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# Minimum density of identifying codes of king grids

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## Abstract

A set  $C \subseteq V(G)$  is an *identifying code* in a graph  $G$  if for all  $v \in V(G)$ ,  $C[v] \neq \emptyset$ , and for all distinct  $u, v \in V(G)$ ,  $C[u] \neq C[v]$ , where  $C[v] = N[v] \cap C$  and  $N[v]$  denotes the closed neighbourhood of  $v$  in  $G$ . The minimum density of an identifying code in  $G$  is denoted by  $d^*(G)$ . In this paper, we study the density of king grids which are strong product of two paths. We show that for every king grid  $G$ ,  $d^*(G) \geq 2/9$ . In addition, we show this bound is attained only for king grids which are strong products of two infinite paths. Given  $k \geq 3$ , we denote by  $\mathcal{K}_k$  the (infinite) king strip with  $k$  rows. We prove that  $d^*(\mathcal{K}_3) = 1/3$ ,  $d^*(\mathcal{K}_4) = 5/16$ ,  $d^*(\mathcal{K}_5) = 4/15$  and  $d^*(\mathcal{K}_6) = 5/18$ . We also prove that  $\frac{2}{9} + \frac{8}{81k} \leq d^*(\mathcal{K}_k) \leq \frac{2}{9} + \frac{4}{9k}$  for every  $k \geq 7$ .

*Keywords:* Identifying code, King grid, Discharging Method.

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## 1 Introduction

Let  $G$  be a graph. The *neighbourhood* of a vertex  $v$  of  $G$ , denoted by  $N(v)$ , is the set of vertices adjacent to  $v$  in  $G$ , and the *closed neighbourhood* of  $v$  is the set  $N[v] = N(v) \cup \{v\}$ . Given a set  $C \subseteq V(G)$ , let  $C[v] = N[v] \cap C$ . We say that  $C$  is an *identifying code* of  $G$  if  $C[v] \neq \emptyset$  for all  $v \in V(G)$ , and  $C[u] \neq C[v]$  for all distinct  $u, v \in V(G)$ . Clearly, a graph  $G$  has an identifying code if and only if it contains no *twins* (vertices  $u, v \in V(G)$  with  $N[u] = N[v]$ ).

Let  $G$  be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer  $r$  and vertex  $v$ , we denote by  $B_r(v)$  the ball of radius  $r$  in  $G$  centered at  $v$ , that is  $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$ . For any set of vertices  $C \subseteq V(G)$ , the *density* of  $C$  in  $G$ , denoted by  $d(C, G)$ , is defined by

$$d(C, G) = \limsup_{r \rightarrow +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|},$$

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where  $v_0$  is an arbitrary vertex in  $G$ . The infimum of the density of an identifying code in  $G$  is denoted by  $d^*(G)$ . Observe that if  $G$  is finite, then  $d^*(G) = |C^*|/|V(G)|$ , where  $C^*$  is a minimum-size identifying code of  $G$ .

The problem of finding low-density identifying codes was introduced in [12] in relation to fault diagnosis in arrays of processors. Particular interest was dedicated to grids as many processor networks have a grid topology. Many results have been obtained on square grids [4,1,9,2,11], triangular grids [12,10], and hexagonal grids [5,7,8]. In this paper, we study *king grids*, which are strong products of two paths. The *strong product* of two graphs  $G$  and  $H$ , denoted by  $G \boxtimes H$ , is the graph with vertex set  $V(G) \times V(H)$  and edge set :

$$\begin{aligned} E(G \boxtimes H) = & \{(a, b)(a, b') \mid a \in V(G) \text{ and } bb' \in E(H)\} \\ & \cup \{(a, b)(a', b) \mid aa' \in E(G) \text{ and } b \in V(H)\} \\ & \cup \{(a, b)(a', b') \mid aa' \in E(G) \text{ and } bb' \in E(H)\}. \end{aligned}$$

