Regularized Optimal Design Problem for a Viscoelastic Plate Vibrating Against a Rigid Obstacle

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Abstract. *We deal with a regularized optimal control problem governed by a nonlinear hyperbolic initial-boundary value problem describing behaviour of a viscoelastic plate vibrating against a rigid obstacle. A variable thickness of a plate plays the role of a control variable. The original problem for the deflection is regularized in order to have the uniqueness of a solution to the state problem and only the existence of an optimal thickness but also necessary optimality conditions.*

Keywords: *Viscoelastic anisotropic plate, variable thickness, rigid foundation, regularization, optimal control, optimality conditions.*

1 Introduction

Shape design optimization problems belong to frequently solved problems with many engineering applications. We deal here with a regularized optimal design problem for a viscoelastic anisotropic plate vibrating against a rigid foundation. A variable thickness of a plate plays the role of a control variable. The corresponding state initial-boundary value contact problem represents one of the most natural problem of mechanics not frequently solved because of the hyperbolic character of the presented evolutional variational inequality. We deal here with a plate made of short memory viscoelastic material. It characterizes constructions made of concrete for example ([7]). The dynamic contact for a viscoelastic bridge in a contact with a fixed road has been solved in [3]. The similar optimal control problems for the beams in a boundary contact are investigated in [1] and [4] respectively.

Due to the variable thickness e and the contact between a bottom of the plate and the obstacle represented by a function Φ the equation for the movement u

* This work was financially supported by the Ministry of Education of Slovak Republic under grant VEGA-1/0159/16 and by grant of Science and Technology Assistance Agency no. APVV-0246-12.

of the middle surface and the complementarity conditions have the form

$$\begin{aligned} &\tfrac{1}{2}\rho e(x)u_{tt} - \tfrac{1}{12}\mathrm{div}\,[e^3(x)\mathrm{grad}\,u_{tt}] + [e^3(x)(A_{ijk\ell}u_{t,x_ix_j} + B_{ijk\ell}u_{x_ix_j})]_{x_kx_\ell} \\ &\quad = F + G, \qquad 0 \le G \perp u - \tfrac{1}{2}e - \Phi \ge 0 \qquad \text{in } (0,T] \times \Omega, \end{aligned}$$

where F and G express a perpendicular force acting on the plate and an unknown contact force respectively. In order to derive not only the existence of optimal variable thickness e but also the necessary optimality conditions we regularize the contact condition using the function

$$\omega \mapsto g_\delta(\omega),\ g_\delta(\omega) = \begin{cases} 0 \text{ for } w \le 0 \\ \frac{6}{\delta^3}\omega^3 - \frac{8}{\delta^4}\omega^4 + \frac{3}{\delta^5}\omega^5 \text{ for } 0 < \omega < \delta \\ \frac{1}{\delta}\omega \text{ for } \omega \ge \delta. \end{cases}$$

in an analogous way as in [5], where the control problem for an elastic beam vibrating against an elastic foundation of Winkler's type was considered. We remark that instead of the function g_δ we can use any not negative nondecreasing function $g \in C^2(\mathbb{R})$ of the variable ω vanishing for $\omega \le 0$ and equaled to $\frac{1}{\delta}\omega$ for $\omega \ge \delta$.

Solving the state problem we apply the Galerkin method in the same way as in [1], where the rigid obstacle acting against a beam is considered or in [2] where the problem for a viscoelastic von Kármán plate vibrating against a rigid obstacle has been solved. The compactness method will be used in solving the minimum problem for a cost functional. We apply the approach from [5] in deriving the optimality conditions.

2 Solving the state problem

2.1 Setting of the state problem

We consider an anisotropic plate short memory viscoelastic plate with the middle surface $\Omega \subset \mathbb{R}^2$. The variable thickness of the plate is expressed by a positive function $x \mapsto e(x),\ x \in \bar{\Omega}$, the positive constant ρ is the density of the material, $A_{ijk\ell},\ B_{ijk\ell}$ are the symmetric and positively definite tensor expressing the viscoelastic and elastic properties of the material. The plate is clamped on its boundary. Let $F : (0,T] \times \Omega \mapsto \mathbb{R}$ be a perpendicular load per a square unit acting on the plate. Let $u_0,\ v_0 : \Omega \mapsto \mathbb{R}$ be the initial displacement and velocity, and

