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Abstract. In this paper, we introduce a new accumulation process, the Semi-Markov Accumulation Process (SMAP). This class of processes extends the framework of continuous-time Markov Additive Processes (MAPs) by allowing the underlying environmental component to be a semi-Markov process instead of a Markov process. Next, we follow an analytic approach to derive a Master Equation formula of the Renewal type that describes the evolution of SMAPs in time. We show that under exponential holding times, a matrix exponential form analogous to the matrix exponent of a MAP is attained. Finally, we consider an application of our results where closed-form solutions are rather easy to achieve.

INTRODUCTION

Stochastic accumulation processes hold a very prominent role in many applications areas such as queueing theory, risk models, manufacturing, etc [1]. One of the most studied accumulation processes in the literature is the class of Markov Additive Processes, itself based on Lévy processes which include the Poisson and Wiener processes. Roughly speaking, a MAP is a two-dimensional Markov process (X, J), where J is also a Markov process, widely referred to as the environmental state. The first component X is additive and the distribution of its increments is modulated by J. In this paper, we extend the framework of continuous-time MAPs by letting J be a semi-Markov process; we call this class Semi-Markov Accumulation Processes. Our goal is to establish a simple formula that captures the evolution of SMAPs in time and allows us to study quantitatively additional performance measures.

Both probabilistic and analytic techniques are typically employed for the performance analysis of MAPs, aiming at their qualitative or quantitative characteristics [2]. Potential measures, occupation densities, and scale functions are only a few interesting examples of performance measures to mention [3]. In principle, the theory of MAPs with one-sided jumps is well-developed [4]. In case of two-sided jumps, exact solutions are available when both jumps follow some phase-type distribution [5]. An intriguing problem is to investigate whether some of the results that hold for MAPs can be generalized also to SMAPs, which is actually the main motivation for our research.

In this paper, we follow an analytic approach to study the transient behaviour of SMAPs. More precisely, via double (Laplace) transforms, we derive a matrix Renewal-type equation –the Master Equation– for the joint probability of the accumulation level and the background environmental state. This Master Equation has a simple representation involving matrices that capture the evolution of the underlying semi-Markov process and the accumulation phenomena. When all holding times are exponential, these matrices have separable variables and Laplace inversion yields an exponential form similar to the matrix exponent of a MAP. Finally, the Master Equation can be further utilized to evaluate moments of SMAPs, which relate to the moments of the aforementioned basic matrices.

The rest of the paper is organized as follows. First, we define formally the class of semi-Markov accumulation processes and introduce the associated terminology and notation. Next, we sketch the calculations to obtain the Master Equation and we compare our formulas with MAPs. Finally, we provide an application of the Master Equation that delivers computationally tractable results.

MODEL DESCRIPTION

Let $\mathcal{E} = \{1, 2, \ldots, N\}$ be a finite state-space and $\{Z_n\}_{n\in\mathbb{N}}$ a discrete Markov chain on $\mathcal{E}$, with one-step transition matrix $\mathbf{P}$; jump probabilities are denoted as p_{ij}. If $\{(Z_n, T_n)\}_{n\geq 0}$ is a *Markov Renewal Sequence* (MRS), the continuous-time process $J(t) := Z_n$, $T_n \leq t < T_{n+1}$, for $n = 0, 1, \ldots$, is a *semi-Markov process*.

