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Abstract. *In the present paper we investigate nonlinear tracking problem under boundary control for the oscillation processes described by Fredholm integro-differential equations. When we investigate this problem we use notion of a weak generalized solution of the boundary value problem. Based on the maximum principle for distributed systems we obtain optimality conditions from which follow the nonlinear integral equation of optimal control and the differential inequality. We have developed an algorithm to construct the optimization problem solution. This solving method of a nonlinear tracking problem is constructive and can be used in applications.*

Keywords: Weak generalized solution, boundary control, functional, maximum principle, nonlinear integral equation, optimization.

1 Introduction

With the emergence the theory control for the systems with distributed parameters, a lot of applied problems described by integral-partial differential equations, integral equations, differential and integral-functional equations ([1], chapter 5, P. 193-197), [2], ([3], chapter 16, P. 410-414), ([4], chapter 1, P. 30-76), became investigate by methods of optimal control theory ([4], chapter 6, P. 356-383), ([5], chapter 2, P. 45-78), ([6], chapter 4, P. 281-309),[7]-[9]. However, the control problems described by the integral-differential equations are little learned. In this paper we investigate the boundary tracking control problem for the elastic oscillations described by the partial Fredholm integral-differential equations in partial derivatives. This problem has a number of specific properties: according to the method of [10] the generalized solution of the problem is built by the solving of countable number of integral equations; the optimal control simultaneously satisfies the two relations in the form of equality and inequality, where the relation in the form of equality leads to a nonlinear integral equation, and the relation in the form of inequality is a differential with regards to the function of the external source.

The sufficient conditions for the unique solvability of specific problems were found, and algorithm was indicated for constructing solutions of nonlinear optimization problems with arbitrary precision in the form of the triple $\left(u^0(t), V^0(t,x), J[u^0(t)]\right)$, where $u^0(t)$ is the optimal control, $V^0(t,x)$ is the optimal process, $J[u^0(t)]$ is the functional's minimum value.

2 Formulation of the optimal control problem and optimality conditions

We consider the optimization problem where it is required to minimize the integral functional

$$J[u(t)] = \int_0^T \int_Q [V(t,x) - \xi(t,x)]^2 dxdt + 2\beta \int_0^T M[t,u(t)]dt, \; \beta > 0 \tag{1}$$

on the set of solutions of the boundary value problem

$$V_{tt} - AV = \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau + g(t,x), \; x \in Q \subset R^n, \; 0 < t \leq T, \tag{2}$$

$$V(0,x) = \psi_1(x), \; V_t(0,x) = \psi_2(x), \; x \in Q, \tag{3}$$

$$\begin{aligned} \Gamma V(t,x) = \sum_{i,j=1}^n a_{i,j}(x)V_{x_j}(t,x)cos(\delta,x_i) + a(x)V(t,x) = \\ = b(t,x)f[t,u(t)], \; x \in \gamma, \; 0 < t < T. \end{aligned} \tag{4}$$

Here A is the elliptic operator defined by the formula

$$AV(t,x) = \sum_{i,j=1}^n \left(a_{i,j}(x)V_{x_j}(t,x)\right)_{x_i} - c(x)V(t,x), \; a_{i,j}(x) = a_{j,i}(x),$$

$$\sum_{i,j=1}^n a_{i,j}(x)\alpha_i\alpha_j \geq c_0 \sum_{i=1}^n \alpha_i^2, \; c_0 > 0$$

δ is a normal vector, emanating from the point $x \in \gamma$; $K(t,\tau)$ is a given function defined in the region $D = \{0 \leq t \leq T, \; 0 \leq \tau \leq T\}$ and satisfying the condition $\int_0^T \int_0^T K^2(t,\tau)dtd\tau < K_0 < \infty$, i.e. $K(t,s) \in H(D)$;

$$\begin{aligned} \psi_1(x) \in H_1(Q), \; \psi_2(x) \in H(Q), \; f_u[t,u(t)] \neq 0, \forall t \in (0,T), \\ \xi(t,x) \in H(Q_T), \; M[t,u(t)] \in H(0,T), \; Q_T = (Q \times T), \end{aligned} \tag{5}$$

are given functions, $a(x) \geq 0, c(x) \geq 0$ are known measurable functions; $H(X)$ is Hilbert space of functions defined on the set of X ; $H_1(X)$ is the first order Sobolev space; $f_u[t,u(t)]$ is the function of boundary source which nonlinearly depends on the control function $u(t) \in H(0,T)$ and it is an element of $H(0,T)$; λ is a parameter, T is a fixed moment of time; $M_u[t,u(t)] \neq 0$ and satisfies the Lipschitz condition with respect to functional argument $u(t) \in H(0,T)$.

