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1     **THE ASYMPTOTIC OF TRANSMISSION EIGENVALUES FOR A DOMAIN**  
2                                     **WITH A THIN COATING**

3                                     H. BOUJLIDA <sup>\*</sup>, H. HADDAR<sup>†</sup>, AND M. KHENISSI <sup>‡</sup>

4     **Abstract.** We consider the transmission eigenvalue problem for a medium surrounded by a thin layer of  
5 inhomogeneous material with different refractive index. We derive explicit asymptotic expansion for the transmission  
6 eigenvalues with respect to the thickness of the thin layer. We prove error estimate for the asymptotic expansion  
7 up to order 1. This expansion can be used to obtain explicit expressions for constant index of refraction.

8     **Key words.** transmission eigenvalues, asymptotic expansions, thin layers, inverse scattering problems

9     **AMS subject classifications.** 35P30, 35P25

10     **1. Introduction.** This work is a contribution to the study of transmission eigenvalues [11, 4,  
11 6] and their relation to the shape and material properties of scatterers. These special frequencies  
12 are associated with the existence of an incident field that does not scatter. They can be equiva-  
13 lently defined as the eigenvalues of a system of two coupled partial differential equations posed on  
14 the inclusion domain. One of these equations refers to the equation satisfied by the total field and  
15 the other one is satisfied by the incident field. The two equations are coupled on the boundary  
16 by imposing that the Cauchy data coincide. This eigenvalue problem can then be formulated as  
17 an eigenvalue problem for a non-selfadjoint compact operator. Although non intuitive, it can be  
18 shown that this problem admits an infinite discrete set of real eigenvalues without finite accumu-  
19 lation points [7, 26]. These special frequencies can be identified from far field data as proved in  
20 [5, 19, 4]. Since they carry information on the material properties of the scatterer, transmission  
21 eigenvalues would then be of interest for the inverse problem of retrieving qualitative information  
22 on the material properties from measured multistatic data [14, 15]. In this perspective, it appears  
23 important to study the dependence of these eigenfrequencies with respect to the material prop-  
24 erties and the geometry. Several works in the literature have addressed this issue by considering  
25 asymptotic regimes and quantifying the dependence of the first leading terms in the asymptotic  
26 expansion of the transmission eigenvalue with respect to the small parameter [10, 8, 21, 16]. We  
27 here consider the case of a scatterer made of a thin coating which corresponds to frequently en-  
28 countered configurations in the stealth technology for instance. The goal is to characterize the  
29 dependence of the first order term on the material properties and the thickness of the coating. A  
30 first work on this topic was done in [10] where the case of coated perfect scatterer is considered.  
31 One proves in particular for the latter case that the first order term depends only on the thickness.  
32 We here address the more complicated configuration of a coated penetrable media. The analysis  
33 indicates that the first order asymptotic resembles to the shape derivative for the buckling plate  
34 equation [17] and contain non trivial dependence on the material properties. More importantly,  
35 this expansion allows us to obtain explicit (approximate) expressions for the thin layer index of  
36 refraction in terms of the thickness of the layer, the transmission eigenvalue for the coated medium  
37 that can be extracted from the measurements and the transmission eigenvalues and eigenvectors  
38 for the coated free medium that can be evaluated numerically. This indeed can be useful for the  
39 solution of the inverse problem.

40     Although the formal derivation follows the systematic procedure using the classical scaled  
41 expansion method (as in [3, 2, 13] for instance), the rigorous justification is much more involved.  
42 For instance the arguments in [10] are hard to extend to the present case since special uniform  
43 estimates have to be obtained for the transmission problem. We restrict ourselves here to the jus-  
44 tification of the first two terms in the asymptotic expansion using the abstract theory developed  
45 in [23, 21]. We follow the procedure developed in [8] for the case of small obstacles asymptotic.  
46 The main technical point in the proof is to obtain the corrector for the main operator, which is

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47 here the biharmonic operator. Our main result provides explicit expansion for simple transmis-  
 48 sion eigenvalues and for multiple transmission eigenvalues that are associated with a generalized  
 49 eigenspace spanned only by eigenvectors.

50 We analyze the problem where the contrast in material properties affect only the lower order  
 51 term in the Helmholtz equation. We finally indicate that although the problem is considered  
 52 only in dimension 2, the results of the main theorem (including the expression of the first order  
 53 asymptotic term) remain true for three dimensions (up to more complicated technicalities in the  
 54 proof related to differential geometry).

55 The paper is organized as follows. We first introduce the transmission eigenvalues and write  
 56 them as the eigenvalues of a non selfadjoint operator. We then present the main result of our paper  
 57 and discuss applications to the inverse problem. We present next the outline of a classical formal  
 58 procedure to obtain the expression of the asymptotic expansion. We give the expression till the  
 59 second order term. We explain in particular why the expression of the second order term would  
 60 have less interest in practice. We then proceed with the main part of the paper that provides  
 61 explicit expressions and an error estimate for the first two terms in the asymptotic expansion.

62 **2. Problem statement and main results.** Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with a  
 63 smooth boundary  $\Gamma$ . We denote by

$$64 \quad \Omega_\epsilon^0 = \{x \in \Omega, \quad d(x, \Gamma) > \epsilon\}$$

65 and its boundary

$$66 \quad \Gamma_\epsilon = \{x \in \Omega, \quad d(x, \Gamma) = \epsilon\} = \partial\Omega_\epsilon^0,$$

67 for  $\epsilon > 0$  a small enough parameter, where  $d(x, \Gamma)$  denotes the distance of a point  $x$  to the boundary  
 $\Gamma$ . Let  $\Omega_\epsilon = \Omega \setminus \overline{\Omega_\epsilon^0}$  be the layer of thickness  $\epsilon$  around  $\Omega_\epsilon^0$  (see Figure 1).

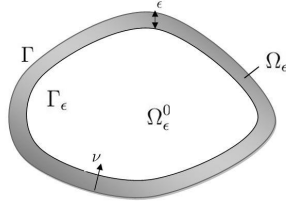


FIG. 1. *Stretch of the geometry*

68

69

We consider the following transmission eigenvalue problem:

$$70 \quad (1) \quad \begin{cases} \Delta w_\epsilon + k_\epsilon^2 n_\epsilon(x) w_\epsilon = 0 & \text{in } \Omega, \\ \Delta v_\epsilon + k_\epsilon^2 v_\epsilon = 0 & \text{in } \Omega, \\ w_\epsilon = v_\epsilon & \text{on } \Gamma, \\ \frac{\partial w_\epsilon}{\partial \nu} = \frac{\partial v_\epsilon}{\partial \nu} & \text{on } \Gamma, \end{cases}$$

71 where  $k_\epsilon$  denotes the unknown eigenfrequency and  $\nu$  the unitary normal to  $\Gamma$  directed to the  
 72 interior of  $\Omega$ . The index of refraction  $n_\epsilon$  is defined as follows:

$$73 \quad n_\epsilon(x) = \begin{cases} n_0(x) & \text{in } \Omega_\epsilon^0, \\ n_1(x) & \text{in } \Omega_\epsilon, \end{cases}$$

74 where  $n_0$  and  $n_1$  are non negative real valued functions  $\in L^\infty(\mathbb{R}^2)$  that are independent from  $\epsilon$ .  
 75 For the sake of simplicity, we assume that the restriction of  $n_0$  and  $n_1$  to  $\Omega_\epsilon$  are constant functions  
 76 along the normal coordinate to  $\Gamma$  for  $\epsilon$  sufficiently small. We finally assume that the function  
 77  $1/(1 - n_\epsilon)$  is either positive definite or negative definite on  $\Omega$ . We remark that this assumption  
 78 also implies that  $1/(1 - n_0)$  is either positive definite or negative definite on  $\Omega$  and that

$$79 \quad (2) \quad 1/|1 - n_\epsilon(x)| \geq \gamma > 0 \quad \text{for a.e. } x \in \Omega$$

with  $\gamma$  being independent from (sufficiently small)  $\epsilon$ .