The *two-way infinite path*, denoted by  $P_{\mathbb{Z}}$ , is the graph with vertex set  $\mathbb{Z}$  and edge set  $\{\{i, i + 1\} \mid i \in \mathbb{Z}\}$ , and the *one-way infinite path*, denoted by  $P_{\mathbb{N}}$ , is the graph with vertex set  $\mathbb{N}$  and edge set  $\{\{i, i + 1\} \mid i \in \mathbb{N}\}$ . A *path* is a connected subgraph of  $P_{\mathbb{Z}}$ . For every positive integer  $k$ ,  $P_k$  is the subgraph of  $P_{\mathbb{Z}}$  induced by  $\{1, 2, \dots, k\}$ . A *king grid* is the strong product of two (finite or infinite) paths. The *plane king grid* is  $\mathcal{G}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{Z}}$ , the *half-plane king grid* is  $\mathcal{H}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{N}}$ , the *quater-plane king grid* is  $\mathcal{Q}_K = P_{\mathbb{N}} \boxtimes P_{\mathbb{N}}$ , and the *king strip of height  $k$*  is  $\mathcal{K}_k = P_{\mathbb{Z}} \boxtimes P_k$ .

In 2002, Charon et al. [3] proved that  $d^*(\mathcal{G}_K)$  is  $2/9$ . They provided the tile depicted in Figure 1, which generates a periodic tiling of the plane with periods  $(0, 6)$  and  $(6, 0)$ , yielding an identifying code  $C_{\infty}$  of the bidimensional infinite king grid with density  $\frac{2}{9}$ .

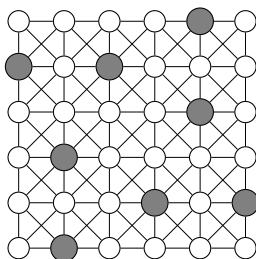


Fig. 1. Tile generating an optimal identifying code of the bidimensional infinite grid. Black vertices are those of the code.

In this paper, using the Discharging Method (see Section 3 of [10] for a detailed presentation of this technique for identifying codes), we provide the following tight general lower bound on the minimum density of identifying

codes of king grids.

**Theorem 1.1** *If  $G$  is a (finite or infinite) king grid, then  $d^*(G) \geq \frac{2}{9}$ .*

Keeping on, we prove the following.

**Theorem 1.2** *If  $G$  is a finite king grid, then  $d^*(G) > \frac{2}{9}$ .*

Finally, we give some bounds for king strips. Pushing further the proof of Theorem 1.1, we prove the following.

**Theorem 1.3** *For every  $k \geq 6$ ,  $d^*(\mathcal{K}_k) \geq \frac{2}{9} + \frac{8}{81k}$ .*

Modifying  $C_\infty$ , we construct identifying codes of  $\mathcal{K}_k$  yielding the following upper bounds.

**Theorem 1.4** *For every  $k \geq 5$ ,*

$$d^*(\mathcal{K}_k) \leq \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \text{if } k \equiv 0 \pmod{3}, \\ \frac{2}{9} + \frac{8}{18k}, & \text{if } k \equiv 1 \pmod{3}, \\ \frac{2}{9} + \frac{7}{18k}, & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

Finally, we show some identifying codes of  $\mathcal{K}_3$ ,  $\mathcal{K}_4$ ,  $\mathcal{K}_5$  and  $\mathcal{K}_6$  (see Figures 2, 3, 4, and 5.) and prove that they are optimal. This yields the following.

**Theorem 1.5**  $d^*(\mathcal{K}_3) = 1/3 = 0.333\dots$      $d^*(\mathcal{K}_4) = 5/16 = 0.3125$   
 $d^*(\mathcal{K}_5) = 4/15 = 0.2666\dots$      $d^*(\mathcal{K}_6) = 5/18 = 0.2777\dots$

Clearly  $d^*(\mathcal{K}_1) = 1/2$  (as  $\mathcal{K}_1 = \mathcal{S}_1 = P_{\mathbb{Z}}$ ) and  $\mathcal{K}_2$  has no identifying code because it has twins. All these results imply that  $\mathcal{G}_K$ ,  $\mathcal{H}_K$  and  $\mathcal{Q}_K$  are the unique king grids having an identifying code with density  $2/9$ . (One can easily derive from  $C_\infty$  identifying codes with density  $2/9$  of  $\mathcal{H}_K$  and  $\mathcal{Q}_K$ ).

