Cahn–Hilliard Approach to Some Degenerate Parabolic Equations with Dynamic Boundary Conditions

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Abstract. In this paper the well-posedness of some degenerate parabolic equations with a dynamic boundary condition is considered. To characterize the target degenerate parabolic equation from the Cahn–Hilliard system, the nonlinear term coming from the convex part of the double-well potential is chosen using a suitable maximal monotone graph. The main topic of this paper is the existence problem under an assumption for this maximal monotone graph for treating a wider class. The existence of a weak solution is proved.

Key words: degenerate parabolic equation, dynamic boundary condition, weak solution, Cahn–Hilliard system.

AMS (MOS) subject classification: 35K65, 35K30, 47J35

1 Introduction

The relationship between the Allen–Cahn equation [2] and the motion by mean curvature is interesting as the singular limit of the following form:

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\varepsilon^2}(u^3 - u) = 0 \quad \text{in } Q := (0,T) \times \Omega,$$

as $\varepsilon \searrow 0$, where $0 < T < +\infty$ and $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$, which is a bounded domain with smooth boundary Γ. For example, Bronsard and Kohn presented a pioneering result in [5], and subsequently many related results have been obtained. A similar concept in this framework, the Cahn–Hilliard system [7], is connected to motion by the Mullins–Sekerka law [18] in the limit of

$$\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q,$$
$$\mu = -\varepsilon \Delta u + \frac{1}{\varepsilon}(u^3 - u) \quad \text{in } Q \tag{1.1}$$

as $\varepsilon \searrow 0$. For both of these, the target problems are sharp interface models in a classical sense and a powerful analysis tool seems to be the method of matched asymptotic expansions (see [1, **?**,**?**] and the references in these papers).

In this paper, we discuss this relation from a different view point. To do so, we begin with the following degenerate parabolic equation:

$$\frac{\partial u}{\partial t} - \Delta\beta(u) = g \quad \text{in } Q, \tag{1.2}$$

where $g : \Omega \to \mathbb{R}$ is a given source. This equation is characterized by the choice of $\beta : \mathbb{R} \to \mathbb{R}$. For example, if we choose β to be a piecewise linear function of the form

$$\beta(r) := \begin{cases} k_s r & r < 0, \\ 0 & 0 \le r \le L, \\ k_\ell(r - L) & r > L; \end{cases} \tag{1.3}$$

where k_s and $k_\ell > 0$ represent the heat conductivities of the solid and liquid regions, respectively, and $L > 0$ is the latent heat constant, then (1.2) is the weak formulation of the Stefan problem, or the "enthalpy formulation," where the unknown u denotes the enthalpy and $\beta(u)$ denotes the temperature. The informant of the sharp interface, in other words the Stefan condition, is hidden in the weak formulation. Another example is the weak formulation of the Hele-Shaw problem. If we choose β to be the inverse of the Heaviside function

$$\mathcal{H}(r) := \begin{cases} 0 & \text{if } r < 0, \\ [0, 1] & \text{if } r = 0, \\ 1 & \text{if } r > 0 \end{cases} \quad \text{for all } r \in \mathbb{R},$$

so that β is the multivalued function $\beta(r) := \mathcal{H}^{-1}(r) = \partial I_{[0,1]}(r)$ for all $r \in [0, 1]$, then (1.2) can be stated as

$$\xi \in \beta(u), \quad \frac{\partial u}{\partial t} - \Delta\xi = g \quad \text{in } Q,$$

where $\partial I_{[0,1]}$ is the subdifferential of the indicator function $I_{[0,1]}$ on the interval $[0, 1]$, the unknown u denotes the order parameter. Details about weak formulations may be found in Visintin [21]. Weak formulations for this kind of sharp interface model are the focus of this paper. Therefore, we use the terms "Stefan problem" and "Hele-Shaw problem" in the sense of weak formulations throughout this paper.

Recently, the author considered the approach to the following Cahn–Hilliard system for the Stefan problem in [13]:

$$\frac{\partial u}{\partial t} - \Delta\mu = 0 \quad \text{in } Q, \tag{1.4}$$

$$\mu = -\varepsilon\Delta u + \beta(u) + \varepsilon\pi(u) - f \quad \text{in } Q, \tag{1.5}$$

with a dynamic boundary condition of the form

$$\frac{\partial u}{\partial t} + \partial_{\boldsymbol{\nu}} \mu - \Delta_\Gamma \mu = 0 \quad \text{on } \Sigma := (0,T) \times \Gamma, \tag{1.6}$$

$$\mu = \varepsilon \partial_{\boldsymbol{\nu}} u - \varepsilon \Delta_\Gamma u + \beta(u) + \varepsilon \pi(u) - f_\Gamma \quad \text{on } \Sigma, \tag{1.7}$$

where the symbol $\partial_{\boldsymbol{\nu}}$ denotes the normal derivative on the boundary Γ outward from Ω, the symbol Δ_Γ stands for the Laplace–Beltrami operator on Γ (see, e.g., [15, Chapter 3]), β is defined by (1.3), and $\pi : \mathbb{R} \to \mathbb{R}$ is a piecewise linear function defined by $\pi(r) := L/2$ if $r < 0$, $\pi(r) := L/2 - r$ if $0 \le r \le L$ and $\pi(r) := -L/2$ if $r > L$. Thanks to this choice, system (1.4)–(1.7) has the structure of a Cahn–Hilliard system. This problem originally comes from [14]. Formally, if we let $\varepsilon \searrow 0$ in (1.4)–(1.7), then we can see that the Cahn–Hilliard system (1.4)–(1.7) converges in a suitable sense to the following Stefan problem with a dynamic boundary condition:

$$\frac{\partial u}{\partial t} - \Delta \beta(u) = -\Delta f \quad \text{in } Q,$$

$$\frac{\partial u}{\partial t} + \partial_{\boldsymbol{\nu}} \beta(u) - \Delta_\Gamma \beta(u) = \partial_{\boldsymbol{\nu}} f_\Gamma - \Delta_\Gamma f_\Gamma \quad \text{on } \Sigma.$$