$$a = \frac{1}{6\rho},\ a_{ijk\ell} = \frac{2}{\rho}A_{ijk\ell},\ b_{ijk\ell} = \frac{2}{\rho}B_{ijk\ell},\ f = \frac{2F}{\rho}$$

be the new mechanical and material characteristics. Then the vertical displacement $u : (0,T] \times \Omega \mapsto \mathbb{R}$ is a solution of the following regularized hyperbolic initial-boundary value problem

$$e(x)u_{tt} - a\frac{1}{12}\mathrm{div}\,[e^3\mathrm{grad}\,u_{tt}] + [e^3(x)(a_{ijk\ell}u_{t,x_ix_j} + b_{ijk\ell}u_{x_ix_j})]_{x_kx_\ell} \\ - g_\delta(\tfrac{1}{2}e(x) + \Phi(x) - u) = f(t,x), \text{ in } (0,T] \times \Omega. \tag{1}$$

$$u(t,\xi) = \frac{\partial u}{\partial \boldsymbol{n}}(t,\xi) = 0,\ t \in (0,T],\ \xi \in \partial\Omega \tag{2}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega. \tag{3}$$

We introduce the Hilbert spaces

$$H \equiv L_2(\Omega),\ H^k(\Omega) = \{y \in H :\ D^\alpha y \in H,\ |\alpha| \le k\},\ k \in \mathbb{N}$$

with the standard inner products $(\cdot,\cdot),\ (\cdot,\cdot)_k$ and the norms $|\cdot|_0,\ \|\cdot\|_k$,

$$\mathring{H}^1(\Omega) = \{y \in H^1(\Omega) :\ y(\xi) = 0,\ \xi \in \partial\Omega \text{ (in the sense of traces)}\}$$

and

$$V \equiv \mathring{H}^2(\Omega) = \{y \in H^2(\Omega) :\ y(\xi) = \frac{\partial y}{\partial \boldsymbol{n}}(\xi) = 0,\ \xi \in \partial\Omega \text{ (in the sense of traces)}\}$$

with the inner product and the norm

$$((y,z)) = \int_\Omega y_{x_ix_j}(x)z_{x_ix_j}(x)\,dx,\ \|y\| = ((y,y))^{1/2},\ y,\, z \in V.$$

We denote by V^* the dual space of linear bounded functionals over V with duality pairing $\langle F,y\rangle_* = F(y),\ F \in V^*,\ y \in V$. It is a Banach space with a norm $\|\cdot\|_*$.
The spaces V, H, V^* form the Gelfand triple meaning the dense and compact embedings

$$V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*.$$

We set $I = (0,T),\ Q = I \times \Omega$. For a Banach space X we denote by $L_p(I;X)$ the Banach space of all functions $y : I \mapsto X$ such that $\|y(\cdot)\|_X \in L_p(0,T),\ p \ge 1$, by $L_\infty(I;X)$ the space of essentially bounded functions with values in X, by $C(\bar{I};X)$ the space of continuous functions $y : \bar{I} \mapsto X,\ \bar{I} = [0,T]$. For $k \in \mathbb{N}$ we denote by $C^k(\bar{I};X)$ the spaces of k-times continuously differentiable functions defined on $\bar{I}$ with values in X. If X is a Hilbert space we set

$$H^k(I;X) = \{v \in C^{k-1}(\bar{I};X) :\ \frac{d^kv}{dt^k} \in L_2(I;X)\}$$

the Hilbert spaces with the inner products

$$(u,v)_{H^k(I,X)} = \int_I [(u,v)_X + \sum_{j=1}^k (u^j,v^j)_X]\,dt,\ k \in \mathbb{N}.$$

We denote by $\dot{w},\ \ddot{w}$ and $\dddot{w}$ the first, the second and the third time derivative of a function $w : I \to X$. In order to derive necessary optimality conditions in the next chapter we assume stronger regularity of data:

$$
\begin{aligned}
&e \in E_{ad} := \Big\{e \in H^2(\Omega):\ 0 < e_{\min} \le e(x) \le e_{\max}\ \forall x \in \bar{\Omega},\ \|e\|_2 \le \hat{e}\Big\};\\
&u_0 \in V \cap H^4(\Omega),\ u_0(x) \ge \tfrac{1}{2}e_{\max} + \Phi(x)\,\forall x \in \Omega;\\
&\Phi \in C(\bar{\Omega}),\ \Phi(\xi) \le 0\ \forall \xi \in \partial\Omega;,\ v_0 \in V,\ f \in H^1(I;H).
\end{aligned}
\tag{4}
$$

