

## Genericity of the strong observability for sampled

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1 **GENERICITY OF THE STRONG OBSERVABILITY FOR SAMPLED**  
2 **SYSTEMS.**

3 SABEUR AMMAR\*, MAJID MASSAOUD†, AND JEAN-CLAUDE VIVALDA ‡

4 **Abstract.** In this paper we prove that, generically, a sampled data system is observable provided  
5 that the number of outputs is greater than the number of inputs plus 1.

6 **Key words.** Observability, Sampled systems, Genericity, Transversality

7 **AMS subject classifications.** 93B07, 93C57

8 **1. Introduction.** In this paper we deal with the genericity of the observability  
9 of sampled data systems. Consider a controlled continuous time system written as

10 (1) 
$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x). \end{cases}$$

11 Given a time  $T$ , to system (1), we relate the following continuous-discrete-time system

12 (2) 
$$\begin{cases} \dot{x}(t) = f(x(t), u_k), t \in [kT, (k+1)T) \\ y_k = h(x(kT)) \end{cases}$$

13 where the control  $u$  is maintained constant on the intervals  $[kT, (k+1)T)$  and the  
14 measurements of the state are made only at each of the times  $0, T, 2T, \dots$ . System (2)  
15 is called the sampled data system related to (1).

16 Many physical processes or industrial devices can be modeled by a system of  
17 continuous-time differential equations as (1). From a mathematical viewpoint, the  
18 time and the state of this system can vary continuously but in practice, a controlled  
19 process is regulated by a digital computer which is not able to record a continuum  
20 of data. This is why control decisions are restricted to be taken at fixed times  
21  $0, T, 2T, \dots$ ; here  $T$  is called the sampling time and is a (generally small) parame-  
22 ter which depends on the instrumentation of the process, on the computing power  
23 and other parameters. For a continuous time system, the resulting situation can be  
24 modeled through the restriction that the applied inputs are constant on the inter-  
25 vals  $[0, T), [T, 2T), \dots$  and the state is (partially) measured only at those fixed times  
26  $0, T, 2T, \dots$ , that is to say we access to the values of the observation function only at  
27 times  $0, T, \dots$

28 For the sake of clarity, the precise assumptions that we make on these systems  
29 are stated in section 1.1 but we recall here the notion of observability. Regarding  
30 system (2), an input  $u^0$  is a sequence  $(u_k)_{k \geq 0}$  with  $u_k \in U$  ( $U$ , the input space). An  
31 input  $u^0$  being given, we denote by  $x(t)$  and  $\bar{x}(t)$  the solutions of (2) starting from  
32  $x_0$  and  $\bar{x}_0$  respectively; we say that system (2) is *observable for  $u^0$*  if for every pair of  
33 initial conditions  $(x_0, \bar{x}_0)$ , there exists an integer  $k$  such that  $h(x(kT)) \neq h(\bar{x}(kT))$ .

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34 The aim of this paper is to prove that, under some conditions on the respective  
35 dimensions of the inputs and the output, generically, the sampled system obtained  
36 from a continuous time system is observable.

37 Two questions can be investigated about observability and sampled systems. The  
38 first one is the problem of the preservation of the observability: if the continuous time  
39 system (1) is observable for any inputs, is it also the case for the sampled system (2)?  
40 The second question is the subject of this paper: given a sampling time  $T$ , how many  
41 are the continuous time systems (1) such that the sampled system (2) is observable?

42 Concerning the first question, the answer for linear systems is well known (see  
43 *e.g.* [18] and also [14] when the sampling time is not constant). If we deal with  
44 nonlinear systems, one would think that the observability of the continuous time  
45 system involves the observability of the sampled one, at least if the sampling time  
46 is chosen small enough. Surprisingly, this is not the case: a counter-example can  
47 be found in [6]; in order to get the observability of the sampled system, to this  
48 natural condition (the observability of the continuous time system), we have to add the  
49 condition of infinitesimal observability (see [11]) together with a technical condition  
50 bearing on the sequence of controls  $u_0, u_1, \dots$ .

51 The aim of this paper lies on a more “philosophical plane”. Due to the importance  
52 of the notion of observability, it is of interest to know “how many” continuous time  
53 systems give rise to observable sampled data systems. To be more precise, in [6]  
54 (systems given on a compact manifold) and in [3, 5] (systems given on  $\mathbf{R}^n$ ), we  
55 gave some natural sufficient conditions bearing on the continuous time system and  
56 under which the sampled system is observable. So, these works have a practical  
57 interest: for a class of continuous time systems our result allows us to decide on  
58 the observability of the sampled system. The present paper intends to prove that  
59 the set of continuous time systems which admit an observable sampled system is  
60 everywhere dense. Knowing that the set of rational numbers is dense in the set of  
61 real numbers does not permit us to decide if a particular given number is rational; in  
62 the same way, the result proven in this paper does not permit to decide if a particular  
63 sampled system is observable. Moreover, this result cannot be deduced from the  
64 abovementioned papers because, while the observability of the continuous time system  
65 is generic (see [11]) the additional conditions in [6, 3, 5] are not; also, in these papers,  
66 the observability is ensured only for sufficiently small sampling time  $T$ .

67 The genericity of the observability has been the subject of some researches in the  
68 last decades. As regards continuous-time systems, the first paper on the subject was  
69 about the genericity of the observability for uncontrolled systems [9]; this work was  
70 generalized to controlled systems by J.-P. Gauthier and I. Kupka, in [10] these authors  
71 proved the genericity of differential observability for systems with more outputs than  
72 inputs. A reference book on this subject is [11]. A related issue is the problem of the  
73 identifiability, in [8], the authors deal with general nonlinear systems which contain  
74 an unknown function, they prove that these (uncontrolled) systems are generically  
75 identifiable if the number of observations is at least three. Regarding the discrete-  
76 time systems, the first paper on the subject was from Aeyels [2], we can cite also [20]  
77 for the uncontrolled case and [7, 4] for the controlled case. In all of these papers,  
78 it is proved that the observability is a generic property provided that the number of  
79 outputs is greater than the number of inputs. Surprisingly, this result is no more valid  
80 for the systems considered in this paper: in the next sections, we shall prove that if  
81 the number of inputs is one and the number of outputs is two, the set of pairs  $(f, h)$   
82 such that system (2) is observable is not dense. Concerning the subject of this paper,  
83 we have also to cite [13], in this paper the authors prove also a result of genericity of

84 the observability for sampled data systems; the systems considered in this paper are  
 85 uncontrolled and the sampling time is not constant but depends on the state of the  
 86 system.

87 The tools used to prove our main result are essentially the same (but applied to  
 88 different situations) as the ones used in the above-mentioned papers, that is to say  
 89 the major theorem of the transversality theory.

90 The paper is organized as follows: in the next section, we state the precise formula-  
 91 tion of the problem we deal with and we recall some useful facts from the transversality  
 92 theory. In section 2, we state the main result of the paper and we introduce the lists  
 93  $L_{2n}$  and  $\bar{L}_{2n}$ , which are built from two initial conditions  $x_0$  and  $\bar{x}_0$  and their images  
 94 under the iterates of  $f$ ; we then introduce five possible configurations for these lists  
 95 (cf section 2.1). We then prove our main theorem for each of these configurations: the  
 96 corresponding results are stated in the three propositions 5, 9 and 10; the conjunc-  
 97 tion of these propositions give the main result. Finally, we give a counter-example in  
 98 order to prove that the observability of the sampled system is not generic in the case  
 99  $d_u = d_y + 1$  (cf. section 5).

100 **1.1. Problem formulation.** We consider two compact manifolds  $X$  and  $U$ ; we  
 101 let  $n = \dim X$  and  $d_u = \dim U$ . As usual we denote by  $T_x X$  the tangent space to  
 102  $X$  at  $x$ , and by  $TX$  the tangent bundle. A parametrized vector field will be a  $C^\infty$   
 103 mapping defined from  $X \times U$  into  $TX$  such that, for every  $u \in U$ ,  $f(\cdot, u)$  is a vector  
 104 field defined on  $X$ . The set of parametrized vector fields defined on  $X$  will be denoted  
 105 by  $\Gamma_U(X)$ . If  $f$  belongs to  $\Gamma_U(X)$ , we denote by  $\varphi_t^u$  the flow generated by the vector  
 106 field  $f(\cdot, u)$  (the parameter  $u$  being fixed); so for every  $x \in X$ , every  $u \in U$ , and every  
 107  $t \geq 0$ , we have

$$108 \quad \varphi_0^u(x) = x \quad \text{and} \quad \frac{d\varphi_t^u(x)}{dt} = f(\varphi_t^u(x), u).$$

109 Let  $u_0, u_1, \dots$  be a sequence of controls (i.e. a sequence of elements of  $U$ ), for  $k \geq 1$ ,  
 110 we denote by  $\underline{u}_k$  the finite sequence  $\underline{u}_k = (u_0, \dots, u_{k-1})$ .

111 Let  $\psi : M \rightarrow N$  be a differentiable mapping between two manifolds  $M$  and  $N$ ,  
 112 the notation  $d\psi(x)$  will stand for the differential of  $\psi$  at  $x$ ; let  $\xi \in T_x M$  be a tangent  
 113 vector,  $d\psi(x) \cdot \xi$  will denote the image of  $\xi$  under  $d\psi(x)$ .

114 Hereafter, together with a parametrized vector field, we consider a  $C^\infty$  mapping  
 115  $h$  from  $X$  to  $\mathbf{R}^{d_y}$  and, given a sampling time  $T > 0$ , we consider the mapping  $\Theta_T^{f,h}$   
 116 defined as

$$117 \quad (3) \quad \Theta_T^{f,h} : X \times U^{2n} \longrightarrow \mathbf{R}^{(2n+1)d_y} \times U^{2n} \\ (x, \underline{u}_{2n}) \longmapsto (h(x_0), h(x_1), \dots, h(x_{2n}), \underline{u}_{2n})$$

118 where the sequence  $(x_0, x_1, \dots, x_{2n})$  is defined recursively by  $x_0 = x$  and  $x_{k+1} =$   
 119  $\varphi_T^{u_k}(x_k)$ . Also, we denote by  $y_i$  and  $\bar{y}_i$ , the values at  $x_i$  and  $\bar{x}_i$  under  $h$ :  $y_i = h(x_i)$   
 120 and  $\bar{y}_i = h(\bar{x}_i)$ .

121 **DEFINITION 1.** We shall say that the sampled data system (2) is strongly observ-  
 122 able if the mapping  $\Theta_T^{f,h}$  defined above is one-to-one.

123 We shall show that, generically, system (2) is strongly observable, to be more  
 124 precise, we endow  $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$  with the Whitney topology and we shall  
 125 prove that the set of pairs  $(f, h)$  such that the mapping  $\Theta_T^{f,h}$  is injective is a residual  
 126 subset of  $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$  provided that  $d_y \geq d_u + 2$  (case  $d_u > 0$ ) or  $d_y \geq 1$   
 127 (case  $d_u = 0$ ). The tools used in this paper come from the transversality theory,  
 128 hereafter, we recall the notion of transversality as well as the Abraham's theorem of  
 129 density [1] which will be intensively in the proof of our main result.

130 DEFINITION 2 (Transversality). *Let  $f$  be a smooth mapping between two smooth*  
 131 *manifolds  $X$  and  $Y$ ,  $W$  a submanifold of  $Y$  and  $x$  a point in  $X$ . We shall say that  $f$*   
 132 *is transversal to  $W$  at  $x$  if either*

- 133 •  $f(x) \notin W$ , or
- 134 •  $f(x) \in W$  and  $T_{f(x)}Y = T_{f(x)}W + df_x(T_xX)$ .

135 *We shall say that  $f$  is transversal to  $W$  if it is transversal to  $W$  at every  $x \in X$ . We*  
 136 *shall use the symbol  $\pitchfork$  to denote the transversality.*

137 Concerning this definition, some elementary conditions show that the second  
 138 equality cannot be satisfied if  $\text{codim } W > \dim X$ . Therefore if  $\text{codim } W > \dim X$ ,  
 139 transversality means non membership: in this case saying that  $f$  is transverse to  $W$   
 140 amounts to saying that  $f(x) \notin W$  for every  $x \in X$ . This trick will be used later in  
 141 the proofs of propositions 9 and 10.

142 We recall also the notion of representation: let  $\mathcal{A}, X$  and  $Y$  be  $C^r$  manifolds  
 143 and  $\rho$  a map from  $\mathcal{A}$  to  $C^r(X, Y)$ . For  $a \in \mathcal{A}$ ,  $\rho_a : X \rightarrow Y$  is the map defined as  
 144  $\rho_a(x) = \rho(a)(x)$ . We say that  $\rho$  is a  $C^r$  representation if the evaluation map:

$$145 \quad \begin{aligned} \text{ev}_\rho : \mathcal{A} \times X &\longrightarrow Y \\ (a, x) &\longmapsto \rho_a(x) = \rho(a)(x) \end{aligned}$$

146 is a  $C^r$  map from  $\mathcal{A} \times X$  to  $Y$ .

147 THEOREM 3 (Transversal density theorem). *Let  $\mathcal{A}, X, Y$  be  $C^r$  manifolds,  $\rho :$   
 148  $\mathcal{A} \rightarrow C^r(X, Y)$  a  $C^r$  representation,  $W \subset Y$  a submanifold (not necessarily closed),  
 149 and  $\text{ev}_\rho : \mathcal{A} \times X \rightarrow Y$  the evaluation map. Define  $\mathcal{A}_W \subset \mathcal{A}$  by:*

$$150 \quad \mathcal{A}_W = \{a \in \mathcal{A} \mid \rho_a \pitchfork W\}$$

151 *Assume that:*

- 152 1.  $X$  has a finite dimension  $n$  and  $W$  has a finite codimension  $q$  in  $Y$ ;
- 153 2.  $\mathcal{A}$  and  $X$  are second countable;
- 154 3.  $r > \max(0, n - q)$ ;
- 155 4.  $\text{ev}_\rho \pitchfork W$ .

156 *Then  $\mathcal{A}_W$  is residual in  $\mathcal{A}$ .*

157 Notice that manifold  $\mathcal{A}$  is not necessarily finite dimensional; it may be a Banach  
 158 space or an open subset of a Banach space.

## 159 2. Main result.

160 THEOREM 4. *Assume that  $d_y \geq d_u + 2$  or that  $d_u = 0$  and  $d_y \geq 1$ , and let  $T > 0$*   
 161 *a given sampling time. Then the set of pairs  $(f, h)$  such that system (2) is strongly*  
 162 *observable is a residual subset of  $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ .*

163 In order to prove this theorem, we need some preliminary results, namely the  
 164 propositions 5, 9 and 10 stated in sections 3 and 4.3–4.4. In these propositions,  
 165 different possible configurations, denoted hereafter by  $\mathbf{C}_0$  through  $\mathbf{C}_4$ , of the lists  
 166  $(x_0, x_1, \dots, x_{2n})$  and  $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{2n})$  are considered (the  $\bar{x}_i$ 's are defined as the  $x_i$ 's,  
 167 cf. (3)). For each of this configuration  $\mathbf{C}_k$  ( $k = 0, \dots, 4$ ), we prove that, generically  
 168  $\Theta_T^{f,h}(x, u_{2n}) \neq \Theta_T^{f,h}(\bar{x}, u_{2n})$  if  $(x, u_{2n}), (\bar{x}, u_{2n})$  is under  $\mathbf{C}_k$  configuration.

169 To be more precise, given two different initial conditions  $x$  and  $\bar{x}$  and an integer  
 170  $s \leq 2n$ , we shall consider the two lists

$$171 \quad (4) \quad L_s = (x_0, \dots, x_s) \quad \text{and} \quad \bar{L}_s = (\bar{x}_0, \dots, \bar{x}_s),$$

173 where the  $\bar{x}_i$ 's are defined as the  $x_i$ 's. Concerning the two lists  $L_{2n}$  and  $\bar{L}_{2n}$ , we  
174 shall examine all the possible situations: the elements of these lists can be pairwise  
175 distinct, some equalities can occur among the elements of the first list while the ones  
176 of the second are pairwise distinct, etc. In the following section, we define five possible  
177 configurations and we prove that, necessarily the above-mentioned lists belong to one  
178 of five configurations.

179 **2.1. The different configurations.** Hereafter we shall give an exhaustive list  
180 of all the possibilities concerning the equalities between the elements of the lists  $L_{2n}$   
181 and  $\bar{L}_{2n}$ ; in the sequel, we shall say that the equalities  $x_i = \bar{x}_j$  and  $x_{i'} = \bar{x}_{j'}$  between  
182 elements of  $L_{2n}$  and elements of  $\bar{L}_{2n}$  are in the same direction if the differences  $i - j$   
183 and  $i' - j'$  have the same sign. Take  $x \neq \bar{x} \in X$ , even if we have to invert the roles of  
184  $L_{2n}$  and  $\bar{L}_{2n}$ , the possible configurations for these lists are:

185 **C<sub>0</sub>** The elements of  $\bar{L}_{2n}$  are pairwise different; moreover the only possible equalities  
186 between elements of  $L_{2n}$  and elements of  $\bar{L}_{2n}$  are all in the “same direction”.  
187 That is to say, let

$$188 \quad I = \{0 \leq i \leq 2n \mid \exists j \in \{0, \dots, 2n\}, x_i = \bar{x}_j\},$$

190 for  $i \in I$ , let  $E_i = \{0 \leq j \leq 2n \mid x_i = \bar{x}_j\}$ , then either for every  $i \in I$ , for  
191 every  $j \in E_i$ ,  $j < i$  or for every  $i \in I$ , for every  $j \in E_i$ ,  $j > i$ .

192 **C<sub>1</sub>** There exist some subscripts  $0 \leq i < p \leq 2n$  and  $0 \leq j, m \leq 2n$  such that

- 193 •  $x_p = x_i$  and  $\bar{x}_m = x_j$ ;
- 194 • there is no equality between the elements of  $L_{p-1}$ ;
- 195 • letting  $q = \max(j, p, m)$ , there is no equality between the elements of  
196  $\bar{L}_{q-1}$ ;
- 197 • the equalities between elements of  $L_{q-1}$  and  $\bar{L}_{q-1}$  have the same direc-  
198 tion.

199 **C<sub>2</sub>** There exist some subscripts  $0 \leq i < p \leq 2n$ , and  $0 \leq j < m \leq 2n$ , with  $m \geq p$   
200 and such that

- 201 •  $x_p = \bar{x}_m$  and  $x_i = \bar{x}_j$  with  $p - i \neq m - j$ ;
- 202 • there is no equalities between the elements of  $L_p$  nor between the ele-  
203 ments of  $\bar{L}_m$ ;
- 204 • the only possible equalities between elements of  $L_{m-1}$  and  $\bar{L}_{m-1}$  are all  
205 in the same direction; moreover if these equalities write  $x_{i_1} = \bar{x}_{j_1}, \dots,$   
206  $x_{i_r} = \bar{x}_{j_r}$  the differences  $i_1 - j_1, \dots, i_r - j_r$  are equal;
- 207 • if  $x_{i'} = \bar{x}_{j'}$  with  $i', j' \leq m$ , then  $i' \geq i$  and  $j' \geq j$ .

208 **C<sub>3</sub>** There exist some subscripts  $0 \leq i < p \leq 2n$ , and  $0 \leq m < j \leq 2n$  with  $p \leq j$  and  
209 such that

- 210 •  $x_p = \bar{x}_m$  and  $x_i = \bar{x}_j$ ;
- 211 • there is no equalities between the elements of  $L_p$  nor between the ele-  
212 ments of  $\bar{L}_j$ ;
- 213 • the only possible equalities between elements of  $L_{j-1}$  and  $\bar{L}_{j-1}$  are  
214 all in the same direction; moreover, if these equalities write  $x_{i_1} =$   
215  $\bar{x}_{j_1}, \dots, x_{i_r} = \bar{x}_{j_r}$  the differences  $i_1 - j_1, \dots, i_r - j_r$  are equal;
- 216 • if  $x_{i'} = \bar{x}_{j'}$  with  $i', j' \leq j$ , then  $i' \geq p$  and  $j' \geq m$ .

217 **C<sub>4</sub>** There exist some subscripts  $0 \leq i < p \leq 2n$ , and  $0 \leq j < m \leq 2n$  with  $m \geq p$   
218 and such that

- 219 •  $x_p = x_i$  and  $\bar{x}_m = \bar{x}_j$ ;
- 220 • there is no equality between one element of  $L_{m-1}$  and one element of  
221  $\bar{L}_{m-1}$ .

222 Denote by  $e_s$  the number of equalities between the elements of  $L_s \cup \bar{L}_s$ , if  $e_{2n} = 0$   
 223 or 1, then we are under the  $\mathbf{C}_0$  configuration.

224 Now, assume that  $e_{2n} \geq 2$ ; if we go from  $L_s \cup \bar{L}_s$  to  $L_{s-1} \cup \bar{L}_{s-1}$  we lose 0, 1 or  
 225 2 equalities. Thus if  $e_{2n} \geq 2$ , there exists a subscript  $m \leq 2n$  such that there exist  
 226 exactly two or exactly three equalities between the elements of  $L_m \cup \bar{L}_m$ ; we denote by  
 227  $s$  the minimal subscript with this property. Notice that, if  $e_s$  is exactly three,  $x_s$  and  
 228  $\bar{x}_s$  must be equal to some elements of  $L_{s-1} \cup \bar{L}_{s-1}$  (because we always have  $x_s \neq \bar{x}_s$ ).