The main goal of this paper is to find the asymptotic expansion of eigenfrequencies  $k_\epsilon$  with respect to  $\epsilon$ . Assuming that  $\frac{1}{1-n_\epsilon} \in L^\infty(\Omega)$ , the transmission eigenvalue problem (1) can be reformulated as the nonlinear eigenvalue problem for  $\lambda_\epsilon := k_\epsilon^2 \in \mathbb{R}$  and  $u_\epsilon := w_\epsilon - v_\epsilon \in H_0^2(\Omega)$  such that

$$(\Delta + \lambda_\epsilon n_\epsilon) \frac{1}{1-n_\epsilon} (\Delta + \lambda_\epsilon) u_\epsilon = 0 \quad \text{in } \Omega,$$

80 which in variational form, after integration by parts, is formulated as finding  $\lambda_\epsilon \in \mathbb{R}$  and non-trivial  
81 function  $u_\epsilon \in H_0^2(\Omega)$  such that

$$82 \quad (3) \quad \int_{\Omega} \frac{1}{1-n_\epsilon} (\Delta u_\epsilon + \lambda_\epsilon u_\epsilon) (\Delta \phi + \lambda_\epsilon n_\epsilon \phi) dx = 0, \quad \forall \phi \in H_0^2(\Omega).$$

83 The space  $H_0^2(\Omega)$  denotes the closure in  $H^2(\Omega)$  of the set of regular compactly supported functions  
84 in  $\Omega$ . We shall work with the reformulation of (3) as a linear eigenvalue problem for a non  
85 selfadjoint compact operator [4]. First observe that (3) can be written as

$$86 \quad (4) \quad A_\epsilon u_\epsilon + \lambda_\epsilon B_\epsilon u_\epsilon + \lambda_\epsilon^2 C_\epsilon u_\epsilon = 0 \quad \text{in } H_0^2(\Omega)$$

87 where

$$88 \quad A_\epsilon : H_0^2(\Omega) \rightarrow H_0^2(\Omega), \quad B_\epsilon : H_0^2(\Omega) \rightarrow H_0^2(\Omega), \quad C_\epsilon : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$$

89 are defined by the Riesz representation theorem as

$$90 \quad (5) \quad (A_\epsilon u_\epsilon, \phi)_{H_0^2(\Omega)} := \int_{\Omega} \frac{1}{1-n_\epsilon} \Delta u_\epsilon \Delta \phi dx,$$

91

$$92 \quad (6) \quad (B_\epsilon u_\epsilon, \phi)_{H_0^2(\Omega)} := \int_{\Omega} \frac{1}{1-n_\epsilon} (u_\epsilon \Delta \phi + n_\epsilon \Delta u_\epsilon \phi) dx,$$

93 and

$$94 \quad (7) \quad (C_\epsilon u_\epsilon, \phi)_{H_0^2(\Omega)} := \int_{\Omega} \frac{n_\epsilon}{1-n_\epsilon} u_\epsilon \phi dx.$$

95 Note that  $A_\epsilon : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$  is a bounded, self-adjoint and invertible operator (thanks to (2)),  
96  $B_\epsilon : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$  is a bounded, compact and self-adjoint operator and  $C_\epsilon : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$   
97 is a (non negative or non positive) bounded, compact and self-adjoint operator. Observe that  
98 since  $A_\epsilon$  is invertible,  $\lambda_\epsilon \neq 0$ . In order to avoid distinguishing the cases of  $1-n_\epsilon$  being positive or  
99 negative we shall abusively set  $C_\epsilon^{\frac{1}{2}} \equiv -(-C_\epsilon^{\frac{1}{2}})$  in the case where  $1-n_\epsilon$  non positive.

100 Setting  $U_\epsilon = (u_\epsilon, \lambda_\epsilon C_\epsilon^{\frac{1}{2}} u_\epsilon)$ , the transmission eigenvalue problem (4) can be transformed into  
101 the linear eigenvalue problem,  $\tau_\epsilon \in \mathbb{R}$ ,  $U_\epsilon \in H_0^2(\Omega) \times H_0^2(\Omega)$  such that

$$102 \quad (8) \quad (\mathcal{T}_\epsilon - \tau_\epsilon I) U_\epsilon = 0, \quad \text{with} \quad \tau_\epsilon = \frac{1}{\lambda_\epsilon},$$

103 for the compact non-selfadjoint operator  $\mathcal{T}_\epsilon : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow H_0^2(\Omega) \times H_0^2(\Omega)$  defined by

$$104 \quad (9) \quad \mathcal{T}_\epsilon = \begin{pmatrix} -A_\epsilon^{-1} B_\epsilon & -A_\epsilon^{-1} C_\epsilon^{\frac{1}{2}} \\ C_\epsilon^{\frac{1}{2}} & 0 \end{pmatrix}.$$

105 We set

$$106 \quad (10) \quad \mathcal{T}_0 = \begin{pmatrix} -A_0^{-1} B_0 & -A_0^{-1} C_0^{\frac{1}{2}} \\ C_0^{\frac{1}{2}} & 0 \end{pmatrix}$$

107 where  $A_0, B_0$  and  $C_0$  are defined by (5), (6) and (7) respectively for  $n_\epsilon = n_0$  in  $\Omega$ . We state here  
 108 the main result of this paper which will be proven in Section 4. In the following a transmission  
 109 eigenvalue  $\lambda_0$  is called simple if the corresponding  $\tau_0 = 1/\lambda_0$  has an algebraic multiplicity equal  
 110 to 1. We refer to Theorem 4.11 for the case where  $\lambda_0$  has an associated eigenspace formed  
 111 only by eigenvectors (and therefore an algebraic multiplicity that coincides with the geometrical  
 112 multiplicity).

THEOREM 2.1. *Assume that  $n_0, n_1 \in C^4(\overline{\Omega})$ . Let  $\lambda_0 \in \mathbb{R}$  be a simple transmission eigenvalue  
 of (3) with  $n_\epsilon = n_0$  in  $\Omega$  and let  $u_0 \in H_0^2(\Omega)$  be an associated eigenfunction. This implies in  
 particular that*

$$\beta_0 := \int_{\Omega} \frac{1}{1-n_0} \left( \lambda_0^2 n_0 |u_0|^2 - |\Delta u_0|^2 \right) dx \neq 0.$$

113 *If we suppose in addition that  $u_0$  and  $A_0^{-1}u_0$  are in  $C^6(\overline{\Omega})$ , then, for sufficiently small  $\epsilon > 0$ , there  
 114 exists a transmission eigenvalue  $\lambda_\epsilon$  of (3) such that*

$$115 \quad \lambda_\epsilon = \lambda_0 + \epsilon \lambda_1 + O(\epsilon^{\frac{3}{2}})$$

116 *where  $\lambda_1$  is given by the following expression*

$$117 \quad \lambda_1 := \frac{\lambda_0}{\beta_0} \int_{\Gamma} \frac{n_0 - n_1}{(1-n_0)^2} |\Delta u_0|^2 ds(x).$$

118 This theorem is an immediate consequence of Theorem 4.8 that is stated and proven in the  
 119 last section of this paper.

120 The formal calculations in Section 3 show that the formula for  $\lambda_1$  is generically valid whenever  
 121  $\beta_0 \neq 0$ . However, we remark that in the case of transmission eigenvalues with multiplicity greater  
 122 than 1, this is not automatically ensured (See Theorem 4.11 for a rigorous expression of  $\lambda_1$  that  
 123 involves all eigenvectors associated with  $\lambda_0$ ).

124 From the practical point of view, this theorem implies in particular that  $\lambda_1$  gives a measure for  
 125 the contrast  $n_0 - n_1$ . For instance, if  $n_1$  is constant and  $n_0$  is constant on  $\Gamma$ , one can approximate  
 126 the value of  $n_1$  using the identity

$$127 \quad (11) \quad n_1|_{\Gamma} = n_0|_{\Gamma} - \frac{\lambda_\epsilon - \lambda_0}{\epsilon \alpha_0} \int_{\Omega} \frac{1}{1-n_0} \left( \lambda_0 n_0 |u_0|^2 - \frac{1}{\lambda_0} |\Delta u_0|^2 \right) dx + O(\epsilon^{\frac{1}{2}})$$

with

$$\alpha_0 := \int_{\Gamma} \frac{|\Delta u_0|^2}{(1-n_0)^2} ds(x).$$

128 For the inverse problem where one would like to determine  $n_1$  from multistatic measurements of  
 129 scattered waves, the value of  $\lambda_\epsilon$  can be approximated using sampling methods as in [5, 4] (see  
 130 also [19] for an alternative approach). The values of  $\lambda_0$  and  $u_0$  can be computed numerically if  
 131 one has a priori knowledge of  $n_0$  and  $\Omega$  (see for instance [12, 18, 20] for numerical methods to  
 132 approximate  $\lambda_0$  and  $u_0$ ). We finally indicate that, although not carefully checked, we conjecture  
 133 that the expression for  $\lambda_1$  remains true in three dimensions (corrections due to the curvature of  $\Gamma$   
 134 only affect higher order terms).