We consider now a quantity $S(t)$ such that while the process $J(t)$ is in state i, $S(t)$ accumulates according to a Lévy process $X_i(t)$ that depends on i [6]. The accumulated quantity may also change when the underlying Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ jumps from one state to another. Writing all these formally, we formulate the following definition:

Definition 1 *Let $\{(Z_n, T_n)\}_{n\geq 0}$ be a MRS, $\{X_i(t)\}_{t\in\mathbb{R}}$ a Lévy process, for each $i \in \mathcal{E}$, and $\{\Delta^n_{ij}\}_{n\in\mathbb{N}}$ an i.i.d. sequence of real-valued r.vs., for each $(i, j) \in \mathcal{E}\times\mathcal{E}$. If $Z_n = i$ and $Z_{n+1} = j$, $\Delta_{Z_n,Z_{n+1}} := \Delta^n_{ij}$. The stochastic process*

$$S(t) := S(T_n) + X_{Z_n}(t) - X_{Z_n}(T_n) + \Delta_{Z_n,Z_{n+1}}\mathbb{1}_{\{t=T_{n+1}\}}, \qquad T_n < t \leq T_{n+1}, \quad n = 0, 1, \ldots$$

with $S(0) = 0$ and $\mathbb{1}$ being the indicator function, is called a semi-Markov accumulation process (SMAP).

For an arbitrary time t, we are interested in the joint distribution of $S(t)$ and $J(t)$, given that $Z_0 = i$ and $T_0 = 0$:

$$\mathbb{P}_i(S(t) \leq x, J(t) = j), \tag{1}$$

where $\mathbb{P}_i(\,\cdot\,) = \mathbb{P}(\cdot \mid Z_0 = i)$, and related conditional expectations denoted with $\mathbb{E}_i$. Before proceeding with the study of the distribution (1), we introduce the related notation in the remaining of this section. Note that we follow closely the terminology of semi-Markov processes; see [7].

Terminology and notation

The conditional probabilities $H_{ij}(t) = \mathbb{P}(T_1 \leq t \mid Z_0 = i, Z_1 = j)$, $i, j \in \mathcal{E}$, are widely known as the holding time distributions. If τ_{ij} is the r.v. with distribution $H_{ij}(t)$, then τ_{ij} represents the time spent in state i until there is a transition to state j. In addition, the r.vs. τ_i with distribution $W_i(t) = \mathbb{P}(T_1 \leq t \mid Z_0 = i)$, $i \in \mathcal{E}$, are called the waiting times. Observe that a waiting time is merely a holding time that is unconditional on the destination state. In matrix notation, we write $\mathbf{H}(t)$ and $\mathbf{W}(t)$, respectively, where $\mathbf{W}(t)$ is a diagonal matrix. The matrix $\mathbf{K}(t)$ formed by the elements $K_{ij}(t) = \mathbb{P}(Z_1 = j, T_1 \leq x \mid Z_0 = i)$ is called the *(global) kernel* of the MRS. It holds that $\mathbf{K}(t) = \mathbf{P} \circ \mathbf{H}(t)$, where $\circ$ is the Hadamard product between two matrices. Moreover, $W_i(t) = \sum_{j\in\mathcal{E}} K_{ij}(t) = \sum_{j\in\mathcal{E}} p_{ij}H_{ij}(t)$.

As we shall see in the next section, our analysis goes through Laplace-Stieltjes transforms (LSTs) and moment generating functions (m.g.f.). Therefore, we also provide here the related notation. More precisely, $\widetilde{\mathbf{H}}(s)$, $\widetilde{\mathbf{W}}(s)$, and $\widetilde{\mathbf{K}}(s)$ correspond to the LSTs of the holding times, waiting times, and kernel matrices, respectively. Moreover, $\widehat{\Delta}_{ij}[z] := \mathbb{E}(e^{z\Delta_{ij}})$, with $\widehat{\mathbf{\Delta}}[z]$ being the matrix equivalent of a m.g.f. Finally, $\psi_i(z)$ is the *characteristic exponent* of $X_i(t)$ [6, p. 12].