229 Assume first that  $e_s = 3$  and denote by  $x_s = z_1$ ,  $\bar{x}_s = z_2$  and  $z_3 = z_4$  the three  
 230 equalities, then

- 231 • if  $z_1 \in L_s$ , and  $z_3, z_4 \in L_s$ , then  $L_{s-1}$  and  $\bar{L}_s$  are under  $\mathbf{C}_1$  (if  $z_2 \in L_s$ ) or  
 232  $\mathbf{C}_4$  configuration (if  $z_2 \in \bar{L}_s$ );
- 233 • if  $z_1 \in L_s$ ,  $z_3 \in L_s$ , and  $z_4 \in \bar{L}_s$ , then  $L_s$  and  $\bar{L}_{s-1}$  are under  $\mathbf{C}_1$  configura-  
 234 tion;
- 235 • if  $z_1 \in L_s$ ,  $z_3 \in \bar{L}_s$ , and  $z_4 \in \bar{L}_s$ , then  $L_s$  and  $\bar{L}_{s-1}$  are under  $\mathbf{C}_4$  configura-  
 236 tion;
- 237 • if  $z_1 \notin L_s$ , then  $L_s$  and  $\bar{L}_{s-1}$  are under  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  or  $\mathbf{C}_3$  configuration (we could  
 238 have to invert the roles of  $L_s$  and  $\bar{L}_s$ ).

239 If  $e_s = 2$  and if the lists  $L_s$  and  $\bar{L}_s$  are not under  $\mathbf{C}_1$ – $\mathbf{C}_4$  configurations, we have two  
 240 cases to consider.

241 In the first case, there exists some subscripts  $0 \leq t < s$  and  $0 \leq t' < s'$  such  
 242 that  $x_t = x_s$  and  $x_{t'} = x_{s'}$ ; if  $s' < s$  we set  $i = t'$  and  $p = s'$ , if  $s' = s$ , we set  
 243  $i = \min(t, t')$  and  $p = \max(t, t')$ . As noticed above, when we go from lists  $L_s$  and  
 244  $\bar{L}_s$  to the lists  $L_{s+1}$  and  $\bar{L}_{s+1}$  we gain 0, 1 or two equalities. Denote by  $\sigma$  the least  
 245 subscript greater than  $s$  such that there exist 3 or 4 equalities among the elements  
 246 of  $L_\sigma$  and  $\bar{L}_\sigma$ ; if such a subscript fails to exist, the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are under  $\mathbf{C}_0$   
 247 configuration. Otherwise, assume that there exists exactly three equalities between  
 248 the elements of  $L_\sigma \cup \bar{L}_\sigma$ , this additional equality can be one of the followings

- 249 •  $\bar{x}_\sigma = x_j$  (with  $j < \sigma$ ), the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are then under  $\mathbf{C}_1$  configuration;
- 250 •  $\bar{x}_\sigma = \bar{x}_j$  (with  $j < \sigma$ ), the lists  $L_p$  and  $\bar{L}_\sigma$  are under  $\mathbf{C}_4$  configuration;
- 251 •  $x_\sigma = \bar{x}_m$  (with  $m < \sigma$ ), the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are then under  $\mathbf{C}_1$  configuration;
- 252 •  $x_\sigma = x_j$  in this case we seek for the least subscript  $\sigma' > \sigma$  (if any) such that  
 253 one get 1 or 2 additional equalities by going from  $L_\sigma \cup \bar{L}_\sigma$  to  $L_{\sigma'} \cup \bar{L}_{\sigma'}$ .

254 If we are in the case where there exist exactly 4 equalities between the elements of  
 255  $L_\sigma \cup \bar{L}_\sigma$ , these equalities can be

- 256 •  $x_\sigma = x_{j_1}$  and  $\bar{x}_\sigma = x_{j_2}$  (with  $j_1, j_2 < \sigma$ ), in this case the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are  
 257 then under  $\mathbf{C}_1$  configuration;
- 258 •  $x_\sigma = x_{j_1}$  and  $\bar{x}_\sigma = \bar{x}_{j_2}$ , the lists  $L_p$  and  $\bar{L}_\sigma$  are then under  $\mathbf{C}_4$  configuration;
- 259 •  $x_\sigma = \bar{x}_{j_1}$  and  $\bar{x}_\sigma = x_{j_2}$ , the lists  $L_p$  and  $\bar{L}_{j_1}$  are then under  $\mathbf{C}_1$  configuration;
- 260 •  $x_\sigma = \bar{x}_{j_1}$  and  $\bar{x}_\sigma = \bar{x}_{j_2}$ , the lists  $L_p$  and  $\bar{L}_\sigma$  are then under  $\mathbf{C}_4$  configuration.

261 In the second case there exist some subscripts  $0 \leq t < s$  and  $0 \leq t' < s' < s$  such that  
 262  $s - t = s' - t'$ ,  $x_s = \bar{x}_t$  and  $x_{s'} = \bar{x}_{t'}$  (without loss of generality, we can exchange the  
 263 roles of  $x$  and  $\bar{x}$ ). Proceeding as in the first case, we prove that either the lists  $L_{2n}$   
 264 and  $\bar{L}_{2n}$  are under  $\mathbf{C}_0$  configuration or there exists a subscript  $s < \sigma \leq 2n$  such that  
 265  $L_\sigma$  and  $\bar{L}_\sigma$  are under one of the configurations  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  or  $\mathbf{C}_3$ .

266 Let the pair  $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$  be fixed; hereafter, we shall say that  
 267 the configuration of the triplet  $(x, \bar{x}, u_{2n})$  ( $x \neq \bar{x}$ ) is  $\mathbf{C}_i$  if the configuration of the  
 268 lists  $L_{2n}$  and  $\bar{L}_{2n}$  related to  $x$  and  $\bar{x}$  is  $\mathbf{C}_i$ . In the sequel, we shall assume that all the  
 269 function spaces (such that  $C^\infty(X, \mathbf{R}^p), \dots$ ) as well as the spaces  $\Gamma(X)$ ,  $\Gamma_U(X)$  and  
 270  $\mathcal{G}_2^U(a)$  are endowed with the  $C^r$  topology, where  $r \in \mathbf{N}^*$ .

271 **2.2. Outline of the proof of Theorem 4.** Without going into the tech-  
 272 nique details, we shall explain our strategy for the proof. Take  $(f, h) \in \Gamma_U(X) \times$   
 273  $C^\infty(X, \mathbf{R}^{d_y})$ , let  $x_0 \neq \bar{x}_0$  be two different initial conditions and  $\underline{u}_{2n+1}$  a sequence  
 274 of controls. The equality  $\Theta_T^{f,h}(x_0, \underline{u}_{2n+1}) = \Theta_T^{f,h}(\bar{x}_0, \underline{u}_{2n+1})$  can be formulated in  
 275 geometric terms. To show where are the difficulties, we make a first attempt by con-  
 276 sidering the simplest way to make this formulation. Consider the mapping, denoted  
 277 by  $r_{f,h}$ , related to the pair  $(f, h)$  and defined as

$$r_{f,h} : \begin{array}{ccc} X^{(2)} \times U^{2n} & \longrightarrow & \mathbf{R}^{(2n+1)d_y} \\ (x, \bar{x}, \underline{u}_{2n+1}) & \longmapsto & (y_0 - \bar{y}_0, \dots, y_{2n} - \bar{y}_{2n}) \end{array}$$

279 where  $y_i = h(x_i)$ ,  $\bar{y}_i = f(\bar{x}_i)$  and the  $x_i$ 's and the  $\bar{x}_i$ 's defined as in (3). The  
 280 equality  $\Theta_T^{f,h}(x_0, \underline{u}_{2n+1}) = \Theta_T^{f,h}(\bar{x}_0, \underline{u}_{2n+1})$  means that  $r_{f,h}(x_0, \bar{x}_0, \underline{u}_{2n})$  belongs to  
 281 the submanifold  $W = \{0\} \subset \mathbf{R}^{(2n+1)d_y}$ . Notice that the codimension of  $W$  is equal  
 282 to  $(2n+1)d_y$  and is greater than  $2n(d_u + 1)$  the dimension of the domain of  $r_{f,h}$ . If  
 283  $r_{f,h}$  is transverse to  $W$ , this inequality on codim  $W$  implies that  $r_{f,h}(x_0, \bar{x}_0, \underline{u}_{2n})$  does  
 284 not belong to  $W$  and therefore that  $\Theta_T^{f,h}(x_0, \underline{u}_{2n+1}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u}_{2n+1})$ . Assume now  
 285 that we are able to prove that, generically,  $r_{f,h}$  is transversal to  $W$ , that is to say  
 286 assume that there exists a residual set  $\mathcal{R}$  such that  $r_{f,h}$  is transversal to  $W$  whenever  
 287  $(f, h)$  belongs to  $\mathcal{R}$ , we then have proved that, generically,  $\Theta_T^{f,h}$  is one-to-one. In order  
 288 to prove that, generically,  $r_{f,h}$  is transversal to  $W$ , we could try to apply Theorem 3  
 289 by proving that the evaluation map related to the representation  $r$  from  $\Gamma_U(X) \times$   
 290  $C^\infty(X, \mathbf{R}^{d_y})$  to  $C^r(X^{(2)} \times U^{2n}, \mathbf{R}^{(2n+1)d_y})$  is transversal to  $W$ . This evaluation map  
 291  $\text{ev}_r$  is defined as  $\text{ev}_r(f, h, x, \bar{x}, \underline{u}_{2n}) = r_{f,h}(x, \bar{x}, \underline{u}_{2n})$ , as it is linear with respect to  
 292  $h$ , its differential with respect to  $h$  is given by  $\eta \mapsto (\eta(x_0) - \eta(\bar{x}_0), \dots, \eta(x_{2n}) -$   
 293  $\eta(\bar{x}_{2n}))$ , if we were able to show that there exists  $\eta$  in  $C^\infty(X, \mathbf{R}^{d_y})$  such that  $\eta(x_i) -$   
 294  $\eta(\bar{x}_i) = \mathfrak{Y}_i$ , for arbitrary vectors  $\mathfrak{Y}_0, \dots, \mathfrak{Y}_{2n}$  of  $\mathbf{R}^{d_y}$ , then we would be done. The  
 295 existence of such an  $\eta$  is generally not ensured but is certainly true if the elements  
 296  $x_0, \dots, x_{2n}, \bar{x}_0, \dots, \bar{x}_{2n}$  are all different or, more generally, if the two lists  $L_{2n}$  and  
 297  $\bar{L}_{2n}$  are under  $\mathbf{C}_0$  configuration. In this case, modifying  $r_{f,h}$  and the definition of  $W$   
 298 as explained in the next section, we can prove that, generically  $r_{f,h}$  is transversal to  
 299  $W$ . Now the two lists are not always under such a configuration, the points  $x_0$  and  $\bar{x}_0$   
 300 could be located on a periodic trajectory of  $f$  or could be singular points. The case of  
 301 singular points shows that we cannot disregard the cases where the two lists  $L_{2n}$  and  
 302  $\bar{L}_{2n}$  are not under configuration  $\mathbf{C}_0$ . Assume that  $x_0$  and  $\bar{x}_0$  are singular points for the  
 303 vector field  $f(\cdot, u_0)$  and take a sequence of identical controls:  $u_0 = u_1 = \dots = u_{2n}$ ,  
 304 then we cannot argue as in the case of  $\mathbf{C}_0$  configuration: a mapping  $\eta$  as above fails  
 305 to exist because  $x_0 = \dots = x_{2n}$  and  $\bar{x}_0 = \dots = \bar{x}_{2n}$ . Notice that this situation is  
 306 unavoidable: on some manifolds, every vector field has at least one singular point;  
 307 this means that the particular configurations  $\mathbf{C}_1$ – $\mathbf{C}_4$  cannot be eliminated by using  
 308 an argument of density.

309 The outline of the rest of this section will be the following: for each configuration  
 310  $\mathbf{C}_0$ – $\mathbf{C}_4$ , we shall prove that there exists a residual subset of  $\Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y})$   
 311 (endowed with the  $C^r$  topology), denoted by  $\mathcal{C}_k^r$  ( $k = 0, \dots, 4$ ), such that if  $(f, h) \in$   
 312  $\mathcal{C}_k^r$  and if  $(x_0, \bar{x}_0, \underline{u}_{2n})$  is in configuration  $\mathbf{C}_k$ , then  $\Theta_T^{f,h}(x_0, \underline{u}_{2n}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u}_{2n})$ .  
 313 Consider the intersection  $\mathcal{C}^r \triangleq \mathcal{C}_0^r \cap \dots \cap \mathcal{C}_4^r$ , which also is a residual subset, and  
 314 take a pair  $(f, h)$  in  $\mathcal{C}^r$ ; let  $x_0 \neq \bar{x}_0$  be two different initial conditions and  $\underline{u}_{2n}$  a finite  
 315 sequence of controls, as  $(x_0, \bar{x}_0, \underline{u}_{2n})$  must be in one of the  $\mathbf{C}_0$ – $\mathbf{C}_4$  configurations, we  
 316 have  $\Theta_T^{f,h}(x_0, \underline{u}_{2n}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u}_{2n})$ . Now taking the intersections of the sets  $\mathcal{C}^r$  for



317  $r = 1, 2, \dots$ , we obtain a residual  $\mathcal{R}$  of  $\Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y})$  endowed with the  $C^\infty$   
 318 topology; this residual is such that if  $(f, h)$  belongs to  $\mathcal{R}$ , then  $\Theta_T^{f,h}$  is one-to-one.

319 In the following section, the existence of the residual sets  $\mathcal{C}_0^r - \mathcal{C}_4^r$  is stated in  
 320 propositions 5, 9 and 10. For the proofs of these propositions, our strategy will be the  
 321 following: we shall consider some submanifold  $W$  together with some representation  
 322  $\rho$ ; the choice of  $W$  and  $\rho$  being related to the considered configuration of the lists  $L_{2n}$   
 323 and  $\bar{L}_{2n}$ . Concerning  $W$  and  $\rho$ , we shall prove the following results:

- 324 • by applying the *Transversal density theorem* [1], we shall see that the set of  
 325 pairs  $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$  which are transversal to  $W$  is dense;
- 326 • we shall prove also that the codimension of  $W$  is greater than the dimension  
 327 of the domain of  $\rho_{f,h}$ , which implies that the range of  $\rho_{f,h}$  does not intersect  
 328  $W$ ;
- 329 • due to our choice of  $W$  saying that  $\rho_{f,h}(x, \bar{x}, \underline{u}_{2n}) \notin W$  will imply that  
 330  $\Theta_T^{f,h}(x, \underline{u}_{2n}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u}_{2n})$ .

331 We shall provide a detailed proof for the  $\mathbf{C}_0$  and  $\mathbf{C}_1$  configurations, the proofs  
 332 for the  $\mathbf{C}_2$  and  $\mathbf{C}_3$  configurations will be omitted because they are very similar to the  
 333 previous ones. Concerning the configurations  $\mathbf{C}_0 - \mathbf{C}_3$ , we only need the assumption  
 334  $d_y \geq d_u + 1$  to prove the existence of the residual subsets  $\mathcal{C}_0 - \mathcal{C}_3$ . We have to consider  
 335 apart the case of  $\mathbf{C}_4$  configuration because to prove the existence of the residual subset  
 336  $\mathcal{C}_4$  we need the assumption  $d_y \geq d_u + 2$ .

337 **3. The triplet  $(x, \bar{x}, \underline{u}_{2n})$  is under configuration  $\mathbf{C}_0$ .** In this section, we  
 338 deal first with the simplest case: the  $\mathbf{C}_0$  configuration.

339 PROPOSITION 5. Assume that  $d_y > d_u$  (the number of observations is greater than  
 340 the number of controls). Denote by  $\mathcal{C}_0^r$  the set of pairs  $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$   
 341 such that  $\Theta_T^{f,h}(x, \underline{u}_{2n+1}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u}_{2n+1})$  whenever the triplet  $(x, \bar{x}, \underline{u}_{2n})$  (with  $x \neq \bar{x}$ )  
 342 is in configuration  $\mathbf{C}_0$ . Then  $\mathcal{C}_0^r$  contains a residual for the  $C^r$  topology.

343 *Proof.* We consider the representation

$$344 \quad \begin{aligned} \rho : \quad \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y}) &\longrightarrow C^r(X^{(2)} \times U^{2n}, (X^{2n+1})^2 \times \mathbf{R}^{(2n+1)d_y}) \\ &\quad (f, h) \qquad \longmapsto \quad \rho_{f,h} \end{aligned}$$

345 where  $\rho_{f,h}$  is the mapping

$$346 \quad \begin{aligned} \rho_{f,h} : \quad X^{(2)} \times U^{2n} &\longrightarrow X^{2n+1} \times X^{2n+1} \times \mathbf{R}^{(2n+1)d_y} \\ &\quad (x, \bar{x}, \underline{u}_{2n}) \longmapsto (x_0, \dots, x_{2n}, \bar{x}_0, \dots, \bar{x}_{2n}, y_0 - \bar{y}_0, \dots, y_{2n} - \bar{y}_{2n}), \end{aligned}$$

347 and where  $y_i = h(x_i)$  and  $\bar{y}_i = h(\bar{x}_i)$ , the  $x_i$ 's and the  $\bar{x}_i$ 's being defined above. We  
 348 consider the submanifold  $W_0$  included in  $(X^{2n+1})^2 \times \mathbf{R}^{(2n+1)d_y}$  defined as follows.  
 349 Submanifold  $W_0$  is the set of those elements  $(a_0, \dots, a_{2n}, \bar{a}_0, \dots, \bar{a}_{2n}, 0, \dots, 0)$  such  
 350 that

- 351 • the elements  $\bar{a}_0, \dots, \bar{a}_{2n}$  are pairwise distinct;
- 352 • we have  $a_k \neq \bar{a}_l$  if  $k > l$ .

353 Notice that the codimension of  $W_0$  is equal to  $(2n+1)d_y$ , as  $d_y > d_u$ , we have  
 354  $(2n+1)d_y \geq (2n+1)d_u + 2n+1$ , so the codimension of  $W_0$  is greater than the  
 355 dimension of  $X^{(2)} \times U^{2n+1}$ , the domain of  $\rho_{f,h}$ .

356 Recall that the evaluation map  $\text{ev}_\rho$  is defined as:

$$357 \quad \text{ev}_\rho(f, h, x, \bar{x}, \underline{u}_{2n}) = \rho_{f,h}(x, \bar{x}, \underline{u}_{2n}).$$

358 We shall see that  $\text{ev}_\rho$  is transversal to  $W_0$  at every given point

359 
$$\mathcal{X} \triangleq (f, h, x, \bar{x}, \underline{u}_{2n}) \in \Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y}) \times X^{(2)} \times U^{2n}.$$

360 Consider a point  $\mathcal{X}$  such that  $\text{ev}_\rho(\mathcal{X}) \in W_0$  and a vector  $(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y})$  that is tangent  
 361 to the codomain of  $\text{ev}_\rho$ , with  $\mathfrak{X}_i \in \mathbf{T}_{x_i}X$ ,  $\bar{\mathfrak{X}}_i \in \mathbf{T}_{\bar{x}_i}X$ , and  $\mathfrak{Y}_i \in \mathbf{R}^{d_y}$  ( $i = 0, \dots, 2n$ );  
 362 we have to prove that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^\infty(X, \mathbf{R}^{d_y})$ ,  $\xi \in \mathbf{T}_xX$ ,  $\bar{\xi} \in \mathbf{T}_{\bar{x}}X$ ,  
 363  $\nu_i \in \mathbf{T}_{u_i}U$  (for  $i = 0, \dots, 2n$ ) and a vector  $\zeta$  in the tangent space to  $W_0$  at  $\text{ev}_\rho(\mathcal{X})$   
 364 such that

365 (5) 
$$(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = d(\text{ev}_\rho)(\mathcal{X}) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu) + \zeta.$$

366 We shall prove this relation with  $\phi = 0$ ,  $\xi = 0$ ,  $\bar{\xi} = 0$  and  $\nu = 0$ . We denote by

367 
$$\alpha_0, \dots, \alpha_{2n}, \bar{\alpha}_0, \dots, \bar{\alpha}_{2n}, 0, \dots, 0$$

368 the components of  $\zeta$ . In the right-hand member of (5), the  $2(2n+1)$  first components  
 369 are equal to  $\alpha_0, \dots, \alpha_{2n}, \bar{\alpha}_0, \dots, \bar{\alpha}_{2n}$ , and can be chosen such that  $\alpha_i = \mathfrak{X}_i$  and  $\bar{\alpha}_i = \bar{\mathfrak{X}}_i$   
 370 ( $i = 0, \dots, 2n$ ). The  $2n+1$  last equations in (5) are

371 (6) 
$$\eta_0 - \bar{\eta}_0 = \mathfrak{Y}_0, \quad \dots, \quad \eta_{2n} - \bar{\eta}_{2n} = \mathfrak{Y}_{2n}$$

373 where we let  $\eta_i \triangleq \eta(x_i)$  and  $\bar{\eta}_i \triangleq \eta(\bar{x}_i)$ . We consider this system as a linear system  
 374 whose unknown are  $\eta_0, \dots, \eta_{2n}, \bar{\eta}_0, \dots, \bar{\eta}_{2n}$ . If  $\text{ev}_\rho(\mathcal{X})$  belongs to the submanifold  
 375  $W_0$ , the points  $\bar{x}_0, \dots, \bar{x}_{2n}$  are pairwise distinct, so the unknown  $\bar{\eta}_0, \dots, \bar{\eta}_{2n}$  can be  
 376 arbitrarily and independently chosen. There could be some equalities between the  
 377 elements of the list  $L_{2n}$  and between an element of  $L_{2n}$  and an element of  $\bar{L}_{2n}$ . If an  
 378 equality such that  $x_k = \bar{x}_l$  exists, then, as  $\text{ev}_\rho(\mathcal{X}) \in W_0$ , we necessarily have  $k < l$   
 379 and  $\eta_k = \bar{\eta}_l$ . The matrix of system (6) is then

380 
$$M = (A \quad | \quad -I_{(2n+1)d_y} + B)$$

381 Where  $I_{(2n+1)d_y}$  is the  $(2n+1)d_y$  dimensional identity matrix and matrix  $B$  is a block  
 382 matrix, whose blocks are 0 or  $d_y$  dimensional identity matrices. Moreover, matrix  $B$   
 383 is upper triangular: if we have an equality like  $x_k = \bar{x}_l$ , then we find in  $B$  the block  
 384  $I_{d_y}$  (identity matrix) at position  $(k, l)$  with  $l > k$ . From the form of matrix  $B$ , we can  
 385 conclude that the rank of  $M$  is equal to  $(2n+1)d_y$  which implies that we can find  
 386  $\bar{\eta}_0, \dots, \bar{\eta}_{2n}, \eta_0, \dots, \eta_{2n}$  such that system (6) has a solution. Denote by  $x_{k_1}, \dots, x_{k_p}$   
 387 the list of the elements of  $L_{2n}$  which are not equal to an element of  $\bar{L}_{2n}$ . It is possible  
 388 to find a solution of (6) such that  $\eta_{k_1} = \dots = \eta_{k_p} = 0$ , for such a solution, it is then  
 389 possible to find a mapping  $\eta$  such that  $\eta(x_i) = \eta_i$  and  $\eta(\bar{x}_i) = \bar{\eta}_i$  ( $i = 0, \dots, 2n$ ).