135 **3. Formal asymptotic expansion.** In this section, we derive the formal asymptotic ex-  
 136 pansion for transmission eigenvalues and give explicit formulas for the terms up to order 2. The  
 137 idea here is to provide a systematic formal way to quickly obtain the explicit expression of  $\lambda_1$  in  
 138 Theorem 2.1 and also higher order terms. The latter turn out to have complicated expressions  
 139 that would be of marginal interest for the solution of the inverse problem mentioned above. This  
 140 formal stage will also be helpful in establishing the rigorous proof based of Osborn's theorem [23].  
 141 It allows one to have an intuition for the expression of the corrector in the asymptotic of the main  
 142 operator  $A_\epsilon$ .

143 We assume the following expansions for the transmission eigenvalues :

$$144 \quad (12) \quad \lambda_\epsilon = \sum_{j=0}^{\infty} \epsilon^j \lambda_j,$$

145 and then follow a classical technique for thin layers asymptotics based on rescaling and asymptotic  
146 expansion with respect to the thickness  $\epsilon$ . We shall mainly follow the approach in [10].

147 **3.1. Scaling.** We assume that the boundary  $\Gamma$  is  $C^\infty$ -smooth, although much less regularity  
148 is needed if we restrict ourselves to only few terms in the expansion. The issue of optimal regularity  
149 assumptions for  $\Gamma$  is not discussed here. However, one can check that at least a  $C^2$  regularity is  
150 needed to get the expression of  $\lambda_1$ . We parametrize  $\Gamma$  as

$$151 \quad \Gamma = \{x_\Gamma(s), s \in [0, L]\},$$

152 with  $L$  being the length of  $\Gamma$  and  $s$  is the curvilinear abscissa. At the point  $x_\Gamma(s)$ , the unit tangent  
153 vector is  $\tau(s) := \frac{dx_\Gamma(s)}{ds}$ , the curvature  $\kappa(s)$  is defined by:

$$154 \quad \frac{d\tau(s)}{ds} = -\kappa(s)\nu(s) \quad \text{or} \quad \frac{d\nu(s)}{ds} = \kappa(s)\tau(s).$$

155 Within these notations, the boundary of  $\Omega_\epsilon^0$  is parametrized as

$$156 \quad \Gamma_\epsilon = \{x_\Gamma(s) + \epsilon\nu(s), s \in [0, L]\}.$$

157 This parametrization of the surface  $\Gamma_\epsilon$  is equivalent to the definition of  $\Gamma_\epsilon$ , for  $\epsilon > 0$  a small enough  
158 parameter.

159 For a function  $u$  defined in  $\Omega_\epsilon$ , we consider  $\tilde{u}$  defined on  $[0, L[\times]0, \epsilon[$  by

$$160 \quad (13) \quad \tilde{u}(s, \eta) = u(\varphi(s, \eta)) \quad \text{where} \quad \varphi(s, \eta) := x_\Gamma(s) + \eta\nu(s).$$

161 Then, the gradient and Laplace operators are expressed in the local coordinates as:

$$162 \quad \nabla u = \left( \frac{1}{(1 + \eta\kappa(s))} \frac{\partial}{\partial s} \tau(s) + \frac{\partial}{\partial \eta} \nu(s) \right) \tilde{u},$$

$$163 \quad (14) \quad \Delta u = \left( \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial s} \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial s} + \frac{\kappa}{(1 + \eta\kappa)} \frac{\partial}{\partial \eta} + \frac{\partial^2}{\partial \eta^2} \right) \tilde{u}.$$

164 To make the formal calculations, we need to separate the thin layer and scaled it with respect  
165 to the thickness so that the equation are posed on a domain independent from  $\epsilon$ . We therefore  
166 rewrite the transmission eigenvalue problem (1) in the following equivalent form

$$167 \quad (15) \quad \begin{cases} \Delta w_\epsilon^+ + k_\epsilon^2 n_1 w_\epsilon^+ = 0 & \text{in} & \Omega_\epsilon, \\ \Delta w_\epsilon^- + k_\epsilon^2 n_0 w_\epsilon^- = 0 & \text{in} & \Omega_\epsilon^0, \\ \Delta v_\epsilon + k_\epsilon^2 v_\epsilon = 0 & \text{in} & \Omega, \\ w_\epsilon^+ = w_\epsilon^-, \quad \frac{\partial w_\epsilon^+}{\partial \nu} = \frac{\partial w_\epsilon^-}{\partial \nu} & \text{on} & \Gamma_\epsilon, \\ w_\epsilon^+ = v_\epsilon & \text{on} & \Gamma, \\ \frac{\partial w_\epsilon^+}{\partial \nu} = \frac{\partial v_\epsilon}{\partial \nu} & \text{on} & \Gamma. \end{cases}$$

168 We denote by  $\xi = \frac{\eta}{\epsilon}$  the stretched normal variable inside  $\Omega_\epsilon$  and define

$$169 \quad \varphi_\epsilon : \mathcal{G} = [0, L[\times]0, 1[ \rightarrow \Omega_\epsilon$$

$$170 \quad (s, \xi) \mapsto \varphi_\epsilon(s, \xi) = x_\Gamma(s) + \epsilon\xi\nu(s).$$

170 Then the expression of the Laplace operator in the scaled layer is:

$$171 \quad (16) \quad \Delta u = \left( \frac{1}{(1 + \xi\epsilon\kappa)} \frac{\partial}{\partial s} \frac{1}{(1 + \xi\epsilon\kappa)} \frac{\partial}{\partial s} + \frac{\kappa}{(1 + \xi\epsilon\kappa)} \frac{\partial}{\partial \xi} + \frac{1}{\epsilon^2} \frac{\partial^2}{\partial \xi^2} \right) \hat{u} =: \Delta_{s, \xi} \hat{u}$$

172 for  $\hat{u}(s, \xi) := u(\varphi_\epsilon(s, \xi))$ .

173 The next step is to write the equation for  $w_\epsilon^+$  in the scaled domain and solve for the asymptotic  
174 expansion of  $w_\epsilon^+$  in terms of the boundary values on  $\Gamma$ . These boundary values are given by the  
175 asymptotic expansion of  $v_\epsilon$ . More specifically, setting  $\hat{w}_\epsilon(s, \xi) := w_\epsilon^+(\varphi_\epsilon(s, \xi))$ , we have that

$$176 \quad (17) \quad \Delta_{s, \xi} \hat{w}_\epsilon + \lambda_\epsilon n_1 \hat{w}_\epsilon = 0 \quad \text{in} \quad \mathcal{G}$$

177 together with the boundary conditions

$$178 \quad (18) \quad \begin{cases} \hat{w}_\epsilon(s, 0) = v_\epsilon(x_\Gamma(s)) & s \in [0, L[, \\ \frac{1}{\epsilon} \frac{\partial \hat{w}_\epsilon}{\partial \xi}(s, 0) = \frac{\partial v_\epsilon}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{cases}$$

179 We assume that

$$180 \quad (19) \quad \hat{w}_\epsilon(s, \xi) = \sum_{j=0}^{\infty} \epsilon^j \hat{w}_j(s, \xi), \quad (s, \xi) \in \mathcal{G} \quad \text{and} \quad v_\epsilon(x) = \sum_{j=0}^{\infty} \epsilon^j v_j(x), \quad x \in \Omega$$

181 for some functions  $\hat{w}_j$  defined on  $\mathcal{G}$  and  $v_j$  defined on  $\Omega$  that are independent from  $\epsilon$ . Multiplying  
182 (17) by  $\epsilon^2(1 + \xi\epsilon\kappa)^3$  and using (12), we obtain

$$183 \quad \sum_{k=0}^5 \epsilon^k A_k \hat{w}_\epsilon = 0,$$

184 where  $(A_k)_{k=0\dots 5}$  are differential operators of order 2 at maximum with the following expressions  
185 for the first fourth terms:

$$186 \quad A_0 = \frac{\partial^2}{\partial \xi^2},$$

$$187 \quad A_1 = 3\xi\kappa \frac{\partial^2}{\partial \xi^2} + \kappa \frac{\partial}{\partial \xi},$$

$$188 \quad A_2 = \frac{\partial^2}{\partial s^2} + 3\xi^2\kappa^2 \frac{\partial^2}{\partial \xi^2} + 2\xi\kappa^2 \frac{\partial}{\partial \xi} + \lambda_0 n_1,$$

$$189 \quad A_3 = \xi^3\kappa^3 \frac{\partial^2}{\partial \xi^2} + \xi^2\kappa^3 \frac{\partial}{\partial \xi} - \xi \frac{\partial \kappa}{\partial s} \frac{\partial}{\partial s} + \xi\kappa \frac{\partial^2}{\partial s^2} + 3\lambda_0 n_1 \xi\kappa + \lambda_1 n_1.$$