This problem is to find a control $u^0(t) \in H(0,T)$, for which the appropriate solution $V^0(t,x)$ of the boundary value problem (2) - (4) deviates little from the given trajectory $\xi(t,x) \in H(Q_T)$ during the entire time $t \in [0,T]$ of the control.

At the same time $u^0(t)$ is called optimal control and $V^0(t,x)$ is the optimal process.

We are looking for a solution of the boundary value problem (2)-(4) in the form of the series

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x), \tag{6}$$

where $z_n(x)$ are generalized eigenfunctions of the boundary value problem [10]

$$D_n(\Phi, z_n) = \int_Q \left(\sum_{i,j=1}^{n} a_{i,j}(x) \Phi_{x_j} z_{nx_i} + c(x) z_n(x) \Phi(t,x) \right) dx + \int_\gamma a(x) z_n(x) \Phi(t,x) dx =$$

$$= \lambda_n^2 \int_Q z_n(x) \Phi(t,x) dx;$$

$$\Gamma z_n(x) = 0,\ x \in \gamma, 0 < t < T, n = 1,2,...,$$

and they form complete orthonormal system in the Hilbert space $H(Q)$, and the corresponding eigenvalues λ_n satisfy the following conditions

$$\lambda_n \leq \lambda_{n+1},\ \forall n = 1,2,3,...,\ \lim_{n\to\infty} \lambda_n = \infty.$$

The Fourier coefficients $V_n(t)$ for each fixed $n = 1,2,3,...,$ satisfy the linear nonhomogeneous Fredholm integral equation of the second type

$$V_n(t) = \lambda \int_0^T K_n(t,s) V_n(s) ds + a_n(t), \tag{7}$$

where

$$K_n(t,s) = \frac{1}{\lambda_n} \int_0^t sin\lambda_n(t-\tau) K(\tau,s) d\tau,$$

$$a_n(t) = \psi_{1n} cos\lambda_n t + \frac{\psi_{2n}}{\lambda_n} sin\lambda_n t + \frac{1}{\lambda_n} \int_0^t sin\lambda_n(t-\tau)[q_n(\tau) + b_n(\tau) f[\tau, u(\tau)]] d\tau. \tag{8}$$

The solution of equation (7) we find ([11],chapter 2, P. 98-110) by the following formula

$$V_n(t) = \lambda \int_0^T R_n(t,s,\lambda) a_n(s) ds + a_n(t), \tag{9}$$

where the resolvent $R_n(t,s,\lambda)$ of the kernel $K_n(s,t)$ is given by

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s),\ \ K_{n,1}(t,s) = K_n(t,s)$$

and the iterated kernels $K_{n,i}(t,s)$ for each $n = 1,2,...$ are defined by the formulas

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta)K_{n,i}(\eta,s)d\eta,\ i = 1,2,3,...,\ K_{n,1}(t,s) = K_n(t,s).$$

Resolvent $R_n(t,s,\lambda)$ is a continuous function when $|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}$ and satisfy the following estimate

$$\int_0^T R_n^2(t,s,\lambda)ds \le \frac{TK_0}{\left(\lambda_n - |\lambda|T\sqrt{K_0}\right)^2}. \qquad (10)$$