THE MASTER EQUATION

For the evaluation of the joint distribution (1), we define the double transform

$$\widetilde{\widehat{F}}_{ij}(z, s) = \int_0^\infty e^{-st}\mathbb{E}_i(e^{zS(t)}\mathbb{1}_{\{J(t)=j\}})dt = \int_0^\infty e^{-st}\int_0^\infty e^{zx}d\mathbb{P}_i(S(t) \leq x, J(t) = j)dt. \tag{2}$$

To calculate $\widetilde{\widehat{F}}_{ij}(z, s)$, we first compute the auxiliary m.g.f.:

$$\widehat{F}^{i,j}_t[z] = \mathbb{E}_i(e^{zS(t)}\mathbb{1}_{\{J(t)=j\}}) = \int_0^\infty e^{zx}d\mathbb{P}_i(S(t) \leq x, J(t) = j), \tag{3}$$

and then compute the Laplace transform of $\widehat{F}^{i,j}_t[z]$ with respect to t. The matrix $\widehat{\boldsymbol{F}}_t[z] := \left(\widehat{F}^{i,j}_t[z]\right)_{(i,j)\in\mathcal{E}\times\mathcal{E}}$ is typically called the *kernel* of the SMAP. Conditioning on the time of T_1, we separate two cases:

- $\tau_i > t$. This means that $S(t) = X_i(t) - X_i(0)$, and consequently, $\widehat{F}^{i,j}_t[z] = \mathbb{E}(e^{z\left(X_i(t)-X_i(0)\right)}) = e^{t\psi_i(z)}$.
- $\tau_i \leq t$. By conditioning on the value of Z_1 and using the strong Markov property, we obtain: $\widehat{F}^{i,j}_t[z] = \sum_{k\in\mathcal{E}} p_{ik}e^{\tau_{ik}\psi_i(z)}\widehat{\Delta}_{ik}[z]\mathbb{E}_k(e^{zS(t-\tau_{ik})}\mathbb{1}_{\{J(t-\tau_{ik})=j\}})$.

Taking into account these points, after some calculations, we prove the following:

Proposition 1 *The matrix* $\widehat{\widetilde{\mathbf{F}}}(z,s)$ *of the double transform of accumulations is a solution of the Master Equation*

$$\widehat{\widetilde{\mathbf{F}}}(z,s) = \mathbf{L}^{-1}(s-\psi(z))\Big(\mathbf{I} - \widetilde{\mathbf{W}}(s-\psi(z))\Big) + \widetilde{\mathbf{K}}(s-\psi(z)) \circ \widehat{\mathbf{\Delta}}[z]\widehat{\widetilde{\mathbf{F}}}(z,s), \tag{4}$$

where $\mathbf{L}^{-1}(s) = s^{-1}\mathbf{I}$ *and* $\mathbf{I}$ *is the identity matrix. The argument* $s - \psi(z)$ *means that the* (i, j) *element of each matrix is evaluated at* $s - \psi_i(z)$.

In case the waiting times are exponential with rates $\lambda_i > 0$, $i \in \mathcal{E}$, the Master Equation (4) can be further simplified. In particular, let $\mathbf{\Lambda} = \mathrm{diag}(\lambda_i)_{i\in\mathcal{E}}$, $\boldsymbol{Q}$ the infinitesimal generator of the underlying continuous-time Markov process on $\mathcal{E}$, and $\mathbf{U}$ the matrix with all elements equal to 1. By using the embedding formula $\boldsymbol{Q} = \mathbf{\Lambda}(\mathbf{P} - \mathbf{I})$, we derive:

$$\Big(s\mathbf{I} - \mathrm{diag}(\psi_i(z))_{i\in\mathcal{E}} - \boldsymbol{Q} \circ \widehat{\mathbf{\Delta}}[z] + \mathbf{\Lambda} \circ (\mathbf{U} - \widehat{\mathbf{\Delta}}[z])\Big)\widehat{\widetilde{\mathbf{F}}}(z,s) = \mathbf{I}. \tag{5}$$

Observe that the terms $\boldsymbol{Q}$ and $\mathbf{\Lambda}$ capture the evolution of the underlying process, whereas $\mathrm{diag}(\psi_i(z))_{i\in\mathcal{E}}$ and $\widehat{\mathbf{\Delta}}[z]$ capture the accumulation phenomena. Since the matrices in the left-hand side of Equation (5) depend only on z (except for $s\mathbf{I}$), Laplace inversion with respect to s returns the next result.

