\mathrm{N}}$ and $\mathbb{R}^K$.

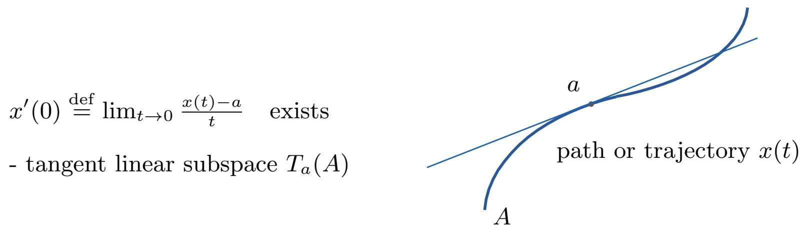
Since the Hadamard semidifferential does not require the linearity of the tangent space, the *a priori smoothness assumption* of the sets A and B can be de facto dropped since the semidifferential only needs to be defined on a tangent cone. Several tangent cones are available in the literature, but the following one is especially well suited for semidifferentials.

Definition 2. The *Bouligand tangent cone* to a set A at a point $a \in \overline{A}$ is

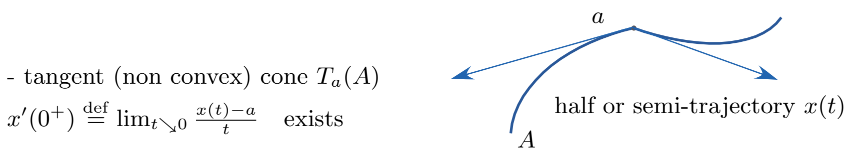
$$T_a A \overset{\text{def}}{=} \left\{ v \in \mathbb{R}^N : \exists \{x_n\} \subset A \text{ and } \{t_n \searrow 0\} \text{ such that } \lim_{n\to\infty} \frac{x_n - a}{t_n} = v \right\}.$$

□

When the boundary ∂A of A is smooth, $T_a A$ is a linear subspace of $\mathbb{R}^N$. However,



the linearity of $T_a A$ puts a severe restriction on the sets A. For instance, the requirement that $T_a A$ be linear rules out a curve in $\mathbb{R}^2$ with kinks.



This naturally leads to the following notions of admissible trajectory.

Definition 3. Given $A \subset \mathbb{R}^{\mathrm{N}}$, an *admissible semi-trajectory* in A at $a \in \overline{A}$ is a function $x : [0, \tau] \to A$, $\tau > 0$, such that the *semi-tangent* at a

$$x'(0^+) \overset{\text{def}}{=} \lim_{t\searrow 0} \frac{x(t) - a}{t}$$

exists. When the limit $x'(0^+)$ exists, it follows that $x(t) \to a$ as $t \searrow 0$. □

An equivalent characterization of the Bouligand's tangent cone is obtained.

Theorem 1. $T_a A = \{x'(0^+) : x$ *is an admissible semi-trajectory in* A *at* $a\}$.

Following Fréchet, we now relax the linearity and formalize the notion of semidifferential for functions $f : A \to B$.

Definition 4 (Geometrical definition)**.** Given $A \subset \mathbb{R}^{\mathrm{N}}$ and $B \subset \mathbb{R}^K$, the function $f : A \to B$ is *Hadamard semidifferentiable at* $a \in A$ if

(i) for each *admissible semi-trajectory* x in A at a, the limit

$$(f \circ x)'(0^+) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{f(x(t)) - f(a)}{t} \text{ exists}$$

(ii) and there exists a (positively homogeneous) function $v \mapsto d_A f(a; v) : T_a A \to T_{f(a)} B$ such that for all *admissible semi-trajectories* x in A at a

$$(f \circ x)'(0^+) = d_A f(a; x'(0^+)).$$

The function $v \mapsto d_A f(a)(v) = d_A f(a; v)$ is referred to as the *(tangential) semidifferential of* f at $a \in A$. It can be shown that $d_A f(a)$ is continuous on $T_a A$. □

This definition has an equivalent analytical counterpart.