390 We have shown that  $\text{ev}_\rho$  is transversal to  $W_0$ . The conclusion of the proposition  
 391 now follows from the application of the *Transversal density theorem* [1]: the set  $\mathcal{O}_0^r$   
 392 of pairs  $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$  such that  $\rho_{f,h}$  is transversal to  $W_0$  is open  
 393 and dense in the  $C^r$  topology (for every  $r > 0$ ). Take now a pair  $(f, h)$  in this set  
 394  $\mathcal{O}_0^r$  and take two initial conditions  $x \neq \bar{x}$ , and a finite sequence of controls  $\underline{u}_{2n}$  such  
 395 that the triplet  $(x, \bar{x}, \underline{u}_{2n})$  is in configuration  $\mathbf{C}_0$ , the mapping  $\rho_{f,h}$  is transversal to  
 396  $W_0$  at  $(x, \bar{x}, \underline{u}_{2n})$  but as  $\text{codim } W_0 > \dim(X^{(2)} \times U^{2n})$ , transversality means that  
 397  $\rho_{f,h}(x, \bar{x}, \underline{u}_{2n}) \notin W_0$ , which implies that at least one of the equalities  $y_j = \bar{y}_j$  is not  
 398 satisfied and so  $\Theta_T^{f,h}(x, \underline{u}_{2n}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u}_{2n})$ .  $\square$

399 **4. The case of the  $C_1$ – $C_4$  configurations.** In the proofs of the next propo-  
400 sitions 9 and 10, we shall have to consider the derivative of  $\varphi_T(x)$  with respect to the  
401 vector field  $f$  (we are no more able to take  $\phi = 0$  as in the proof of Proposition 5).  
402 This is why we need to state a technical lemma, which bears on the computation of  
403 these derivatives. To prove this lemma, we have to take into account the periodic  
404 trajectories of a vector field; these trajectories have some generic properties—which  
405 we intend to use in the proof of Lemma 8—, which are stated in the Kupka-Smale  
406 theorem. Now the Kupka-Smale theorem has not been stated for parametrized vector  
407 fields, so we will show that it can be generalized to those vector fields.

408 **4.1. Periodic trajectories and the Kupka-Smale theorem.** Take a param-  
409 etrized vector field  $f \in \Gamma_U(X)$ ,  $u \in U$ ,  $x \in X$ , and assume that  $x$  belongs to a periodic  
410 trajectory of the vector field  $f(\cdot, u)$ . Then there exists  $\pi_0 > 0$  such that  $\varphi_{\pi_0}^u(x) = x$ ,  
411 this implies that  $d\varphi_{\pi_0}^u(x) \cdot f(x, u) = f(x, u)$ , so 1 is an eigenvalue of  $A \triangleq d\varphi_{\pi_0}^u(x)$ . In  
412 the sequel we shall have to consider expressions like  $\text{Id} + A + \dots + A^k$  and we shall  
413 need that this sum of linear mappings be invertible; this is certainly true if, apart  
414 from 1, the other eigenvalues have modulus different from 1. The theorem of Kupka-  
415 Smale [15, 17] asserts that this is generically the case for a vector field. Let  $a > 0$ ,  
416 hereafter, we denote by  $\mathcal{G}_2(a)$  the subset of  $\Gamma(X)$  of those vector fields  $f$  such that

- 417 • if  $x$  is a singular point of  $f$  (i.e.  $f(x) = 0$ ), then for every  $t \neq 0$ ,  $d\varphi_t(x) :$   
418  $\mathbb{T}_x X \rightarrow \mathbb{T}_x X$  has no complex eigenvalue of modulus 1;
- 419 • if  $x$  belongs to a periodic trajectory of  $f$  with period  $0 < \pi_0 \leq a$ , then,  
420 denoting by  $1, \lambda_2, \dots, \lambda_n$  the eigenvalues of  $d\varphi_{\pi_0}(x)$ , we have  $|\lambda_i| \neq 1$  for  
421  $i = 2, \dots, n$ .

422 Hereafter, recall that the manifolds  $X$  and  $U$  are assumed to be compact. We  
423 have

424 **THEOREM 6 (Kupka-Smale).** *Let  $a > 0$ , the set  $\mathcal{G}_2(a)$  is residual; moreover for*  
425 *the  $C^r$  topology ( $r < +\infty$ ),  $\mathcal{G}_2(a)$  is open and dense.*

426 This theorem can be generalized to parametrized vector fields, namely we have.

427 **THEOREM 7.** *Let  $a > 0$ , the set  $\mathcal{G}_2^U(a)$  of parametrized vector fields such that*  
428  *$f(\cdot, u) \in \mathcal{G}_2(a)$  for every  $u \in U$  is a residual; moreover  $\mathcal{G}_2^U(a)$  is open and dense for*  
429 *the  $C^r$  topology.*

430 This theorem can be proved by adapting the steps of the proof of the Kupka-  
431 Smale’s theorem which can be found in [1]. Owing to lack of space, we do not write  
432 here the proof of Theorem 7 but we give a sketch of this proof in Appendix A;  
433 moreover, the reader is referred to [19] where this result is proved in the case where  
434 the dimension of  $U$  is 1.

435 **4.2. Technical lemma.** The proofs of propositions 9 and 10 below will follow  
436 the same scheme as the proof of Prop. 5. Nevertheless, in the following propositions,  
437 in order to prove that the mapping  $\text{ev}_\rho$  is transversal to some submanifold  $W$ , we  
438 shall have to consider the derivative of the flow with respect to a vector field. Thus,  
439 before going further, we recall a result which will be used in the proof of Lemma 8  
440 and propositions 9 and 10. Take two vector fields  $f$  and  $\phi$  defined on  $X$  and denote by  
441  $\varphi_t$  and  $\varphi_t^\lambda$  ( $\lambda \in \mathbf{R}$ ) the flows related to  $f$  and  $f + \lambda\phi$  respectively. In [1, *Perturbation*  
442 *theorem*], the following formula is proved: for every  $x \in X$ , we have

$$443 \quad (7) \quad \left. \frac{d}{d\lambda} \varphi_t^\lambda(x) \right|_{\lambda=0} = \int_0^t d\varphi_\sigma \circ \phi \circ \varphi_{t-\sigma}(x) d\sigma.$$

444 Obviously, this formula can be extended to the case of parametrized vector fields.  
 445 Consider  $f$  and  $\phi$  in  $\Gamma_U(X)$  and denote by  $\varphi_t^{u,\lambda}$  the flow generated by the vector field  
 446  $f(\cdot, u) + \lambda \phi(\cdot, u)$  (with  $u$  fixed). Starting from an initial condition  $x_0$ , consider now  
 447 the sequence  $x_0^\lambda, x_1^\lambda, \dots$  defined recursively as  $x_0^\lambda = x_0$  and  $x_{i+1}^\lambda = \varphi_T^{u_i, \lambda}(x_i^\lambda)$ , then  
 448 applying formula (7), we deduce easily that

$$449 \quad \left. \frac{d}{d\lambda} x_{i+1}^\lambda \right|_{\lambda=0} = J_i + \delta_i(J_{i-1}) + \dots + \delta_1(J_0)$$

450 where

$$451 \quad J_k = \int_0^T d\varphi_\sigma^{u_k}(\varphi_{T-\sigma}^{u_k}(x_k)) \cdot \phi(\varphi_{T-\sigma}^{u_k}(x_k), u_k) d\sigma;$$

452 the integral  $J_k$  belongs to the tangent space of  $X$  at  $\varphi_T^{u_k}(x_k) = x_{k+1}$ . Moreover the  
 453  $\delta_k$ 's are the mappings defined as

$$454 \quad \delta_k = d(\varphi_T^{u_i} \circ \dots \circ \varphi_T^{u_k})(x_k).$$

455 We state now a preliminary result which will be used in the proofs of propositions 9  
 456 and 10. We shall say that these lists are under  $\mathbf{C}'_3$  (resp.  $\mathbf{C}'_4$ ) configuration if

- 457 • they are under  $\mathbf{C}_3$  (resp.  $\mathbf{C}_4$ ) configuration and if
- 458 • there exists a subscript  $k \in \{i, \dots, p-1\} \cup \{m, \dots, j-1\}$  (resp.  $k \in \{i, \dots, p-1\} \cup \{j, \dots, m-1\}$ ) such that  $u_k \neq u_{p-1}$ .

460 The proof of the following lemma is postponed in Appendix B.

461 LEMMA 8. *Let  $\mathfrak{X}_p \in T_{x_p}X$  be an arbitrary tangent vector to  $X$  at  $x_p$ . Assume*  
 462 *that the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are under  $\mathbf{C}_1$  configuration, then one can find a vector field*  
 463  *$\phi \in \Gamma_U(X)$  such that we have*

$$464 \quad \left. \frac{dx_p^\lambda}{d\lambda} \right|_{\lambda=0} = \mathfrak{X}_p, \quad \left. \frac{dx_i^\lambda}{d\lambda} \right|_{\lambda=0} = 0.$$

466 *If these lists are under  $\mathbf{C}_2$ ,  $\mathbf{C}'_3$  or  $\mathbf{C}'_4$  configuration, then one can find a vector field*  
 467  *$\phi \in \Gamma_U(X)$  such that we have*

$$468 \quad \left. \frac{dx_p^\lambda}{d\lambda} \right|_{\lambda=0} = \mathfrak{X}_p, \quad \left. \frac{dx_i^\lambda}{d\lambda} \right|_{\lambda=0} = 0, \quad \left. \frac{d\bar{x}_m^\lambda}{d\lambda} \right|_{\lambda=0} = 0, \quad \left. \frac{d\bar{x}_j^\lambda}{d\lambda} \right|_{\lambda=0} = 0.$$

470 **4.3. The triplet  $(x, \bar{x}, \underline{u}_{2n})$  is under one of the configurations  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  or**  
 471  **$\mathbf{C}_3$ .**

472 PROPOSITION 9. *Assume that  $d_y > d_u$  (the number of observations is greater*  
 473 *than the number of controls). For  $k = 1, 2, 3$ , denote by  $\mathcal{C}_k^r$  the subset of pairs  $(f, h) \in$   
 474  $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$  such that  $\Theta_T^{f,h}(x, \underline{u}_{2n+1}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u}_{2n+1})$  whenever the triplet  
 475  $(x, \bar{x}, \underline{u}_{2n})$  (with  $x \neq \bar{x}$ ) is in configuration  $\mathbf{C}_k$ . Then each subset  $\mathcal{C}_k^r$  contains a  
 476 residual for the  $C^r$  topology.*

477 Hereafter, we write the proof of this proposition only in the case of  $\mathbf{C}_1$  configuration,  
 478 the case proofs for the other configurations being very similar.

479 **4.3.1. Proof of the proposition in the case of  $\mathbf{C}_1$  configuration.** If the  
 480 triplet  $(x, \bar{x}, \underline{u}_{2n})$  is in the  $\mathbf{C}_1$  configuration, there exist subscripts  $i, j, p, m$  such that  
 481  $x_p = x_i$  and  $\bar{x}_m = x_j$ .

482 We choose four subscripts  $0 \leq i, j, p, m \leq 2n$  such that  $i < p$ ; letting  $q =$   
483  $\max(i, j, p, m)$ , we consider the representation  $\rho$

$$484 \quad \rho : \begin{array}{ccc} \mathcal{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y}) & \longrightarrow & C^r(X^{(2)} \times U^q, X^{q+1} \times X^{q+1} \times \mathbf{R}^{q d_y}) \\ (f, h) & \longmapsto & \rho_{f,h} \end{array}$$

485 defined through the mapping  $\rho_{f,h}$

$$486 \quad \rho_{f,h} : \begin{array}{ccc} X^{(2)} \times U^q & \longrightarrow & X^{q+1} \times X^{q+1} \times \mathbf{R}^{q d_y} \\ (x, \bar{x}, \underline{u}_q) & \longmapsto & (x_0, \dots, x_q, \bar{x}_0, \dots, \bar{x}_q, y_0 - \bar{y}_0, \dots, y_{q-1} - \bar{y}_{q-1}). \end{array}$$

487 We fix two lists of subscripts (possibly empty)  $i_1 < \dots < i_r < q$  and  $j_1 < \dots < j_r < q$   
488 such that the signs of  $i_1 - j_1, \dots, i_r - j_r$  are the same ; if  $j, m < q$ , one has  $i_k = j$   
489 and  $j_k = m$  for some subscript  $k$ . We consider also the submanifold

$$490 \quad V_m^{j,i,p} \subset X^{q+1} \times X^{q+1} \times \mathbf{R}^{(2n+1)d_y}$$

491 defined as follows:  $V_m^{j,i,p}$  is the set of those elements

$$492 \quad (a_0, \dots, a_q, \bar{a}_0, \dots, \bar{a}_q, 0, \dots, 0)$$

493 such that

- 494 • we have the equalities  $\bar{a}_m = a_j$  and  $a_i = a_p$ ;
- 495 • the elements of  $\{\bar{a}_0, \dots, \bar{a}_{q-1}\}$  are pairwise different;
- 496 •  $a_{i'} \neq \bar{a}_{j'}$  if  $(i', j') \neq (i_k, j_k)$  ( $i', j' < q, k = 1, \dots, r$ ).

497 Notice that the number of submanifold having the above properties is finite, moreover  
498 the codimension of  $V_m^{j,i,p}$  is equal to  $2n+q d_y$  and is greater or equal to  $2n+q(d_u+1) >$   
499  $2n+q d_u$ ; so the codimension of  $V_m^{j,i,p}$  is greater than the dimension of the domain of  
500  $\rho_{f,h}$ .

501 We shall show that  $\text{ev}_\rho$  is transversal to  $V_m^{j,i,p}$ . Let  $\mathcal{X} \triangleq (f, h, x, \bar{x}, \underline{u}_q)$  be a  
502 point such that  $\text{ev}_\rho(\mathcal{X}) \in V_m^{j,i,p}$  and take a vector  $(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y})$  which is tangent to the  
503 codomain of  $\text{ev}_\rho$  at  $\text{ev}_\rho(\mathcal{X})$ , with  $\mathfrak{X}_k \in T_{x_k} X$ ,  $\bar{\mathfrak{X}}_k \in T_{\bar{x}_k} X$  ( $k = 0, \dots, q$ ) and  $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$   
504 ( $k = 0, \dots, q-1$ ). We have to prove that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^r(X, \mathbf{R}^{d_y})$ ,  
505  $\xi \in T_x X$ ,  $\bar{\xi} \in T_{\bar{x}} X$ ,  $\nu_l \in T_{u_l} U$  (for  $l = 0, \dots, m-1$ ) and a vector  $\zeta$  in the tangent  
506 space to  $V_m^{j,i,p}$  at  $\text{ev}_\rho(\mathcal{X})$  such that

$$507 \quad (8) \quad (\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = d(\text{ev}_\rho)(\mathcal{X}) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu) + \zeta.$$

508 We shall prove this relation with  $\xi = 0$  and  $\nu = 0$ . We denote by

$$509 \quad \alpha_0, \dots, \alpha_q, \bar{\alpha}_0, \dots, \bar{\alpha}_q, \beta_0, \dots, 0, \dots, 0$$

510 the components of  $\zeta$ .

511 Among the  $2q+2$  first equations in (8), the ones corresponding to subscripts  
512 different from  $j, i$  and  $p$  (first  $q+1$  equations) or  $m$  (last  $q+1$  equations) are trivial  
513 because the corresponding tangent vectors  $\alpha_k$  ( $k \neq j, i, p$ ) and  $\bar{\alpha}_k$  ( $k \neq m$ ) can be  
514 arbitrarily chosen. Thus we focus on the following four equations

$$515 \quad (9) \quad \alpha_j + A_j = \mathfrak{X}_j, \quad \alpha_i + A_i = \mathfrak{X}_i, \quad \alpha_p + A_p = \mathfrak{X}_p, \quad \bar{\alpha}_m + \bar{A}_m + \bar{\xi}_m = \bar{\mathfrak{X}}_m.$$

517 Here  $A_l$  (resp.  $\bar{A}_l$ ),  $l = 0, \dots, q$ , denotes the derivative of  $x_l^\lambda$  (resp.  $\bar{x}_l$ ) with respect  
518 to  $\lambda$  evaluated at  $\lambda = 0$  while  $\bar{\xi}_j$  is defined recursively as

$$519 \quad (10) \quad \bar{\xi}_0 = \bar{\xi}, \quad \bar{\xi}_{k+1} = d\varphi_T^{u_k}(\bar{x}_k) \cdot \bar{\xi}_k, \quad k = 0, \dots, 2n-1.$$

521 Notice also that, as  $\zeta$  is a tangent vector to  $W_r^{j,i,p}$ , necessarily,  $\alpha_i = \alpha_p$  and  $\alpha_j = \bar{\alpha}_m$ .  
522 As we are under  $\mathbf{C}_1$  configuration, we can apply the preliminary lemma 8, thus we  
523 can find a vector field  $\phi$  such that  $A_i = 0$  while  $A_p$  is equal to an arbitrary tangent  
524 vector in  $T_{x_p}X$  (notice that we cannot guarantee that  $A_j = 0$  because we do not  
525 know the position of  $j$  with respect to  $i$  and  $p$ ). Taking into account that  $\alpha_p = \alpha_i$   
526 and  $\bar{\alpha}_m = \alpha_j$ , the equations (9) then rewrite

$$527 \quad (11) \quad \alpha_j + A_j = \mathfrak{X}_j, \quad \alpha_i = \mathfrak{X}_i, \quad \alpha_i + A_p = \mathfrak{X}_p, \quad \alpha_j + \bar{A}_m + \bar{\xi}_m = \mathfrak{X}_m,$$

529 a solution to system (11) is then

$$530 \quad \alpha_i = \mathfrak{X}_i, \quad A_p = \mathfrak{X}_p - \mathfrak{X}_i, \quad \alpha_j = \mathfrak{X}_j - A_j, \quad \bar{\xi}_m = \mathfrak{X}_m - \bar{A}_m - \mathfrak{X}_j + A_j$$

532 Notice that, once  $A_p$  has been chosen,  $A_j$  and  $\bar{A}_m$  are fixed (they depend on  $\mathfrak{X}_p - \mathfrak{X}_i$ ),  
533 moreover  $\bar{\xi}$  can be chosen in such a way that  $\bar{\xi}_m$  is equal to an arbitrary tangent  
534 vector.

535 As for the last  $2q$  equations, they can be written

$$536 \quad (12) \quad \eta_0 - \bar{\eta}_0 - \bar{\chi}_0 = \mathfrak{Y}_0, \quad \dots, \quad \eta_{q-1} - \bar{\eta}_{q-1} - \bar{\chi}_{q-1} = \mathfrak{Y}_{q-1}$$

538 where  $\bar{\chi}_k = dh(x_k) \cdot \bar{\xi}_k$ ,  $\eta_k = \eta(x_k)$ , and  $\bar{\eta}_k = \eta(\bar{x}_k)$  ( $k = 0, \dots, q-1$ ). We regard  
539 this system as a linear system. As  $\bar{x}_m = x_j$  is the only equality between the  $x_k$ 's  
540 and the  $\bar{x}_k$ 's, we can consider that the unknowns for this system are  $\eta_0, \dots, \eta_{q-1}$  and  
541  $\bar{\eta}_0, \dots, \bar{\eta}_{q-1}$ ; the matrix of this system then writes

$$542 \quad (A \mid -I_{q d_y} + B)$$

543 here  $I_{q d_y}$  denotes the  $q d_y$  dimensional identity matrix while  $B$  is a  $d_y \times d_y$  block  
544 matrix which is a strictly upper or strictly lower triangular matrix. Thus  $I_{q d_y} - B$  is  
545 an upper or lower-triangular block matrix, the elements of the diagonal being equal  
546 to the  $d_y \times d_y$  identity matrix, hence  $-I_{q d_y} + B$  is non singular, which proves that  
547 system (12) admits a solution.

548 This proves that  $\text{ev}_p$  is transversal to  $V_m^{j,i,p}$ ; we achieve the proof of Proposition 9  
549 as the one of Proposition 5.

550 **4.4. The triplet  $(x, \bar{x}, \underline{u}_{2n})$  is under one configuration  $\mathbf{C}_4$ .** We examine  
551 now the case of  $\mathbf{C}_4$  configuration, in this case, the assumption  $d_y \geq d_u + 1$  is no more  
552 sufficient.

553 **PROPOSITION 10.** *Assume that  $d_u = 0$  and  $d_y \geq 1$ , or  $d_u > 0$  and  $d_y \geq d_u + 2$   
554 (the number of observations is greater than the number of controls plus one). Denote  
555 by  $\mathcal{C}_4^r$  the set of pairs  $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$  such that  $\Theta_T^{f,h}(x, \underline{u}_{2n+1}) \neq$   
556  $\Theta_T^{f,h}(\bar{x}, \underline{u}_{2n+1})$  whenever the triplet  $(x, \bar{x}, \underline{u}_{2n})$  (with  $x \neq \bar{x}$ ) is in configuration  $\mathbf{C}_4$ .  
557 Then  $\mathcal{C}_4^r$  contains a residual for the  $C^r$  topology.*

558 *Proof of Proposition 10.* In the sequel, we shall say that the lists  $L_{2n}$  and  $\bar{L}_{2n}$   
559 are under  $\mathbf{C}_4'$  configuration if they are under  $\mathbf{C}_4$  configuration but not under  $\mathbf{C}_4'$   
560 configuration. We shall consider these two subcases separately.