The symmetric and positively definite fourth-order tensors $a_{ijk\ell},\ b_{ijk\ell}$ fulfil

$$
\begin{aligned}
&a_{ijk\ell} = a_{k\ell ij} = a_{jik\ell},\ b_{ijk\ell} = b_{k\ell ij} = b_{jik\ell},\\
&\alpha_0 > 0,\ \alpha_0\,\varepsilon_{ij}\varepsilon_{ij} \le a_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell} \le \alpha_1\,\varepsilon_{ij}\varepsilon_{ij}\ \forall\ \{\varepsilon_{ij}\} \in \mathbb{R}^{2\times 2}_{sym},\\
&\beta_0 > 0,\ \beta_0\,\varepsilon_{ij}\varepsilon_{ij} \le b_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell} \le \beta_1\,\varepsilon_{ij}\varepsilon_{ij}\ \forall\ \{\varepsilon_{ij}\} \in \mathbb{R}^{2\times 2}_{sym},
\end{aligned}
\tag{5}
$$

where the Einstein summation convention is employed and $\mathbb{R}^{2\times 2}_{sym}$ is the set of all second-order symmetric tensors. For $e,\ u, y \in H^2(\Omega)$ we define bilinear forms $A(e),\ B(e)$ by

$$
A(e)(u,y) = e^3 a_{ijk\ell} u_{ij} y_{k\ell},\ B(e)(u,y) = e^3 b_{ijk\ell} u_{ij} y_{k\ell}.
$$

Definition 1. *A function u is a weak solution of the problem* (1)-(3) *if $\ddot{u} \in L_2(I;\mathring{H}^1(\Omega)),\ \dot{u} \in L_2(I;V)$, there hold the identity*

$$
\begin{aligned}
&\int_Q \big[e(x)\ddot{u}y + ae^3(x)\nabla\ddot{u}\cdot\nabla y + A(e;\dot{u},y) + B(e;u,y)\big]\,dx\,dt\\
&= \int_Q \big[g_\delta(\tfrac{1}{2}e(x) + \Phi(x) - u) + f(t,x)\big]\,y\,dx\,dt\ \forall\, y \in L_2(I;V)
\end{aligned}
\tag{6}
$$

and the initial conditions

$$
u(0) = u_0,\ \dot{u}(0) = v_0. \tag{7}
$$

2.2 Existence and uniqueness of the state problem

We verify the existence and uniqueness of a weak solution.

Theorem 1. *There exists a unique solution u of the problem (6),(7) such that $u \in C^1(\bar{I};V)$, $\ddot{u} \in L_2(I;V) \cap L_\infty(I;\mathring{H}^1(\Omega)) \cap C^1(\bar{I};H),\ \dddot{u} \in L_2(I;H)$ and there hold the estimates*

$$
\begin{aligned}
&\|u\|_{C(\bar{I},V)}\| + \|\dot{u}\|_{L_2(I,V)} + \|\dot{u}\|_{C(\bar{I},\mathring{H}^1(\Omega))}\\
&\le C_0(\alpha_0,\alpha_1,\beta_0,\beta_1,e_{\min},e_{\max},\hat{e},u_0,v_0,f),
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&\|\dot{u}\|_{C(\bar{I},V)} + \|\ddot{u}\|_{L_2(I,V)} + \|\ddot{u}\|_{L_\infty(I,\mathring{H}^1(\Omega))} + \|\dddot{u}\|_{L_2(I,H)}\\
&\le C_1(\delta,\alpha_0,\alpha_1,\beta_0,\beta_1,e_{\min},e_{\max},\hat{e},u_0,v_0,f).
\end{aligned}
\tag{9}
$$

Proof. Let $\{w_i \in V \cap H^4(\Omega);\ i \in \mathbb{N}\}$ be a basis of V. We introduce the Galerkin approximation u_m of a solution in a form

$$u_m(t) = \sum_{i=1}^{m} \alpha_i(t) w_i,\ \ \alpha_i(t) \in \mathbb{R},\ \ i = 1, ..., m,\ \ m \in \mathbb{N},$$
$$\int_\Omega \big[e(x)\ddot{u}_m w_i + ae^3(x)\nabla\ddot{u}_m \cdot \nabla w_i + A(e)(\dot{u}_m, w_i) + B(e)(u_m, w_i)\big]\, dx =$$
$$\int_\Omega \big[g_\delta(\tfrac{1}{2}e(x) + \Phi(x) - u_m) + f(t)\big] w_i\, dx,\ \ i = 1, ..., m;$$
$$u_m(0) = u_{0m},\ \ \dot{u}_m(0) = v_{0m};\ \ u_{0m} \to u_0 \text{ in } H^4(\Omega),\ v_{0m} \to v_0 \text{ in } V.$$