Here, we should take care of the difference between the order and position of ε in (1.1) and (1.5) even when $\beta(u) = u^3$ and $\pi(u) = -u$. In [13], β is assumed to satisfy the following condition:

> β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, and is a subdifferential $\beta = \partial \widehat{\beta}$ of some proper, lower semicontinuous, and convex function $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = 0$ with some effective domain $D(\beta)$. This implies $\beta(0) = 0$. Moreover, there exist two constants c, $\tilde{c} > 0$ such that
> $$\widehat{\beta}(r) \ge c|r|^2 - \tilde{c} \quad \text{for all } r \in \mathbb{R}. \tag{1.8}$$

It is easy to see that (1.2) represents a large number of problems, including the porous media equation, the nonlinear diffusion equation of Penrose–Fife type, the fast diffusion equation, and so on. However, to apply this approach from the Cahn–Hilliard system to these wider classes of the degenerate parabolic equation, the growth condition (1.8) is too strong (see, e.g. [12]). Therefore, in this paper based on the essential idea from [10], we relax the assumption in (1.8). This is the different point from the previous work [13]. See also [3, **?**] for related problems of interest.

Notation. Let $H := L^2(\Omega)$, $V := H^1(\Omega)$, $H_\Gamma := L^2(\Gamma)$ and $V_\Gamma := H^1(\Gamma)$ with the usual norms $|\cdot|_H$, $|\cdot|_V$, $|\cdot|_{H_\Gamma}$, $|\cdot|_{V_\Gamma}$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{H_\Gamma}$, $(\cdot,\cdot)_{V_\Gamma}$, respectively, and let $\boldsymbol{H} := H \times H_\Gamma$, $\boldsymbol{V} := \{(z, z_\Gamma) \in V \times V_\Gamma : z_\Gamma = z_{|\Gamma} \text{ a.e. on } \Gamma\}$ and $\boldsymbol{W} := H^2(\Omega) \times H^2(\Gamma)$. Then $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ are Hilbert spaces with the inner product

$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{H}} := (u, z)_H + (u_\Gamma, z_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{u}, \boldsymbol{z} \in \boldsymbol{H},$$

and the related norm is analogously defined as one of $\boldsymbol{V}$ or $\boldsymbol{W}$. Define $m : \boldsymbol{H} \to \mathbb{R}$ by

$$m(\boldsymbol{z}) := \frac{1}{|\Omega| + |\Gamma|} \left\{ \int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma \right\} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},$$

where $|\Omega| := \int_\Omega 1 dx$ and $|\Gamma| := \int_\Gamma 1 d\Gamma$. The symbol $\boldsymbol{V}^*$ denotes the dual space of $\boldsymbol{V}$, and the pair $\langle \cdot, \cdot \rangle_{\boldsymbol{V}^*, \boldsymbol{V}}$ denotes the duality pairing between $\boldsymbol{V}^*$ and $\boldsymbol{V}$. Moreover, define the bilinear form $a(\cdot, \cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{u}, \boldsymbol{z}) := \int_\Omega \nabla u \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{u}, \boldsymbol{z} \in \boldsymbol{V},$$

where ∇_Γ denotes the surface gradient on Γ (see, e.g., [15, Chapter 3]). We introduce the subspace $\boldsymbol{H}_0 := \{\boldsymbol{z} \in \boldsymbol{H} : m(\boldsymbol{z}) = 0\}$ of $\boldsymbol{H}$ and $\boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0$, with their norms $|\boldsymbol{z}|_{\boldsymbol{H}_0} := |\boldsymbol{z}|_{\boldsymbol{H}}$ for all $\boldsymbol{z} \in \boldsymbol{H}_0$ and $|\boldsymbol{z}|_{\boldsymbol{V}_0} := a(\boldsymbol{z}, \boldsymbol{z})^{1/2}$ for all $\boldsymbol{z} \in \boldsymbol{V}_0$. Then the duality mapping $\boldsymbol{F} : \boldsymbol{V}_0 \to \boldsymbol{V}_0^*$ is defined by $\langle \boldsymbol{F}\boldsymbol{z}, \tilde{\boldsymbol{z}} \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} := a(\boldsymbol{z}, \tilde{\boldsymbol{z}})$ for all $\boldsymbol{z}, \tilde{\boldsymbol{z}} \in \boldsymbol{V}_0$ and the inner product in $\boldsymbol{V}_0^*$ is defined by $(\boldsymbol{z}_1^*, \boldsymbol{z}_2^*)_{\boldsymbol{V}_0^*} := \langle \boldsymbol{z}_1^*, \boldsymbol{F}^{-1} \boldsymbol{z}_2^* \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0}$ for all $\boldsymbol{z}_1^*, \boldsymbol{z}_2^* \in \boldsymbol{V}_0^*$. Moreover, define $\boldsymbol{P} : \boldsymbol{H} \to \boldsymbol{H}_0$ by $\boldsymbol{P}\boldsymbol{z} := \boldsymbol{z} - m(\boldsymbol{z})\boldsymbol{1}$ for all $\boldsymbol{z} \in \boldsymbol{H}$, where $\boldsymbol{1} := (1, 1)$. Thus we obtain the dense and compact embeddings $\boldsymbol{V}_0 \hookrightarrow\hookrightarrow \boldsymbol{H}_0 \hookrightarrow\hookrightarrow \boldsymbol{V}_0^*$. See [8, **?**] for further details.