191 Inserting the ansatz (19) in (17) and (18) we obtain after equating the terms of same order in  $\epsilon$   
192 and using the convention  $\hat{w}_j = v_j = 0$  for  $j < 0$ ,

$$193 \quad (20) \quad \begin{cases} \frac{\partial^2}{\partial \xi^2} \hat{w}_j = - \sum_{k=1}^5 A_k \hat{w}_{j-k} & \text{in } \mathcal{G}, \\ \hat{w}_j(s, 0) = v_j(x_\Gamma(s)) & s \in [0, L[, \\ \frac{\partial \hat{w}_j}{\partial \xi}(s, 0) = \frac{\partial v_{j-1}}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{cases}$$

194 These equations can be solved inductively to get the expressions of  $\hat{w}_j$  in terms of the boundary  
195 values of  $v_l$ ,  $l \leq j$ . One gets for  $j = 0, 1, 2$  and 3

$$196 \quad \hat{w}_0(s, \xi) = v_0(x_\Gamma(s)),$$

$$197 \quad \hat{w}_1(s, \xi) = \frac{\partial v_0}{\partial \nu}(x_\Gamma(s))\xi + v_1(x_\Gamma(s)),$$

(21)

$$198 \quad \hat{w}_2(s, \xi) = -\frac{\xi^2}{2} \left( \kappa \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial^2 w_0^-}{\partial s^2}(x_\Gamma(s)) + \lambda_0 n_1 w_0^-(x_\Gamma(s)) \right) + \frac{\partial v_1}{\partial \nu}(x_\Gamma(s))\xi + v_2(x_\Gamma(s)),$$

200 and

$$201 \quad \hat{w}_3(s, \xi) = \frac{\xi^3}{6} \left( -2\kappa^2 \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) - 3\kappa \frac{\partial^2 w_0^-}{\partial s^2}(x_\Gamma(s)) - \kappa \lambda_0 n_1 w_0^-(x_\Gamma(s)) + \lambda_0 n_1 \frac{\partial v_0}{\partial \nu}(x_\Gamma(s)) \right)$$

$$202 \quad + \frac{\xi^3}{6} \left( \frac{\partial^3 v_0}{\partial s^2 \partial \nu}(x_\Gamma(s)) - \kappa \frac{\partial w_0^-}{\partial s}(x_\Gamma(s)) \right)$$

(22)

$$203 \quad + \frac{\xi^2}{2} \left( \kappa \frac{\partial v_1}{\partial \nu}(x_\Gamma(s)) + \lambda_0 n_1 v_1(x_\Gamma(s)) + \lambda_1 n_1 w_0^-(x_\Gamma(s)) \right) + \frac{\partial v_2}{\partial \nu}(x_\Gamma(s))\xi + v_3(x_\Gamma(s)).$$

204

205 Now, we also postulate the following expansion for  $w_\epsilon^-$ :

$$206 \quad (23) \quad w_\epsilon^-(x) = \sum_{j=0}^{\infty} \epsilon^j w_j^-(x)$$

207 with  $w_j^- : \Omega \rightarrow \mathbb{R}$  are functions independent of  $\epsilon$ . Then  $(w_j^-, v_j)$  solves

$$208 \quad (24) \quad \begin{cases} \Delta w_j^- + \lambda_0 n_0 w_j^- = - \sum_{l=1}^j \lambda_l n_0 w_{j-l}^- & \text{in } \Omega, \\ \Delta v_j + \lambda_0 v_j = - \sum_{l=1}^j \lambda_l v_{j-l} & \text{in } \Omega. \end{cases}$$

209 Note that the functions  $w_j^-$  are defined in all  $\Omega$  and not only  $\Omega_\epsilon^0$  and therefore (23) gives a extension  
210 of  $w_\epsilon^-$  to all  $\Omega$ . The continuity conditions at  $\Gamma$  can be written as

$$211 \quad \tilde{w}_\epsilon^-(s, \epsilon) = \hat{w}_\epsilon(s, 1) \text{ and } \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, \epsilon) = \frac{1}{\epsilon} \frac{\partial \hat{w}_\epsilon}{\partial \xi}(s, 1)$$

212 where  $\tilde{w}_\epsilon^-$  is defined from  $w_\epsilon^-$  using the local change of variables (13) in a neighborhood of  $\Gamma$ .  
213 Using Taylor's expansion (up to the second order, which is sufficient to compute the first three  
214 terms in the asymptotic expansion) we get

$$215 \quad (25) \quad \tilde{w}_\epsilon^-(s, \epsilon) = \tilde{w}_\epsilon^-(s, 0) + \epsilon \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, 0) + \frac{\epsilon^2}{2} \frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s, 0) + o(\epsilon^2) = \hat{w}_\epsilon(s, 1)$$

216 and

$$217 \quad (26) \quad \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, \epsilon) = \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, 0) + \epsilon \frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s, 0) + \frac{\epsilon^2}{2} \frac{\partial^3 \tilde{w}_\epsilon^-}{\partial \eta^3}(s, 0) + o(\epsilon^2) = \frac{1}{\epsilon} \frac{\partial \hat{w}_\epsilon}{\partial \xi}(s, 1).$$

218 Injecting (19) and (23) into (25) and (26), we respectively obtain the following continuity conditions  
219 on  $\Gamma$ ,

$$220 \quad w_0^-(x_\Gamma(s)) = \hat{w}_0(s, 1),$$

$$221 \quad (27) \quad w_1^-(x_\Gamma(s)) + \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) = \hat{w}_1(s, 1),$$

$$222 \quad w_2^-(x_\Gamma(s)) + \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{1}{2} \frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) = \hat{w}_2(s, 1),$$

224 and

$$225 \quad 0 = \frac{\partial \hat{w}_0}{\partial \xi}(s, 1),$$

$$226 \quad (28) \quad \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) = \frac{\partial \hat{w}_1}{\partial \xi}(s, 1),$$

$$227 \quad \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) = \frac{\partial \hat{w}_2}{\partial \xi}(s, 1).$$

229 System (24) coupled with the boundary conditions (28) and (27) provide an inductive way to  
230 determine  $(w_j^-, v_j)$ . We obtain the set of equations satisfied by these terms after substituting the  
231 expressions of  $\hat{w}_j(s, 1)$  given by (21),(22). We hereafter summarize the set of equations obtained  
232 for  $(w_j^-, v_j)$  and how to use them to get the expressions of  $\lambda_j$ ,  $j = 0, 1, 2$ .

233 We first obtain that the couple  $(w_0^-, v_0)$  solves

$$234 \quad (29) \quad \begin{cases} \Delta w_0^- + \lambda_0 n_0 w_0^- = 0 & \text{in } \Omega, \\ \Delta v_0 + \lambda_0 v_0 = 0 & \text{in } \Omega, \\ w_0^- - v_0 = 0 & \text{on } \Gamma, \\ \frac{\partial w_0^-}{\partial \nu} - \frac{\partial v_0}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$



235 This means in particular that  $\lambda_0$  is a transmission eigenvalue for the limiting problem where the  
236 thin layer is removed. We then obtain that the couple  $(w_1^-, v_1)$  satisfies

$$237 \quad (30) \quad \begin{cases} \Delta w_1^- + \lambda_0 n_0 w_1^- = -\lambda_1 n_0 w_0^- & \text{in } \Omega, \\ \Delta v_1 + \lambda_0 v_1 = -\lambda_1 v_0 & \text{in } \Omega, \\ w_1^- - v_1 = 0 & \text{on } \Gamma, \\ \frac{\partial w_1^-}{\partial \nu} - \frac{\partial v_1}{\partial \nu} = \lambda_0 (n_0 - n_1) w_0^- & \text{on } \Gamma. \end{cases}$$

Since  $\lambda_0$  is an eigenvalue of the associated homogeneous system, this problem is solvable only if a compatibility condition is satisfied by the right hand sides. This compatibility condition can be obtained by multiplying the first equation with  $\overline{w_0^-}$  and the second equation with  $\overline{v_0}$ , taking the difference then integrating by parts and using (29). One ends up with

$$\lambda_1 = \frac{\int_{\Gamma} \lambda_0 (n_0 - n_1) |w_0^-|^2 ds(x)}{\int_{\Omega} (n_0 |w_0^-|^2 - |v_0|^2) dx}$$