561 **Configuration  $\mathbf{C}_4'$ .** In this case there exist subscripts  $0 \leq i < p \leq 2n$  and  $0 \leq$   
562  $j < m \leq 2n$  such that  $x_i = x_p$  and  $\bar{x}_j = \bar{x}_m$ ; moreover there exists a subscripts  
563  $k \in \{i, \dots, p-1\} \cup \{j, \dots, m-1\}$  such that  $u_k \neq u_{p-1}$ . Without loss of generality,  
564 we can assume that  $m \geq p$ .

565 We choose four subscripts  $0 \leq i < p \leq 2n$  and  $0 \leq j < m \leq 2n$  ( $m \geq p$ ), as well  
 566 as a subscript  $k_0 \in \{i, \dots, p-1\} \cup \{j, \dots, m-1\}$ . We consider the representation  $\rho$   
 567 defined on  $\mathcal{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y})$  through the mapping  $\rho_{f,h}$  as

$$568 \quad \begin{aligned} \rho_{f,h} : X^{(2)} \times U_{(k_0, p-1)}^m &\longrightarrow X^{m+1} \times X^{m+1} \times \mathbf{R}^{m d_y} \\ (x, \bar{x}, \underline{u}_m) &\longmapsto (x_0, \dots, x_m, \bar{x}_0, \dots, \bar{x}_m, y_0 - \bar{y}_0, \dots, y_{m-1} - \bar{y}_{m-1}). \end{aligned}$$

569 Together with  $\rho$ , we consider the submanifold

$$570 \quad Z_{j,m}^{i,p} \subset X^{m+1} \times X^{m+1} \times \mathbf{R}^{m d_y}$$

571 defined as the set of those elements

$$572 \quad (a_0, \dots, a_m, \bar{a}_0, \dots, \bar{a}_m, 0, \dots, 0)$$

573 such that

- 574 • we have the equalities  $a_i = a_p$  and  $\bar{a}_j = \bar{a}_m$ ;
- 575 • the above equalities are the only ones between the elements of  $L_p \cup \bar{L}_m$ .

576 Notice first that the codimension of  $Z_{j,m}^{i,p}$  is equal to  $2n + m d_y$  and is greater than  
 577  $2n + m d_u$  which is greater than the dimension of the domain of  $\rho_{f,h}$ .

578 We shall show that  $\text{ev}_\rho$  is transversal to  $Z_{j,m}^{i,p}$ . Let  $\mathcal{X} \triangleq (f, h, x, \bar{x}, \underline{u}_m)$  be a  
 579 point such that  $\text{ev}_\rho(\mathcal{X}) \in Z_{j,m}^{i,p}$  and take a tangent vector  $(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y})$  with  $\mathfrak{X}_k \in T_{x_k} X$ ,  
 580  $\bar{\mathfrak{X}}_k \in T_{\bar{x}_k} X$  ( $k = 0, \dots, m$ ), and  $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$  ( $k = 0, \dots, m-1$ ). We have to prove  
 581 that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^r(X, \mathbf{R}^{d_y})$ ,  $\xi \in T_x X$ ,  $\bar{\xi} \in T_{\bar{x}} X$ ,  $\nu_k \in T_{u_k} U$  (for  
 582  $j = 0, \dots, m-1$ ) and a vector  $\zeta$  in the tangent space to  $Z_{j,m}^{i,p}$  at  $\text{ev}_\rho(\mathcal{X})$  such that

$$583 \quad (13) \quad (\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = \text{d}(\text{ev}_\rho)(f, h, x, \bar{x}, \underline{u}_m) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu) + \zeta.$$

584 We shall prove this relation with  $\xi = 0$ ,  $\bar{\xi} = 0$  and  $\nu = 0$ . We denote by

$$585 \quad \alpha_0, \dots, \alpha_m, \bar{\alpha}_0, \dots, \bar{\alpha}_m, 0, \dots, 0$$

586 the components of  $\zeta$ ; notice that, as  $\zeta$  is a tangent vector to  $Z_{j,m}^{i,p}$ , we have  $\alpha_i = \alpha_p$   
 587 and  $\bar{\alpha}_j = \bar{\alpha}_m$ .

588 To prove that Equation (13) admits a solution, the reasoning is analogous to the  
 589 one made in the proofs of the previous propositions but here, we have to apply twice  
 590 Lemma 8. Hereafter, given a vector field  $\phi$  we denote by  $x_i^{\lambda, \phi}$ , the sequence defined  
 591 recursively as follows

$$592 \quad x_0^{\lambda, \phi} = x_0, \quad x_{i+1}^{\lambda, \phi} = \varphi_T^{u_i, \lambda}(x_i^{\lambda, \phi})$$

594 where  $\varphi_t^{u_i, \lambda}$  denotes the flow related to the vector field  $f(\cdot, u_i) + \lambda\phi$ . According to  
 595 Lemma 8, there exist a vector field  $\phi_0$  such that the derivatives of  $x_i^{\lambda, \phi_0}$ ,  $\bar{x}_j^{\lambda, \phi_0}$  and  
 596  $\bar{x}_m^{\lambda, \phi_0}$  with respect to  $\lambda$  are all zero while the derivative of  $x_p^{\lambda, \phi_0}$  can be arbitrarily  
 597 chosen. As there exists  $k_0$  such that  $u_{k_0} \neq u_{p-1}$ , we can also apply this lemma by  
 598 replacing  $u_{p-1}$  by  $u_{k_0}$  in the lemma. For example, if  $j \leq k_0 < m$ , and, assuming  
 599 without loss of generality, that  $k_0$  is the greatest subscript less than  $m$  such that  
 600  $u_{k_0} \neq u_{p-1}$ , we deduce from Lemma 8 that there exists a vector field  $\phi_1$  such that the  
 601 derivatives of  $x_i^{\lambda, \phi_1}$ ,  $x_p^{\lambda, \phi_1}$  and  $\bar{x}_j^{\lambda, \phi_1}$  with respect to  $\lambda$  are all zero while the derivative



602 of  $\bar{x}_{k_0+1}^{\lambda, \phi_1}$  can be arbitrarily chosen. Noticing that

$$603 \quad \bar{x}_m^{\lambda, \phi_1} = \begin{cases} \bar{x}_{k_0+1}^{\lambda, \phi_1} & \text{if } k_0 = m - 1, \\ \varphi_{(m-k_0)T}^{u_{p-1}}(\bar{x}_{k_0}^{\lambda, \phi_1}) & \text{if } k_0 < m - 1; \end{cases}$$

604 we see that the derivative of  $\bar{x}_m^{\lambda, \phi_1}$  with respect to  $\lambda$  can be arbitrarily chosen.

605 We chose now the vector field  $\phi \in \Gamma_U(X)$  as follows

- 606 •  $\phi(\cdot, u_{p-1}) = \phi_0$  and  $\phi(\cdot, u_{k_0}) = \phi_1$ ;
- 607 •  $\phi(\cdot, u_k) \equiv 0$  if  $u_k \notin \{u_{p-1}, u_{k_0}\}$ .

608 Clearly the derivatives of  $x_i^{\lambda, \phi}$  and  $\bar{x}_j^{\lambda, \phi}$  with respect to  $\lambda$  are zero while the derivatives  
609 of  $x_p^{\lambda, \phi}$  and  $\bar{x}_m^{\lambda, \phi}$  can be arbitrarily chosen.

610 As in the previous configurations, as regard the first  $2m + 2$  equations in (13), we  
611 have to consider only the four following ones

$$612 \quad (14) \quad \alpha_i + A_i = \bar{\mathfrak{X}}_i, \quad \alpha_p + A_p = \bar{\mathfrak{X}}_p, \quad \bar{\alpha}_j + \bar{A}_j = \bar{\mathfrak{X}}_j, \quad \bar{\alpha}_m + \bar{A}_m = \bar{\mathfrak{X}}_m$$

614 where the  $A_k$ 's (resp. the  $\bar{A}_k$ 's) denote the derivatives of the  $x_k^{\lambda, \phi}$  (resp. of the  $\bar{x}_k^{\lambda, \phi}$ )  
615 with respect to  $\lambda$ ; so we have  $A_i = 0$  and  $\bar{A}_j = 0$ . Notice also that, from the definition  
616 of  $Z_{j,m}^{i,p}$ , it follows that  $\alpha_i = \alpha_p$  and  $\bar{\alpha}_j = \bar{\alpha}_m$ . Taking into account these equalities,  
617 the solution to the equations (14) is then

$$618 \quad \alpha_i = \bar{\mathfrak{X}}_i, \quad A_p = \bar{\mathfrak{X}}_p - \bar{\mathfrak{X}}_j, \quad \bar{\alpha}_j = \bar{\mathfrak{X}}_j, \quad \bar{A}_m = \bar{\mathfrak{X}}_m - \bar{\mathfrak{X}}_j.$$

620 As regard the last  $m$  equalities in (13), the proof is the same as the one of  
621 Proposition 9: the two lists  $L_{m-1}$  and  $\bar{L}_{m-1}$  are disjoint and the elements of  $\bar{L}_{m-1}$   
622 are pairwise distinct, so one can find a function  $\eta \in C^\infty(X, \mathbf{R}^{d_y})$  such that  $\eta(x_k) = 0$   
623 for  $k = 0, \dots, m - 1$  while the values of  $\eta$  at  $\bar{x}_k$  ( $k = 0, \dots, m - 1$ ) can be chosen  
624 arbitrarily.

625 *Configuration  $\mathbf{C}_4''$ .* In this case, we have  $u_i = \dots u_{p-1} = u_j = \dots u_{m-1}$ ; the  
626 equalities  $x_i = x_p$  and  $\bar{x}_j = \bar{x}_m$  then imply that the trajectories of the vector field  
627  $f(\cdot, u_{p-1})$  are periodic.

628 We choose some subscripts  $0 \leq i < p \leq 2n$  and  $0 \leq j < m \leq 2n$ , without loss  
629 of generality, we assume that  $m \geq p$ . We consider first t representation  $\rho$  defined on  
630  $\mathcal{G}_2^U(a) \times C^r(X, \mathbf{R})$  through the mapping  $\rho_{f,h}$  defined as

$$631 \quad \rho_{f,h} : \begin{array}{ccc} X^{(2)} \times U_{(i,p,j,m)}^m \times \mathbf{R}_+^* & \longrightarrow & X^{m+1} \times X^{m+1} \times \mathbf{R}^{m d_y} \\ (x, \bar{x}, \underline{u}_m, t) & \longmapsto & (x_0, \dots, x_{p-1}, \varphi_t^{u_{p-1}}(x_i), x_{p+1}, \dots, x_m, \\ & & \bar{x}_0, \dots, \bar{x}_m, y_0 - \bar{y}_0, \dots, y_{m-1} - \bar{y}_{m-1}). \end{array}$$

632 where  $U_{(i,p,j,m)}^m$  is the submanifold of  $U^m$  defined as the set of those  $\underline{u}_m$  such that  
633  $u_i = \dots = u_{p-1} = u_j = \dots = u_{m-1}$ .

634 Together with  $\rho$ , we consider the submanifold  $Z_{j,m}^{i,p}$  defined as in the previous  
635 case. We shall prove that  $\text{ev}_\rho$  is transversal to  $Z_{j,m}^{i,p}$ . Let  $\mathcal{X} \triangleq (f, h, x, \bar{x}, \underline{u}_m, t)$  be a  
636 point such that  $\text{ev}_\rho(\mathcal{X}) \in Z_{j,m}^{i,p}$  and take a tangent vector  $(\bar{\mathfrak{X}}, \bar{\mathfrak{X}}, \mathfrak{Y})$  with  $\bar{\mathfrak{X}}_k \in T_{x_k} X$ ,  
637  $k \in \{0, \dots, m\} \setminus \{p\}$ ,  $\bar{\mathfrak{X}}_p \in T_{x_i} X$ ,  $\bar{\mathfrak{X}}_k \in T_{\bar{x}_k} X$ ,  $k = 0, \dots, m$  and  $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$ ,  $k =$   
638  $0, \dots, m - 1$ . Notice that  $\text{ev}_\rho(\mathcal{X}) \in Z_{j,m}^{i,p}$  implies that  $x_i$  and  $\bar{x}_j$  belong to periodic  
639 trajectories of the vector field  $f(\cdot, u_{p-1})$ .

640 We have to prove that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^r(X, \mathbf{R}^{d_y})$ ,  $\xi \in T_x X$ ,  
641  $\bar{\xi} \in T_{\bar{x}} X$ ,  $\nu_k \in T_{u_k} U$ ,  $\tau \in \mathbf{R}$ , and a vector  $\zeta$  in the tangent space to  $W_{j,m}^{i,p}$  at  $\text{ev}_\rho(\mathcal{X})$



642 such that

$$643 \quad (15) \quad (\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = d(\text{ev}_\rho)(\mathcal{X}) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu, \tau) + \zeta.$$

644 We shall prove this relation with  $\nu = 0$  and  $\bar{\xi} = 0$ . We denote by

$$645 \quad \alpha_0, \dots, \alpha_m, \bar{\alpha}_0, \dots, \bar{\alpha}_m, 0, \dots, 0$$

646 the components of  $\zeta$ ; notice that, as  $\zeta$  is a tangent vector to  $Z_{j,m}^{i,p}$ , we have  $\alpha_i = \alpha_p$   
 647 and  $\bar{\alpha}_j = \bar{\alpha}_m$ .

648 As in the previous cases, in order to prove that the first  $2m + 2$  equations in (15)  
 649 can be satisfied, it is sufficient to focus our attention to the four following equations

$$650 \quad \alpha_i + \xi_i + A_i = \mathfrak{X}_i, \quad \alpha_p + \xi_p + A_p + \tau f(x_i, u_{p-1}) = \mathfrak{X}_p,$$

$$651 \quad \bar{\alpha}_j + \bar{A}_j = \bar{\mathfrak{X}}_j, \quad \bar{\alpha}_m + \bar{A}_m = \bar{\mathfrak{X}}_m,$$

653 the notations are the same as in the previous cases except for  $A_p$  and  $\xi_p$ :

$$654 \quad A_p = \left. \frac{d\varphi_t^{u_{p-1}, \lambda}(x_i)}{d\lambda} \right|_{\lambda=0}, \quad \xi_p = d\varphi_t^{u_{p-1}}(x_i) \cdot \xi_i.$$

656 We can apply Lemma 8: there exists  $\phi$  such that  $\bar{A}_j = 0$  while  $\bar{A}_m$  can be arbitrarily  
 657 chosen. Here we cannot ensure that we also have  $A_i = A_p = 0$  because  $x_i$  could  
 658 belong to the periodic trajectory of  $f(\cdot, u_{p-1})$  passing through  $\bar{x}_j$ , so we first choose  
 659  $\bar{\alpha}_j = \bar{\mathfrak{X}}_j$  and  $\bar{A}_m = \bar{\mathfrak{X}}_m - \bar{\mathfrak{X}}_j$ . Now as  $\varphi_t^{u_{p-1}}(x_i) = x_i$ , we have  $t = q\pi_0$  where  $\pi_0$   
 660 denotes the prime period of the periodic trajectory of  $f(\cdot, u_{p-1})$  passing through  $x_i$ ;  
 661 thus  $d\varphi_t^{u_{p-1}}(x_i) = (d\varphi_{\pi_0}^{u_{p-1}}(x_i))^q$ . As before, due to the fact that  $f$  belongs to  $\mathcal{G}_2^U(a)$ ,  
 662 the linear mapping

$$663 \quad (\xi_i, \tau) \mapsto ((d\varphi_{\pi_0}^v(x_i))^q - \text{Id}) \cdot \xi_i + \tau f(x_i, v)$$

664 is onto. Thus we can find  $\xi_i$  and  $\tau$  such that

$$665 \quad ((d\varphi_{\pi_0}^v(x_i))^q - \text{Id}) \cdot \xi_i + \tau f(x_i, v) = \mathfrak{X}_p - \mathfrak{X}_i + A_p - A_i.$$

666 we take also  $\alpha_i = \mathfrak{X}_i - A_i$ , with these choices of  $\alpha_i$ ,  $\xi_i$  and  $\tau$ , we see that the two first  
 667 equations are also satisfied.

668 As regards the last  $m$  equations in (15), we argue as in the previous case.

669 At this point, the application of the Transversal Density Theorem, shows that  $\mathcal{R}_4$ ,  
 670 the set of pairs  $(f, h)$  in  $\mathcal{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y})$  such that  $\rho_{f,h}$  is transversal to the finite  
 671 set of submanifolds  $Z_{j,m}^{i,p}$ , is a residual. Now, we have to compute the codimension of  
 672  $Z_{j,m}^{i,p}$ , it is equal to  $2n + m d_y$  and is greater or equal to  $2n + m d_u + m$ ; this codimension  
 673 is greater than the dimension of the domain of  $\rho_{f,h}$  if  $m \geq 2$  or  $d_y \geq d_u + 2$ . In this  
 674 case, to be transversal to  $Z_{j,m}^{i,p}$  means non membership and we can conclude the proof  
 675 of Proposition 10 as for the previous propositions. If  $m = 1$ ,  $d_u = 0$  and  $d_y \geq 1$ ,  $\text{ev}_\rho$  is  
 676 still transversal to submanifold  $Z_{j,m}^{i,p}$  but  $\text{codim}(Z_{0,1}^{0,1})$  is then equal to the dimension  
 677 of the domain of  $\rho_{f,h}$ , so we need an additional argument to conclude in this case.  
 678 Hereafter, we shall see that, in this particular case, although the codimension of  $Z_{0,1}^{0,1}$  is  
 679 equal to the dimension of the domain of  $\rho_{f,h}$ , transversality implies non membership.

680 Case where  $m = 1$ ,  $d_u = 0$  and  $d_y \geq 1$ . Take  $(f, h) \in \mathcal{R}_4^r$ , then  $\rho_{f,h}$  is transversal  
681 to  $W_{0,1}^{0,1}$ . Assume that  $x_0 \neq \bar{x}_0$  are two points of  $X$  such that  
682 • there exists some  $t > 0$  such that  $\varphi_t(x_0) = x_0$ ,  $\varphi_T^u(\bar{x}_0) = \bar{x}_0$ ;  
683 •  $h(x_0) = h(\bar{x}_0)$ ;  
684 then there exist  $\xi_0 \in T_{x_0}X$ ,  $\bar{\xi}_0 \in T_{\bar{x}_0}X$ ,  $\tau \in \mathbf{R}$ ,  $\alpha_0$  and  $\bar{\alpha}_0$  in the tangent spaces to  
685  $X$  at  $x_0$  and  $\bar{x}_0$  such that the following equations are satisfied

$$686 \quad (16) \quad \begin{cases} \alpha_0 + \xi_0 = \mathfrak{X}_0, & \alpha_0 + d\varphi_t(x_0) \cdot \xi_0 + \tau f(x_0, u) = \mathfrak{X}_1, \\ \bar{\alpha}_0 + \bar{\xi}_0 = \bar{\mathfrak{X}}_0, & \bar{\alpha}_0 + d\varphi_T(\bar{x}_0) \cdot \bar{\xi}_0 = \bar{\mathfrak{X}}_1 \\ dh(x_0) \cdot \xi_0 - dh(\bar{x}_0) \cdot \bar{\xi}_0 = \mathfrak{Y} \end{cases}$$

687 whatever  $\mathfrak{X}_0$ ,  $\mathfrak{X}_1$ ,  $\bar{\mathfrak{X}}_0$ ,  $\bar{\mathfrak{X}}_1$  and  $\mathfrak{Y}$  tangent vectors to the appropriate spaces. Clearly  
688 the four first equations in this system are equivalent to the two following ones

$$689 \quad (d\varphi_t(x_0) - \text{Id}) \cdot \xi_0 + \tau f(x_0, u) = \mathfrak{X}_1 - \mathfrak{X}_0,$$

$$690 \quad (d\varphi_T(\bar{x}_0) - \text{Id}) \cdot \bar{\xi}_0 = \bar{\mathfrak{X}}_1 - \bar{\mathfrak{X}}_0.$$

692 As 1 is an eigenvalue of the linear mapping  $d\varphi_T(\bar{x}_0)$ , clearly the second equation of  
693 this system cannot be satisfied. This implies that if  $\varphi_T(x_0) = x_0$  and  $\varphi_T(\bar{x}_0) = \bar{x}_0$ ,  
694 the point  $\rho_{f,h}(x_0, \bar{x}_0)$  cannot belong to  $W_{0,1}^{0,1}$ , which means that we must have  $h(x_0) \neq$   
695  $h(\bar{x}_0)$ . This achieve the proof of Proposition 10.  $\square$

696 **5. The case  $d_u \geq 1$  and  $d_y = d_u + 1$ . Counterexample.** We shall exhibit  
697 here a simple counterexample which shows that if  $d_y = 2$  and  $d_u = 1$ , then the  
698 conclusion of our main result (Theorem 4) is no more true. That it to say, we exhibit  
699 a pair  $(f_0, h_0)$  such that for every  $(f, h)$  in some neighborhood of  $(f_0, h_0)$ , the related  
700 mapping  $\Theta_T^{f,h}$  is not injective. We recall hereafter, the isotopy theorem [1] which will  
701 be used to prove some optimality of our main result.

702 **THEOREM 11** (Transversal isotopy theorem). *Let  $\mathcal{A}$ ,  $Z$  and  $Y$  be  $C^{r+1}$  manifolds*  
703 *( $r \geq 1$ ),  $\rho : \mathcal{A} \rightarrow C^{r+1}(Z, Y)$  a  $C^{r+1}$  representation,  $W \subset Y$  a submanifold and*  
704  *$a_0 \in \mathcal{A}$  a point. For  $a \in \mathcal{A}$  let  $W_a = \rho_a^{-1}(W)$ . Assume that*

- 705 1.  $W$  is closed in  $Y$ ;
- 706 2.  $Z$  is compact and  $C^{r+3}$ ;
- 707 3.  $\rho_{a_0}$  is transversal to  $W$ .