Theorem 2 (Analytical definition)**.** *Given $A \subset \mathbb{R}^{\mathrm{N}}$ and $B \subset \mathbb{R}^K$, the function $f : A \to B$ is* Hadamard semidifferentiable at $a \in A$ *if and only if there exists a (positively homogeneous) function $v \mapsto d_A f(a; v) : T_a A \to T_{f(a)} B$ such that for all $v \in T_a A$ and all sequences $\{x_n\} \subset A$ and $\{t_n \searrow 0\}$ such that $(x_n - a)/t_n \to v$*

$$\lim_{n \to \infty} \frac{f(x_n) - f(a)}{t_n} = d_A f(a; v).$$

With the above definitions, the two important properties are preserved: continuity of f at a and the *chain rule*. The previous definitions extend to subsets A of topological vector spaces X, but we have to be careful and retain the abstract notions that are really meaningful. For shapes and geometries, the subset A will be a *complete metric space* with or without a group structure in a *surrounding Banach or Fréchet space*. We consider *Courant metrics* and the metric space of characteristic functions. Oriented distance functions can also be considered.

3 Metric Group of Characteristic Functions

Consider the metric Abelian group of characteristic functions on $\mathbb{R}^{\mathrm{N}}$

$$\mathrm{X}(\mathbb{R}^{\mathrm{N}}) = \left\{ \chi_\Omega : \Omega \subset \mathbb{R}^{\mathrm{N}} \text{ Lebesgue measurable} \right\} \subset L^\infty(\mathbb{R}^{\mathrm{N}}).$$

It is a *closed subset without interior* of the Banach space $L^\infty(\mathbb{R}^{\mathrm{N}})$ and of the Fréchet spaces $L^p_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$, $1 \le p < \infty$. The analog would be the sphere in $\mathbb{R}^3$.

3.1 Velocity Method

For the *velocity method*, consider the following *continuous trajectory* in $X(\mathbb{R}^{\mathrm{N}})$

$$t \mapsto \chi_{T_t(V)(\Omega)} : [0,1] \to X(\mathbb{R}^{\mathrm{N}}), \quad \frac{dT_t(V)}{dt} = V(t) \circ T_t(V),\ T_0(V) = I.$$

The *semitangent* at χ_Ω is obtained by considering the limit of the differential quotient $\left(\chi_{T_t(V)(\Omega)} - \chi_\Omega\right)/t \in L^\infty(\mathbb{R}^{\mathrm{N}})$ which does not exist in $L^\infty(\mathbb{R}^{\mathrm{N}})$, but also not in $L^p_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$, $1 \le p < \infty$.

To get a derivative consider the *distribution* associated with $\chi_{T_t(V)(\Omega)}$

$$\phi \mapsto \int_{\mathbb{R}^{\mathrm{N}}} \chi_{T_t(V)(\Omega)}\, \phi\, dx = \int_{T_t(V)(\Omega)} \phi\, dx = \int_\Omega \phi \circ T_t \ \det DT_t\, dx : \mathcal{D}(\mathbb{R}^{\mathrm{N}}) \to \mathbb{R}$$

If $V \in C^{0,1}(\overline{\mathbb{R}^{\mathrm{N}}}, \mathbb{R}^{\mathrm{N}})$, then

$$\left.\frac{d}{dt}\right|_{t=0^+} \int_\Omega \phi \circ T_t \ \det DT_t\, dx = \int_\Omega \mathrm{div}\ (V(0)\, \phi)\ dx = \int_{\mathbb{R}^{\mathrm{N}}} \chi_\Omega\, \mathrm{div}\ (V(0)\, \phi)\ dx$$

(see, for instance, [7, Thm. 4.1, Chapter 9, p. 483]). The bilinear function

$$(\phi, V) \mapsto \int_{\mathbb{R}^{\mathrm{N}}} \chi_\Omega\, \mathrm{div}\ (V(0)\, \phi)\ dx : H^1_0(\mathbb{R}^{\mathrm{N}}) \times C^{0,1}(\overline{\mathbb{R}^{\mathrm{N}}}, \mathbb{R}^{\mathrm{N}}) \to \mathbb{R}$$

is continuous. This generates the continuous linear mapping $V \mapsto \nabla\chi_\Omega \cdot V : C^{0,1}(\overline{\mathbb{R}^{\mathrm{N}}}, \mathbb{R}^{\mathrm{N}}) \to H^{-1}(\mathbb{R}^{\mathrm{N}})$

$$(\nabla\chi_\Omega \cdot V)\, \phi \overset{\mathrm{def}}{=} \int_{\mathbb{R}^{\mathrm{N}}} \chi_\Omega\, \mathrm{div}\ (V(0)\, \phi)\ dx,$$