238 which coincides with the expression of given in Theorem 2.1 expressed in terms of  $u_0 = w_0^- - v_0$ .  
239 Although not covered by the analysis of convergence, we also provide the expression of the third  
240 term in the asymptotic expression. One get that the couple  $(w_2^-, v_2)$  solves

$$241 \quad (31) \quad \begin{cases} \Delta w_2^- + \lambda_0 n_0 w_2^- = -\lambda_1 n_0 w_1^- - \lambda_2 n_0 w_0^- & \text{in } \Omega, \\ \Delta v_2 + \lambda_0 v_2 = -\lambda_1 v_1 - \lambda_2 v_0 & \text{in } \Omega, \\ w_2^- - v_2 = h_1 & \text{on } \Gamma, \\ \frac{\partial w_2^-}{\partial \nu} - \frac{\partial v_2}{\partial \nu} = h_2 & \text{on } \Gamma, \end{cases}$$

242 where

$$243 \quad h_1 = -\frac{1}{2} \frac{\partial^2 w_0^-}{\partial \nu^2} - \frac{1}{2} \lambda_0 (n_0 - n_1) w_0^-$$

244 and

$$245 \quad h_2 = \kappa \frac{\partial^2 w_0^-}{\partial \nu^2} - \frac{7\kappa}{2} \frac{\partial^2 w_0^-}{\partial s^2} + \left(2\kappa^2 + \lambda_0 (n_0 + \frac{n_1}{2})\right) \frac{\partial w_0^-}{\partial \nu} - \frac{3\kappa}{2} \frac{\partial w_0^-}{\partial s} + \frac{3}{2} \frac{\partial^3 w_0^-}{\partial \nu \partial s^2}$$

$$246 \quad + \left(\lambda_1 (2n_1 - n_0) + \lambda_0 (\kappa (\frac{n_1}{2} - n_0))\right) w_0^- - \frac{\partial^2 w_1^-}{\partial \nu^2} + \kappa \frac{\partial w_1^-}{\partial \nu}.$$

248 Writing the compatibility condition for (31), we obtain the following formula for  $\lambda_2$

$$249 \quad \lambda_2 \int_{\Omega} \frac{1}{1-n_0} \left( \frac{1}{\lambda_0} |\Delta u_0|^2 - \lambda_0 n_0 |u_0|^2 \right) dx = -\lambda_1^2 \int_{\Omega} \left( \frac{1}{\lambda_0} \Delta u_0 \bar{u}_0 + \frac{1}{1-n_0} |u_0|^2 \right) dx$$

$$250 \quad - \lambda_1 \int_{\Omega} \frac{1}{1-n_0} \left( u_1 \Delta \bar{u}_0 + n_0 \Delta u_1 \bar{u}_0 + 2n_0 \lambda_0 u_1 \bar{u}_0 \right) dx$$

$$251 \quad (32) \quad + \int_{\Gamma} h_1 \frac{\partial}{\partial \nu} \left( \frac{1}{1-n_0} (\Delta + \lambda_0) \bar{u}_0 \right) ds(x) - \int_{\Gamma} h_2 \left( \frac{1}{1-n_0} (\Delta + \lambda_0) \bar{u}_0 \right) ds(x).$$

253 This complicated expression shows in particular a nonlinear dependence of  $\lambda_2$  in terms of  $n_1$ . It  
254 suggests that the use of  $\lambda_2$  for solutions to the inverse problem of determining  $n_1$  may not be  
255 appropriate.

256 **4. Convergence analysis.** The main goal of this section is to prove Theorem 2.1 that  
257 provides a rigorous mathematical justification of the formal asymptotic expansion for simple real  
258 transmission eigenvalues up to the first order. The proof is split into several steps. The first one is  
259 to establish the convergence in norm of the operator  $\mathcal{T}_\epsilon$  to  $\mathcal{T}_0$ . This ensures the convergence of  $\lambda_\epsilon$   
260 to  $\lambda_0$ . In order to get to the term of order 1 in  $\epsilon$ , we shall apply the Osborn theorem which requires

261 for instance a characterization of the pointwise asymptotic expansion of  $\mathcal{T}_\epsilon(U)$  up to order 1 in  $\epsilon$   
 262 (for some given function  $U \in H_0^2(\Omega) \times H_0^2(\Omega)$ ). The latter can be obtained from the asymptotic  
 263 expansions of  $A_\epsilon^{-1}u$ ,  $B_\epsilon u$  and  $C_\epsilon u$  for some  $u \in H_0^2(\Omega)$ . The difficult part to get the expansion of  
 264  $A_\epsilon^{-1}u$  since for the two others, the first order terms are vanishing. This critical result is provided  
 265 by Lemma 4.5.

266 In all the following we use the notation

$$267 \quad (f, g) := (f, g)_{H_0^2(\Omega)} = \int_{\Omega} \Delta f \Delta g dx \text{ and } \|g\| := (g, g)_{H_0^2(\Omega)}^{\frac{1}{2}}.$$

268 For an operator  $A : V \rightarrow V$ ,  $\|A\|$  denotes the operator norm. To simplify the writing,  $C$  will  
 269 denote a generic constant whose value may change but remains independent from  $\epsilon$  as  $\epsilon \rightarrow 0$ .

270 **4.1. Pointwise convergence of the spectrum of  $\mathcal{T}_\epsilon$ .** In this first step, we prove pointwise  
 271 convergence of the spectrum of the operator  $\mathcal{T}_\epsilon$  to the spectrum of  $\mathcal{T}_0$ . This is a direct consequence  
 272 of the following convergence in norm [23, 8].

273 **THEOREM 4.1.** *Assume that  $n_0 \in C^2(\bar{\Omega})$ . Let  $\mathcal{T}_\epsilon$  and  $\mathcal{T}_0$  be defined by (9) and (10) respectively.  
 274 Then  $\mathcal{T}_\epsilon$  converges to  $\mathcal{T}_0$  in the operator norm.*

275 *Proof.* The proof follows from Lemma 4.2 and Lemma 4.4 below, using the definition of  $\mathcal{T}_\epsilon$   
 276 and  $\mathcal{T}_0$ .  $\square$

277 In the first lemma we prove norm convergence for  $B_\epsilon$  and  $C_\epsilon$ .

278 **LEMMA 4.2.** *Let  $B_\epsilon$ ,  $C_\epsilon$ ,  $B_0$  and  $C_0$  be the operators defined by (6) and (7). Then, for suffi-  
 279 ciently small  $\epsilon$ ,*

$$280 \quad (33) \quad \|B_\epsilon - B_0\| \leq C\epsilon^{\frac{1}{2}} \text{ and } \|C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}\| \leq C\epsilon.$$

281 *Proof.* From the definitions of  $B_\epsilon$  and  $B_0$ , we have that for  $u, \phi \in H_0^2(\Omega)$

$$282 \quad \begin{aligned} ((B_\epsilon - B_0)u, \phi) &= \int_{\Omega} \frac{1}{1 - n_\epsilon} (u\Delta\phi + n_\epsilon\Delta u\phi) dx - \int_{\Omega} \frac{1}{1 - n_0} (u\Delta\phi + n_0\Delta u\phi) dx \\ 283 \quad &= \int_{\Omega_\epsilon} \left( \frac{1}{1 - n_1} - \frac{1}{1 - n_0} \right) (u\Delta\phi + \Delta u\phi) dx. \end{aligned}$$

285 Therefore,

$$286 \quad |((B_\epsilon - B_0)u, \phi)| \leq C \left( \|u\|_{L^\infty(\Omega)} \|\Delta\phi\|_{L^1(\Omega_\epsilon)} + \|\phi\|_{L^\infty(\Omega)} \|\Delta u\|_{L^1(\Omega_\epsilon)} \right).$$

288 Using the Sobolev embedding theorem and the Cauchy Schwartz inequality, we get

$$289 \quad |((B_\epsilon - B_0)u, \phi)| \leq C\epsilon^{\frac{1}{2}} (\|u\|_{H_0^2(\Omega)} \|\phi\|_{H_0^2(\Omega)}).$$

290 By choosing  $\phi = (B_\epsilon - B_0)u$ , we get

$$291 \quad \|(B_\epsilon - B_0)u\|_{H_0^2(\Omega)} \leq C\epsilon^{\frac{1}{2}} \|u\|_{H_0^2(\Omega)}.$$