708 *Then there is an open neighborhood  $N$  of  $a_0$  in  $\mathcal{A}$  such that, for  $a \in N$ , there is a  $C^r$*   
709 *diffeomorphism  $F_a : Z \rightarrow Z$  such that  $F_a(W_{a_0}) = W_a$  and  $F_a$  is  $C^r$  isotopic to the*  
710 *identity.*

711 We shall apply the transversal isotopy theorem 11 to the following situation: we  
712 take

- 713 •  $\mathcal{A} = \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ , where  $X$  and  $U$  are compact manifolds,  $\dim U =$   
714  $d_u > 0$  with  $d_y = d_u + 1$ ;
- 715 •  $Z = X^2 \times U \times S^1$ ;
- 716 •  $Y = X^3 \times \mathbf{R}^{d_y}$ .

717 We consider also a representation  $\rho$  which is slightly different from the one which has  
718 been used in the proof of Proposition 10; we define this representation through  $\rho_{f,h}$   
719 as

$$720 \quad \rho_{f,h} : \begin{array}{ccc} X^2 \times U \times S^1 & \longrightarrow & X^3 \times \mathbf{R}^{d_y} \\ (x, \bar{x}, u, s) & \longmapsto & (x, \varphi_T^u(x), \varphi_{\gamma(s)}^u(\bar{x}), h(x) - h(\bar{x})) \end{array}$$

721 here  $\gamma \in C^\infty(S^1, \mathbf{R})$ .

722 The submanifold  $W \subset X^3 \times \mathbf{R}^{d_y}$  is then defined as

$$723 \quad W = \{ (z_1, z_2, z_3, 0) \in X^3 \times \mathbf{R}^{d_y} \mid z_1 = z_2 = z_3 \}.$$

724 Consider a pair  $(f_0, h_0)$  such that  $\rho_{f_0, h_0}$  is transversal to  $W$  and such that  $W_{f_0, h_0}$   
725 is nonempty. Applying the *Transversal isotopy theorem 11*, we deduce that there  
726 exists a neighborhood  $N$  of  $(f_0, h_0)$  such that if  $(f, h) \in N$ ,  $W_{f, h} = F(W_{f_0, h_0})$  with  $F$   
727 a diffeomorphism from  $Z$  to  $Z$ . Thus  $W_{f, h}$  is nonempty; notice that we can assume  
728 that  $\rho_{f, h}$  is transversal to  $W$  for every pair  $(f, h) \in N$ , this is a direct consequence of  
729 the *Openness of Transversal Intersection Theorem [1]* applied to this situation with  
730  $K = Z$ . Let  $(x, \bar{x}, u, s)$  be an element of  $W_{f, h}$ , we shall show that  $x \neq \bar{x}$ : arguing by  
731 contradiction, we shall see that if we have the equality  $x = \bar{x}$ , then  $\rho_{f, h}$  cannot be  
732 transversal to  $W$  at  $(x, \bar{x}, u, s)$ . We introduce some notations

$$733 \quad \begin{aligned} A &= d\varphi_T^u(x), & \bar{A} &= d\varphi_{\gamma(s)}^u(\bar{x}), \\ 734 \quad B &= \left. \frac{\partial \varphi_T^v(x)}{\partial v} \right|_{v=u}, & \bar{B} &= \left. \frac{\partial \varphi_{\gamma(s)}^v(\bar{x})}{\partial v} \right|_{v=u}, \\ 735 \quad C &= dh(x), & \bar{C} &= dh(\bar{x}). \end{aligned}$$

737 Consider now the following “matrix”  $M$

$$738 \quad M = \begin{pmatrix} A - \text{Id} & 0 & B & 0 \\ -\text{Id} & \bar{A} & \bar{B} & f(x, u) \\ C & -\bar{C} & 0 & 0 \end{pmatrix}.$$

739 Arguing as in the proof of Proposition 10, we see that the transversality of  $\rho_{f, h}$  to  
740  $W$  at  $(x, \bar{x}, u, s)$  is equivalent to the invertibility of the square matrix  $M$ . Since we  
741 assume that  $x = \bar{x}$ , we have  $C = \bar{C}$ , which implies that the determinant of  $M$  is equal  
742 to the one of the following matrix  $M'$ :

$$743 \quad M' = \begin{pmatrix} A - \text{Id} & 0 & B & 0 \\ \bar{A} - \text{Id} & \bar{A} & \bar{B} & f(x, u) \\ 0 & -\bar{C} & 0 & 0 \end{pmatrix}.$$

744 Now as  $f(x, u)$  belongs to the kernels of  $A - \text{Id}$  and  $\bar{A} - \text{Id}$  (because  $\varphi_{\gamma(t)}^u(\bar{x}) = x$ ), the  
745 vector  $(f(x, u), 0, 0, 0)^T$  belongs to the kernel of  $M'$  which implies that the determinant  
746 of  $M'$  is zero : we have reached a contradiction. As a consequence, if  $(f, h) \in N$ , there  
747 exists  $x \neq \bar{x}, u$  and  $s$  such that  $h(x) = h(\bar{x})$ ; moreover, the trajectory of  $f(\cdot, u)$  passing  
748 through  $x$  is periodic and  $\bar{x}$  belongs to this trajectory, so we have  $\varphi_{kT}^u(x) = x$  and  
749  $\varphi_{kT}^u(\bar{x}) = \bar{x}$ .

750 We give now an explicit example of a pair  $(f_0, h_0)$  such that  $\rho_{f_0, h_0}$  is transversal  
751 to  $W$ . In what follows, for the sake of simplicity and without loss of generality, we  
752 assume that  $T = 2\pi$ . The manifold  $X$  will be equal to the circle  $S^1$  and the set of  
753 controls  $U$  will also be equal to the circle  $S^1$ . We denote by  $u_1$  and  $u_2$  (resp.  $s_1$  and  
754  $s_2$ ) the components of  $u$  (resp. of  $s$ ) and we consider the following vector field:

$$755 \quad f_0(x, u) = R_1 \cdot x + u_1 R_2 \cdot x$$

756 with  $R_1$  and  $R_2$  the following skew-symmetric matrices

$$757 \quad R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix},$$

758 with  $\beta \in (0, 1)$ .

760 Mapping  $h_0$  is defined by:

$$761 \quad h_0 : X \longrightarrow \mathbf{R}^2 \\ x \longmapsto (x_1, x_1 x_2),$$

762 while function  $\gamma$  is defined by:

$$763 \quad \gamma(s) = \left( -\frac{1}{2} + \frac{1}{4} s_1 \right) 2\pi.$$

764 *The set  $\mathbf{W}_{f_0, h_0}$ .* Let  $x = (x_1, x_2)$  and  $\bar{x} = (\bar{x}_1, \bar{x}_2)$  be two points in  $S^1$ , and  
765  $u = (u_1, u_2) \in S^1$  a control and  $s \in S^1$ ; assume that  $\rho_{f_0, h_0}(x, \bar{x}, u, s) \in W$ . Letting  
766  $z = x_1 + ix_2 \in \mathbf{C}$ , the equality  $\varphi_T^u(x) = x$  is equivalent to  $e^{i(1+u_1\beta)2\pi} z = z$  which is  
767 equivalent to  $1 + u_1\beta = k$  with  $k \in \mathbf{Z}$ . Now, as  $\beta \in (0, 1)$ , we have  $|u_1\beta| < 1$  and so  
768 the equality  $u_1\beta = k - 1 \in \mathbf{Z}$  is possible if and only if  $u_1 = 0$ . From the definition of  
769  $h$ , we can easily see that the equality  $h(x) = h(\bar{x})$  amounts to  $x = \bar{x}$  or  $x_1 = \bar{x}_1 = 0$ .  
770 If  $x = \bar{x}$ , as  $u_1 = 0$ , the equality  $\varphi_{\gamma(s)}^u(\bar{x}) = x$  is possible only if  $\gamma(s) \in \mathbf{Z}$ , but from the  
771 definition of  $\gamma$ , we have  $\gamma(s) \in [-3/4, -1/4]$ . Thus  $x \neq \bar{x}$  and  $x_1 = \bar{x}_1 = 0$ , so, taking  
772 into account that  $u_1 = 0$ , the equality  $\varphi_{\gamma(s)}^u(\bar{x}) = x$  is equivalent to the following ones

$$773 \quad -\sin(\gamma(s))\bar{x}_2 = 0, \quad \cos(\gamma(s))\bar{x}_2 = x_2 = -\bar{x}_2.$$

775 These two equalities are true iff  $\gamma(s) = (2k + 1)\pi$  with  $k \in \mathbf{Z}$ , which is equivalent to  
776  $s_1 = 4k + 4$ , as  $|s_1| \leq 1$ , this is possible only if  $s_1 = 0$ .

777 In conclusion,  $\mathbf{W}_{f_0, h_0}$  is the set consisting of the following eight elements

$$778 \quad \mathbf{W}_{f_0, h_0} = \{((0, \varepsilon_0), (0, -\varepsilon_0), (0, \varepsilon_1), (0, \varepsilon_2))\}.$$

779 where  $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ .

780 *The transversality of  $\rho_{f_0, h_0}$ .* Take  $(x, \bar{x}, u, s) \in X \times X \times U \times S^1$  an element such  
781 that  $\rho_{f_0, h_0}(x, \bar{x}, u, s)$  belongs to  $W$ . Thus we have

$$782 \quad (17) \quad x = (0, \varepsilon_0), \quad \bar{x} = (0, -\varepsilon_0), \quad u = (0, \varepsilon_1), \quad s = (0, \varepsilon_2)$$

784 with  $\varepsilon_i \in \{-1, 1\}$  ( $i = 0, 1, 2$ ). Let  $(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{Y})$  be a tangent vector to  $X^3 \times \mathbf{R}^2$  at  
785  $\rho_{f_0, h_0}(x, \bar{x}, u, s)$ . Thus,  $\mathfrak{X}_1, \mathfrak{X}_2$  and  $\mathfrak{X}_3$  are tangent to  $S^1$  at  $(0, \varepsilon_0)$  while  $\mathfrak{Y}$  is a vector  
786 in  $\mathbf{R}^2$ . We write a tangent vector to submanifold  $W$  as  $\zeta = (\alpha, \alpha, \alpha, 0)$  where  $\alpha$  is a  
787 tangent vector to  $S^1$  at  $(0, \varepsilon_0)$ . We have to prove that there exists  $\xi$  (resp.  $\bar{\xi}$ ) in the  
788 tangent space to  $S^1$  at  $(0, \varepsilon_0)$  (resp.  $(0, -\varepsilon_0)$ ) as well as  $\nu$  a vector in the tangent space  
789 to  $S^1$  at  $(0, \varepsilon_1)$  and  $\sigma$  a vector tangent to  $S^1$  at  $s$  such that the following equalities  
790 are satisfied

$$791 \quad (18) \quad \mathfrak{X}_1 = \xi + \alpha,$$

$$792 \quad (19) \quad \mathfrak{X}_2 = d\varphi_T^u(x) \cdot \xi + \frac{\partial \varphi_T^u(x)}{\partial u} \cdot \nu + \alpha,$$

$$793 \quad (20) \quad \mathfrak{X}_3 = d\varphi_{\gamma(s)}^u(\bar{x}) \cdot \bar{\xi} + \frac{\partial \varphi_{\gamma(s)}^u(\bar{x})}{\partial u} \cdot \nu + \frac{\partial \varphi_{\gamma(s)}}{\partial s} \cdot \sigma + \alpha,$$

$$794 \quad (21) \quad \mathfrak{Y} = dh(x) \cdot \xi - dh(\bar{x}) \cdot \bar{\xi}.$$

796 We examine first the fourth equation (21), taking into account that  $\xi = (\xi_1, 0)^\top$   
 797 and  $\bar{\xi} = (\bar{\xi}_1, 0)^\top$ , it writes

$$798 \begin{pmatrix} 1 & 0 \\ \varepsilon_0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -\varepsilon_0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{Y}_1 \\ \mathfrak{Y}_2 \end{pmatrix}.$$

799 Clearly, the solution of this equation is

$$800 (22) \quad \xi_1 = \frac{\varepsilon_0 \mathfrak{Y}_2 + \mathfrak{Y}_1}{2}, \quad \bar{\xi}_1 = \frac{\varepsilon_0 \mathfrak{Y}_2 - \mathfrak{Y}_1}{2}.$$

802 Taking into account that  $\mathfrak{x}_i = (\mathfrak{x}_i^1, 0)^\top$  and  $\alpha = (\alpha_1, 0)^\top$ , the solution of the first  
 803 equation (18) is then given by (22) and by the equality

$$804 (23) \quad \alpha_1 = \mathfrak{x}_1^1 - \frac{\varepsilon_0 \mathfrak{Y}_2 + \mathfrak{Y}_1}{2}.$$

805 Concerning the second equation (19), we notice that  $d\varphi_T^u(x) = \text{Id}$  and that

$$806 \frac{\partial \varphi_T^u(x)}{\partial u} = \begin{pmatrix} -2\pi \beta \varepsilon_0 & 0 \\ 0 & 0 \end{pmatrix}$$

807 so the solution of equation (19) is given by (22), (23) and by

$$808 (24) \quad \nu_1 = -\frac{\mathfrak{x}_2^1 - \mathfrak{x}_1^1}{2\pi \beta \varepsilon_0}.$$

809 So far, the values of  $\xi$ ,  $\bar{\xi}$  and  $\nu$  are fixed, therefore showing that equation (20)  
 810 comes down showing that the third term in the right-hand member of (20) can take  
 811 arbitrary value. This third term writes

$$812 \frac{\partial \varphi_{\gamma(s)}}{\partial s} \cdot \sigma = \frac{\pi}{2} \sigma_1 f(\varphi_{\gamma(s)}^u(x)) = \frac{\pi}{2} \sigma_1 \begin{pmatrix} \varepsilon_0 \\ 0 \end{pmatrix},$$

813 obviously, thanks to a suitable choice of  $\sigma_1$ , this expression can be made equal to an  
 814 arbitrary tangent vector  $\tilde{\mathfrak{x}}_3$  of  $S^1$  at  $x$ .

815 This achieves the proof of the transversality of  $\rho_{f_0, h_0}$  to  $W$ .

816 **Appendix A. Proof of Theorem 7.** The proof of the Kupka-Smale theorem  
 817 can obviously be found in the original papers [15, 16] and [17] but the reader can also  
 818 find a very detailed proof of this result in [1]. For the proof of the generalization of  
 819 this theorem, we shall follow closely the arguments given in this book.

### 820 **A.1. The different sets in the theorem of Kupka-Smale.**

821 **G1** Set  $\mathcal{G}_1$  is the set of vector fields whose critical points are elementary. A point  $x$   
 822 is **critical** for the vector field  $f$  if  $f(x) = 0$ , it is elementary if the differential

$$823 d\varphi_t(x) : T_x X \longrightarrow T_x X$$

824 has no complex eigenvalue of modulus 1 for every  $t \neq 0$ .

825 **GΔ** Let  $a$  be a positive number, set  $\mathcal{G}\Delta(a)$  is the set of vector fields  $f$  such that if  $\mathcal{O}$   
 826 is a closed orbit of  $f$  with period  $0 < \tau \leq a$ , then this period is **transversal**,  
 827 that is to say, the eigenvalue 1 of the differential

$$828 d\varphi_\tau(x) : T_x X \longrightarrow T_x X$$

829 has an algebraic multiplicity equal to 1 (notice that, in this case,  $f(x)$  is an  
 830 eigenvector of  $d\varphi_\tau(x)$ ).

831 **G3/2** Set  $\mathcal{G}_{3/2}(a)$  is defined as  $\mathcal{G}_{3/2}(a) \triangleq \mathcal{G}_1 \cap \mathcal{G}\Delta(a)$ .  
832 **G2** Set  $\mathcal{G}_2(a)$  is the set of vector fields included in  $\mathcal{G}_1$  and whose periods  $0 < \tau \leq a$   
833 of closed orbits are elementary. A period is **elementary** if the modulus of  
834 eigenvalues of  $d\varphi_\tau(x)$  that are distinct from 1, are different from 1.

835 A consequence of the Kupka-Smale theorem is that all of these sets are open and  
836 dense for the  $C^r$  topology in  $\Gamma(X)$ .

837 We recall that  $\Gamma_U(X)$  denotes the set of parametrized vector fields over  $X$ ; we  
838 define the sets  $\mathcal{G}_1^U, \mathcal{G}_\Delta^U, \dots$  as the sets of vector fields belonging to one of the above  
839 categories for every  $u \in U$ . For instance we define  $\mathcal{G}_1^U$  as the set of parametrized vector  
840 fields  $f$  such that  $f(\cdot, u) \in \mathcal{G}_1$  for every  $u \in U$ . Notice that every vector field  $f \in \Gamma(X)$   
841 can be regarded as a parametrized vector field, so we can write  $\Gamma(X) \subset \Gamma_U(X)$ ,  
842  $\mathcal{G}_1 \subset \mathcal{G}_1^U$ , etc.

843 The tangent bundle  $T(X \times U)$  is diffeomorphic to the cartesian product  $TX \times TU$ ,  
844 so a vector field  $f$  defined on  $X \times U$  can be regarded as a pair  $(f_1, f_2)$  of two vector  
845 fields defined respectively on  $X$  and  $U$ . We have then the following result.

846 **LEMMA 12.** *The mapping  $\Pi$  defined as*

$$847 \quad \begin{aligned} \Pi: \Gamma(X \times U) &\longrightarrow \Gamma_U(X) \\ (f_1, f_2) &\longmapsto f_1 \end{aligned}$$

848 *is open for the  $C^r$  topology (for any  $r > 0$ ).*

849 *Proof.* The set  $C^\infty(X \times U, TX \times TU)$  is diffeomorphic to  $C^\infty(X \times U, TX) \times$   
850  $C^\infty(X \times U, TU)$  (see [12, Prop. 3.6]); we deduce easily that  $\Gamma(X \times U)$  is diffeomorphic  
851 to  $\Gamma_U(X) \times \Gamma_X(U)$ . On the other hand the first projection map from  $\Gamma_U(X) \times \Gamma_X(U)$   
852 to  $\Gamma_U(X)$  is an open and continuous mapping, it follows that  $\Pi$  is continuous and open  
853 as a composite mapping of an open and continuous mapping with a diffeomorphism.  $\square$

854 As a consequence of this lemma, the direct image of an open and dense subset of  
855  $\Gamma(X \times U)$  is an open and dense subset of  $\Gamma_U(X)$ .

856 We recall also that a mapping  $f: E \rightarrow F$  between topological spaces is called  
857 quasi-open if, for every open subset  $\mathcal{O} \subset E$ ,  $f(\mathcal{O})$  has a non empty interior.

858 We shall need also, the following lemma.

859 **LEMMA 13.** *The mapping  $\Phi$  defined as*

$$860 \quad \begin{aligned} \Phi: \Gamma_U(X) \times U &\longrightarrow \Gamma(X) \\ (f, u) &\longmapsto f_u \end{aligned}$$

861 *is continuous. Here  $f_u$  denotes the vector field defined on  $X$  by  $f_u(x) = f(x, u)$ .*

862 **A.2. The set  $\mathcal{G}_1^U$  is open and dense in  $\Gamma_U(X)$ .** The proof of the Kupka-  
863 Smale theorem 6 as made in [1] needs seven steps. Beforehand, one proves that  $\mathcal{G}_1^U$  is  
864 open and dense for the  $C^r$  topology. The proof follows closely the one provided in [1,  
865 page 98ff], in this book, the authors introduce a submanifold  $W$  of the 1-jet bundles  
866  $J^1(X, TX)$  as follows. Consider a chart  $(P, \varphi)$  of  $X$ , to this chart we relate, as usual,  
867 the chart  $(Q, \psi)$  of  $TX$ , where  $Q = \pi^{-1}(P)$  (here  $\pi: TX \rightarrow X$  denotes the canonical  
868 projection) and  $\psi(v) = (\pi(v), \bar{v})$  with  $\bar{v}$  the local expression of  $v \in TX$  in the chart  
869  $(P, \varphi)$ . Define then the chart  $\tau_{P,Q}$  as

$$870 \quad \begin{aligned} \tau_{P,Q}: J^1(P, Q) &\longrightarrow P' \times Q' \times \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n) \\ \sigma &\longmapsto (x, \psi \circ f \circ \varphi^{-1}(x), A) \end{aligned}$$

871 where  $P' = \varphi(P)$ ,  $Q' = \psi(Q)$  and  $x$  is the source of the 1-jet  $\sigma$ . The submanifold  
872  $W$  is defined as the set of 1-jets such that  $\psi \circ f \circ \varphi^{-1}(x) = 0$  and  $A$  has at least

873 one eigenvalue with real part zero (clearly, this definition does not depend on the  
874 particular choice of charts). To  $W$ , one relates the subset  $M$  of  $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$  of linear  
875 maps having an eigenvalue with real part zero. In [1], the proof of the density of  
876  $\mathcal{G}^1(X)$  relies on the fact that  $M$  is closed and is a finite union of submanifolds of  
877 codimension  $\geq 1$ , which implies that  $\text{codim } W \geq n + 1$ . In our case, we consider the  
878 same 1-jet bundle but we replace  $X$  by  $X \times U$ , so we can see  $A$  as a linear map from  
879  $\mathbf{R}^{n+d_u}$  to  $\mathbf{R}^{n+d_u}$ , we can identify  $A$  with its matrix written in the canonical basis  
880 of  $\mathbf{R}^{n+d_u}$  and we write  $A$  as a block matrix  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$  with  $A_{11} \in \mathbf{R}^{n \times n}$  and  
881  $A_{22} \in \mathbf{R}^{d_y} \times \mathbf{R}^{d_y}$ . We modify the definition of  $W$  as follows:  $W$  is the set of 1-jets  
882 such that  $A_{11}$  has at least one eigenvalue with real part zero. In a similar way as  
883 above, we can prove that  $W$  has a codimension  $\geq n + d_u + 1$ . Consider now the set  
884  $\mathcal{G}'_1(X \times U)$  of vector fields  $f$  defined on  $X \times U$  such that the 1-jet of  $f$  does not belong  
885 to  $W$ ; arguing as in [1], the fact that  $\text{codim } W \geq n + d_u + 1$ , proves that the set is open  
886 and dense. As  $\Pi$  is an open mapping (cf Lemma 12), and as  $\Pi(\mathcal{G}'_1(X \times U)) = \mathcal{G}_1^U$ ,  
887 we have proved that  $\mathcal{G}_1^U$  is also open and dense.