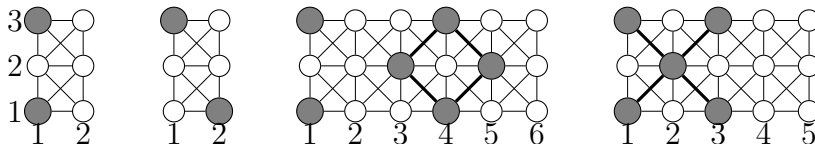


Fig. 2. Four tiles generating optimal identifying codes of  $\mathcal{K}_3$  (density  $1/3$ )

## 2 Sketches of proofs

**Sketch of proof of Theorem 1.1.** Let  $G$  be a king grid and  $C$  an identifying code of  $G$ . We shall prove that  $d(C, G) \geq 2/9$ . For this, we use the Discharging

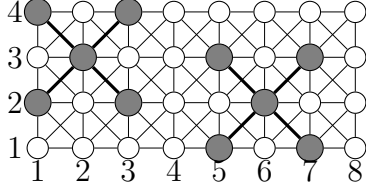


Fig. 3. Tile generating an optimal identifying code of  $\mathcal{K}_4$  (density  $5/16$ )

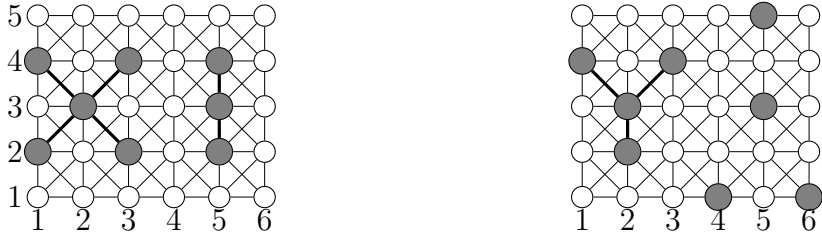


Fig. 4. Two tiles generating optimal identifying codes of  $\mathcal{K}_5$  (density  $4/15$ )



Fig. 5. Two tiles generating optimal identifying codes of  $\mathcal{K}_6$  (density  $5/18$ )

Method. The initial charge of a vertex  $v$  is 1 if  $v \in C$  and 0 otherwise. We then apply some local discharging rules. We shall prove that the final charge of every vertex in  $C$  is at least  $2/9$ . This would imply the result.

We set  $U = V(G) \setminus C$ . Given  $X \subseteq V(G)$  and  $1 \leq i \leq 9$ , we denote by  $X_i$  (resp.  $X_{\geq i}$ ) the set of vertices in  $X$  having exactly  $i$  vertices (resp. at least  $i$  vertices) in their identifier. An  $X$ -vertex is a vertex in  $X$ . A vertex is *full* if its eight neighbours in  $\mathcal{G}_K$  are in  $G$ ; otherwise it is a *side vertex*.

We first establish the following properties of  $C$ .

- (i) Two  $C_2$ -vertices are not adjacent.
- (ii) Every  $C$ -vertex has at most one neighbour in  $U_1$ .
- (iii) Every full  $C_2$ -vertex has at least three neighbours in  $U_{\geq 3}$ .
- (iv) Every full  $C_3$ -vertex has a neighbour in  $U_{\geq 3}$ .

- (v) Every  $C_1$ -vertex  $(a, b)$  has no neighbour in  $U_1$  and at most six neighbours in  $U_2$ . Furthermore, if it has six neighbours in  $U_2$ , then either  $\{(a-1, b-2), (a-2, b-1), (a+2, b+1), (a+1, b+2)\} \subseteq C$  or  $\{(a+1, b-2), (a+2, b-1), (a-2, b+1), (a-1, b+2)\} \subseteq C$ .