where $\nabla\chi_\Omega$ is the distributional gradient of χ_Ω. The support of $\nabla\chi_\Omega \cdot V$ is in Γ, the boundary of Ω. So, the tangent space to $X(\mathbb{R}^{\mathrm{N}})$ (considered as a subset of the space of distributions) at χ_Ω contains the linear subspace

$$\left\{\nabla\chi_\Omega \cdot V : V \in C^{0,1}(\overline{\mathbb{R}^{\mathrm{N}}}, \mathbb{R}^{\mathrm{N}})\right\} \subset H^{-1}(\mathbb{R}^{\mathrm{N}}) \subset \mathcal{D}(\mathbb{R}^N)'$$

of functions in $H^{-1}(\mathbb{R}^{\mathrm{N}})$. When Ω is an open set with Lipschitz boundary

$$\left.\frac{d}{dt}\right|_{t=0^+} \int_{\mathbb{R}^{\mathrm{N}}} \chi_{T_t(V)(\Omega)}\, \phi\, dx = \int_\Gamma V(0) \cdot n_\Gamma\, \phi\, dH_{N-1}$$

is a bounded measure, where H_d is the d-dimensional Hausdorff measure.

3.2 Topological Derivative via Dilatations

The rigorous introduction of the *topological derivative* in 1999 by Sokołowski and Zòchowski [14]) (see also the book by Novotny-Sokołowski [13]) opened a broader spectrum of notions of "differential" with respect to a set. The set Ω is topologically perturbed by introducing a *small hole* around a point $a \in \Omega$, that is, a dilatation of a. This idea can be readily extended to some families of closed subsets E of Ω of dimension d, $1 \le d \le N-1$, for which $H_d(E)$ is finite.

Given $\Omega \subset \mathbb{R}^N$ open, we consider several examples where m_N denotes the Lebesgue measure in $\mathbb{R}^N$. The *distance function* $d_E(x)$ of x to a subset $E \subset \mathbb{R}^N$ and the *r-dilatation of E* are defined as

$$d_E(x) \overset{\mathrm{def}}{=} \inf_{y \in E} |x - y|, \quad E_r \overset{\mathrm{def}}{=} \{x \in \mathbb{R}^N : d_E(x) \le r\}. \tag{3.1}$$

Example 1. $E = \{a\}$, $a \in \mathbb{R}^3$, $\dim E = 0$. The *r-dilatation* of E is $\bar{B}_r(a)$,

$$t \stackrel{\text{def}}{=} \mathrm{m}_3(\bar{B}_r(a)) = \alpha_3\, r^3, \quad \alpha_3 = 4\pi/3 = \text{ volume of unit ball in } \mathbb{R}^3$$
$$\phi \mapsto \phi(a) : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R} \text{ is a distribution.}$$

Assuming that $\bar{B}_r(a) \subset \Omega$ for some $r > 0$, the perturbed sets will be

$$t \mapsto \Omega_t \stackrel{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash \bar{B}_{\sqrt[3]{t/\alpha_3}}(a).$$

Given $\phi \in \mathcal{D}(\mathbb{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned} \frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right] &= -\frac{1}{\mathrm{m}_3(\bar{B}_{\sqrt[3]{t/\alpha_3}}(a))} \int_{\bar{B}_{\sqrt[3]{t/\alpha_3}}(a)} \chi_\Omega\, \phi\, dx \\ &= -\frac{1}{\mathrm{m}_3(\bar{B}_r(a))} \int_{\bar{B}_r(a)} \chi_\Omega\, \phi\, dx \to -\phi(a). \end{aligned}$$

This distribution is a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_\Omega \phi\, dx\right] \to -\rho\phi(a).$$

□

Example 2. Let $A \subset \mathbb{R}^3$ be an open set of class $C^{1,1}$, ∂A compact, $\dim \partial A = 2$, and $b_A(x) \stackrel{\text{def}}{=} d_A(x) - d_{\mathbb{R}^3 \backslash A}(x)$ be the *oriented distance function*, then

$$\exists \varepsilon > 0 \text{ such that } b_A \in C^{1,1}(U_\varepsilon(\partial A)), \quad U_\varepsilon(\partial A) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^3 : |b_A(x)| < \varepsilon\right\}$$
$$\text{projection onto } \partial A : p_{\partial A}(x) = x - b_A(x)\, \nabla b_A(x), \quad H_2(\partial A) < \infty.$$