### 888 A.3. The seven steps of the proof.

889 *Step 1: for every  $a > 0$ , the set  $\mathcal{G}_2^U(a)$  is open .* First notice that  $\mathcal{G}_2^U(a) \neq \emptyset$   
890 because  $\mathcal{G}_2(a) \subset \mathcal{G}_2^U(a)$ ; moreover, we shall use the fact that  $\mathcal{G}_2(a)$  is open (proven  
891 in [1]).

892 Let  $f \in \mathcal{G}_2^U(a)$  and  $u \in U$  be fixed, denote by  $f_u$  the vector fields in  $\Gamma(X)$  defined  
893 by  $f_u(x) = f(x, u)$ ;  $f_u$  belongs to  $\mathcal{G}_2(a)$ , hence there exists  $\mathcal{N}_u$  a neighborhood of  $f_u$  in  
894  $\Gamma(X)$  with  $\mathcal{N}_u \subset \mathcal{G}_2(a)$ . As the mapping  $\Phi$  is continuous (cf Lemma 13), there exists  
895  $\mathcal{N}_u^U$  a neighborhood of  $f$  in  $\Gamma_U(X)$  and  $\mathcal{V}_u$  neighborhood of  $u$  such that  $\Phi(g, v) \in \mathcal{N}_u$   
896 for every  $(g, v) \in \mathcal{N}_u^U \times \mathcal{V}_u$ . The neighborhoods  $\mathcal{V}_u$  cover  $U$ , as  $U$  is compact, there  
897 exists a finite subcover:  $U \subset \bigcup_{i=1}^N \mathcal{V}_{u_i}$ , let  $\mathcal{N}^U = \bigcap_{i=1}^N \mathcal{N}_{u_i}^U$ ;  $\mathcal{N}^U$  is a neighborhood  
898 of  $f$  in  $\Gamma_U(X)$ . Let  $(g, v) \in \mathcal{N}^U \times U$ , there exists  $1 \leq i \leq N$  such that  $v \in \mathcal{V}_{u_i}$ , on  
899 the other hand  $g \in \mathcal{N}^U \subset \mathcal{N}_{u_i}^U$ , therefore  $\Phi(g, v) = g_v$  belongs to  $\mathcal{G}_2(a)$ .

900 In conclusion, every element  $g$  of  $\mathcal{N}^U$  is such that  $g_v \in \mathcal{G}_2(a)$  for every  $v$  in  $U$ ,  
901 therefore  $f \in \mathcal{N}^U \subset \mathcal{G}_2^U(a)$ , we have showed that  $\mathcal{G}_2^U(a)$  is open.

902 *Step 2: for every  $a > 0$ , the set  $\mathcal{G}_{3/2}^U$  is open .* The proof is very similar to the  
903 previous one.

904 *Step 3: if  $f \in \mathcal{G}_1^U$ , there exists a neighborhood  $\mathcal{N}$  of  $f$  with  $\mathcal{N} \subset \mathcal{G}_1^U$  and there*  
905 *exists  $a_0 > 0$  such that  $\mathcal{G}_2^U(a_0) \cap \mathcal{N} = \mathcal{N}$  . .* Let  $f \in \mathcal{G}_1^U$ , for every  $u \in U$ ,  $f_u \in \mathcal{G}_1$ ,  
906 so there exists a neighborhood  $\mathcal{N}_u \subset \mathcal{G}_1$  of  $f_u$  in  $\mathcal{G}_2(a)$  ( $f_u \in \mathcal{N}_u \subset \mathcal{G}_1$ ) and there  
907 exists  $a_u > 0$  such that  $\mathcal{N}_u \subset \mathcal{G}_2(a_u)$ . As in step 1, as  $\Phi$  is continuous, we construct  
908 a family of neighborhoods  $\mathcal{N}_u^U$  of  $f$  in  $\Gamma_U(X)$  and a family  $\mathcal{V}_u$  of neighborhoods of  
909  $u$  such that for every  $(g, v) \in \mathcal{N}_u^U \times \mathcal{V}_u$ , the vector field  $g_v$  belongs to  $\mathcal{N}_u \subset \mathcal{G}_1$  and  
910  $\mathcal{N}_u \subset \mathcal{G}_2(a_u)$ . Notice that, since  $\mathcal{G}_1^U$  is open, we can assume that  $\mathcal{N}_u^U \subset \mathcal{G}_1^U$ .

911 Due to the compactness of  $U$ , we have  $U \subset \bigcup_{i=1}^N \mathcal{V}_{u_i}$ ; let

$$912 \quad \mathcal{N} = \bigcap_{i=1}^N \mathcal{N}_{u_i}^U, \quad a_0 = \min_{1 \leq i \leq N} a_{u_i} .$$

914 Let  $(g, v) \in \mathcal{N}^U \times U$ , there exists  $1 \leq i \leq N$  such that  $v \in \mathcal{V}_{u_i}$  and we have  $g \in \mathcal{N} \subset$   
915  $\mathcal{N}_{u_i}^U$ , therefore  $\Phi(g, v) = g_v \in \mathcal{N}_{u_i}$  and  $\mathcal{N}_{u_i} \subset \mathcal{G}_2(a_{u_i})$ , but, clearly,  $\mathcal{G}_2(a_{u_i}) \subset \mathcal{G}_2(a_0)$   
916 since  $a_0 \leq a_{u_i}$ . Thus, we have showed that

$$917 \quad f \in \mathcal{N} \subset \mathcal{G}_1^U \cap \mathcal{G}_2^U(a_0),$$

918 which achieves the proof.

919 *Step 4: if  $\mathcal{N} \subset \mathcal{G}_1^U$  is open and  $\mathcal{G}_2^U(a) \cap \mathcal{N}$  is dense in  $\mathcal{N}$  then  $\mathcal{G}_{3/2}^U(\frac{3}{2}a) \cap \mathcal{N}$  is*  
920 *dense in  $\mathcal{N}$ .* This step is based on lemmas 31.7 and 31.8 in [1, p. 102]. Lemma 30.7  
921 asserts that if  $x$  is a point located on a closed orbit of a vector field  $f \in \Gamma(X)$ , given  
922 a tangent vector  $v \in T_x X$ , there exists a vector field  $g \in \Gamma(X)$  such that, denoting  
923 by  $\tau$  the prime period of the closed trajectory and by  $\varphi_t^\lambda$  the flow related to  $f + \lambda g$ ,  
924 we have

$$925 \quad \left. \frac{d\varphi_\tau^\lambda(x)}{d\lambda} \right|_{\lambda=0} = v.$$

926 As pointed out in section 4.2, Lemma 31.7 is still true for parametrized vector fields.  
927 Moreover, Lemma 31.8 can be proved in exactly the same way for parametrized vec-  
928 tor fields, the representation  $\rho$  being modified as follows (we define it through the  
929 evaluation map):

$$930 \quad (25) \quad \begin{array}{ccc} \text{ev}_\rho : \Gamma_U(X) \times U \times X \times \mathbf{R}_+^* & \longrightarrow & X \times \mathbf{R}_+^* \times X \\ (f, u, x, t) & \longmapsto & (x, t, \varphi_t^u(x)) \end{array}$$

931 Finally, we can prove that  $\text{ev}_\rho$  is transverse to  $\Delta$  on  $\mathcal{G}_2^U(a) \times X \times U \times [0, \frac{3}{2} + \varepsilon]$  in  
932 the same way as for the proof written in [1] because  $X \times U$  is compact; we can then  
933 conclude exactly as in [1].

934 *Step 5: if  $\mathcal{N} \subset \mathcal{G}_1^U$  is open and if  $\mathcal{G}_{3/2}^U(a) \cap \mathcal{N}$  is dense in  $\mathcal{N}$  then  $\mathcal{G}_2^U(a) \cap \mathcal{N}$*   
935 *is dense in  $\mathcal{N}$ .* In [1], the proof of this step is needs four lemmas. We shall briefly  
936 indicate how they can be extended to the case of parametrized vector fields. In the  
937 sequel, we shall identify the tangent bundle  $T(X \times U)$  with the product  $TX \times TU$ .  
938 Take a parametrized vector field,  $f \in \Gamma_U(X)$  and consider the related vector field  
939  $\tilde{f}$  defined on  $X \times U$  as  $\tilde{f}_1(x, u) = f(x, u)$  and  $\tilde{f}_2(x, u) = 0$ . Assume that  $x \in X$   
940 belongs to a periodic trajectory, denoted by  $\gamma$ , of  $f(\cdot, u)$ , then  $(x, u)$  belongs to the  
941 periodic trajectory  $\tilde{\gamma} \triangleq \gamma \times \{u\}$  of  $\tilde{f}$ . We can apply the tangent perturbation lemma  
942 (Lemma 32.4 in [1]), to  $\tilde{f}$  and  $\tilde{\gamma}$ : we then obtain a vector field  $\tilde{g}$  defined on  $X \times U$   
943 that is zero on  $\tilde{\gamma}$  and zero outside some neighborhood of  $\tilde{\gamma}$ . Moreover if we denote by  
944  $\tilde{\varphi}^\lambda$  the flow of  $\tilde{f} + \lambda \tilde{g}$ , we have  $\left. \frac{d}{d\lambda} \{d\varphi_\tau(x, u)\} \right|_{\lambda=0} = \tilde{A}$  where  $\tilde{A}$  is endomorphism of  
945  $T_x X \times T_u U$  that vanishes at  $(f(x, u), 0)^T$ . In particular, if we choose  $\tilde{A}$  such that its  
946 second component is zero, then looking at the proof of Lemma 32.4, we see that,  $\tilde{g}_2$ ,  
947 the second component of  $\tilde{g}$ , is zero. This lemma is therefore still valid for parametrized  
948 vector fields.

949 Consider now a parametrized vector field  $f \in \mathcal{G}_{3/2}^U(a)$  and assume that  $x_0$  be a  
950 point on a periodic trajectory  $\gamma_0$  of  $f(\cdot, u_0)$ , if  $u$  is closed to  $u_0$ , then the vector field  
951  $f(\cdot, u)$  is closed to  $f(\cdot, u_0)$ . Invoking Lemma 24.4 in [1], we know that there exists  
952 some neighborhood of  $x_0$  such that the vector field  $f(\cdot, u)$  admits a unique periodic  
953 trajectory in this neighborhood if  $u$  is closed enough to  $u_0$ . We can be more specific  
954 by using the implicit function theorem. Take a submanifold  $Y \subset X$  of codimension  
955 1, passing through  $x_0$  and which intersects  $\gamma_0$  transversally at  $x_0$ . We consider the  
956 Poincaré map  $\mathcal{P}$ , related to the vector field  $f(\cdot, u_0)$ , defined in a neighborhood of  $x_0$   
957 in  $Y$ . This map is also defined for  $u$  closed enough to  $u_0$ , that is to say, there exists  
958 neighborhoods  $N_1 \subset Y$  of  $x_0$  in  $Y$  and  $N_2$  of  $u_0$  such that if  $(x, u) \in N_1 \times N_2$ , there  
959 exists a first time  $\tau_u > 0$  such that  $\varphi_{\tau_u}^u(x, u) \in N_1$ . Now,  $\mathcal{P}(x_0, u_0) = x_0$  and we can  
960 invoke the implicit function theorem to prove the existence of  $x_u$  and  $\tau_u$  defined for  
961  $u$  in some neighborhood of  $u_0$  included in  $N_2$  and such that  $\mathcal{P}(x_u, u) = x_u$  (as well  
962 as  $\varphi_{\tau_u}^u(x_u) = x_u$ ). To see why we can use this theorem, we use the assumption that  
963  $f \in \mathcal{G}_{3/2}^U(a)$ . We know that 1 is an eigenvalue of  $d\varphi_{\tau_{u_0}}^{u_0}(x_0)$  with multiplicity one and



964 that the other eigenvalues of  $d\varphi_{\tau u_0}^{u_0}(x_0)$  are different from 1. The vector space  $T_{x_0}X$   
965 can then be written as  $T_{x_0}X = \mathbf{R}f(x_0, u_0) \oplus E$  where  $E$  is the  $n - 1$  dimensional  
966 subspace of  $T_{x_0}X$  generated by the eigenvectors of  $d\varphi_{\tau u_0}^{u_0}(x_0)$  that are different from  
967 1. If we choose the submanifold  $Y$  such as its tangent space at  $x_0$  is equal to  $E$ , we  
968 then see that the eigenvalues of the differential of  $\mathcal{P}$  with respect to the first variable  
969  $x$  are the  $n - 1$  eigenvalues (counted with multiplicities) of  $d\varphi_{\tau u_0}^{u_0}(x_0)$  that are different  
970 from 1. In conclusion, there exists  $x_u \in X$ ,  $\tau_u > 0$  defined in a neighborhood of  $u_0$ ,  
971 which depend smoothly on  $u$  and which satisfy  $\varphi_{\tau_u}^u(x_u, u) = x_u$ .

972 Now the  $M$ -structure lemma (Lemma 31.10 in [1]) can be generalized as follows.

973 LEMMA 14. *Let  $E$  be a finite dimensional Banach space. Consider  $u \mapsto L_u$  a*  
974 *smooth mapping defined from a neighborhood  $N$  of  $u_0 \in U$  to  $\mathcal{L}(E, E)$ , and  $u \mapsto v_u$ ,*  
975 *a smooth mapping defined from  $N$  to  $E \setminus \{0\}$ . Assume that  $L_u$  is transversal for  $v_u$*   
976 *(i.e.  $L_u v_u = v_u$  and the algebraic multiplicity of the eigenvalue 1 is equal to 1). Then*  
977 *for every  $u \in N$ , there exists  $A_u \in \mathcal{L}(E, E)$  such that*

978 1.  $A_u v_u = 0$ ;

979 2. for every mapping  $\mathcal{L} : I \times U \rightarrow \mathcal{L}(E, E)$  ( $I$  denotes the interval  $(0, b)$ ,  $b > 0$ )  
980 satisfying the three conditions

981 (a)  $\mathcal{L}(0, u) = L_u$ ;

982 (b)  $\left. \frac{\partial}{\partial s} \mathcal{L}(s, u) \right|_{s=0} = A_u$ ;

983 (c) for every  $s \in I$ ,  $\mathcal{L}(s, u)$  is transversal for  $v_u$

984 there exists  $\varepsilon > 0$  and  $N' \subset N$  (with  $N'$  open neighborhood of  $u_0$ ) such that  $\mathcal{L}(s, u)$   
985 is elementary for  $v_u$  for every  $(s, u) \in (0, \varepsilon) \times N'$  (i.e.  $\mathcal{L}(s, u)$  is transversal for  $v_u$   
986 and has no complex eigenvalue of modulus 1, except 1). Moreover  $u \mapsto A_u$  is smooth.

987 We sketch a proof of this lemma.

988 *Proof.* We consider the set, denoted by  $W$ , defined as follows

989 
$$W = \{ (B, u) \in L(E, E) \times N \mid B v_u = v_u \}.$$

990 It is not difficult to see that  $W$  is a submanifold of  $\mathcal{L}(E, E) \times N$  of codimension  $\dim E$   
991 (the mapping  $(B, u) \mapsto B v_u - v_u$  is a submersion).

992 Then, consider  $M$  the subset of  $\mathcal{L}(E, E)$  defined as in the proof of Lemma 31.10  
993 in [1]. Recall that  $M$  is a finite union of submanifolds of  $\mathcal{L}(E, E)$  ( $M = \bigcup_{i=1}^k M_i$ ),  
994 hence,  $\mathcal{M} \triangleq M \times N$  is a finite union of submanifolds of  $\mathcal{L}(E, E) \times N$  ( $\mathcal{M} = \bigcup_{i=1}^k M_i \times$   
995  $N$ ); moreover every  $M_i \times N$  has a codimension  $\geq 1$  in  $W$ .

996 Denoting by  $X_u$  the point  $(L_u, u)$  of  $W$ , there exists a tangent vector  $\xi_u \in T_{X_u}W$   
997 but  $\xi_u \notin T_{X_u}M_i \times N$  ( $i = 1, \dots, k$ ); moreover  $\xi_u$  depends smoothly on  $u$ . This is  
998 possible because each  $M_i \times N$  has a codimension  $\geq 1$  in  $W$ . The tangent vector  $\xi_u$   
999 may be written  $\xi_u = (A_u, v)$ , where  $A_u$  can be regarded as an element of  $\mathcal{L}(E, E)$   
1000 (and  $v \in T_u U$ ). Now, due to the property of  $A_u$ , we can conclude that there exist  
1001  $\varepsilon > 0$  such that  $\mathcal{L}(s, u) \notin M \times N$  as soon as  $0 < s < \varepsilon$ .  $\square$

1002 Returning to the vector field  $f \in \mathcal{G}_{3/2}^U(a)$ , we argue now as in Lemma 31.12, in  
1003 order to prove the existence of a neighborhood  $N_1$  of  $\gamma_{u_0}$  and a neighborhood  $N_2$  of  $u_0$   
1004 such that, for every  $u \in N_2$ , the vector field  $f(\cdot, u)$  admits a unique periodic trajectory  
1005  $\gamma_u$  located in  $N_1$ . Moreover, there exists a parametrized vector field  $g \in \Gamma_U(X)$  such  
1006 that  $g(\cdot, u)|_{\gamma_u} = 0$ ,  $g(\cdot, u)|_{X \setminus N_1} = 0$ ,  $g(\cdot, u) \equiv 0$  for  $u \in U \setminus N_2$ , and  $\gamma_u$  is an elementary  
1007 periodic orbit for  $f(\cdot, u) + \lambda g(\cdot, u)$  for sufficiently small  $\lambda \in \mathbf{R}_+^*$ . We can find such a  
1008 vector field  $g$  for every periodic trajectory of  $f(\cdot, u_0)$  (the number of these trajectories

1009 is finite), and so, we can conclude as in lemma 31.13 in [1], that there exists a vector  
 1010 field  $g$  such that the critical elements of  $f(\cdot, u) + \lambda g(\cdot, u)$  of period  $\leq a$  are exactly the  
 1011 same as the critical elements of  $f(\cdot, u)$  of period  $\leq a$  for every  $u \in N_2$ . Moreover the  
 1012 periodic trajectories of  $f(\cdot, u) + \lambda g(\cdot, u)$  are elementary for  $u \in N_2$ .

1013 Now, the open neighborhoods  $N_2$  cover  $U$ , which is compact. So there exist a  
 1014 finite set of elements  $u_0, \dots, u_s$  together with neighborhoods  $N_2^0, \dots, N_2^s$  of the  $u_i$ 's,  
 1015 there exist vector fields  $g_0, \dots, g_s$  as in the above lemma. Take  $\rho_0, \dots, \rho_s$  a partition  
 1016 of unity of  $U$  subordinated to the covering  $N_2^0, \dots, N_2^s$  and consider the vector field  
 1017  $g \triangleq \rho_0 g_0 + \dots + \rho_s g_s$ , we claim that  $f + \lambda g$  belong to  $\mathcal{G}_2^U(a)$  for every positive  $\lambda$   
 1018 sufficiently small.

1019 We conclude this step as in [1].

1020 *Step 6: If  $\mathcal{N} \subset \mathcal{G}_1^U$  is an open set, and  $\mathcal{G}_2^U(a) \cap \mathcal{N}$  is dense, then  $\mathcal{G}_2^U(\frac{3}{2}a) \subset \mathcal{N}$   
 1021 is dense..* This step results trivially from steps 4 and 5.

1022 *Step 7: If  $f \in \mathcal{G}_1^U$ , there exists an open neighborhood  $\mathcal{N}$  of  $f$  in  $\mathcal{G}_1^U$  such that  
 1023  $\mathcal{G}_2^U(a) \cap \mathcal{N} \subset \mathcal{N}$  is dense..* From step 3, if  $f \in \mathcal{G}_1^U$ , there exists  $a_0 > 0$  and a  
 1024 neighborhood  $\mathcal{N}$  of  $f$  in  $\mathcal{G}_1^U$  such that  $\mathcal{G}_2^U(a_0)\mathcal{N} = \mathcal{N}$ . In particular, this implies  
 1025 that  $\mathcal{G}_2^U(a_0)\mathcal{N}$  is dense in  $\mathcal{N}$ . Hence iterating step 5  $k$  times, one has that  $\mathcal{G}_2^U(\frac{3^k}{2^k}a_0) \cap$   
 1026  $\mathcal{N}$  is dense in  $\mathcal{N}$ . As one can find  $k$  such that  $\frac{3^k}{2^k}a_0 > a$ , we are done.

## 1027 Appendix B. Proof of Lemma 8.

1028 The derivative of  $x_p^\lambda$  with respect to  $\lambda$  can be written as

$$1029 \quad (26) \quad \frac{d}{d\lambda} x_p^\lambda \Big|_{\lambda=0} = A_p + B_p,$$

1031 where  $A_p$  is zero or a sum of terms of the form  $\delta_{j_k}^p(J_{j_k})$ , with  $J_{j_k}$  is an integral that  
 1032 can be written as

$$1033 \quad (27) \quad J_{j_k} = \int_0^T d\varphi_\sigma^{u_{p-1}}(\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})) \cdot \phi(\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}), u_{p-1}) d\sigma,$$

1034 where the subscripts  $j_k$  in  $A_p$  are less or equal to  $p-1$  and are such that  $u_{j_k} = u_{p-1}$ ;  
 1035 moreover  $\delta_{j_k}^p$  is the mapping defined as

$$1036 \quad \delta_{j_k}^a = \begin{cases} \text{Id} & \text{if } j_k = p-1, \\ d(\varphi_T^{u_{p-1}} \circ \dots \circ \varphi_T^{u_{j_k+1}})(x_{j_k+1}) & \text{if } j_k < p-1. \end{cases}$$

1037 As for the term  $B_p$ , it is zero or the sum of terms  $\delta_{j_k}^p(J'_{j_k})$ , where the  $J'_{j_k}$  are  
 1038 integrals that we write

$$1039 \quad J'_{j_k} = \int_0^T d\varphi_\sigma^{u_{j_k}}(\varphi_{T-\sigma}^{u_{j_k}}(x_{j_k})) \cdot \phi(\varphi_{T-\sigma}^{u_{j_k}}(x_{j_k}), u_{j_k}) d\sigma,$$

1040 with  $u_{j_k} \neq u_{p-1}$ .