2 Existence of the weak solution

In this section, we state an existence theorem for the weak solution of a degenerate parabolic equation with a dynamic boundary condition of the following form:

$$\xi \in \beta(u), \quad \frac{\partial u}{\partial t} - \Delta \xi = g \quad \text{a.e. in } Q,$$
$$\xi_\Gamma \in \beta(u_\Gamma), \quad \xi_\Gamma = \xi_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}} \xi - \Delta_\Gamma \xi_\Gamma = g_\Gamma \quad \text{a.e. on } \Sigma,$$
$$u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma,$$

where β, g, g_Γ, u_0 and $u_{0\Gamma}$ satisfy the following assumptions:

(A1) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, and is a subdifferential $\beta = \partial \widehat{\beta}$ of some proper, lower semicontinuous, and convex function $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = 0$ in some effective domain $D(\beta)$. This implies that $\beta(0) = 0$;
(A2) $\boldsymbol{g} \in L^2(0, T; \boldsymbol{H}_0)$;
(A3) $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{H}$ with $m_0 \in \mathrm{int} D(\beta)$, and the compatibility conditions $\widehat{\beta}(u_0) \in L^1(\Omega), \widehat{\beta}(u_{0\Gamma}) \in L^1(\Gamma)$ hold.

We remark that the growth condition of $\widehat{\beta}$ in (A1) and the regularity of $\boldsymbol{u}_0$ in (A3) are relaxations from a previous related result [13] (cf. (1.8)).

Theorem 2.1. Under assumptions (A1)–(A3)*, there exists at least one pair* $(\boldsymbol{u}, \boldsymbol{\xi})$ *of functions* $\boldsymbol{u} \in H^1(0,T;\boldsymbol{V}^*) \cap L^2(0,T;\boldsymbol{H})$ *and* $\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V})$ *such that* $\xi \in \beta(u)$ *a.e. in* Q*,* $\xi_\Gamma \in \beta(u_\Gamma)$ *and* $\xi_\Gamma = \xi_{|_\Gamma}$ *a.e. on* Σ*, and that satisfy*

$$\begin{aligned}\langle u'(t), z\rangle_{V^*,V} + \langle u'_\Gamma(t), z_\Gamma\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Omega \nabla\xi(t)\cdot\nabla z dx + \int_\Gamma \nabla_\Gamma \xi_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma \\ = \int_\Omega g(t) z dx + \int_\Gamma g_\Gamma(t) z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V} \qquad (2.1)\end{aligned}$$

for a.a. $t \in (0,T)$ *with* $u(0) = u_0$ *a.e. in* Ω *and* $u_\Gamma(0) = u_{0\Gamma}$ *a.e. on* Γ*.*

The continuous dependence is completely the same as in a previous result [13, Theorem 2.2]. Therefore, we devolve the uniqueness problem on [13].

3 Proof of the main theorem

In this section, we prove the main theorem. The strategy of the proof is similar to that of [13, Theorem 2.1]. However, to relax the assumption we use a different uniform estimate. Let us start with an approximate problem. Recall the Yosida approximation $\beta_\lambda : \mathbb{R} \to \mathbb{R}$ and the related Moreau–Yosida regularization $\widehat{\beta}_\lambda$ of $\widehat{\beta} : \mathbb{R} \to \mathbb{R}$ (see, e.g., [4]). We see that $0 \le \widehat{\beta}_\lambda(r) \le \widehat{\beta}(r)$ for all $r \in \mathbb{R}$. Moreover, we define the following proper, lower semicontinuous, and convex functional $\varphi : \boldsymbol{H}_0 \to [0, +\infty]$:

$$\varphi(\boldsymbol{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{1}{2}\displaystyle\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{V}_0, \\ +\infty & \text{otherwise.} \end{cases}$$

The subdifferential $\partial\varphi$ on $\boldsymbol{H}_0$ is characterized by $\partial\varphi(\boldsymbol{z}) = (-\Delta z, \partial_{\boldsymbol{\nu}} z - \Delta_\Gamma z_\Gamma)$ with $\boldsymbol{z} \in D(\partial\varphi) = \boldsymbol{W} \cap \boldsymbol{V}_0$ (see, e.g., [9, Lemma C]). By virtue of the well-known theory of evolution equations (see, e.g., [8, **?**,**?**,**?**]), for each $\varepsilon \in (0,1]$ and $\lambda \in (0,1]$, there exist $\boldsymbol{v}_{\varepsilon,\lambda} \in H^1(0,T;\boldsymbol{H}_0) \cap L^\infty(0,T;\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{W})$ and $\boldsymbol{\mu}_{\varepsilon,\lambda} \in L^2(0,T;\boldsymbol{V})$ such that

$$\begin{aligned}\lambda \boldsymbol{v}'_{\varepsilon,\lambda}(t) + \boldsymbol{F}^{-1}\big(\boldsymbol{v}'_{\varepsilon,\lambda}(t)\big) + \varepsilon\partial\varphi\big(\boldsymbol{v}_{\varepsilon,\lambda}(t)\big) \\ = \boldsymbol{P}\big(-\boldsymbol{\beta}_\lambda\big(\boldsymbol{u}_{\varepsilon,\lambda}(t)\big) - \varepsilon\boldsymbol{\pi}\big(\boldsymbol{u}_{\varepsilon,\lambda}(t)\big) + \boldsymbol{f}(t)\big) \quad \text{in } \boldsymbol{H}_0 \qquad (3.1)\end{aligned}$$

for a.a. $s \in (0,T)$ with $\boldsymbol{v}_{\varepsilon,\lambda}(0) = \boldsymbol{v}_{0\varepsilon}$ in $\boldsymbol{H}_0$, where $\boldsymbol{v}_{0\varepsilon} \in \boldsymbol{V}_0$ solves the auxiliary problem $\boldsymbol{v}_{0\varepsilon} + \varepsilon\partial\varphi(\boldsymbol{v}_{0\varepsilon}) = \boldsymbol{v}_0$ in $\boldsymbol{H}_0$ so that there exists a constant $C > 0$ such that

$$\begin{aligned} |\boldsymbol{v}_{0\varepsilon}|^2_{\boldsymbol{H}_0} \le C, \quad \varepsilon|\boldsymbol{v}_{0\varepsilon}|^2_{\boldsymbol{V}_0} \le C, \qquad (3.2) \\ \int_\Omega \widehat{\beta}(v_{0\varepsilon} + m_0) dx \le C, \quad \int_\Gamma \widehat{\beta}(v_{0\varepsilon} + m_0) d\Gamma \le C.\end{aligned}$$