A solution originally existing only locally can be prolonged to the whole time interval I with the *a priori* estimates

$$\begin{aligned} &\|u_m\|_{C(\bar{I},V)} + \|\dot{u}_m\|_{L_2(I,V)} + \|\dot{u}_m\|_{C(\bar{I},\mathring{H}^1(\Omega))} \\ &\le C_2(\alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat{e}, u_0, v_0, f). \end{aligned} \tag{10}$$

Better estimates can be achieved after differentiating the Galerkin equation with respect to t:

$$\begin{aligned} &\|\dot{u}_m\|_{C(\bar{I},V)} + \|\ddot{u}_m\|_{L_2(I,V)} + \|\ddot{u}_m\|_{C(\bar{I},\mathring{H}^1(\Omega))} \\ &\le C_3(\delta, \alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat{e}, u_0, v_0, f). \end{aligned} \tag{11}$$

We proceed with the convergence of the Galerkin approximation. Applying the estimates (10), (11), the Aubin-Lions compact imbedding theorem [9], Sobolev imbedding theorems and the interpolation theorems in Sobolev spaces [8] we obtain for a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) a function $u \in C(\bar{I}, V)$ with $\dot{u} \in L_\infty(I, V),\ \ddot{u} \in L_\infty(I, \mathring{H}^1(\Omega))$ and the convergences

$$\begin{aligned} &\ddot{u}_m \rightharpoonup^* \ddot{u} && \text{in } L_\infty(I, \mathring{H}^1(\Omega)), \\ &\ddot{u}_m \rightharpoonup \ddot{u} && \text{in } L_2(I, V), \\ &\dot{u}_m \rightharpoonup^* \dot{u} && \text{in } L_\infty(I; V), \\ &u_m \to u && \text{in } C(\bar{I}; V), \\ &u_m \to u && \text{in } C^1(\bar{I}; H^{2-\varepsilon}(\Omega))\ \forall\, \varepsilon > 0, \\ &u_m \to u && \text{in } C^1(\bar{I}; C(\bar{\Omega})). \end{aligned} \tag{12}$$

The convergence process (12) implies that a function u fulfils for a.e. $t \in I$

$$\begin{aligned} &\int_\Omega \big[e\ddot{u}w + ae^3(x)\nabla\ddot{u} \cdot \nabla w + A(e; \dot{u}, w) + B(e; u, w)\big]\, dx \\ &= \int_\Omega \big[g_\delta(\tfrac{1}{2}e(x) + \Phi(x) - u) + f\big]\, w\, dx,\ \ \forall\, w \in V. \end{aligned} \tag{13}$$

The identity (6) follows directly after setting $w \equiv y(t, \cdot),\ y \in L_2(I; V)$ in (13). The estimate (10) together with the convergences (12) implies the estimate (8).

Due to the differentiability of $g_\delta,\ f$ we obtain the third time derivative $\dddot{u} \in L_2(I;H)$ fulfilling

$$\begin{aligned}&\int_Q \left[\dddot{u}\left(e(x)y - \operatorname{div}\left(e^3(x)\nabla y\right)\right) + A(e;\ddot{u},y) + B(e;\dot{u},y)\right]\,dx\,dt\\&= \int_Q \left[-g'_\delta(\tfrac{1}{2}e(x) + \Phi(x) - u)\dot{u} + \dot{f}(t,x)\right] y\,dx\,dt\ \forall\, y \in L_2(I;V).\end{aligned} \tag{14}$$

The estimate (9) is then the consequence of (11) together with the convergences (12) and the relation (14). The proof of the uniqueness can be performed in a standard way using the Gronwall lemma.

Remark 1. The constant $C_0(\alpha,\beta,e_{\min},e_{\max},\hat{e},u_0,v_0,f,q)$ in the estimate (8) does not depend on δ for $\delta \in (0,\delta_0)$. It is possible to derive the existence of a variational solution u of the original problem with the rigid obstacle in a similar way as in [2], where the method of penalization was applied. We can use the limit of a subsequence of solutions $\{u_{\delta_n}\},\ \delta_n \to 0+$ to the problem (6), (7) for $\delta \equiv \delta_n,\ n \in \mathbb{N}$ instead of the sequence of penalized solutions.