292 The proof is similar for the second inequality. For  $u, \phi \in H_0^2(\Omega)$ , we have

$$293 \quad \begin{aligned} ((C_\epsilon - C_0)u, \phi) &= \int_{\Omega_\epsilon} \left( \frac{1}{1 - n_1} - \frac{1}{1 - n_0} \right) u\phi dx \leq C \left( |\Omega_\epsilon| \|u\|_{L^\infty(\Omega)} \|\phi\|_{L^\infty(\Omega)} \right) \end{aligned}$$

295 From the Sobolev embedding theorem, we obtain

$$296 \quad ((C_\epsilon - C_0)u, \phi) \leq C\epsilon \left( \|u\|_{H_0^2(\Omega)} \|\phi\|_{H_0^2(\Omega)} \right).$$

297 By choosing  $\phi = (C_\epsilon - C_0)u$ , we have

$$298 \quad (34) \quad \|(C_\epsilon - C_0)u\|_{H_0^2(\Omega)} \leq C\epsilon \|u\|_{H_0^2(\Omega)}.$$

299 Using the square root Lemma in [24] and the fact that  $C_\epsilon^n$  converges to  $C_0^n$  at the same order  
 300  $O(\epsilon)$ , we conclude that  $C_\epsilon^{\frac{1}{2}}$  converges to  $C_0^{\frac{1}{2}}$  at the same order  $O(\epsilon)$ . Hence we have

$$301 \quad (35) \quad \|(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u\|_{H_0^2(\Omega)} \leq C\epsilon \|u\|_{H_0^2(\Omega)}. \quad \square$$

302 Now we show convergence in the  $H_0^2(\Omega)$  norm for  $A_\epsilon^{-1}f$  assuming smoothness of  $f$ . This will be  
 303 useful in the proof of Lemma 4.4 since the operators  $B_\epsilon$  and  $C_\epsilon$  are regularizing.

304 **LEMMA 4.3.** *Let  $A_\epsilon$  and  $A_0$  be defined by (5) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively and  $f \in H_0^2(\Omega)$ .  
 305 If  $A_0^{-1}f \in \mathcal{C}^2(\overline{\Omega})$ , then for sufficiently small  $\epsilon$ ,*

$$306 \quad (36) \quad \|A_\epsilon^{-1}f - A_0^{-1}f\| \leq C\epsilon^{\frac{1}{2}}.$$

307 *Proof.* For a fixed  $f \in H_0^2(\Omega)$ , define  $z_\epsilon$  and  $z_0$  in  $H_0^2(\Omega)$  as  $z_\epsilon = A_\epsilon^{-1}f$  and  $z_0 = A_0^{-1}f$ . Since  
 308  $A_\epsilon z_\epsilon = A_0 z_0 = f$ , we have that for  $\phi \in H_0^2(\Omega)$

$$309 \quad (37) \quad (A_\epsilon(z_\epsilon - z_0), \phi) = (A_0 z_0 - A_\epsilon z_0, \phi) = \int_{\Omega_\epsilon} \left( \frac{1}{1-n_0} - \frac{1}{1-n_1} \right) \Delta z_0 \Delta \phi dx.$$

310 If  $z_0 \in \mathcal{C}^2(\overline{\Omega})$ , we get

$$311 \quad \int_{\Omega_\epsilon} \left( \frac{1}{1-n_0} - \frac{1}{1-n_\epsilon} \right) \Delta z_0 \Delta \phi dx \leq C \|\Delta z_0\|_\infty \int_{\Omega_\epsilon} \Delta \phi dx \leq C\epsilon^{\frac{1}{2}} \|\phi\|_{H_0^2(\Omega)}.$$

313 Thus, we have shown that

$$314 \quad (A_\epsilon(z_\epsilon - z_0), \phi) \leq C\epsilon^{\frac{1}{2}} \|\phi\|_{H_0^2(\Omega)}.$$

315 By plugging in  $\phi = z_\epsilon - z_0$ , we obtain the desired convergence using the coercivity of  $A_\epsilon$ .  $\square$

316 **LEMMA 4.4.** *Assume that  $n_0 \in \mathcal{C}^2(\overline{\Omega})$ . Let  $A_\epsilon, B_\epsilon, C_\epsilon, A_0, B_0$  and  $C_0$  be defined by (5), (6)  
 317 and (7) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively. Then for sufficiently small  $\epsilon$ ,*

$$318 \quad \|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \xrightarrow{\epsilon \rightarrow 0} 0 \quad \text{and} \quad \|A_\epsilon^{-1}C_\epsilon^{\frac{1}{2}} - A_0^{-1}C_0^{\frac{1}{2}}\| \xrightarrow{\epsilon \rightarrow 0} 0.$$

319 *Proof.* From (37), we have that for  $f, \phi \in H_0^2(\Omega)$  and with  $z_\epsilon = A_\epsilon^{-1}f$  and  $z_0 = A_0^{-1}f$

$$320 \quad (A_\epsilon(z_\epsilon - z_0), \phi) \leq C \|\Delta A_0^{-1}f\|_{L^2(\Omega_\epsilon)} \|\phi\|_{H_0^2(\Omega)}.$$

322 Furthermore,

$$323 \quad \|A_\epsilon^{-1}B_\epsilon f - A_0^{-1}B_0 f\|_{H_0^2(\Omega)} \leq \|(A_\epsilon^{-1} - A_0^{-1})B_0 f\|_{H_0^2(\Omega)} + \|A_\epsilon^{-1}(B_\epsilon - B_0)f\|_{H_0^2(\Omega)}$$

$$324 \quad (38) \quad \leq C \|\Delta A_0^{-1}B_0 f\|_{L^2(\Omega_\epsilon)} + \|A_\epsilon^{-1}\| \| (B_\epsilon - B_0) \| \|f\|_{H_0^2(\Omega)}.$$

326 For estimating  $\|\Delta A_0^{-1}B_0 f\|_{L^2(\Omega_\epsilon)}$ , observe that  $B_0 u \in H_0^2(\Omega)$  is the weak solution

$$327 \quad \Delta \Delta B_0 u = \Delta \left( \frac{n_0}{1-n_0} u \right) + \frac{1}{1-n_0} \Delta u \text{ in } \Omega.$$

328 Classical regularity results [22, 25] and the fact that  $n_0 \in \mathcal{C}^2(\overline{\Omega})$  imply that  $B_0 u \in H^4(\Omega) \cap H_0^2(\Omega)$   
 329 and therefore

$$330 \quad \|\Delta A_0^{-1}B_0 f\|_{H^1(\Omega)} \leq C \|f\|_{H^2(\Omega)}.$$

331 By the Sobolev embedding theorem, this implies that

$$332 \quad \|\Delta A_0^{-1}B_0 f\|_{L^p(\Omega)} \leq C \|f\|_{H^2(\Omega)},$$

333 for  $p > 2$ . Let  $\tilde{p} = \frac{p}{2} > 1$  and  $q$  such that  $\frac{1}{\tilde{p}} + \frac{1}{q} = 1$ .

$$334 \quad (39) \quad \|\Delta A_0^{-1}B_0 f\|_{L^2(\Omega_\epsilon)}^2 \leq \|\Delta A_0^{-1}B_0 f\|_{L^p(\Omega)}^2 |\Omega_\epsilon|^{\frac{1}{q}} \leq C\epsilon^{\frac{1}{2q}} \|f\|_{H^2(\Omega)}.$$

336 From (33) we obtain

$$337 \quad (40) \quad \|A_\epsilon^{-1}\| \| (B_\epsilon - B_0) \| \|f\|_{H_0^2(\Omega)} \leq C\epsilon^{\frac{1}{2}} \|f\|_{H_0^2(\Omega)}.$$

338 Using (38), (39) and (40) we have that

$$339 \quad \|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \xrightarrow{\epsilon \rightarrow 0} 0.$$

340 The second convergence result follows from similar arguments.  $\square$

341 Now we would like to obtain explicit formula for the correction term in the asymptotic expansion  
 342 for the operator  $\mathcal{T}_\epsilon$ . More precisely, we define explicit formula for the corrector term associated  
 343 with  $A_\epsilon^{-1} - A_0^{-1}$ .