Moreover, $\boldsymbol{u}_{\varepsilon,\lambda} := \boldsymbol{v}_{\varepsilon,\lambda} + m_0\mathbf{1}$, $m_0 := m(\boldsymbol{u}_0)$ and $\mathbf{1} := (1,1)$, and $\boldsymbol{\beta}_\lambda(\boldsymbol{z}) := (\beta_\lambda(z), \beta_\lambda(z_\Gamma))$ and $\boldsymbol{\pi}(\boldsymbol{z}) := (\pi(z), \pi(z_\Gamma))$ for all $\boldsymbol{z} \in \boldsymbol{H}$, where $\pi : D(\pi) = \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with a Lipschitz constant L_π that breaks the monotonicity in $\beta + \varepsilon\pi$; $\boldsymbol{f} \in L^2(0,T; D(\partial\varphi))$ is the solution of $\boldsymbol{g}(t) = \partial\varphi(\boldsymbol{f}(t))$ in $\boldsymbol{H}_0$ for a.a. $t \in (0,T)$. Namely, from [9, Lemma C], we can choose $\boldsymbol{f}(t) := (f(t), f_\Gamma(t))$ to satisfy

$$\begin{cases} -\Delta f(t) = g(t) & \text{a.e. in } \Omega, \\ \partial_{\boldsymbol{\nu}} f(t) - \Delta_\Gamma f_\Gamma(t) = g_\Gamma(t) & \text{a.e. on } \Gamma, \end{cases} \quad \text{for a.a. } t \in (0,T). \tag{3.3}$$

3.1 Uniform estimates for approximate solutions

The key strategy in the proof is to obtain uniform estimates independent of $\varepsilon > 0$ and $\lambda > 0$, after which we consider the limiting procedures $\lambda \searrow 0$ and $\varepsilon \searrow 0$. Recall (3.1) in the equivalent form

$$\boldsymbol{v}'_{\varepsilon,\lambda}(s) + \boldsymbol{F}\big(\boldsymbol{P}\boldsymbol{\mu}_{\varepsilon,\lambda}(s)\big) = \mathbf{0} \quad \text{in } \boldsymbol{V}_0^*, \tag{3.4}$$

$$\boldsymbol{\mu}_{\varepsilon,\lambda}(s) = \lambda\boldsymbol{v}'_{\varepsilon,\lambda}(s) + \varepsilon\partial\varphi\big(\boldsymbol{v}_{\varepsilon,\lambda}(s)\big) + \boldsymbol{\beta}_\lambda\big(\boldsymbol{u}_{\varepsilon,\lambda}(s)\big) + \varepsilon\boldsymbol{\pi}\big(\boldsymbol{u}_{\varepsilon,\lambda}(s)\big) - \boldsymbol{f}(s) \quad \text{in } \boldsymbol{H} \tag{3.5}$$

for a.a. $s \in (0,T)$. Moreover, if we put $\varepsilon_0 := \min\{1, 1/(4L_\pi^2)\}$, then we have:

Lemma 3.1. There exist positive constants M_1, M_2 independent of $\varepsilon \in (0,\varepsilon_0]$ and $\lambda \in (0,1]$ such that

$$\lambda\big|\boldsymbol{v}_{\varepsilon,\lambda}(t)\big|_{\boldsymbol{H}_0}^2 + \big|\boldsymbol{v}_{\varepsilon,\lambda}(t)\big|_{\boldsymbol{V}_0^*}^2 \le M_1,$$

$$\frac{\varepsilon}{2}\int_0^t \big|\boldsymbol{v}_{\varepsilon,\lambda}(s)\big|_{\boldsymbol{V}_0}^2 ds + 2\int_0^t \big|\widehat{\beta}_\lambda\big(u_{\varepsilon,\lambda}(s)\big)\big|_{L^1(\Omega)} ds + 2\int_0^t \big|\widehat{\beta}_\lambda\big(u_{\Gamma,\varepsilon,\lambda}(s)\big)\big|_{L^1(\Gamma)} ds \le M_2$$

for all $t \in [0,T]$.

Proof. Multiplying (3.1) by $\boldsymbol{v}_{\varepsilon,\lambda}(s) \in \boldsymbol{V}_0$, we have

$$\begin{aligned} &\lambda\big(\boldsymbol{v}'_{\varepsilon,\lambda}(s), \boldsymbol{v}_{\varepsilon,\lambda}(s)\big)_{\boldsymbol{H}_0} + \big(\boldsymbol{v}'_{\varepsilon,\lambda}(s), \boldsymbol{v}_{\varepsilon,\lambda}(s)\big)_{\boldsymbol{V}_0^*} + \varepsilon\big(\partial\varphi\big(\boldsymbol{v}_{\varepsilon,\lambda}(s)\big), \boldsymbol{v}_{\varepsilon,\lambda}(s)\big)_{\boldsymbol{H}_0} \\ &\quad + \big(\boldsymbol{P}\boldsymbol{\beta}_\lambda\big(\boldsymbol{v}_{\varepsilon,\lambda}(s) + m_0\mathbf{1}\big), \boldsymbol{v}_{\varepsilon,\lambda}(s)\big)_{\boldsymbol{H}_0} = \big(\boldsymbol{f}(s) - \varepsilon\boldsymbol{P}\boldsymbol{\pi}\big(\boldsymbol{v}_{\varepsilon,\lambda}(s) + m_0\mathbf{1}\big), \boldsymbol{v}_{\varepsilon,\lambda}(s)\big)_{\boldsymbol{H}_0} \end{aligned}$$