Further, taking into account (8) and (9) solution of the boundary value problem (2)–(5) can be written as

$$V(t,x) = \sum_{n=1}^{\infty}(\psi_n(t,\lambda) + \frac{1}{\lambda_n}\int_0^T \varepsilon_n(t,\eta,\lambda)b_n(\eta)f(\eta,u(\eta))d\eta)z_n(x), \qquad (11)$$

where

$$\psi_n(t,\lambda) = \psi_{1n}(\cos\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\cos\lambda_n s ds) +$$
$$+\frac{\psi_{2n}}{\lambda_n}(\sin\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n s ds) + \frac{1}{\lambda_n}\int_0^T \varepsilon_n(t,\eta,\lambda)g_n(\eta)d\eta, \qquad (12)$$

$$\varepsilon_n(t,\eta,\lambda) = \begin{cases} \sin\lambda_n(t-\eta) + \int_\eta^T R_n(t,s,\lambda)sin\lambda_n(s-\eta)ds,\ 0 \le \eta \le t, \\ \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds,\ t \le \eta \le T. \end{cases} \qquad (13)$$

The function (11) is an element of Gilbert space $H(Q_T)$ and weak generalized solution of boundary problem (2) –(5).

According to condition (5) each control $u(t)$ uniquely defines the controlled process $V(t,x)$. Therefore for the solution $V(t,x)+\Delta V(t,x)$ of boundary value problem (2) - (4) corresponds the control $u(t)+\Delta u(t)$, where $\Delta V(t,x)$ is the increment corresponding to the increment $\Delta u(t)$. By the method of to the maximum principle ([4], chapter 6, P. 356-383), ([5], chapter 2, P. 45-78) the increment of functional (1) can be written as

$$\Delta J[u] = J[u+\Delta u] = -\int_0^T \Delta\Pi[t,V(t,x),\omega(t,x),u(t)]dt + \int_0^T\int_Q \Delta V^2(t,x)]dxdt;$$

where

$$\Pi[t,V(t,x),\omega(t,x),u(t)] = \int_\gamma \omega(t,x)b(t,x)dxf[t,u(t)] - 2\beta M[t,u(t)],$$

and the function $\omega(t,x)$ is the solution of the adjoint boundary value problem

$$\omega_{tt} - A\omega = \lambda\int_0^T K(t,\tau)\omega(\tau,x)d\tau - 2[V(t,x)-\xi(t,x)],\ x \in Q, 0 < t < T,$$
$$\omega(T,x) = 0, \omega_t(T,x) = 0,\ x \in Q, \qquad (14)$$
$$\omega)(t,x) = 0,\ x \in \gamma,\ 0 < t < T.$$

According to the maximum principle for systems with distributed parameters ([4], chapter 6, P. 356-383), ([5], chapter 2, P. 45-78), the optimal control is determined by the relations

$$2\beta \frac{M_u[t,u(t)]}{f_u[t,u(t)]} = \sum_{n=1}^{\infty} b_n(t)\omega_n(t), \tag{15}$$

$$f_u[t,u(t)] \left(\frac{M_u[t,u(t)]}{f_u[t,u(t)]} \right)_u > 0, \tag{16}$$

which are called *optimality conditions*.

3 Solution of the adjoint boundary-value problem

We are looking for a solution of the adjoint boundary value problem (14) in the form of the series

$$\omega(t,x) = \sum_{n=1}^{\infty} \omega_n(t) z_n(x). \tag{17}$$

The Fourier coefficients $\omega_n(t)$ for each fixed $n = 1,2,3,...$, satisfy the linear nonhomogeneous Fredholm integral equation of the second type

$$\omega_n(t) = \lambda \int_0^T B_n(s,t)\omega_n(s)ds - \frac{2}{\lambda_n}\int_t^T sin\lambda_n(\tau - t)[V_n(\tau) - \xi_n(\tau)]d\tau, \tag{18}$$

where

$$B_n(s,t) = \frac{1}{\lambda_n}\int_t^T sin\lambda_n(\tau - t)K(s,\tau)d\tau.$$

The solution of equation (18) we find ([11],chapter 2, P. 98-110) by the following formula

$$\omega_n(t) = \lambda \int_0^T P_n(s,t,\lambda)\left(-\frac{2}{\lambda_n}\int_s^T sin\lambda_n(\tau - s)[V_n(\tau) - \xi_n(\tau)]d\tau \right) ds-$$