Let $\mathcal{K}$ be a closed convex set in $L_2(I;V)$ of the form

$$\mathcal{K} := \{y \in L_2(I;V);\ \dot{y} \in L_2(I;\mathring{H}^1(\Omega)),\ y \geq \tfrac{1}{2}e + \Phi\}. \tag{15}$$

A function $u \in \mathcal{K}$ such that $\dot{u} \in L_2(I;V)$ and $u(0,\cdot) = u_0$ solves the initial value problem for a nonstationary variational inequality

$$\begin{aligned}&\int_Q \big(A(e;\dot{u},y-u) + B(e;u,y-u) - ae^3\nabla\dot{u}\cdot\nabla(\dot{y}-\dot{u}) - e\dot{u}(\dot{y}-\dot{u})\big)\,dx\,dt\\&\quad + \int_\Omega \big(ae^3\nabla\dot{u}\cdot\nabla(y-u) + e\dot{u}(y-u)\big)(T,\cdot)\,dx\\&\quad \geq \int_\Omega \big(a\nabla v_0\cdot(\nabla y(0,\cdot) - \nabla u_0) + v_0(y(0,\cdot)-u_0)\big)\,dx\\&\qquad + \int_Q f(y-u)\,dx\,dt \quad \forall y \in \mathcal{K}.\end{aligned} \tag{16}$$

3 Optimal control problem

3.1 The existence of an optimal thickness

We consider a cost functional $J : L_2(I;V) \times H^2(\Omega) \mapsto \mathbb{R}$ fulfilling the assumption

$$u_n \rightharpoonup u \text{ in } L_2(I;V),\ e_n \rightharpoonup e \text{ in } H^2(\Omega) \Rightarrow J(u,e) \leq \liminf_{n\to\infty} J(u_n,e_n) \tag{17}$$

and formulate

Optimal control problem $\mathcal{P}$: To find a control $e_* \in E_{ad}$ such that

$$J(u(e_*),e_*) \leq J(u(e),e)\ \forall e \in E_{ad}, \tag{18}$$

where $u(e)$ is a (unique) weak solution of the Problem (1)-(3).

Theorem 2. *There exists a solution of the Optimal control problem $\mathcal{P}$.*

Proof. We use the weak lower semicontinuity property of the functional J and the compactness of the admissible set E_{ad} of thicknesses in the space $C(\bar{\Omega})$. Let $\{e_n\} \subset E_{ad}$ be a minimizing sequence for (18). i.e.

$$\lim_{n\to\infty} J(u(e_n); e_n) = \inf_{e\in E_{ad}} J(u(e), e). \tag{19}$$

The set E_{ad} is convex and closed and hence a weakly closed in $H^2(\Omega)$ as the closed convex set. Then there exists a subsequence of $\{e_n\}$ (denoted again by $\{e_n\}$) and an element $e_* \in E_{ad}$ such that

$$e_n \rightharpoonup e_* \text{ in } H^2(\Omega),\ e_n \to e_* \text{ in } C(\bar{\Omega}). \tag{20}$$

The *a priori* estimates (8), Sobolev imbedding theorems and the Ascoli theorem on uniform convergence on $\bar{I}$ imply the existence of a function $u^* \in C(\bar{I}; V)$ such that $\dot{u} \in L_\infty(I; V) \cap C(\bar{I}; \mathring{H}^1(\Omega)),\ \ddot{u} \in L_\infty(I; \mathring{H}^1(\Omega))$ and the convergences

$$\begin{aligned} &\ddot{u}(e_n) \rightharpoonup^* \ddot{u}^* \text{ in } L_\infty(I; \mathring{H}^1(\Omega)),\\ &\dot{u}(e_n) \rightharpoonup^* \dot{u}^* \text{ in } L_\infty(I; V),\ \dot{u}(e_n) \to \dot{u}^* \text{ in } C(\bar{I}; \mathring{H}^1(\Omega)),\\ &u(e_n) \rightharpoonup^* u^* \text{ in } L_\infty(I; V),\ u(e_n) \to u^* \text{ in } C(\bar{I}; C^1(\bar{\Omega})) \end{aligned} \tag{21}$$