for a.a. $s \in (0,T)$. Using the definition of the subdifferential, we see that

$$\begin{aligned} &\frac{\lambda}{2}\frac{d}{ds}\big|\boldsymbol{v}_{\varepsilon,\lambda}(s)\big|_{\boldsymbol{H}_0}^2 + \frac{1}{2}\frac{d}{ds}\big|\boldsymbol{v}_{\varepsilon,\lambda}(s)\big|_{\boldsymbol{V}_0^*}^2 + \frac{\varepsilon}{2}\big|\boldsymbol{v}_{\varepsilon,\lambda}(s)\big|_{\boldsymbol{V}_0}^2 \\ &\quad + \big|\widehat{\beta}_\lambda\big(u_{\varepsilon,\lambda}(s)\big)\big|_{L^1(\Omega)} + \big|\widehat{\beta}_\lambda\big(u_{\Gamma,\varepsilon,\lambda}(s)\big)\big|_{L^1(\Gamma)} \\ &\le \big(|\Omega| + |\Gamma|\big)\widehat{\beta}(m_0) + \frac{1}{2}\big|\boldsymbol{v}_{\varepsilon,\lambda}(s)\big|_{\boldsymbol{V}_0^*}^2 + \big|\boldsymbol{f}(s)\big|_{\boldsymbol{V}_0}^2 + L_\pi^2\varepsilon^2\big|\boldsymbol{v}_{\varepsilon,\lambda}(s)\big|_{\boldsymbol{V}_0}^2 \quad \text{for a.a. } s \in (0,T). \end{aligned}$$

Taking $\varepsilon \in (0,\varepsilon_0]$ and using the Gronwall inequality, we obtain the existence of M_1 and M_2 independent of $\varepsilon \in (0,\varepsilon_0]$ and $\lambda \in (0,1]$ satisfying the conclusion. □

Lemma 3.2. There exists a positive constant M_3, independent of $\varepsilon \in (0, \varepsilon_0]$ and $\lambda \in (0,1]$, such that

$$2\lambda \int_0^t \bigl|\boldsymbol{v}'_{\varepsilon,\lambda}(s)\bigr|_{\boldsymbol{H}_0}^2 ds + \int_0^t \bigl|\boldsymbol{v}'_{\varepsilon,\lambda}(s)\bigr|_{\boldsymbol{V}_0^*}^2 ds + \varepsilon \bigl|\boldsymbol{v}_{\varepsilon,\lambda}(t)\bigr|_{\boldsymbol{V}_0}^2 + 2\bigl|\widehat{\beta}_\lambda\bigl(u_{\varepsilon,\lambda}(t)\bigr)\bigr|_{L^1(\Omega)} + 2\bigl|\widehat{\beta}_\lambda\bigl(u_{\Gamma,\varepsilon,\lambda}(t)\bigr)\bigr|_{L^1(\Gamma)} \le M_3,$$

$$\int_0^t \bigl|\boldsymbol{P}\boldsymbol{\mu}_{\varepsilon,\lambda}(s)\bigr|_{\boldsymbol{V}_0}^2 ds \le M_3 \quad \textit{for all } t \in [0,T].$$

Proof. Multiplying (3.1) by $\boldsymbol{v}'_{\varepsilon,\lambda}(s) \in \boldsymbol{H}_0$, we have

$$\lambda\bigl|\boldsymbol{v}'_{\varepsilon,\lambda}(s)\bigr|_{\boldsymbol{H}_0}^2 + \frac{1}{2}\bigl|\boldsymbol{v}'_{\varepsilon,\lambda}(s)\bigr|_{\boldsymbol{V}_0^*}^2 + \varepsilon\frac{d}{ds}\varphi\bigl(\boldsymbol{v}_{\varepsilon,\lambda}(s)\bigr) + \frac{d}{ds}\int_\Omega \widehat{\beta}_\lambda\bigl(u_{\varepsilon,\lambda}(s)\bigr)dx + \frac{d}{ds}\int_\Gamma \widehat{\beta}_\lambda\bigl(u_{\Gamma,\varepsilon,\lambda}(s)\bigr)d\Gamma \le L_\pi^2\varepsilon^2\bigl|\boldsymbol{v}_{\varepsilon,\lambda}(s)\bigr|_{\boldsymbol{V}_0}^2 + \bigl|\boldsymbol{f}(s)\bigr|_{\boldsymbol{V}_0}^2 \quad \text{for a.a. } s \in (0,T).$$

Integrating this over $(0,t)$ with respect to s, we see that there exists a positive constant M_3, independent of $\varepsilon \in (0,\varepsilon_0]$ and $\lambda \in (0,1]$, such that the first estimate holds. Next, multiplying (3.4) by $\boldsymbol{P}\boldsymbol{\mu}_{\varepsilon,\lambda}(s) \in \boldsymbol{V}_0$ and integrating the resultant over $(0,t)$ with respect to s, we obtain the second estimate. □

The previous two lemmas are essentially the same as [13, Lemmas 3.1 and 3.2]. The next uniform estimate is the point of emphasis in this paper.