$$-\frac{2}{\lambda_n}\int_t^T sin\lambda_n(\tau - t)[V_n(\tau) - \xi_n(\tau)]d\tau, \tag{19}$$

where the resolvent $P_n(s,t,\lambda)$ of the kernel $B_n(s,t)$ is given by

$$P_n(s,t,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} B_{n,i}(s,t), \;\; B_{n,1}(s,t) = B_n(s,t),$$

$$B_{n,i+1}(s,t) = \int_0^T B_n(\eta,t)B_{n,i}(s,\eta)d\eta, \; i = 1,2,3,...$$

and it is continuous function when $|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}$ and satisfy the following estimate

$$\int_0^T P_n^2(s,t,\lambda)d\tau \le \frac{TK_0}{\left(\lambda_n - |\lambda|T\sqrt{K_0}\right)^2}. \tag{20}$$

Further, taking into account (17) and (19) solution of the adjoint boundary value problem can be written as

$$\omega(t,x) = -2\{-h(t,x) + E(t,x)\}, \tag{21}$$

where

$$h(t,x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^T b_n(t) E_n(s,t,\lambda) l_n(\tau,\lambda) d\tau z_n(x),$$

$$E(t,x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^T \left(\int_0^T b_n(t) E_n(t,\tau,\lambda) \varepsilon_n(\tau,\eta,\lambda) b_n(\eta) d\tau \right) f(\eta,u(\eta)) d\eta z_n(x),$$

$$E_n(t,\tau,\lambda) = \begin{cases} \lambda \int_0^{\tau} \frac{1}{\lambda_n} P_n(s,t,\lambda) sin\lambda_n(\tau - s) ds, \ 0 \le \tau \le t, \\ \frac{1}{\lambda_n} sin\lambda_n(\tau - t) + \lambda \int_0^{\tau} \frac{1}{\lambda_n} P_n(s,t,\lambda) sin\lambda_n(\tau - s) ds, \ t \le \tau \le T, \end{cases}$$

$$l_n(t,\lambda) = \xi_n(t) - \psi_n(t,\lambda).$$

By means of the direct calculations we have proved the following lemmas:

Lemma 1. *The function $h(t,x)$ is an element of the space $H(Q_T)$.*

Lemma 2. *Function $E(t,x)$ is an element of $H(Q_T)$.*

Based on the Lemma 1 and Lemma 2 from (21) it follows that solution of adjoint boundary value problem (14) $\omega(t,x)$ is an element of the Hilbert space $H(Q_T)$.

4 Nonlinear integral equation of optimal control

We find the optimal control according to optimality conditions (15) and (16). We substitute in (15) the solution of adjoint boundary-value problem (14) defined by (21).

We rewrite the equality (15) in the form of

$$\beta \frac{M_u(t,u)}{f_u(t,u)} + \sum_{n=1}^{\infty} \frac{1}{\lambda_n} b_n(t) \int_0^T \int_0^T E_n(t,\tau,\lambda) \varepsilon_n(\tau,\eta,\lambda) d\tau b_n(\eta) f(\eta,u(\eta)) d\eta =$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} b_n(t) \int_0^T E_n(t,\tau,\lambda) l_n(\tau,\lambda) d\tau. \tag{22}$$

Thus, the optimal control is defined as the solution of a nonlinear integral equation (22) and at the same time (15) and (16) should be carried out. Condition (5) restricts the class of functions $f[t,u(t)]$ of external influences. Therefore, we assume that the function $f[t,u(t)]$ satisfies (16) for any control $u(t) \in H(0,T)$, i.e. the optimization problem is considered in class $\{f(t,u(t))\}$ of functions satisfying (16). Nonlinear integral equation (22) is solved according to the procedure of work [7],[9]. We set

$$\beta \frac{M_u(t,u)}{f_u(t,u)} = p(t). \tag{23}$$

According to condition (16) control function $u(t)$ is uniquely determined from equality (23), i.e. there is a such function φ that ([12], chapter 8, P. 467-480)

$$u(t) = \varphi(t, p(t), \beta). \tag{24}$$

By (23) and (24) we rewrite the equation (22) in the operator form

$$p(t) + G[p, \lambda] = h(t, \lambda) \tag{25}$$

where

$$G[p, \lambda] = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} b_n(t) \int_0^T \left(\int_0^T E_n(t, \tau, \lambda) \varepsilon(\tau, \eta, \lambda) d\tau \right) \times$$
$$\times b_n(\eta) f[\eta, \varphi(\eta, p(\eta), \beta)] d\eta,$$
$$h(t, \lambda) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} b_n(t) \int_0^T E_n(t, \tau, \lambda) l_n(\tau, \lambda) d\tau.$$