344 **4.2. Corrector term for  $A_\epsilon^{-1} - A_0^{-1}$ .** In this subsection, we construct a corrector function  
 345 and use it to estimate the convergence rate of  $z_\epsilon = A_\epsilon^{-1}u$  for  $u \in H_0^2(\Omega)$ . Let  $z_0 = A_0^{-1}u \in H_0^2(\Omega)$ ,  
 346 i.e  $z_0 \in H_0^2(\Omega)$  solution of

$$347 \quad (41) \quad \Delta \frac{1}{1-n_0} \Delta z_0 = \Delta \Delta u \quad \text{in} \quad \Omega.$$

348 Inspired by the formal calculations on the previous section, namely problem (30), we define  $z_1$   
 349 solution of

$$350 \quad (42) \quad \begin{cases} \Delta \frac{1}{1-n_0} \Delta z_1 = 0 & \text{in} \quad \Omega, \\ z_1 = 0 & \text{on} \quad \Gamma, \\ \frac{\partial z_1}{\partial \nu} = \left( \frac{1-n_1}{1-n_0} - 1 \right) \Delta z_0 & \text{on} \quad \Gamma. \end{cases}$$

351 We expect that  $z_\epsilon = z_0 + \epsilon z_1 + O(\epsilon^2)$  in  $\Omega_\epsilon^0$ . We extend  $z_1$  in  $\Omega_\epsilon$  as  $\tilde{z}_1^\epsilon$  defined by

$$352 \quad (43) \quad \tilde{z}_1^\epsilon = \begin{cases} z_1 & \text{in} \quad \Omega_\epsilon^0, \\ z_1 - \psi & \text{in} \quad \Omega_\epsilon \end{cases}$$

353 where  $\psi$  is a polynomial of order  $\leq 3$  and satisfying the boundary conditions:

$$354 \quad (44) \quad \begin{cases} \psi = 0, \quad \frac{\partial \psi}{\partial \nu} = \left( \frac{1-n_1}{1-n_0} - 1 \right) \Delta z_0 & \text{on} \quad \Gamma, \\ \psi = \frac{\partial \psi}{\partial \nu} = 0 & \text{on} \quad \Gamma_\epsilon. \end{cases}$$

355 This gives the following expression of  $\psi$  (that plays the role  $\hat{w}_2$  in the formal calculations)

$$356 \quad \psi(x) = \psi(\varphi(s, \epsilon \xi)) = \hat{\psi}(s, \xi) = \epsilon \left( \frac{1-n_1}{1-n_0} - 1 \right) \Delta z_0(\varphi(s, 0)) \xi(1-\xi)^2.$$

The choice of  $\psi$  ensures in particular that  $\tilde{z}_1^\epsilon \in H_0^2(\Omega)$ . To simplify the notation we set

$$m := \left( \frac{1}{1-n_0} - \frac{1}{1-n_1} \right).$$

357 Now we have the following Lemma.

358 **LEMMA 4.5.** Assume that  $n_0$  and  $n_1$  are in  $C^4(\bar{\Omega})$ . Let  $u \in H_0^2(\Omega)$  then set  $z_\epsilon = A_\epsilon^{-1}u$  and  
 359  $z_0 = A_0^{-1}u$ . We define  $\tilde{z}_1^\epsilon$  as in (43) and assume that  $z_0 \in C^6(\bar{\Omega})$ . Then we have, for sufficiently  
 360 small  $\epsilon$ ,

$$361 \quad \|z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon\|_{H_0^2(\Omega)} \leq C\epsilon^{\frac{3}{2}}.$$

362 *Proof.* For any  $\phi \in H_0^2(\Omega)$  we have that

$$363 \quad (45) \quad (A_\epsilon(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon), \phi) = (A_\epsilon(z_\epsilon - z_0), \phi) - \epsilon (A_\epsilon \tilde{z}_1^\epsilon, \phi).$$

364 We recall that

$$365 \quad (A_\epsilon(z_\epsilon - z_0), \phi) = \int_{\Omega_\epsilon} \left( \frac{1}{1-n_0} - \frac{1}{1-n_1} \right) \Delta z_0 \Delta \phi dx.$$

367 Furthermore, we have that

$$368 \quad (46) \quad (A_\epsilon \tilde{z}_1^\epsilon, \phi) = \int_{\Omega_\epsilon^0} \frac{1}{1-n_0} \Delta z_1 \Delta \phi dx + \int_{\Omega_\epsilon} \frac{1}{1-n_1} \Delta(z_1 - \psi) \Delta \phi dx.$$

369 Using the fact that  $\Delta \frac{1}{1-n_0} \Delta z_1 = 0$  and the Green formula yield,

$$370 \quad (A_\epsilon \tilde{z}_1^\epsilon, \phi) = \int_{\Gamma_\epsilon} m \Delta z_1 \frac{\partial \phi}{\partial \nu} ds(x) + \int_{\Gamma_\epsilon} \left( \frac{\partial}{\partial \nu} \left( \frac{1}{1-n_1} \Delta z_1 \right) - \frac{\partial}{\partial \nu} \left( \frac{1}{1-n_0} \Delta z_1 \right) \right) \phi ds(x) \\ 371 \quad + \int_{\Gamma_\epsilon} \frac{1}{1-n_1} \Delta \psi \frac{\partial \phi}{\partial \nu} ds(x) - \int_{\Gamma_\epsilon} \frac{\partial}{\partial \nu} \left( \frac{1}{1-n_1} \Delta \psi \right) \phi ds(x) - \int_{\Omega_\epsilon} \Delta \frac{1}{1-n_1} \Delta \psi \phi dx.$$

373 Using the expression of  $\psi$  we have

$$\begin{aligned}
374 \quad & \frac{1}{1-n_1} \Delta \psi|_{\Gamma_\epsilon} = \frac{1}{1-n_1} \Delta_{s,\xi} \tilde{\psi}(s,1) = \frac{2}{\epsilon} m \Delta z_0(\varphi(s,0)), \\
375 \quad & \frac{\partial}{\partial \nu} \left( \frac{1}{1-n_1} \Delta \psi \right) |_{\Gamma_\epsilon} = \frac{\partial}{\partial \eta} \left( \frac{1}{1-n_1} \Delta \psi \right) |_{\Gamma_\epsilon} = \frac{1}{1-n_1} \frac{1}{\epsilon} \frac{\partial}{\partial \xi} (\Delta_{s,\xi} \tilde{\psi})(s,1) = \frac{6}{\epsilon^2} m \Delta z_0(\varphi(s,0)).
\end{aligned}$$

376 We then get after substitution of these expressions

$$\begin{aligned}
377 \quad & (A_\epsilon \tilde{z}_1^\epsilon, \phi) = \int_{\Gamma_\epsilon} m \left( \Delta z_1(\varphi(s,\epsilon)) + \frac{2}{\epsilon} \Delta z_0(\varphi(s,0)) \right) \frac{\partial \phi}{\partial \nu} ds(x) \\
378 \quad & - \int_{\Gamma_\epsilon} m \left( \frac{\partial}{\partial \nu} (\Delta z_1(\varphi(s,\epsilon))) + \frac{6}{\epsilon^2} \Delta z_0(\varphi(s,0)) \right) \phi ds(x) - \int_{\Omega_\epsilon} \Delta \frac{1}{1-n_1} \Delta \psi \phi dx \\
379 \quad & = \int_{\Gamma_\epsilon} \phi_1^\epsilon \frac{\partial \phi}{\partial \nu} ds(x) - \int_{\Gamma_\epsilon} \phi_2^\epsilon \phi ds(x) - \int_{\Omega_\epsilon} \Delta \frac{1}{1-n_1} \Delta \psi \phi dx \\
380 \quad &
\end{aligned}$$

381 where we have set

$$382 \quad \phi_1^\epsilon(s) := m \Delta z_1(\varphi(s,\epsilon)) + \frac{2}{\epsilon} m \Delta z_0(\varphi(s,0)),$$

$$383 \quad \phi_2^\epsilon(s) := m \frac{\partial}{\partial \nu} (\Delta z_1(\varphi(s,\epsilon))) + \frac{6}{\epsilon^2} m \Delta z_0(\varphi(s,0))$$

384 using the parametrization of the curve  $\Gamma_\epsilon$ ,  $s \mapsto \varphi(s,\epsilon)$  with  $\varphi$  defined by (13). Using this  
385 parametrization and setting  $\tilde{\phi}(s,\eta) := \phi(\varphi(s,\eta))$  in  $\Omega_\epsilon$  we have

$$386 \quad \int_{\Gamma_\epsilon} \phi_1^\epsilon \frac{\partial \phi}{\partial \nu} ds(x) = \int_0^L \phi_1^\epsilon \frac{\partial \tilde{\phi}}{\partial \eta}(s,\epsilon) (1 + \epsilon \kappa) ds = \int_0^L \int_0^\epsilon \phi_1^\epsilon \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s,\eta) (1 + \epsilon \kappa) ds d\eta.$$