Lemma 3.3. There exists positive constant M_4, independent of $\varepsilon \in (0,1]$ and $\lambda \in (0,1]$, such that

$$\bigl|\boldsymbol{u}_{\varepsilon,\lambda}(t)\bigr|_{\boldsymbol{H}}^2 \le M_4\left(1 + \frac{\lambda}{\varepsilon}\right), \quad \bigl|\boldsymbol{v}_{\varepsilon,\lambda}(t)\bigr|_{\boldsymbol{H}_0}^2 \le M_4\left(1 + \frac{\lambda}{\varepsilon}\right),$$

$$\lambda\bigl|\boldsymbol{v}_{\varepsilon,\lambda}(t)\bigr|_{\boldsymbol{V}_0}^2 + \varepsilon\int_0^t \bigl|\partial\varphi\bigl(\boldsymbol{v}_{\varepsilon,\lambda}(s)\bigr)\bigr|_{\boldsymbol{H}_0}^2 ds \le M_4\left(1 + \frac{\lambda}{\varepsilon}\right) \quad \textit{for all } t \in [0,T].$$

Proof. Multiplying (3.4) by $\boldsymbol{v}_{\varepsilon,\lambda}(s) \in \boldsymbol{V}_0$ and using the fact $(d/ds)m(\boldsymbol{u}_{\varepsilon,\lambda}(s)) = 0$, we have

$$\bigl(\boldsymbol{u}'_{\varepsilon,\lambda}(s), \boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr)_{\boldsymbol{H}} + a\bigl(\boldsymbol{\mu}_{\varepsilon,\lambda}(s), \boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr) = 0$$

for a.a. $s \in (0,T)$ (see [13, Remark 3]). On the other hand, multiplying (3.5) by $\partial\varphi(\boldsymbol{v}_{\varepsilon,\lambda}(s)) \in \boldsymbol{H}_0$ and integrating by parts, we have

$$a\bigl(\boldsymbol{\mu}_{\varepsilon,\lambda}(s), \boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr) = \frac{\lambda}{2}\frac{d}{ds}a\bigl(\boldsymbol{u}_{\varepsilon,\lambda}(s), \boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr) + \varepsilon\bigl|\partial\varphi\bigl(\boldsymbol{v}_{\varepsilon,\lambda}(s)\bigr)\bigr|_{\boldsymbol{H}_0}^2 + \int_\Omega \beta'_\lambda\bigl(u_{\varepsilon,\lambda}(s)\bigr)\bigl|\nabla u_{\varepsilon,\lambda}(s)\bigr|^2 dx + \int_\Gamma \beta'_\lambda\bigl(u_{\Gamma,\varepsilon,\lambda}(s)\bigr)\bigl|\nabla_\Gamma u_{\Gamma,\varepsilon,\lambda}(s)\bigr|^2 d\Gamma + \varepsilon\bigl(\boldsymbol{\pi}\bigl(\boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr), \partial\varphi\bigl(\boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr)\bigr)_{\boldsymbol{H}} + \bigl(\partial\varphi\bigl(\boldsymbol{f}(s)\bigr), \boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr)_{\boldsymbol{H}}$$

for a.a. $s \in (0,T)$. Using the Lipschitz continuity of π and (3.3), we see that there exists a positive constant C_π such that

$$\frac{d}{ds}\left|\boldsymbol{u}_{\varepsilon,\lambda}(s)\right|_{\boldsymbol{H}}^2+\lambda\frac{d}{ds}\left|\boldsymbol{v}_{\varepsilon,\lambda}(s)\right|_{\boldsymbol{V}_0}^2+\varepsilon\left|\partial\varphi\bigl(\boldsymbol{v}_{\varepsilon,\lambda}(s)\bigr)\right|_{\boldsymbol{H}_0}^2 \le C_\pi\bigl(\left|\boldsymbol{u}_{\varepsilon,\lambda}(s)\right|_{\boldsymbol{H}}^2+1\bigr)+\left|\boldsymbol{g}(s)\right|_{\boldsymbol{H}_0}^2$$

for a.a. $s \in (0,T)$. Then, using (3.2) and the Gronwall inequality, we deduce that

$$\begin{aligned}\left|\boldsymbol{u}_{\varepsilon,\lambda}(t)\right|_{\boldsymbol{H}}^2 + \lambda\left|\boldsymbol{v}_{\varepsilon,\lambda}(t)\right|_{\boldsymbol{V}_0}^2 &\le \left\{|\boldsymbol{v}_{0\varepsilon} + m_0\mathbf{1}|_{\boldsymbol{H}}^2 + \lambda|\boldsymbol{v}_{0\varepsilon}|_{\boldsymbol{V}_0}^2 + C_\pi T + |\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H}_0)}^2\right\} e^{C_\pi T} \\ &\le \left\{2C + 2|m_0|^2\bigl(|\Omega| + |\Gamma|\bigr) + \frac{\lambda}{\varepsilon}C + C_\pi T + |\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H}_0)}^2\right\} e^{C_\pi T}\end{aligned}$$

for all $t \in [0,T]$. That is, there exists a positive constant M_4 independent of $\varepsilon \in (0,1]$ and $\lambda \in (0,1]$ such that the uniform estimates hold. □

Lemma 3.4. There exists positive constant M_5, independent of $\varepsilon \in (0,1]$ and $\lambda \in (0,1]$, such that

$$\int_0^t \left|\boldsymbol{\mu}_{\varepsilon,\lambda}(s)\right|_{\boldsymbol{V}_0}^2 ds \le M_5\left(1+\frac{\lambda}{\varepsilon}\right),$$

$$\int_0^t \left|\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_{\varepsilon,\lambda}(s)\bigr)\right|_{\boldsymbol{H}}^2 ds \le M_5\left(1+\frac{\lambda}{\varepsilon}\right) \quad \text{for all } t \in [0,T].$$

Using Lemmas 3.1 to 3.3, the proofs of these uniform estimates are completely the same as those for [9, Lemmas 4.3 and 4.4]. Therefore, we omit the proof.