Now we investigate the questions of unique solvability of the operator equation (25).

Lemma 3. *The function $p(t)$ is an element of the space $H(0,T)$.*

Proof. By (23) we have the estimate

$$sup \left| \frac{M_u(t,u)}{f_u(t,u)} \right| \le N, \ \forall t \in [0,T].$$

Since $u(t) \in H(0,T)$, the statement of the lemma follows by the inequality

$$\int_0^T |p(t)|^2 dt \le \beta^2 \int_0^T \left| \frac{M_u(t,u)}{f_u(t,u)} \right|^2 dt \le \beta^2 N^2 T^2 < \infty.$$

Lemma 4. *The operator $G[p(t)]$ which defined by the formula (25) maps the space H(0,T) into itself, i.e. it is an element of the space $H(0,T)$.*

Proof. Taking into account the following estimations

$$\int_0^T \varepsilon_n^2(t, \eta, \lambda) d\eta \le 2T \left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda| \sqrt{T^2 K_0} \right)^2} \right),$$

$$\int_0^T E_n^2(t, \tau, \lambda) d\tau \le \frac{2T}{\lambda_n^2} \left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda| \sqrt{T^2 K_0} \right)^2} \right),$$

we obtain the assertion of lemma from following inequality

$$\int_0^T G^2[p, \lambda] = \int_0^T \left\{ \sum_{n=1}^{\infty} \frac{1}{\lambda_n} b_n(t) \int_0^T \left(\int_0^T E_n(t, \tau, \lambda) \varepsilon_n(\tau, \eta, \lambda) d\tau \right) \times \right.$$

$$\times b_n(\eta) f[\eta, \varphi(\eta, p(\eta), \beta)]d\eta \Big\}^2 dt \leq \int_0^T \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} b_n^2(t) \times$$

$$\times \sum_{n=1}^{\infty} \left\{ \int_0^T \int_0^T E_n(t,\tau,\lambda)\varepsilon_n(\tau,\eta,\lambda)d\tau b_n(\eta) f[\eta, \varphi(\eta, p(\eta), \beta)]d\eta \right\}^2 dt \leq$$

$$\leq \int_0^T \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} b_n^2(t) \sum_{n=1}^{\infty} \int_0^T \int_0^T E_n^2(t,\tau,\lambda)d\tau \int_0^T \varepsilon_n^2(\tau,\eta,\lambda)d\tau b_n^2(\eta)d\eta \times$$

$$\times \int_0^T f^2[\eta, \varphi(\eta, p(\eta), \beta)]d\eta dt \leq \int_0^T \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} b_n^2(t) \times$$

$$\times \sum_{n=1}^{\infty} \int_0^T \frac{2T}{\lambda_n^2} \left\{ 1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\sqrt{T^2 K_0}\right)^2} \right\} 2T \left\{ 1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_n - |\lambda|\sqrt{T^2 K_0}\right)^2} \right\} b_n^2(\eta) d\eta \times$$

$$\times \int_0^T f^2[\eta, \varphi(\eta, p(\eta), \beta)]d\eta dt \leq \frac{1}{\lambda_1^2} \left(\int_0^T \sum_{n=1}^{\infty} b_n^2(t) dt \right)^2 \frac{(2T)^2}{\lambda_1^2} \times$$

$$\times \left\{ 1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\sqrt{T^2 K_0}\right)^2} \right\}^2 \|f[\eta, \varphi(\eta, p(\eta), \beta)]\|_{H(0,T)}^2 < \infty.$$