1041 We write the derivative of  $x_i^\lambda$  and, if we are under  $\mathbf{C}_2$ ,  $\mathbf{C}'_3$  or  $\mathbf{C}'_4$  configurations,  
 1042 the ones of  $\bar{x}_j^\lambda$  and  $\bar{x}_m^\lambda$ , in the same way; we denote by  $\bar{J}_{j_k}$  and  $\bar{J}'_{j_k}$  the integrals  
 1043 appearing in the derivatives of  $\bar{x}_j^\lambda$  and  $\bar{x}_m^\lambda$ . We shall see that  $\phi$  can be chosen such  
 1044 that

- 1045 • the terms  $B_p$ ,  $B_i$ ,  $\bar{B}_j$  and  $\bar{B}_m$  are zero;
- 1046 • all the integrals  $\bar{J}_{j_k}$  that occur in the terms  $\bar{A}_j$  and  $\bar{A}_m$  are zero;
- 1047 • all the integrals  $J_{j_k}$  are zero but the one corresponding to the subscript  $j_k =$   
 1048  $p-1$ , which can be arbitrarily chosen.

1049 To this end, the perturbation  $\phi$  that we shall consider will be zero outside some  
 1050 neighborhood of  $u_{p-1}$ ; to be more precise, given  $\phi_0$  a vector field defined on  $X$ , it is  
 1051 possible to find  $\phi \in \Gamma_U(X)$  such that

- 1052 •  $\phi(\cdot, u_{p-1}) = \phi_0$ ;
- 1053 • for  $j = 0, \dots, 2n$ ,  $\phi(\cdot, u_j) \equiv 0$  as soon as  $u_j \neq u_{p-1}$ .

1054 With this choice of  $\phi$ , we have  $B_p = 0$ , and  $B_i = 0$  as well as  $\bar{B}_j = 0$ , and  $\bar{B}_m = 0$  if  
 1055 we are under  $\mathbf{C}_2$ ,  $\mathbf{C}'_3$  or  $\mathbf{C}'_4$  configuration.

1056 We split the points  $x_{j_k}$  (resp.  $\bar{x}_{j_k}$ ) that appear under the integrals  $J_{j_k}$  (resp.  
 1057  $\bar{J}_{j_k}$ ) into two classes: the first class, denoted by  $\mathcal{P}_1$ , contains the points  $x_{j_k}$  and  $\bar{x}_{j_k}$   
 1058 that belong to the trajectory of the vector field  $f(\cdot, u_{p-1})$  passing through  $x_{p-1}$  (and  
 1059 so  $\mathcal{P}_1$  contains the point  $x_{p-1}$  itself), the second class, denoted by  $\mathcal{P}_2$ , contains the  
 1060 points  $x_{j_k}$  and  $\bar{x}_{j_k}$  that do not belong to this trajectory. Denote by  $\mathcal{T}_1$  and  $\mathcal{T}_2$  the  
 1061 union of these trajectories restricted to the interval  $[0, T]$ , namely

$$1062 \quad \mathcal{T}_i = \{ \varphi_t^{u_{p-1}}(z) \mid t \in [0, T], z \in \mathcal{P}_i \} \quad i = 1, 2.$$

1063 The sets  $\mathcal{T}_1$  and  $\mathcal{T}_2$  being disjoint and compact, let  $\mathcal{U}_1$  and  $\mathcal{U}_2$  be two open sets  
 1064 of  $X$  such that  $\mathcal{T}_i \subset \mathcal{U}_i$  ( $i = 1, 2$ ) and  $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ . In the sequel, we shall  
 1065 assume that the vector field  $\phi_0$  is zero when restricted to  $\mathcal{U}_2$ ; this implies that the  
 1066 integrals  $J_{j_k}$  (resp.  $\bar{J}_{j_k}$ ) such that  $x_{j_k}$  (resp.  $\bar{x}_{j_k}$ ) belongs to  $\mathcal{P}_2$  are zero. Denote  
 1067 by  $j_1, \dots, j_a$  ( $0 \leq j_1 < \dots < j_a = p-1$ ) the subscripts such that  $x_{j_k}$  belongs to  
 1068  $\mathcal{P}_1$  and let  $t_1, \dots, t_a$  be such that  $x_{j_k} = \varphi_{t_k}^{u_{p-1}}(x_{p-1})$  ( $k = 1, \dots, a$ ). Denote also by  
 1069  $l_1 < \dots < l_b < \max(m, j)$  the subscripts such that  $\bar{x}_{l_k}$  belongs to  $\mathcal{P}_1$  and let  $t'_1, \dots, t'_b$   
 1070 be such that  $\bar{x}_{l_k} = \varphi_{t'_k}^{u_{p-1}}(x_{p-1})$  ( $k = 1, \dots, b$ ).

1071 Notice that, excepted when  $j_k = j_a$ , we cannot have  $t_k = 0$  since this would imply  
 1072 that  $x_{p-1} = x_{j_k}$  with  $j_k < p-1$ , which is not possible under  $\mathbf{C}_1$ – $\mathbf{C}_4$  configuration.  
 1073 Also, all the  $t'_k$  ( $k = 1, \dots, b$ ) are non zero, because, if there existed a subscript  $k$   
 1074 such that  $t'_k = 0$ , then, as  $u_{l_k} = u_{p-1}$ , we would have  $\bar{x}_{l_{k+1}} = x_p$ , which implies  
 1075 that we are under  $\mathbf{C}_2$  or  $\mathbf{C}'_3$  configuration and that  $l_k + 1 = m$ . Thus we have  
 1076  $\bar{x}_{m-1} = x_{p-1}$ , if we are under  $\mathbf{C}'_2$  configuration this implies  $(m-1) - (p-1) = i - j$   
 1077 and so  $m - p = i - j$ , which contradicts the definition of the  $\mathbf{C}_2$  configuration. If we  
 1078 are under  $\mathbf{C}'_3$  configuration, we found a pair  $(i', j') \triangleq (p-1, m-1)$  such that  $x_{i'} = \bar{x}_{j'}$   
 1079 with  $i' - j' = p - m$  and  $i' < p$ , which contradicts the definition of  $\mathbf{C}_3$  configuration.

1080 The trajectory related to the vector field  $f(\cdot, u_{p-1})$  passing through  $x_{p-1}$  may be  
 1081 periodic or aperiodic, we have to distinguish between these two cases.

1082 *The trajectory passing through  $x_{p-1}$  is not periodic.* Assume that the trajectory  
 1083 of the vector field  $f(\cdot, u_{p-1})$  passing through  $x_{p-1}$  is not periodic. Taking into account  
 1084 that, with our notations,  $\phi(\cdot, u_{p-1}) = \phi_0$ , the terms  $J_{j_k}$  appearing in  $A_p$  and (possibly)  
 1085 in  $A_i$  can also be written as

$$1086 \quad J_{j_k} = d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{T-t_k} d\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) d\sigma.$$

1087 In the same way the integrals  $\bar{J}_{l_k}$  appearing (possibly) in  $\bar{A}_m$  and  $\bar{A}_j$  write

$$1088 \quad \bar{J}_{l_k} = d\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \int_{-t'_k}^{T-t'_k} d\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) d\sigma.$$

1089 We set

$$1090 \quad T_{\min} = \min(\{t_k \mid k = 1, \dots, a\} \cup \{t'_k \mid k = 1, \dots, b\})$$

$$1091 \quad T_{\max} = \max(\{T + t_k \mid k = 1, \dots, a\} \cup \{T + t'_k \mid k = 1, \dots, b\})$$

1093 and we introduce the set  $\mathcal{T} = \{\varphi_t^{u_{p-1}}(x_{p-1}) \mid T_{\min} \leq t \leq T_{\max}\}$ . For  $z = \varphi_t^{u_{p-1}}(x_{p-1})$   
 1094 in  $\mathcal{T}$ , we define  $\phi_0(z)$  as

$$1095 \quad \phi_0(z) = \mu(t) d\varphi_{t-T}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p$$

1096 where  $\mu$  is a smooth function defined on  $[T_{\min}, T_{\max}]$  and  $\mathfrak{X}_p$  is an arbitrary vector  
 1097 tangent to  $X$  at  $x_p$ . As the trajectory passing through  $x_{p-1}$  is not periodic,  $\phi_0$  is  
 1098 unambiguously defined on  $\mathcal{T}$ , moreover  $\phi_0$  extends to a smooth vector field defined on  
 1099 the whole manifold  $X$  (and which is zero on  $\mathcal{U}_2$ ). With this choice of  $\phi_0$ , the integrals  
 1100 occurring in  $A_i$  and  $A_p$  write

$$1101 \quad J_{j_k} = \left( \int_{-t_k}^{T-t_k} \mu(T-\sigma) d\sigma \right) d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p = \left( \int_{t_k}^{T+t_k} \mu(\sigma) d\sigma \right) d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p;$$

1102 while the integrals occurring in the terms  $\bar{A}_j$  and  $\bar{A}_m$  write

$$1103 \quad \bar{J}_{l_k} = \left( \int_{-t'_k}^{T-t'_k} \mu(T-\sigma) d\sigma \right) d\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p = \left( \int_{t'_k}^{T+t'_k} \mu(\sigma) d\sigma \right) d\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p;$$

1104 here all the  $t_k$ 's are non-zero but  $t_a$ , and all the  $t'_k$ 's are non zero. Choose now a  
 1105 smooth function  $M$  defined on  $[T_{\min}, T_{\max}]$  and such that

- 1106 •  $M(T) = M(2T) = \dots = M(cT) = 1$ ; here  $c$  denotes the integer part of  
 1107  $T_{\max}/T$ ;
- 1108 •  $M(T+t_k) = M(t_k) = M(T+t'_{k'}) = M(t'_{k'}) = 0$ ,  $k = 1, \dots, a-1$ ,  $k' = 1, \dots, b$   
 1109 where  $t_k, t'_k \neq \alpha T$ ,  $\alpha = 1, \dots, c$ ;
- 1110 • and  $M(0) = 0$ ;

1111 We take now  $\mu(t) = \frac{dM(t)}{dt}$ , with this choice of  $\mu$ , all the integrals  $J_{j_k}$  and  $\bar{J}_{l_k}$  are  
 1112 zero except  $J_{p-1}$  which is equal to  $\mathfrak{X}_p$ .

1113 *Point  $x_{p-1}$  is singular..* In this case, we have  $x_{p-1} = x_p$ , so necessarily we are  
 1114 under  $\mathbf{C}_1$  or  $\mathbf{C}'_4$  configuration. The vector field  $\phi_0$  is then chosen such that  $\phi_0$  is zero  
 1115 outside an open neighborhood  $\mathcal{N}$  of  $x_{p-1}$ , this neighborhood being chosen so small  
 1116 that the trajectories of  $f(\cdot, u_{p-1})$  passing through the points  $x_{j_k}$  ( $j_k \neq p-1$ ) and  $\bar{x}_{l_k}$   
 1117 do not cross  $\mathcal{N}$ . With this choice of  $\phi_0$  all the integrals  $J_{j_k}$  and  $\bar{J}_{l_k}$  are zero but the  
 1118 one which correspond to  $j_k = p-1$  which is equal to

$$1119 \quad J_{p-1} = \int_0^T d\varphi_s^{u_{p-1}}(x_{p-1}) \cdot \phi_0(x_{p-1}) ds.$$

1120 Now  $d\varphi_s^{u_{p-1}}(x_{p-1}) = e^{sA}$  where  $A$  is the differential of  $f$  at  $x_{p-1}$ ; notice that, as  
 1121  $f \in \mathcal{G}_2^U(a)$ ,  $A$  does not have any purely imaginary eigenvalue (and so is invertible).  
 1122 Hence we can compute explicitly integral  $J_{p-1}$ :

$$1123 \quad J_{p-1} = A^{-1}(e^{TA} - \text{Id}) \cdot \phi_0(x_{p-1});$$

1124 as  $e^{TA}$  does not admit 1 as an eigenvalue,  $e^{TA} - \text{Id}$  is invertible, which proves that  
 1125  $J_{p-1}$  can be made equal to any tangent vector of  $T_{x_{p-1}}$  thanks to an appropriate  
 1126 choice of  $\phi_0(x_{p-1})$ .

1127 *The trajectory passing through  $x_{p-1}$  is periodic .* We shall show now the same  
 1128 result in the case when the trajectory passing through  $x_{p-1}$  is periodic; in other words,  
 1129 we assume that the mapping  $t \mapsto \varphi_t^{u_{p-1}}(x_{p-1})$  is periodic and we denote by  $\pi_0$  its  
 1130 prime period. In this case, the function  $\mu$  which appears in the definition of  $\phi_0$  must  
 1131 be periodic. Writing  $T = q\pi_0 + \tau$  with  $q \in \mathbf{N}$  and  $0 \leq \tau < \pi_0$ , we have

$$\begin{aligned}
 1132 \quad J_{j_k} &= \int_0^T d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma \\
 1133 \quad &= \sum_{l=0}^{q-1} \int_{l\pi_0}^{(l+1)\pi_0} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma + \int_{q\pi_0}^T d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma \\
 1134 \quad &= \sum_{l=0}^{q-1} d\varphi_{l\pi_0}^{u_{p-1}}(\varphi_{T-\pi_0}^{u_{p-1}}(x_{j_k})) \cdot \int_0^{\pi_0} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma \\
 1135 \quad (28) \quad &\quad \quad \quad + d\varphi_{q\pi_0}^{u_{p-1}}(\varphi_T^{u_{p-1}}(x_{j_k})) \int_0^\tau d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma . \\
 1136
 \end{aligned}$$

1137 Now the  $x_{j_k}$ 's in the above integrals are such that  $x_{j_k} = \varphi_{t_k}^{u_{p-1}}(x_{p-1})$ . Notice  
 1138 that, due to the periodicity of the trajectory passing through  $x_{p-1}$ , we can assume  
 1139 that  $0 \leq t_k < \pi_0$ . Writing the  $x_{j_k}$ 's in terms of  $x_{p-1}$ , the above integrals between 0  
 1140 and  $\pi_0$  can be re-written as.

$$1141 \quad \int_0^{\pi_0} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma = d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\pi_0-t_k} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) d\sigma .$$

1142 In the same way, the integral between 0 and  $\tau$  in (28) can be written

$$1143 \quad \int_0^\tau d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma = d\varphi_{t_k}^{u_{p-1}}(x_p) \int_{-t_k}^{\tau-t_k} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) d\sigma .$$

1144 It follows from these considerations and from the equality  $\varphi_T^{u_{p-1}}(x_{p-1}) = x_p$ , that we  
 1145 can write  $J_{j_k}$  under the form

$$\begin{aligned}
 1146 \quad J_{j_k} &= \sum_{l=0}^{q-1} d\varphi_{l\pi_0+t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\pi_0-t_k} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma \\
 1147 \quad &\quad \quad + d\varphi_{q\pi_0+t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\tau-t_k} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma \\
 1148 \quad &= d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \left( Q \cdot \int_{-t_k}^{\pi_0-t_k} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma \right. \\
 1149 \quad (29) \quad &\quad \quad \left. + \delta^q \cdot \int_{-t_k}^{\tau-t_k} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma \right); \\
 1150
 \end{aligned}$$

1151 where we let

$$1152 \quad (30) \quad \delta = d\varphi_{\pi_0}^{u_{p-1}}(x_p) \quad \text{and} \quad Q = \sum_{l=0}^{q-1} \delta^l .$$

1153 We introduce also the following notation, let  $t \in [0, \pi_0]$  and denote by  $I_t$  the integral

$$1154 \quad I_t = \int_0^t d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma .$$

1155 We now rewrite  $J_{j_k}$  in terms of integrals  $I_t$ , from (29), we get

1156 • if  $t_k < \tau$ ,

$$1157 \quad (31) \quad J_{j_k} = d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot ((Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0-t_k} + \delta^q \cdot I_{\tau-t_k}).$$

1158 • if  $\tau \leq t_k$

$$1159 \quad (32) \quad J_{j_k} = d\varphi_{t_k}^{u_{p-1}}(x_p) \cdot (Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0-t_k} + \delta^{q-1} \cdot I_{\pi_0+\tau-t_k});$$

1160 notice that, in each case,  $\pi_0 - t_k$  and  $\pi_0 + \tau - t_k$  belong to  $[0, \pi_0]$ ;

1161 As regards the integrals  $\bar{J}_{l_k}$  ( $k = 1, \dots, b$ ), analogous computations lead to the same  
1162 above formulas (31) and (32).

1163 We choose now  $\phi_0$  as follows, for  $z = \varphi_{-\sigma}^{u_{p-1}}(x_p)$  with  $\sigma \in [0, \pi_0]$ , we define  $\phi_0(z)$   
1164 as

$$1165 \quad (33) \quad \phi_0(z) = d\varphi_{-\sigma}^{u_{p-1}}(x_p) \cdot \vartheta(\sigma)$$

1166 where  $\vartheta : \mathbf{R} \rightarrow \mathbb{T}_{x_p} X$  is a  $\pi_0$ -periodic mapping to be determined; notice first that, as  
1167 we want  $\phi_0$  to be  $C^r$ , we must have  $\vartheta(0) = \vartheta(\pi_0) = 0$  as well as  $\vartheta^{(l)}(0) = \vartheta^{(l)}(\pi_0) = 0$   
1168 for  $l = 1, \dots, r$ .

1169 We deal first with the special case  $\tau = 0$ , in this case  $T = q\pi_0$  and we have  
1170  $x_{p-1} = x_p$ , so the lists  $L_p$  and  $\bar{L}_m$  must be in the  $\mathbf{C}_1$  or  $\mathbf{C}'_4$  configuration. If we are  
1171 under  $\mathbf{C}'_4$  configuration and if there exists a subscript  $j_1 < p-1$  such that  $u_{j_1} = u_{p-1}$   
1172 and  $x_{j_1} = \varphi_{t_1}^{u_{p-1}}(x_{p-1})$ , then  $x_{j_1} = x_{j_1+1}$  which is impossible from the definition of  
1173 the  $\mathbf{C}'_4$  configuration. If there exists a subscript  $l_1 < m$  such that  $u_{l_1} = u_{p-1}$  and  
1174  $\bar{x}_{l_1} = \varphi_{t_1}^{u_{p-1}}(x_{p-1})$ , then we would have  $\bar{x}_{l_1} = \bar{x}_{l_1+1}$ ; as the lists  $L_p$  and  $\bar{L}_m$  are in the  
1175  $\mathbf{C}_4$  configuration, this implies that  $l_1 = m-1$  and  $u_{m-1} = u_{p-1}$  which is incompatible  
1176 with the definition of the  $\mathbf{C}'_4$  configuration. We conclude that the terms  $A_{p-1}$ ,  $\bar{A}_j$   
1177 and  $\bar{A}_m$  are zero in this case and that  $A_p$  is equal to  $Q \cdot I_{\pi_0}$ ; so, with our choice of  
1178  $\phi_0$ , we have

$$1179 \quad A_p = Q \cdot \int_0^{\pi_0} \vartheta(\sigma) d\sigma$$

1180 and it is obviously possible to find a periodic function  $\vartheta$  satisfying the above con-  
1181 straints and whose integral over the interval  $[0, \pi_0]$  is equal to  $Q^{-1} \cdot \mathfrak{X}_p$ ; clearly this  
1182 choice of  $\vartheta$  is also possible if we are under  $\mathbf{C}_1$  configuration.

1183 We assume now that  $\tau \neq 0$ ; we deal first with the case of  $\mathbf{C}_1$  configuration. We  
1184 introduce the following sets.

$$1185 \quad \mathbf{T}_\alpha = \{ t_k, k = 1, \dots, a-1 \mid t_k \equiv -\alpha\tau \pmod{\pi_0} \}.$$

1186 If  $\mathbf{T}_1 = \emptyset$ , we set  $\alpha_0 = 0$ , if not, we denote by  $\alpha_0$  the largest integer  $\alpha$  such that  
1187  $\mathbf{T}_1 \neq \emptyset, \mathbf{T}_2 \neq \emptyset, \dots, \mathbf{T}_\alpha \neq \emptyset$ . If  $\alpha_0 \neq 0$ , we introduce the integers  $\gamma_\alpha = [\alpha\tau/\pi_0] + 1$   
1188 (where  $[x]$  denotes the integer part of  $x$ ); so if  $t_k \in \mathbf{T}_\alpha$ , we have  $t_k = \gamma_\alpha\pi_0 - \alpha\tau$ .

1189 It could happen that  $\pi_0$  is divisible by  $\tau$ ; hereafter, we distinguish two cases.

1190 First we assume that there does not exist any  $\alpha \leq \alpha_0$  such that  $(\alpha+1)\tau \equiv 0$   
1191  $(\text{mod } \pi_0)$ ; this assumption implies that, for every pair  $0 \leq \alpha < \alpha' \leq \alpha_0$  we have  
1192  $\alpha'\tau - \alpha\tau \not\equiv 0 \pmod{\pi_0}$ . Thus, there exists a mapping  $V : \mathbf{R} \rightarrow \mathbb{T}_{x_p} X$  such that

1193 •  $V$  is  $\pi_0$ -periodic and  $V(0) = 0, V^{(l)}(0) = 0$  for  $l = 1, \dots, r+1$ ;

1194 •  $V(\tau) = \delta^{-q} \cdot \mathfrak{X}_p$ , and

$$1195 \quad V(2\tau) = \begin{cases} \delta^{-q-1} \cdot V(\tau) & \text{if } \pi_0 < 2\tau \\ \delta^{-q} \cdot V(\tau) & \text{if } \pi_0 > 2\tau, \end{cases}$$

1196  $\vdots$

$$1197 \quad V((\alpha_0 + 1)\tau) = \begin{cases} \delta^{-q-1} \cdot V(\alpha_0\tau) & \text{if } \gamma_{\alpha_0}\pi_0 < (\alpha_0 + 1)\tau \\ \delta^{-q} \cdot V(\alpha_0\tau) & \text{if } \gamma_{\alpha_0}\pi_0 > (\alpha_0 + 1)\tau; \end{cases}$$

1198

1199 •  $V(\pi_0 - t_k) = 0$ ,  $V(\tau - t_k) = 0$  (case  $t_k < \tau$ ) or  $V(\pi_0 + \tau - t_k) = 0$  (case  
1200  $\tau \leq t_k$ ) if  $t_k \notin \mathbf{T}_1 \cup \dots \cup \mathbf{T}_{\alpha_0}$  (notice that, for such a  $t_k$ ,  $\pi_0 - t_k$  and  $\pi_0 + \tau - t_k$   
1201 do no more belong to  $\mathbf{T}_1 \cup \dots \cup \mathbf{T}_{\alpha_0}$ ).