3.2 Limiting procedure

From the previous uniform estimates, we can consider the limit as $\lambda \searrow 0$. More precisely, for each $\varepsilon \in (0,\varepsilon_0]$, there exists a subsequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with $\lambda_k \searrow 0$ as $k \to +\infty$ and a quadruplet $(\boldsymbol{v}_\varepsilon, \boldsymbol{v}_\varepsilon^*, \boldsymbol{\mu}_\varepsilon, \boldsymbol{\xi}_\varepsilon)$ of $\boldsymbol{v}_\varepsilon \in H^1(0,T;\boldsymbol{V}_0^*)\cap L^\infty(0,T;\boldsymbol{V}_0)\cap L^2(0,T;\boldsymbol{W})$, $\boldsymbol{v}_\varepsilon^* \in L^2(0,T;\boldsymbol{H}_0)$, $\boldsymbol{\mu}_\varepsilon \in L^2(0,T;\boldsymbol{V})$, $\boldsymbol{\xi}_\varepsilon \in L^2(0,T;\boldsymbol{H})$, such that

$$\begin{gathered}\boldsymbol{v}_{\varepsilon,\lambda_k} \to \boldsymbol{v}_\varepsilon \quad \text{weakly star in } L^\infty(0,T;\boldsymbol{H}_0), \quad \boldsymbol{v}_{\varepsilon,\lambda_k} \to \boldsymbol{v}_\varepsilon \quad \text{weakly in } L^2(0,T;\boldsymbol{V}_0), \\ \lambda_k\boldsymbol{v}'_{\varepsilon,\lambda_k} \to \mathbf{0} \quad \text{in } L^2(0,T;\boldsymbol{H}_0), \quad \boldsymbol{v}'_{\varepsilon,\lambda_k} \to \boldsymbol{v}'_\varepsilon \quad \text{weakly in } L^2(0,T;\boldsymbol{V}_0^*), \\ \boldsymbol{u}_{\varepsilon,\lambda_k} \to \boldsymbol{u}_\varepsilon := \boldsymbol{v}_\varepsilon + m_0\mathbf{1} \quad \text{weakly star in } L^\infty(0,T;\boldsymbol{V}), \\ \partial\varphi(\boldsymbol{v}_{\varepsilon,\lambda_k}) \to \boldsymbol{v}_\varepsilon^* \quad \text{weakly in } L^2(0,T;\boldsymbol{H}_0), \quad \boldsymbol{\mu}_{\varepsilon,\lambda_k} \to \boldsymbol{\mu}_\varepsilon \quad \text{weakly in } L^2(0,T;\boldsymbol{V}), \\ \boldsymbol{\beta}_{\lambda_k}(\boldsymbol{u}_{\varepsilon,\lambda_k}) \to \boldsymbol{\xi}_\varepsilon \quad \text{weakly in } L^2(0,T;\boldsymbol{H}) \quad \text{as } k \to +\infty.\end{gathered}$$

From the compactness theorem (see, e.g., [20, Section 8, Corollary 4]), this gives

$$\begin{gathered}\boldsymbol{v}_{\varepsilon,\lambda_k} \to \boldsymbol{v}_\varepsilon \quad \text{in } C\bigl([0,T];\boldsymbol{H}_0\bigr), \quad \boldsymbol{u}_{\varepsilon,\lambda_k} \to \boldsymbol{u}_\varepsilon \quad \text{in } C\bigl([0,T];\boldsymbol{H}\bigr), \\ \boldsymbol{\pi}(\boldsymbol{u}_{\varepsilon,\lambda_k}) \to \boldsymbol{\pi}(\boldsymbol{u}_\varepsilon) \quad \text{in } C\bigl([0,T];\boldsymbol{H}\bigr) \quad \text{as } k \to +\infty.\end{gathered}$$

Moreover, from the demi-closedness of $\partial\varphi$ and [4, Proposition 2.2], we see that $\boldsymbol{v}_\varepsilon^* = \partial\varphi(\boldsymbol{v}_\varepsilon)$ in $L^2(0,T;\boldsymbol{H}_0)$ and $\boldsymbol{\xi}_\varepsilon \in \boldsymbol{\beta}(\boldsymbol{u}_\varepsilon)$ in $L^2(0,T;\boldsymbol{H})$. From these facts, we deduce from (3.4) and (3.5) that

$$\boldsymbol{v}_\varepsilon'(t) + \boldsymbol{F}\big(\boldsymbol{P}\boldsymbol{\mu}_\varepsilon(t)\big) = \boldsymbol{0} \quad \text{in } \boldsymbol{V}_0^*, \tag{3.6}$$

$$\boldsymbol{\xi}_\varepsilon(t) \in \boldsymbol{\beta}\big(\boldsymbol{u}_\varepsilon(t)\big), \quad \boldsymbol{\mu}_\varepsilon(t) = \varepsilon\partial\varphi\big(\boldsymbol{v}_\varepsilon(t)\big) + \boldsymbol{\xi}_\varepsilon(t) + \varepsilon\boldsymbol{\pi}\big(\boldsymbol{u}_\varepsilon(t)\big) - \boldsymbol{f}(t) \quad \text{in } \boldsymbol{H} \tag{3.7}$$

for a.a. $t \in (0,T)$, with $\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_{0\varepsilon}$ in $\boldsymbol{H}$. We also have the regularity $\boldsymbol{u}_\varepsilon \in H^1(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})$. Now, taking the limit inferior as $\lambda \searrow 0$ on the uniform estimates, $\lambda/\varepsilon \searrow 0$ for all $\varepsilon \in (0,\varepsilon_0]$, and we therefore obtain the same kind of uniform estimates as in the previous lemmas independent of $\varepsilon \in (0,\varepsilon_0]$.