Consider the *shell or sandwich of thickness* $t = 2r$ *around* $E = \partial A$ and, for $0 < r < \varepsilon$, the *r-dilatation* $E_r \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^3 : |b_A(x)| \le r\right\} = \left\{x \in \mathbb{R}^3 : d_{\partial A}(x)| \le r\right\}$,

$$t = 2r = \alpha_1\, r, \quad \alpha_1 = 2 = \text{ volume of the unit ball in } \mathbb{R}^1$$
$$\phi \mapsto \int_E \phi\, dH_2 : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R} \text{ is a distribution.}$$

Assuming that $U_\varepsilon(\partial A) \subset \Omega$, the perturbed sets for $0 < t < \varepsilon$ are

$$t \mapsto \Omega_t \stackrel{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash E_{t/2}.$$

Given $\phi \in \mathcal{D}(\mathbb{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned} \frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right] &= -\frac{1}{t} \int_{E_{t/2}} \chi_\Omega\, \phi\, dx \\ &= -\frac{1}{\alpha_1 r} \int_{E_r} \chi_\Omega\, \phi\, dx \to -\int_E \phi\, dH_2. \end{aligned}$$

This distribution is a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_{\Omega} \phi\, dx\right] \to -\rho \int_E \phi\, dH_2.$$

When $E = \partial A$, we create a new connected component (cf. Figure 1). This

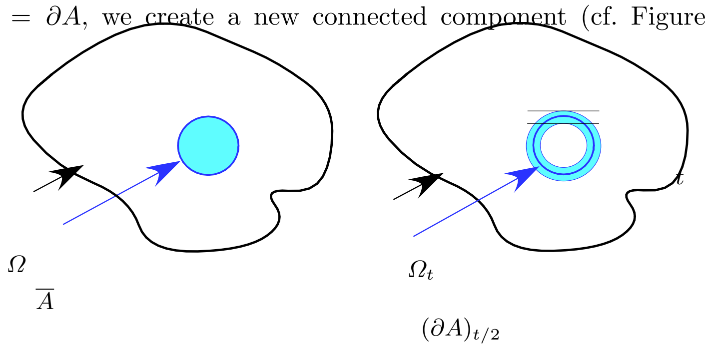


Fig. 1. For $E = \partial A$, Ω_t has two connected components

construction extends to a set of class $C^{1,1}$ with compact boundary in $\mathbb{R}^N$. $\square$

Example 3. Let $A = \partial A \subset \mathbb{R}^N$, $N \geq 3$, be a compact C^2-submanifold for which there exists $\varepsilon > 0$ such that $d_A^2 \in C^2(U_\varepsilon(A))$, then

$$\text{projection onto } \partial A : p_A(x) = x - \frac{1}{2}\nabla d_A^2(x), \quad Dp_A(x) = I - \frac{1}{2}D^2 d_A^2(x),$$
$$\operatorname{Im} Dp_A(x) = \text{tangent space at } x \in A, \quad \dim A(x) = \dim(\operatorname{Im} Dp_A(x)).$$

Let $\dim A = d$ and $H_d(A) < \infty$ for some d, $0 < d < N-1$. Given $E = A$ and $0 < r < \varepsilon$, consider the r-*dilatation* E_r of E,

$$t = \alpha_{N-d}\, r^{N-d}, \quad \alpha_{N-d} = \text{ volume of the unit ball in } \mathbb{R}^{N-d}$$
$$\phi \mapsto \int_E \phi\, dH_d : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R} \text{ is a distribution.}$$

Assuming that $U_\varepsilon(A) \subset \Omega$, the perturbed set for $0 < r < \varepsilon$ will be

$$t \mapsto \Omega_t \overset{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash E_{\sqrt[N-d]{t/\alpha_{N-d}}}.$$

Given $\phi \in \mathcal{D}(\mathbb{R}^N)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned}\frac{1}{t}\left[\int_{\Omega_t} \phi\, dm_N - \int_\Omega \phi\, dm_N\right] &= -\frac{1}{t}\int_{E_{\sqrt[N-d]{t/\alpha_{N-d}}}} \chi_\Omega\, \phi\, dm_N \\ &= -\frac{1}{\alpha_{N-d}\, r^{N-d}} \int_{E_r} \chi_\Omega\, \phi\, dm_N \to -\int_E \phi\, dH_d.\end{aligned}$$