1202 We take then the mapping  $\vartheta$  equal to the derivative of  $V$ , with this choice, all the  
1203 integrals  $J_{j_k}$  are zero but  $J_{p-1}$  which is equal to  $\mathfrak{X}_p$ ; this implies that  $A_p$  is equal to  
1204  $\mathfrak{X}_p$  while we have  $A_i = 0$ .

1205 *Case where  $\tau$  divides  $\pi_0$ .* Assume now that there exists  $\alpha \leq \alpha_0$  such that  $(\alpha +$   
1206  $1)\tau \equiv 0 \pmod{\pi_0}$ , then necessarily  $\alpha = \alpha_0$  (if  $\alpha < \alpha_0$ , the time  $t_\alpha = 0$  would belong  
1207 to  $\mathbf{T}_{\alpha+1}$ ). There exists a subscript  $j_k$  such that  $x_{j_k} = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_\tau^{u_{p-1}}(x_{p-1}) =$   
1208  $x_p$ , as  $j_k < p$ , from this equality and from the definition of configuration  $\mathbf{C}_1$ , we  
1209 deduce that  $j_k = i$  and that  $u_i = u_{p-1}$ . Arguing by induction, assume that, for some  
1210  $0 \leq r < p - 1 - i$ , we have  $u_i = \dots = u_{i+r} = u_{p-1}$  and  $x_i = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}), \dots, x_{i+r} =$   
1211  $\varphi_{-(\alpha_0-r)\tau}^{u_{p-1}}(x_{p-1})$ . There exists a subscript  $j_k$  such that  $x_{j_k} = \varphi_{-(\alpha_0-r-1)\tau}^{u_{p-1}}(x_{p-1}) =$   
1212  $\varphi_\tau^{u_{p-1}}(x_{i+r}) = x_{i+r+1}$ , as above, this equality implies that  $u_{i+r+1} = u_{p-1}$ . We have  
1213 proved that  $u_i = u_{i+1} = \dots = u_{p-1}$  and that  $\alpha_0 = p - i - 1$ . As  $x_p = \varphi_{(p-i)T}^{u_{p-1}}(x_i)$ , we  
1214 re-write  $A_p$  as follows:

$$1215 \quad A_p = \int_0^{(p-i)T} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma + d\varphi_{(p-i)T}^{u_{p-1}}(x_p) \cdot A_i.$$

1216 As  $(\alpha_0 + 1)\tau \equiv 0 \pmod{\pi_0}$ , there exists  $r \in \mathbf{N}$  such that  $(p - i)T = q'\pi_0$ , thus the  
1217 first term in this new expression of  $A_p$  can be re-written as

$$1218 \quad \int_0^{(p-i)T} d\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) d\sigma = Q' \cdot I_{\pi_0}$$

1219 where  $Q' = \text{Id} + \delta + \dots + \delta^{q'-1}$ . We shall see that  $\phi_0$  can be chosen such that  $A_i = 0$   
1220 and  $A_p$  equal to any tangent vector field. Hereafter, we call chain of length  $c$  ( $c \geq 2$ )  
1221 a sequence of  $c$  pairs  $((x_{j_{k_1}}, t_{k_1}), \dots, (x_{j_{k_c}}, t_{k_c}))$  such that

- 1222 • the points  $x_{j_{k_1}}, \dots, x_{j_{k_c}}$  belong to  $\mathcal{P}_1$  and the subscripts  $j_{k_1}, \dots, j_{k_c}$  are pair-  
1223 wise distinct;
- 1224 • the times  $t_{k_1}, \dots, t_{k_c}$  belong to  $\mathbf{T}_1 \cup \dots \cup \mathbf{T}_{\alpha_0} \cup \{0\}$  and are such that

$$1225 \quad t_{k_2} \equiv t_{k_1} + \tau \pmod{\pi_0}, \dots, t_{k_c} \equiv t_{k_1} + (c - 1)\tau \pmod{\pi_0}.$$

1226 Notice that two chains are either disjoint or equal. The chain

$$1227 \quad \mathbf{c}_0 \triangleq ((x_i, \tau), (x_{i+1}, 2\tau), \dots, (x_{p-1}, (\alpha_0 + 1)\tau))$$

1228 has a length equal to  $\alpha_0 + 1$ ; the lengths of all the other chains are less than  $\alpha_0 + 1$   
1229 because otherwise, we could find at least two equalities between the elements of  $L_p$ .

1230 We choose vector field  $\phi_0$  as in the case where  $(\alpha_0 + 1)\tau \not\equiv 0 \pmod{\pi_0}$ , but the  
 1231 mappings  $\vartheta$  and  $V$  are chosen as follows:

- 1232 •  $\vartheta(t) = V'(t)$  and  $V(t) = V_0(t) + tv_0$  where  $V_0$  is  $\pi_0$ -periodic with  $V_0(\pi_0) =$   
 1233  $V_0(0) = 0$ ; notice that this choice of  $\vartheta$  implies  $I_t = V_0(t) + tv_0$ ;
- 1234 •  $V'_0(0) = V'_0(\pi_0) = -v_0$  and  $V_0^{(k)} = V_0^{(k)}(\pi_0) = 0$  with  $k = 2, \dots, r + 1$ ;
- 1235 •  $v_0 = \frac{1}{\pi_0} \cdot (Q')^{-1} \mathfrak{X}_p$ ;
- 1236 • if  $((x_{j_{k_1}}, t_{k_1}), \dots, (x_{j_{k_c}}, t_{k_c}))$  is a chain of maximal length distinct from the  
 1237 chain  $\mathfrak{c}_0$ , we must have

$$1238 \quad (34) \quad \begin{cases} (Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_{k_s}} + \delta^q \cdot I_{\tau - t_{k_s}} = 0, & \text{if } t_{k_s} < \tau, \\ Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_{k_s}} + \delta^{q-1} \cdot I_{\pi_0 + \tau - t_{k_s}} = 0, & \text{if } \tau \leq t_{k_s}. \end{cases}$$

1239 These equalities can also be written as

$$1240 \quad \begin{cases} (Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot (V_0(-t_{k_s}) + (\pi_0 - t_{k_s})v_0) \\ \quad + \delta^q \cdot (V_0(\tau - t_{k_s}) + (\tau - t_{k_s})v_0) = 0, & \text{if } t_{k_s} < \tau, \\ Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot (V_0(-t_{k_s}) + (\pi_0 - t_{k_s})v_0) \\ \quad + \delta^{q-1} \cdot (V_0(\tau - t_{k_s}) + (\pi_0 + \tau - t_{k_s})v_0) = 0, & \text{if } \tau \leq t_{k_s}. \end{cases}$$

1241 As  $c < \alpha_0 + 1$ , the numbers  $\tau - t_{k_1}, -t_{k_1}, \dots, -t_{k_1} - (c-1)\tau$  are pairwise dis-  
 1242 tinct modulo  $\pi_0$ , the values of  $V_0$  at these points can be chosen independently  
 1243 from each other; this proves that the above equalities can be achieved.

- 1244 • If  $t_k$  does not belong to a chain, nor is the case for  $\tau - t_k$ , so equality (34)  
 1245 can be satisfied thanks to an appropriate choice of  $V(-t_k)$  and  $V(\tau - t_k)$ .

1246 With this choice of  $\vartheta$ , taking into account the formulas (31) and (32), we see that we  
 1247 have  $A_p = \mathfrak{X}_p$  and  $A_i = 0$ .

1248 If we are under the configurations  $\mathbf{C}_2$ ,  $\mathbf{C}'_3$  or  $\mathbf{C}'_4$ , the sets  $\mathbf{T}_\alpha$  are defined as  
 1249 follows.

$$1250 \quad \mathbf{T}_\alpha = \{t_k, k = 1, \dots, a-1 \mid t_k \equiv -\alpha\tau \pmod{\pi_0}\} \\ 1251 \quad \cup \{t'_k, k = 1, \dots, b \mid t'_k \equiv -\alpha\tau \pmod{\pi_0}\}, \\ 1252$$

1254 the integer  $\alpha_0$  being defined as above. If there does not exist any  $\alpha \leq \alpha_0$  such that  
 1255  $(\alpha + 1)\tau \equiv 0 \pmod{\pi_0}$ , the reasoning is exactly the same than in the the case of  $\mathbf{C}_1$   
 1256 configuration: choosing  $\phi_0$  as explained above, we get  $A_i = 0$ ,  $\bar{A}_m = 0$  and  $\bar{A}_j = 0$   
 1257 while  $A_p$  can be arbitrarily chosen.

1258 Assume now that there exists  $\alpha$  such that  $(\alpha + 1)\tau \equiv 0 \pmod{\pi_0}$ , then, as above,  
 1259 necessarily  $\alpha = \alpha_0$ . As we are under  $\mathbf{C}_2$ ,  $\mathbf{C}'_3$  or  $\mathbf{C}'_4$  configuration, the definition of  
 1260 a chain takes into account the elements of  $\bar{L}_{2n}$ : we call chain of length  $c$  a sequence  
 1261  $((z_1, \mathfrak{t}_{k_1}), \dots, (z_c, \mathfrak{t}_{k_c}))$  such that

- 1262 • the points  $z_1, \dots, z_c$  belong to  $\mathcal{P}_1$ ;
- 1263 • the times  $\mathfrak{t}_{k_1}, \dots, \mathfrak{t}_{k_c}$  belong to  $\mathbf{T}_1 \cup \dots \cup \mathbf{T}_{\alpha_0} \cup \{0\}$  and are such that

$$1264 \quad \mathfrak{t}_{k_2} \equiv \mathfrak{t}_{k_1} + \tau \pmod{\pi_0}, \dots, \mathfrak{t}_{k_c} \equiv \mathfrak{t}_{k_1} + (c-1)\tau \pmod{\pi_0};$$

- 1265 • if  $\mathfrak{t}_k = t_k$  (resp.  $\mathfrak{t}_k = t'_k$ ), then  $z_k = x_{j_k}$  (resp.  $z_k = \bar{x}_{l_k}$ ); moreover the  
 1266 subscripts  $j_k$  (resp.  $l_k$ ) related to those  $x_{j_k}$  (resp.  $\bar{x}_{l_k}$ ) are pairwise distinct.

1267 If we are under configuration  $\mathbf{C}'_4$ , the only possible chain of length  $\alpha_0 + 1$  is either

$$1268 \quad \mathfrak{c}_0 = ((x_i, \tau), \dots, (x_{p-1}, (\alpha_0 + 1)\tau)),$$



1269 or  
1270

$$\bar{\mathbf{c}}_0 = ((\bar{x}_j, \tau), \dots, (\bar{x}_{m-1}, (\alpha_0 + 1)\tau)).$$

1271 Notice that, from the definition of  $\mathbf{C}'_4$  configuration, these two chains,  $\mathbf{c}_0$  and  $\bar{\mathbf{c}}_0$ ,  
1272 cannot coexist. All the other chains have a length less than  $\alpha_0 + 1$ . To show this  
1273 fact, let  $((z_1, \mathbf{t}_{k_1}), \dots, (z_c, \mathbf{t}_{k_c}))$  be a chain such that  $c = \alpha_0 + 1$ . All the elements  
1274  $(z_1, \dots, z_c)$  belong either to  $L_{p-1}$  or to  $\bar{L}_{m-1}$  (if not, we could find an equality between  
1275 an element of  $L_{p-1}$  and an element of  $\bar{L}_{m-1}$ ). If all the  $z_k$ 's belong to  $L_{p-1}$ , then  
1276 we have  $z_{c+1} = z_1$ ; from the definition of  $\mathbf{C}'_4$  configuration, it follows that  $z_1 = x_i$   
1277 and  $z_{c+1} = x_p$ . If all the  $z_k$ 's are in  $\bar{L}_{m-1}$ , we obtain that  $z_1 = \bar{x}_j$ ,  $z_{c+1} = \bar{x}_m$  and  
1278  $u_j = \dots = u_{m-1} = u_{p-1}$ . As all the chain but  $\mathbf{c}_0$  have a length less than  $\alpha_0 + 1$ , we  
1279 can conclude, as for  $\mathbf{C}_1$  configuration, that there exists a function  $\mathcal{V}$  defining a vector  
1280 field  $\phi_0$  which ensures that  $A_i = 0$ ,  $\bar{A}_j = 0$ , and  $\bar{A}_m = 0$  while  $A_p$  can be arbitrarily  
1281 chosen.

1282 We shall see that a chain of length  $\alpha_0 + 1$  is not possible under  $\mathbf{C}_2$  or  $\mathbf{C}'_3$  config-  
1283 uration. Assume that we are not under  $\mathbf{C}_4$  configuration and denote by

$$1284 \quad \mathbf{c}_0 = ((z_1, \mathbf{t}_1), \dots, (z_{\alpha_0+1}, \mathbf{t}_{\alpha_0+1}))$$

1285 a chain with length  $\alpha_0 + 1$ ; notice that, from the definition of a chain, we have

$$1286 \quad (35) \quad z_{k+1} = \varphi_\tau^{u_{p-1}}(z_k) = \varphi_T^{u_{p-1}}(z_k).$$

1287 There exists  $0 \leq \alpha_1 \leq \alpha_0$  such that  $\mathbf{t}_{k_1} \equiv -\alpha_1\tau \pmod{\pi_0}$  so, from the definition of a  
1288 chain, we have

$$1289 \quad \mathbf{t}_{k_1} \equiv -\alpha_1\tau \pmod{\pi_0}, \quad \mathbf{t}_{k_2} \equiv -(\alpha_1 - 1)\tau \pmod{\pi_0}, \dots,$$

$$1290 \quad \mathbf{t}_{k_{\alpha_0+1}} \equiv -(\alpha_1 - \alpha_0)\tau \pmod{\pi_0}$$

1292 therefore  $\mathbf{t}_{k_{\alpha_1+1}} \equiv 0 \pmod{\pi_0}$  and  $z_{\alpha_1+1} = x_{p-1}$ . If  $\alpha_1 < \alpha_0$ , this equality implies  
1293  $z_{\alpha_1+2} = x_p$ ; if  $\alpha_1 = \alpha_0$ , we have  $z_1 = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_\tau^{u_{p-1}}(x_{p-1}) = x_p$ . Thus,  
1294 in chain  $\mathbf{c}_0$ , there exists an element  $z_i$  equal to  $x_p$ , this implies that we cannot be  
1295 under  $\mathbf{C}_2$  configuration because, in this case we can neither have  $z_i = x_{j_k}$  because  
1296  $j_k < p$  nor  $z_i = x_p = \bar{x}_m = \bar{x}_{l_k}$  because  $l_k < m$ . Thus we are under  $\mathbf{C}'_3$  configuration,  
1297 reordering the elements of the chain, we can assume that

$$1298 \quad \mathbf{t}_{k_1} \equiv -\alpha_0\tau \pmod{\pi_0}, \mathbf{t}_{k_2} \equiv -(\alpha_0 - 1)\tau \pmod{\pi_0}, \dots, \mathbf{t}_{k_{\alpha_0+1}} \equiv 0 \pmod{\pi_0}.$$

1299 We have  $z_1 = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_\tau^{u_{p-1}}(x_{p-1}) = x_p$ , so, as all the subscripts  $j_k$  are less  
1300 than  $p$ , we have  $z_1 = \bar{x}_m \in \bar{L}_{2n}$ . Let  $r$  be the greatest subscript such that  $z_1, \dots, z_r \in$   
1301  $\bar{L}_{2n}$ ; from (35), we have  $z_r = \bar{x}_{m+r-1}$  and  $z_{r+1} = x_{j_k}$  for some subscript  $j_k < p$ , so  
1302  $x_{j_k} = \bar{x}_{m+r}$ ; from the definition of configuration  $\mathbf{C}'_3$  this implies that we cannot have  
1303  $m + r < j$ , so  $m + r = j$  and  $z_{r+1} = x_i$ . Let  $s$  be the greatest subscript greatest  
1304 than or equal to  $r + 1$  such that  $z_{r+1}, \dots, z_s \in L_{2n}$ . We have  $z_s = x_{s+i-r-1}$  and we  
1305 claim that  $z_s$  is the last element of  $\mathbf{c}_0$  because if there exist  $z_{s+1} \in \bar{L}_{2n}$ , then we have  
1306  $z_{s+1} = \varphi_\tau^{u_{p-1}}(z_s) = x_{s+i-r}$ ; from the definition of configuration  $\mathbf{C}'_3$  this implies that  
1307  $s + i - r = p$  and  $z_{s+1} = \bar{x}_m$  which contradicts the definition of a chain. The element  
1308  $z_s$  being the last element of the chain, we have  $z_s = \varphi_{\alpha_0\tau}^{u_{p-1}}(z_1) = \varphi_{-\tau}^{u_{p-1}}(\bar{x}_m) = x_{p-1}$ ,  
1309 therefore chain  $\mathbf{c}_0$  can be written

$$1310 \quad \mathbf{c}_0 = ((\bar{x}_m, -\alpha_0\tau), \dots, (\bar{x}_{j-1}, -(\alpha_0 - j + m + 1)\tau), (x_i, -(\alpha_0 - j + m)\tau), \dots, (x_{p-1}, 0))$$

1311 from which we deduce that  $u_i = \dots = u_{p-1} = u_m = \dots = u_{j-1}$  which contradicts the  
1312 definition of  $\mathbf{C}'_3$ .

- 1314 [1] Ralph Abraham and Joel Robbin. *Transversal Mappings and Flows*. W. A. Benjamin, New  
1315 York, 1967.
- 1316 [2] Dirk Aeyels. Generic observability of differentiable systems. *SIAM Journal on Control and*  
1317 *Optimization*, 19(5):595–603, 1981.
- 1318 [3] Sabeur Ammar, Hind Feki, and Jean-Claude Vivalda. Observability under sampling for bilinear  
1319 system. *International Journal of Control*, 87(2):312–319, 2014.
- 1320 [4] Sabeur Ammar, Mohamed Mabrouk, and Jean-Claude Vivalda. On the genericity of the dif-  
1321 ferential observability of controlled discrete-time systems. *SIAM Journal on Control and*  
1322 *Optimization*, 46(6):2182–2198, 2007.
- 1323 [5] Sabeur Ammar, Majid Massaoud, and Jean-Claude Vivalda. Observability under sampling for  
1324 nonlinear systems. *Asian Journal of Control*, 18(4):1269–1278, 2016.
- 1325 [6] Sabeur Ammar and Jean-Claude Vivalda. On the preservation of observability under sampling.  
1326 *Systems & Control Letters*, 52(1):7–15, 2004.
- 1327 [7] Sabeur Ammar and Jean-Claude Vivalda. On the genericity of the observability of con-  
1328 trolled discrete-time systems. *ESAIM: Control, Optimisation and Calculus of Variations*,  
1329 11(02):161–179, 2005.
- 1330 [8] Éric Busvelle and Jean-Paul Gauthier. On determining unknown functions in differential sys-  
1331 tems, with an application to biological reactors. *ESAIM: Control, Optimisation and Cal-  
1332 culus of Variations*, 9:509–551, 2003.
- 1333 [9] Jean-Paul Gauthier, Hassan Hammouri, and Ivan Kupka. Observers for nonlinear systems.  
1334 In *Proceedings of the 30th IEEE Conference on Decision and Control*, volume 2, pages  
1335 1483–1489, dec 1991.
- 1336 [10] Jean-Paul Gauthier and Ivan Kupka. Observability for systems with more outputs than inputs  
1337 and asymptotic observers. *Mathematische Zeitschrift*, 223:47–78, 1996.
- 1338 [11] Jean-Paul Gauthier and Ivan Kupka. *Deterministic Observation Theory and Applications*.  
1339 Cambridge University Press, Cambridge, 2001.
- 1340 [12] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities*. Graduate Texts in  
1341 Mathematics. Springer Verlag, 1986.
- 1342 [13] Jeremy P. Huke and David S. Broomhead. Embedding theorems for non-uniformly sampled  
1343 dynamical systems. *Nonlinearity*, 20(9):2205, 2007.
- 1344 [14] Gerhard Kreisselmeier. On sampling without loss of observability/controllability. *Automatic*  
1345 *Control, IEEE Transactions on*, 44(5):1021–1025, may 1999.
- 1346 [15] Ivan Kupka. Contribution à la théorie des champs génériques. *Contributions to Differential*  
1347 *Equations*, 2:457–484, 1963.
- 1348 [16] Ivan Kupka. Addendum et corrections au mémoire: “Contributions à la théorie des champs  
1349 génériques”. *Contributions to Differential Equations*, 3:411–420, 1964.
- 1350 [17] Stephen Smale. Stable manifolds for differential equations and diffeomorphisms. *Ann. Scuola*  
1351 *Norm. Sup. Pisa (3)*, 17:97–116, 1963.
- 1352 [18] Eduardo D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*.  
1353 Texts in Applied Mathematics. Springer, New York, 1998.
- 1354 [19] Jorge Sotomayor. Generic one-parameter families of vector fields on two-dimensional manifolds.  
1355 *Inst. Hautes Études Sci. Publ. Math.*, 43(1):5–46, 1974.
- 1356 [20] Jean-Claude Vivalda. On the genericity of the observability of uncontrolled discrete nonlinear  
1357 systems. *SIAM Journal on Control and Optimization*, 42(4):1509–1522, 2003.