Proof of Theorem 2.1. By using the estimates for $\boldsymbol{v}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{\mu}_\varepsilon$ and $\boldsymbol{\xi}_\varepsilon$, there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ as $k \to +\infty$ and functions $\boldsymbol{v} \in H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;\boldsymbol{H}_0)$, $\boldsymbol{u} \in H^1(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{H})$, $\boldsymbol{\mu} \in L^2(0,T;\boldsymbol{V})$ and $\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{H})$ such that

$$\begin{gathered}
\boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{v} \quad \text{weakly star in } L^\infty(0,T;\boldsymbol{H}_0), \\
\boldsymbol{u}_{\varepsilon_k} \to \boldsymbol{u} = \boldsymbol{v} + m_0\boldsymbol{1} \quad \text{weakly star in } L^\infty(0,T;\boldsymbol{H}), \quad \varepsilon_k\boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{0} \quad \text{in } L^\infty(0,T;\boldsymbol{V}_0), \\
\boldsymbol{v}_{\varepsilon_k}' \to \boldsymbol{v}' \quad \text{weakly in } L^2(0,T;\boldsymbol{V}_0^*), \quad \boldsymbol{u}_{\varepsilon_k}' \to \boldsymbol{u}' \quad \text{weakly in } L^2(0,T;\boldsymbol{V}^*), \\
\boldsymbol{\mu}_{\varepsilon_k} \to \boldsymbol{\mu} \quad \text{weakly in } L^2(0,T;\boldsymbol{V}), \quad \boldsymbol{\xi}_{\varepsilon_k} \to \boldsymbol{\xi} \quad \text{weakly in } L^2(0,T;\boldsymbol{H}), \\
\varepsilon_k\boldsymbol{\pi}(\boldsymbol{u}_{\varepsilon_k}) \to \boldsymbol{0} \quad \text{in } L^\infty(0,T;\boldsymbol{H}) \quad \text{as } k \to +\infty.
\end{gathered}$$

From the Ascoli–Arzelà theorem, we also have

$$\boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{v} \quad \text{in } C\big([0,T];\boldsymbol{V}_0^*\big), \quad \boldsymbol{u}_{\varepsilon_k} \to \boldsymbol{u} \quad \text{in } C\big([0,T];\boldsymbol{V}^*\big) \quad \text{as } k \to +\infty.$$

Now, multiplying (3.7) by $\boldsymbol{\eta} \in L^2(0,T;\boldsymbol{V})$ and integrating over $(0,T)$, we obtain

$$\begin{aligned}
\int_0^T \big(\boldsymbol{\mu}_{\varepsilon_k}(t),\boldsymbol{\eta}(t)\big)_{\boldsymbol{H}} dt &= \varepsilon_k \int_0^T a\big(\boldsymbol{v}_{\varepsilon_k}(t),\boldsymbol{\eta}(t)\big) dt + \int_0^T \big(\boldsymbol{\xi}_{\varepsilon_k}(t),\boldsymbol{\eta}(t)\big)_{\boldsymbol{H}} dt \\
&\quad + \varepsilon_k \int_0^T \big(\boldsymbol{\pi}\big(\boldsymbol{u}_{\varepsilon_k}(t)\big),\boldsymbol{\eta}(t)\big)_{\boldsymbol{H}} dt - \int_0^T \big(\boldsymbol{f}(t),\boldsymbol{\eta}(t)\big)_{\boldsymbol{H}} dt.
\end{aligned} \tag{3.8}$$

Letting $k \to \infty$, we obtain

$$\int_0^T \big(\boldsymbol{\mu}(t),\boldsymbol{\eta}(t)\big)_{\boldsymbol{H}} dt = \int_0^T \big(\boldsymbol{\xi}(t) - \boldsymbol{f}(t),\boldsymbol{\eta}(t)\big)_{\boldsymbol{H}} dt \quad \text{for all } \boldsymbol{\eta} \in L^2(0,T;\boldsymbol{V}),$$

namely, $\boldsymbol{\mu} = \boldsymbol{\xi} - \boldsymbol{f}$ in $L^2(0,T;\boldsymbol{H})$. This implies the regularity of $\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V})$, that is, $\xi_\Gamma = \xi_{|\Gamma}$ a.e. on Σ. Next, we take $\boldsymbol{\eta} := \boldsymbol{u}_{\varepsilon_k} \in L^2(0,T;\boldsymbol{V})$ in (3.8), so that

$$\limsup_{k\to+\infty}\int_0^T \big(\boldsymbol{\xi}_{\varepsilon_k}(t),\boldsymbol{u}_{\varepsilon_k}(t)\big)_{\boldsymbol{H}}dt \le \int_0^T \langle \boldsymbol{u}(t),\boldsymbol{\mu}(t)\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}dt + \int_0^T \big(\boldsymbol{f}(t),\boldsymbol{u}(t)\big)_{\boldsymbol{H}}dt$$
$$= \int_0^T \big(\boldsymbol{\xi}(t),\boldsymbol{u}(t)\big)_{\boldsymbol{H}}dt.$$

Thus, applying [4, Proposition 2.2] we have $\boldsymbol{\xi} \in \boldsymbol{\beta}(\boldsymbol{u})$ in $L^2(0,T;\boldsymbol{H})$, and so we obtain $\xi \in \beta(u)$ a.e. in Ω. $\xi_\Gamma \in \beta(u_\Gamma)$ a.e. on Γ. Finally, letting $k \to +\infty$ and applying Hahn–Banach extension theorem of bounded linear functional on $\boldsymbol{V}$ to $V \times V_\Gamma$, then we see that (3.6) gives (2.1) for a.a. $t \in (0,T)$, with $u(0) = u_0$ a.e. in Ω and $u_\Gamma(0) = u_{0\Gamma}$ a.e. on Γ. □

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