Lemma 5. *Suppose that the conditions*

$$\|f[t,u(t)] - f[t,\bar{u}(t)]\|_{H(0,T)} \leq f_0 \|u(t) - \bar{u}(t)\|_{H(0,T)},\ f_0 > 0,$$

$$\|\varphi[t, p(t), \beta] - \varphi[t, \bar{p}(t), \beta]\|_{H(0,T)} \leq \phi_0(\beta) \|p(t) - \bar{p}(t)\|_{H(0,T)},\ \varphi_0(\beta) > 0$$

are satisfied. Then if the condition

$$\gamma = C_0 f_0 \varphi_0(\beta) < 1,$$

is met, the operator $G[p,\lambda]$ is contractive. Here

$$C_0 = \frac{2T}{\lambda_1^2} \left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\sqrt{T^2 K_0}\right)^2} \right) \|b(t,x)\|_{H(\gamma_T)}^2.$$

Proof. The proof of this theorem follows from Lemma 4 by the following inequality, i.e. the following inequality is fulfilled

$$\|G[p,\lambda] - G[p,\tilde{p},\lambda]\|_{H(0,T)}^2 \leq C_0^2 \|f[t,u(t)] - f[t,\tilde{u}(t)]\|_{H(0,T)}^2 \leq$$

$$\leq C_0^2 f_0^2 \|\varphi[t,p(t),\beta] - \varphi[t,\tilde{p}(t),\beta]\|_{H(0,T)}^2 \leq C_0^2 f_0^2 \varphi_0^2(\beta) \|p(t) - \tilde{p}(t)\|_{H(0,T)}^2.$$

Theorem 1. *Suppose that conditions (5), lemma 5 and $|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}$ are satisfied. Then operator equation (25) has a unique solution in the space $H(0,T)$.*

Proof. According to Lemmas 3 and 4, operator equation (25) can be considered in the space $H(0,T)$. According to Lemma 5 operator $G(p)$ is contractive. Since the Hilbert space $H(0,T)$ is a complete metric space, by the theorem on principle of contracting mappings ([12], chapter 1, P. 43-53) the operator $G(p)$ has a unique fixed point, i.e. operator equation (25) has unique solution.

The solution of operator equation (25) can be found by the method of successive approximations, i.e. n th approximation of the solution is found by the formula

$$p_n = h - G[p_{n-1}], \;\; n = 1,2,3,..,$$

where $p_0(t)$ is an arbitrary element of the space $H(0,T)$. For the exact solution $\bar{p}(t) = \lim\limits_{n\to\infty} p_n(t)$ we have the following estimate

$$\|\bar{p}(t) - p_n(t)\| \leq \frac{\gamma^n}{1-\gamma}\|h - G[p_0(t)] - p_0\|_{H(0,T)}$$

or when $h = p_0(t)$

$$\|\bar{p}(t) - p_n(t)\|_{H(0,T)} \leq \frac{\gamma^n}{1-\gamma}\|G[\vartheta_0]\|_{H(0,T)},$$

where $0 < \gamma < 1$ is the contraction constant. The exact solution can be found as the limit of the approximate solutions, i.e. substituting this solution in (24) we find the optimal control

$$u^0(t) = \varphi[t, \bar{p}(t), \beta].$$

We find the optimal process $V^0(t,x)$, i.e. the solution of boundary value problem (2)-(4), corresponding to the optimal control $u^0(t,x)$, according to (6) from the formula

$$V^0(t,x) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)a_n^0(s)ds - a_n^0(t)\right)z_n(x),$$

$$a_n^0(t) = \psi_{1n}cos\lambda_n t + \frac{\psi_{2n}}{\lambda_n}sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)[q_n(\tau) + b_n(\tau)f[\tau, u^0(\tau)]]d\tau.$$

The minimum value of the functional (1) is calculated by the formula

$$J[u^0(t)] = \int_0^T\int_Q \left[V^0(t,x) - \xi(t,x)\right]^2 dxdt + 2\beta\int_0^T M[t, u^0(t)]dt.$$

The found triple $(u^0(t), V^0(t,x), J[u^0(t)]$ is a solution of the nonlinear optimization problem.

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