Lemma 1 *The kernel of a SMAP with exponential waiting times has a matrix-exponential form* $\widehat{\boldsymbol{F}}_t[z] = e^{tA[z]}$ *with*

$$A[z] = \mathrm{diag}(\psi_i(z))_{i\in\mathcal{E}} + \boldsymbol{Q} \circ \widehat{\mathbf{\Delta}}[z] - \mathbf{\Lambda} \circ (\mathbf{U} - \widehat{\mathbf{\Delta}}[z]). \tag{6}$$

Remark 1 *The matrix* $A[z]$ *is similar to the matrix exponent of a MAP, i.e.* $\Psi[z] = \mathrm{diag}(\psi_i(z))_{i\in\mathcal{E}} + \boldsymbol{Q} \circ \widehat{\mathbf{\Delta}}[z]$. *The difference comes from the fact that the diagonal elements of* $\widehat{\mathbf{\Delta}}[z]$ *are identically equal to 1 for MAPs.*

Finally, the kth moment of the SMAP can be evaluated by differentiating k times the Master Equation (4) with respect to z and evaluating it at $z = 0$. The final expression for the kth moment of the SMAP involves moments of the Lévy processes and the jumps, as well as derivatives of the matrices $\widetilde{\mathbf{W}}(s)$ and $\widetilde{\mathbf{K}}(s)$, all up to the order k.

APPLICATION

In this section, we specialize our results to a particular instance of accumulation. Our application is inspired by fluid embedding [8], a technique that is commonly used in MAPs to eliminate phase-type jumps of one direction and resort to the theory of MAPs with one-sided jumps. More precisely, as shown in Figure 1, if the jumps of one direction are phase-type, they can be replaced by linear stretches of slope one. If $\mathcal{E}$ is the state-space of the original MAP, the auxiliary MAP (after fluid embedding) has an augmented state-space $\mathcal{E}' = \mathcal{E} \cup \{1, \dots, m\}$, where m is the number of phases of the phase-type jumps. Thus, for an arbitrary fixed time τ in the original model, it holds that $\tau = \int_0^{\tau'} \mathbb{1}_{\{J(t)\in\mathcal{E}\}}dt$, where τ' is the corresponding time of τ in the auxiliary model. In other words, τ is the occupation time of the fluid embedded MAP in the states $\mathcal{E}$.

In many problems, it is important to retrieve the original time; a classical example from risk theory is the Gerber-Shiu function, which is expressed in terms of the ruin time (first passage time to negative values). Therefore, we explain here how to model occupation times as SMAPs and we derive exact solutions for the one-dimensional case.

First, let us consider a two-sided Lévy process X_t where the positive jumps are of phase-type with representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ and arrive according to a Poisson process with rate λ. By applying fluid embedding to the phase-type jumps, we generate a spectrally negative MAP (i.e. with only negative jumps) with state-space $\mathcal{E}' = \{0, 1, \dots, m\}$ (0 is the only state of X_t). To evaluate the occupation time in state 0 of this augmented MAP during an interval $[0, t]$, we must define $S(t) = \int_0^t \mathbb{1}_{\{J(u)=0\}}du$. Moreover, $\psi_i(z) = z\delta_{0i}$, $i \in \mathcal{E}'$, $\widehat{\mathbf{\Delta}}[z] = \mathbf{U}$, and $\boldsymbol{Q}$ is the block-matrix $\begin{pmatrix} -\lambda & \lambda\boldsymbol{\alpha} \\ \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}$, where δ_{ij} is the Kronecker delta, $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{e}$, and $\boldsymbol{e}$ is the column vector with all elements equal to one. Therefore, from Equation (5), we obtain

$$\widehat{\widetilde{\mathbf{F}}}(z,s) = \Big(s\mathbf{I} - \mathrm{diag}(z, 0, \dots, 0) - \boldsymbol{Q}\Big)^{-1}. \tag{7}$$