for a chosen subsequence. Functions $u_n \equiv u(e_n)$ solve the initial value state problem (6), (7) for $e \equiv e_n$. We verify that u_* solves the problem (6), (7) with $e \equiv e_*$. The previous convergences together with the Lipschitz continuity of g_δ imply

$$\begin{aligned} &e_n\ddot{u}_n \rightharpoonup e_*\ddot{u}_* \text{ in } L_2(Q),\ e_n^3\ddot{u}_{n,i} \rightharpoonup e_*^3\ddot{u}_{*i} \text{ in } L_2(Q),\ i = 1, 2;\\ &e_n^3\dot{u}_{n,ij} \rightharpoonup e_*^3\dot{u}_{*ij} \text{ in } L_2(Q),\ e_n^3 u_{n,ij} \rightharpoonup e_*^3 u_{*ij} \text{ in } L_2(Q),\ i, j \in \{1,2\};\\ &g_\delta(\tfrac{1}{2}(e_n + \Phi - u_n) \rightharpoonup g_\delta(\tfrac{1}{2}(e_* + \Phi - u_*) \text{ in } L_2(Q). \end{aligned}$$

Then $u_* \equiv u(e_*)$ and hence

$$u(e_n) \rightharpoonup u(e_*) \text{ in } L_2(I; V), \quad e_n \rightharpoonup e \text{ in } H^2(\Omega).$$

Property (17) together with (19) then imply that

$$J(u(e_*), e_*) = \min_{e\in E_{ad}} J(u(e), e)$$

and the proof is complete.

3.2 Necessary optimality conditions

We introduce the Banach space $\mathcal{W} = \{w \in H^1(I; V) :\ \ddot{w} \in L_2(I; V^*)\}$ with a norm

$$\|w\|_{\mathcal{W}} = \|w\|_{H^1(I;V)} + \|\ddot{w}\|_{L_2(I;V^*)}$$

and operators $\mathcal{A}(e) : \mathcal{W} \to L_2(I;V^*)$, $\mathcal{B}(e) : H^2(\Omega) \to L_2(I;V^*)$ by

$$\begin{aligned}\langle\langle \mathcal{A}(e)z, y\rangle\rangle = &\int_Q \ddot{z}[ey - a\,\mathrm{div}\,(e^3\nabla y)]\,dx\,dt + \\ &\int_Q [A(e)(\dot{z}, y) + B(e)(z, y) + g'_\delta(\omega(e))zy]\;dx\,dt,\end{aligned} \tag{22}$$

$$\begin{aligned}\langle\langle \mathcal{B}(e)h, y\rangle\rangle = &\int_Q h\left[\ddot{u}(e)y - 3a\,div\,(e^2\nabla y)\right]\;dx\,dt + \\ &\int_Q h\left[A'(e)(\dot{z}, y) + B'(e)(z, y) - \frac{1}{2}g'_\delta(\omega(e))y\right]\;dx\,dt,\end{aligned} \tag{23}$$

$$\omega(e) = \tfrac{1}{2}e + \Phi - u(e),\; y \in L_2(I;V).$$

In a similar way as in [5] or [6] the following theorem about Fréchet differentiability of the mapping $e \mapsto u(e)$ can be verified.

Theorem 3. *The mapping* $u(\cdot) : E_{ad} \to \mathcal{W}$ *is Fréchet differentiable and its derivative*
$z \equiv z(h) = u'(e)h \in \mathcal{W},\; h \in H^2(\Omega)$ *fulfils for every* $e \in E_{ad}$ *uniquely the operator equation*

$$\mathcal{A}(e)z = -\mathcal{B}(e)h,\; z(0) = \dot{z}(0) = 0 \tag{24}$$

Proof. The existence of a solution z to the equation (24) can be verified using the standard Galerkin method and its uniqueness by the Gronwall lemma.

We proceed with the differentiability of $e \mapsto z(e)$:

Let $h \in H^2(\Omega)$ with $e + h \in E_{ad}$ and

$$r(h) = u(e + h) - u(e) - z(h)$$

We have

$$\mathcal{A}(e)r(h) = \mathcal{A}(e)[u(e + h) - u(e)] + \mathcal{B}(e)h,$$

and verify

$$r(h) = o(h) \text{ i.e. } \lim_{\|h\|_2 \to 0} \frac{r(h)}{\|h\|_2} = 0 \;\Rightarrow\; z(h) = u'(e)h.$$

In order to derive necessary optimality conditions we assume that the cost functional
$J(\cdot,\cdot) : L_2(I;V) \times H^2(\Omega) \to \mathbb{R}$ is Fréchet differentiable.