A *defective vertex* is a vertex in  $C_1$  with six neighbours in  $U_2$ . Let  $v = (a, b)$  be a defective vertex. The *team* of  $v$  is a set among  $\{(a-1, b-2), (a-2, b-1), (a+2, b+1), (a+1, b+2)\}$  and  $\{(a+1, b-2), (a+2, b-1), (a-2, b+1), (a-1, b+2)\}$  which is included in  $C$ . By Property (v), the team exists. Moreover, by Property (i), at least two vertices of the team are in  $C_{\geq 3}$ . Those vertices are the *partners* of  $v$ .

We apply the following discharging rules.

- (R1) Every  $C$ -vertex sends  $\frac{2}{9i}$  to each of its neighbours in  $U_i$ .  
(R2) Every defective vertex receives  $\frac{1}{54}$  from each of its partners.

Using the above properties, we then prove that the final charge of every vertex  $v$  is at least  $2/9$ .  $\square$

**Sketch of proof of Theorem 1.2.** We only need to prove that, at the end of the proof of Theorem 1.1, one vertex has final charge greater than  $2/9$ . To do so we shall prove that there is a side  $C$ -vertex or a  $C_{\geq 3}$ -vertex and check that such a vertex has final charge at least  $\frac{2}{9} + \frac{1}{27}$ .  $\square$

**Sketch of proof of Theorem 1.3.** Using the Discharging Method, we prove that in average, for every column, there is an extra charge of at least  $\frac{4}{81}$  on the three top vertices and an extra charge of at least  $\frac{4}{81}$  on the three top vertices.  $\square$

**Sketch of proof of Theorem 1.5.** The  $b$ th row of  $\mathcal{K}_k$  is  $R_b = \{(a, b) \mid a \in \mathbb{Z}\}$ . We have  $d(C, \mathcal{K}_k) = \frac{1}{k} \sum_{i=1}^k d(C, R_i)$ . We show that if  $C$  is an identifying code of  $\mathcal{K}_k$  ( $k \geq 3$ ), then  $d(C, R_1) + d(C, R_2) \geq 1/2$ ,  $d(C, R_k) + d(C, R_{k-1}) \geq 1/2$ ,  $d(C, R_3) \geq 1/3$  and  $d(C, R_{k-2}) \geq 1/3$ . One easily derives that if  $C$  is an identifying code of  $\mathcal{K}_5$  (resp.  $\mathcal{K}_6$ ), then  $d(C, \mathcal{K}_5) \geq 4/15$ . (resp.  $d(C, \mathcal{K}_6) \leq 5/18$ .)

To prove lower bounds on  $d^*(\mathcal{K}_k)$  for  $k \in \{3, 4\}$ , we use the Discharging Method on the columns  $Q_a = \{(a, b) \mid 1 \leq b \leq k\}$ . Let  $C$  be an identifying code of  $\mathcal{K}_k$ . We set the initial charge of every integer  $a \in \mathbb{Z}$  to  $\text{chrg}_0(a) = |Q_a \cap C|$ . We say that  $a \in \mathbb{Z}$  is *satisfied* if its charge is least  $q_k$  and *unsatisfied* otherwise, where  $q_3 = 1$  and  $q_4 = 5/4$ . We apply five discharging rules, Rule  $i$  for  $i = 1$  to 5 one after another. We denote by  $\text{chrg}_i(a)$  the charge of  $a$  after applying Rule  $i$ .

Rule  $i$  : every unsatisfied  $a \in \mathbb{Z}$  receives  $\min\{\text{chrg}_{i-1}(a-i) - q_k, q_k - \text{chrg}_{i-1}(a)\}$

from  $a - i$ , if  $a - i$  is satisfied (before Rule  $i$ ).

Finally, we prove that, after these rules, every integer  $a \in \mathbb{Z}$  is satisfied. This implies  $d(C, \mathcal{K}_k) \geq q_k/k$ .  $\square$

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