This distribution is a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi \, \mathrm{dm}_N - \int_{\Omega} \phi \, \mathrm{dm}_N\right] \to -\rho \int_E \phi \, dH_d.$$

□

4 Generalization and Concluding Remarks

In section 3.2 we considered the *Minkowski content* $M^d(E)$ of closed subsets E of $\mathbb{R}^N$ (of *positive reach*) such that

$$M^d(E) \stackrel{\text{def}}{=} \lim_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}} = H_d(E), \quad 0 \le d \le N, \tag{4.1}$$

and the associated distribution (measure)

$$\phi \mapsto \int_E \phi \, dH_d = \lim_{r \searrow 0} \frac{1}{\alpha_{N-d}\, r^{N-d}} \int_{E_r} \phi \, dm_N : \mathcal{D}(\mathbb{R}^{\mathcal{N}}) \to \mathbb{R}\,. \tag{4.2}$$

Choosing the volume $t = \alpha_{N-d}\, r^{N-d}$ of the ball of radius r in $\mathbb{R}^{N-d}$ as the auxiliary variable, that is, $r = (t/\alpha_{N-d})^{1/(N-d)}$,

$$\phi \mapsto \int_E \phi \, dH_d = \lim_{t \searrow 0} \frac{1}{t} \int_{E_{(t/\alpha_{N-d})^{1/(N-d)}}} \phi \, dm_N : \mathcal{D}(\mathbb{R}^{\mathcal{N}}) \to \mathbb{R}\,. \tag{4.3}$$

Given a Lebesgue measurable $\Omega \subset \mathbb{R}^N$, we considered the perturbation

$$\Omega_t = \Omega \backslash E_r \tag{4.4}$$

and obtained a continuous trajectory $t \mapsto \chi_{\Omega_t}$ in $\mathrm{X}(\mathbb{R}^N)$ such that

$$\chi_{\Omega_t} \to \chi_{\Omega \backslash E} \text{ in } L^p_{\text{loc}}(\mathbb{R}^N),\ 1 \le p < \infty.$$

If $m_N(E) = 0$, then $\chi_{\Omega_t} \to \chi_{\Omega}$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $1 \le p < \infty$.

Such a construction extends to *dilatations* of *d-rectifiable* compact sets (see Federer [9]) and to *H_d-rectifiable sets E* verifying a certain *density condition* (see Ambrosio et al [2, Dfn. 2.57, p. 80] and [1, pp. 730–731]).

Another family of closed sets is provided by the extension of the *Steiner formula* by Federer [8, Thm. 5.6, p. 455] to closed sets A of *positive reach.* Given $E \subset \partial A$ closed and $0 \le r < \operatorname{reach}(A)$, define the *orthogonal r-dilatation* of E: $E^A_r \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^N : d_A(x) \le r \text{ and } p_A(x) \in E \right\}$, where p_A is the projection onto A. Then $\lim_{r \searrow 0} \mathrm{m}_N(E^A_r)/(\alpha_{N-d} r^{N-d})$ is a Radon measure for some d, $0 \le d \le N$.

The emerging point of view is to consider the elements of the group $\mathrm{X}(\mathrm{R}^{\mathrm{N}})$ of characteristic functions χ_Ω of Lebesgue measurable subsets $\Omega \subset \mathbb{R}^N$ as a subset of measures in the space of distributions $\mathcal{D}(\mathbb{R}^N)'$:

$$\phi \mapsto \int_{\mathbb{R}^N} \chi_\Omega \, \phi \, dx = \int_\Omega \phi \, dx : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}, \quad X(\mathrm{R}^{\mathrm{N}}) \subset \mathcal{D}(\mathbb{R}^N)'. \tag{4.5}$$

It is conjectured that the tangent cone $T_{\chi_\Omega}\mathrm{X}(\mathbb{R}^\mathrm{N})$ is contained in $\mathcal{D}(\mathbb{R}^N)'$. In section 3.1 the velocities generate *tangents* that are distributions in $H^1(\mathbb{R}^N)'$; in section 3.2 the compact subsets E generate *semi-tangents* that are bounded measures. As a result, $T_{\chi_\Omega}\mathrm{X}(\mathbb{R}^\mathrm{N})$ is not a linear space and it does not only contain measures, but we don't know how big it is. We could also attempt to characterize the tangent space to a family of measures.

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