The optimal control problem can be expressed in a form

$$j(e_*) = \min_{e \in E_{ad}} j(e),\; j(e) = J(u(e), e). \tag{25}$$

The functional j in (25) is Fréchet differentiable and its derivative in $e_* \in E_{ad}$ has the form

$$\langle j'(e_*), h\rangle = \langle\langle J_u(u(e_*), e_*), u'(e_*)h\rangle\rangle + \langle J_e(u(e_*), e_*), h\rangle_{-2},\; h \in H^2(\Omega) \tag{26}$$

with the duality pairings $\langle\langle\cdot,\cdot\rangle\rangle$, $\langle\cdot,\cdot\rangle_{-2}$ between $L_2(I;V)^*$ and $L_2(I;V)$, $(H^2(\Omega))^*$ and $H^2(\Omega)$ respectively.

The optimal thickness $e_* \in E_{ad}$ fulfils the variational inequality

$$\langle j'(e_*), e - e_*\rangle_{-2} \geq 0 \;\forall e \in E_{ad}. \tag{27}$$

which can be expressed in a form

$$\langle\langle J_u(u(e_*), e_*), u'(e_*)(e - e_*)\rangle\rangle + \langle J_e(u(e_*), e_*), e - e_*\rangle_{-2} \geq 0 \;\forall e \in E_{ad}. \tag{28}$$

Applying Theorem 3 we obtain necessary optimality conditions in a form of a system with an adjoint state p:

Theorem 4. *The optimal thickness e_*, the corresponding state (deflection) $u^* \equiv u(e_*)$ and the adjoint state $p^* \equiv p(e_*)$ are solutions of the initial value problem*

$$\begin{aligned}
&\int_Q \left[e_* u^*_{tt} y + a e^3 \nabla u^*_{tt} \cdot \nabla y + A(e)(u^*_t, y) + B(e)(u^*, y)\right] dx\,dt \\
&\quad = \int_Q \left[g_\delta(\tfrac{1}{2}e + \Phi - u^*) + f(t,x)\right] y\,dx\,dt \;\forall\, y \in L_2(I;V), \\
&\quad u^*(0) = u_0,\; u^*_t(0) = v_0, \\
&\quad \mathcal{A}(e_*)p_* = -J_u(u^*, e_*);\; p^*(T) = p^*_t(T) = 0, \\
&\quad \langle\langle \mathcal{B}(e_*)(e - e_*), p^*\rangle\rangle + \langle J_e(u^*, e_*), e - e_*\rangle_{-2} \geq 0 \;\forall e \in E_{ad}.
\end{aligned}$$

Remark 2. If the partial derivative $e \mapsto J_e(u(e), e)$ is strongly monotone i.e.

$$\langle J_e(u(e_1), e_1) - J_e(u(e_2), e_2), e_1 - e_2\rangle_2 \geq N\|e\|_2^2 \;\forall e_1,\, e_2 \in H^2(\Omega),\; N > 0,$$

then it is possible after using the variational inequality (28) to obtain for sufficiently large N the uniqueness of the Optimal control e_*.

Remark 3. We have mentioned in Remark 1 that there is a sequence $\{\delta_n\}$, $\delta_n \to 0+$ such that a corresponding sequence of regularized solutions u_{δ_n} of (6), (7) converges to a solution u of the original problem. If $e_* \equiv e_*(\delta_n)$ is a sequence of optimal thicknesses tending to some $\tilde{e_*} \in E_{ad}$ then it is an open question if $\tilde{e_*} \in E_{ad}$ is a solution of the corresponding Optimal control problem connected with the Problem (16). In this case there is no uniqueness of solutions and hence this Optimal control problem has the form

Optimal control problem $\tilde{\mathcal{P}}$: To find a couple $\{\tilde{u}_*, \tilde{e}_*\} \in \mathcal{U} \times E_{ad}$ such that

$$J(\tilde{u}_*, \tilde{e}_*) \leq J(u, e) \;\forall \{u, e\} \in \mathcal{U} \times E_{ad},$$

where

$$\mathcal{U} = \{u \in \mathcal{K};\; \dot{u} \in L_2(I;V),\; u(\cdot, 0) = u_0 \text{ and}(16) \text{ holds}\}.$$

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