Obviously, we only care about the $(1, 1)$ element of the matrix $\widehat{\widetilde{\mathbf{F}}}(z,s)$, since the only state that has an actual meaning in the original model is the 0 one. By using properties of determinants, it is easy to verify that

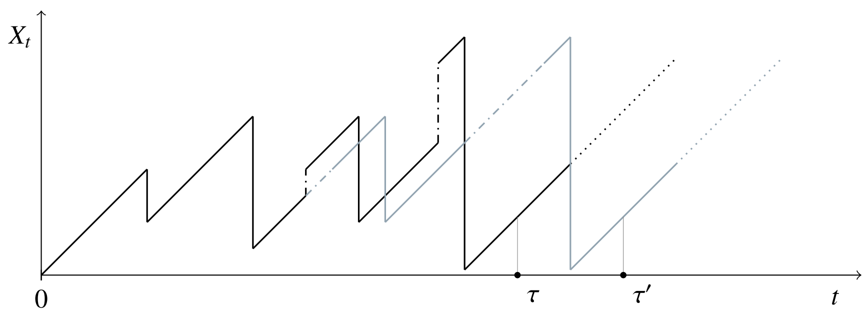

FIGURE 1. Sample path of a MAP with two-sided jumps (black line), where the positive phase-type jumps (dashed-dotted line) are replaced by linear stretches of slope 1, resulting in a spectrally-negative MAP with an augmented state-space (gray line).

$$\widehat{\widehat{F}}_{11}(z,s) = \frac{\det(s\mathbf{I}-\boldsymbol{T})}{\det(s\mathbf{I}-\boldsymbol{Q}) - z\det(s\mathbf{I}-\boldsymbol{T})} = \frac{\det(s\mathbf{I}-\boldsymbol{T})/\det(s\mathbf{I}-\boldsymbol{Q})}{1 - z\det(s\mathbf{I}-\boldsymbol{T})/\det(s\mathbf{I}-\boldsymbol{Q})} = \sum_{n=0}^{\infty} z^n \left(\frac{\det(s\mathbf{I}-\boldsymbol{T})}{\det(s\mathbf{I}-\boldsymbol{Q})}\right)^{n+1}. \tag{8}$$

Observe that $\det(s\mathbf{I}-\boldsymbol{T})/\det(s\mathbf{I}-\boldsymbol{Q})$ is a rational function with respect to s. Its denominator has degree $m+1$ and a pole at 0 (all other poles are negative), while its numerator is a polynomial of degree m with all roots negative. Thus, this rational function corresponds to a phase-type distribution that has an atom at infinity; let $\overline{F^{*n}}(t)$ be the nth convolution of its tail distribution. Laplace inversion with respect to s gives $\widehat{F}_t^{1,1}[z]$ as a series expansion with respect to z involving the convolutions $\overline{F^{*n}}(t)$. An alternative way is to use perturbation analysis to find a first order approximation with respect to z for the roots of $\det(s\mathbf{I}-\boldsymbol{Q}) - z\det(s\mathbf{I}-\boldsymbol{T})$. Assuming all roots $\rho_j + zc_j$, $j = 0,\ldots,m$, have multiplicity 1, Laplace inversion with respect to s will lead to a mixture of exponential terms $e^{-t(\rho_j+zc_j)}$. Further Laplace inversion of the latter terms with respect to z concerns time-shifted expressions of the Dirac measure.

CONCLUSIONS

To sum up, the derived Master Equation can be utilized to evaluate numerically performance measures related to SMAPs. Even if analytical Laplace inversion is impossible for either of the parameters s and z, we can use numerical Laplace inversion instead in order to recover the distribution of accumulation. Finally, we can potentially use the Master Equation to examine also qualitative characteristics of SMAPs.

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