388 From the definition of  $\phi_1^\epsilon$  we then get for  $\phi \in H_0^2(\Omega)$ ,

$$389 \quad \int_{\Gamma_\epsilon} \phi_1^\epsilon \frac{\partial \phi}{\partial \nu} ds(x) = \frac{2}{\epsilon} \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s,0)) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s,\eta) ds d\eta + O(\epsilon^{\frac{1}{2}}) \|\phi\|_{H^2(\Omega)}.$$

391 Here and in all the following  $O(\epsilon^r)$  denotes a function such that  $O(\epsilon^r) \leq C \epsilon^r$  for a constant  $C$   
392 independent from the test function  $\phi$  but that may depend on  $\|z_0\|_{C^6(\overline{\Omega})}$ . Using Taylor's expansion  
393 we also get for  $\phi \in H_0^2(\Omega)$ ,

$$\begin{aligned}
394 \quad & \int_{\Gamma_\epsilon} \phi_2^\epsilon \phi ds(x) = \frac{\epsilon}{2} \int_0^L \int_0^\epsilon \phi_2^\epsilon \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s,\eta) (1 + \epsilon \kappa) ds d\eta + O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)} \\
395 \quad & = \frac{3}{\epsilon} \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s,0)) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s,\eta) ds d\eta + O(\epsilon^{\frac{1}{2}}) \|\phi\|_{H^2(\Omega)} \\
396 \quad &
\end{aligned}$$

where the last equality is obtained after substituting the expression of  $\phi_2^\epsilon$ . One ends up with

$$\epsilon \int_{\Gamma_\epsilon} \left( \phi_1^\epsilon \frac{\partial \phi}{\partial \nu} - \phi_2^\epsilon \phi \right) ds(x) = - \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s,0)) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s,\eta) ds d\eta + O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)}.$$

397 Equation (45) then gives

$$\begin{aligned}
398 \quad & (A_\epsilon(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon), \phi) = \int_{\Omega_\epsilon} m \Delta z_0 \Delta \phi dx - \epsilon (A_\epsilon \tilde{z}_1^\epsilon, \phi) \\
399 \quad & = \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s,\eta)) \Delta \phi(\varphi(s,\eta)) (1 + \eta \kappa) ds d\eta - \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s,0)) \frac{\partial^2 \phi}{\partial \eta^2}(\varphi(s,\eta)) ds d\eta \\
400 \quad & - \epsilon \int_{\Omega_\epsilon} \Delta \frac{1}{1-n_1} \Delta \psi \phi dx + O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)}. \\
401 \quad &
\end{aligned}$$

402 We use the expression of the Laplacien in local coordinates

$$403 \quad (1 + \eta \kappa) \Delta \phi(\varphi(s,\eta)) = \frac{\partial}{\partial s} \left( \frac{1}{1 + \eta \kappa} \frac{\partial \tilde{\phi}}{\partial s}(s,\eta) \right) + \kappa \frac{\partial \tilde{\phi}}{\partial \eta}(s,\eta) + (1 + \eta \kappa) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s,\eta)$$

404 to make the decomposition

$$\begin{aligned}
405 \quad & \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s, \eta)) \Delta \phi(\varphi(s, \eta)) (1 + \eta \kappa) ds d\eta = \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s, \eta)) \frac{\partial}{\partial s} \left( \frac{1}{1 + \eta \kappa} \frac{\partial \tilde{\phi}}{\partial s}(s, \eta) \right) \\
406 \quad & + \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s, \eta)) \left( \kappa \frac{\partial \tilde{\phi}}{\partial \eta}(s, \eta) + \eta \kappa \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s, \eta) \right) + \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s, \eta)) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s, \eta). \\
407
\end{aligned}$$

408 To estimate the first term, we integrate by parts on  $[0, L[$ , we obtain

$$\begin{aligned}
409 \quad & \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s, \eta)) \frac{\partial}{\partial s} \left( \frac{1}{1 + \eta \kappa} \frac{\partial \tilde{\phi}}{\partial s}(s, \eta) \right) d\eta ds \\
410 \quad & = - \int_0^L \int_0^\epsilon \frac{1}{1 + \eta \kappa} \frac{\partial}{\partial s} (m \Delta z_0(\varphi(s, \eta))) \frac{\partial \tilde{\phi}}{\partial s}(s, \eta) ds d\eta \\
411 \quad & = -\epsilon \int_0^L \int_0^1 m \frac{1}{1 + \epsilon \xi \kappa} \frac{\partial}{\partial s} \Delta z_0(\varphi(s, \epsilon \xi)) \frac{\partial \tilde{\phi}}{\partial s}(s, \epsilon \xi) ds d\xi \\
412 \quad & = -\epsilon \int_0^L \int_0^1 m \frac{\partial}{\partial s} \Delta z_0(\varphi(s, 0)) \left( \int_0^{\epsilon \xi} \frac{\partial^2 \tilde{\phi}}{\partial \eta \partial s}(s, \eta) d\eta \right) ds d\xi + O(\epsilon^2) \|\phi\|_{H^2(\Omega)} \\
413 \quad & = O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)}.
\end{aligned}$$

415 For the last term we proceed similarly to obtain

$$\begin{aligned}
416 \quad & \int_0^L \int_0^\epsilon m \Delta z_0(\varphi(s, \eta)) \frac{\partial \tilde{\phi}}{\partial \eta}(s, \eta) ds d\eta = \int_0^L \int_0^1 m \Delta z_0(\varphi(s, \epsilon \xi)) \frac{\partial \tilde{\phi}}{\partial \eta}(s, \epsilon \xi) \epsilon ds d\xi \\
417 \quad & = \epsilon \int_0^L \int_0^1 m \Delta z_0(\varphi(s, 0)) \left( \int_0^{\epsilon \xi} \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s, \eta) d\eta \right) ds d\xi + O(\epsilon^2) \|\phi\|_{H^2(\Omega)} = O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)} \\
418
\end{aligned}$$

420 Observing in addition that

$$421 \quad \int_0^L \int_0^\epsilon \eta \kappa m (\Delta z_0(\varphi(s, \eta)) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s, \eta)) ds d\eta = O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)},$$

422 one ends up with

$$\begin{aligned}
423 \quad & (A_\epsilon(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon), \phi) = \int_0^L \int_0^\epsilon m \left( \Delta z_0(\varphi(s, \eta)) - \Delta z_0(\varphi(s, 0)) \right) \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s, \eta) ds d\eta \\
424 \quad & - \epsilon \int_{\Omega_\epsilon} \Delta \frac{1}{1 - n_1} \Delta f \psi \phi dx + O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)}. \\
425
\end{aligned}$$

To conclude we just observe that the two remaining terms are also of the form  $O(\epsilon^{\frac{3}{2}}) \|\phi\|_{H^2(\Omega)}$ . For the first term, we simply use a Taylor expansion for  $\Delta z_0$  while for the second one we just use that, due to the regularity of  $n_0$  and  $n_1$ ,

$$\Delta \frac{1}{1 - n_1} \Delta \psi \in L^\infty(\Omega).$$

426 In conclusion,

$$427 \quad (A_\epsilon(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon), \phi) \leq C \epsilon^{\frac{3}{2}} \|\phi\|_{H^2(\Omega)}.$$

428 Choosing  $\phi = z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon$ , since the coercivity constant associated with  $A_\epsilon$  is independent from  
429  $\epsilon$ , we get

$$430 \quad \|z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon\|_{H_0^2(\Omega)} \leq C \epsilon^{\frac{3}{2}}$$

431 which ends the proof.  $\square$

432 **LEMMA 4.6.** *Assume that  $n_0$  and  $n_1$  are in  $C^4(\overline{\Omega})$ . If  $u \in C^6(\overline{\Omega}) \cap H_0^2(\Omega)$ , then for sufficiently  
433 small  $\epsilon$ ,*

$$434 \quad (47) \quad \|B_0(A_\epsilon^{-1} - A_0^{-1})u\|_{H_0^2(\Omega)} \leq C\epsilon \text{ and } \|C_0^{\frac{1}{2}}(A_\epsilon^{-1} - A_0^{-1})u\|_{H_0^2(\Omega)} \leq C\epsilon$$

435 where  $C$  independent of  $\epsilon$ .

436 *Proof.* From the estimate of Lemma 4.5 we have that

$$437 \quad \|z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon\|_{L^2(\Omega)} \leq C\epsilon^{\frac{3}{2}}.$$

438 Since  $\epsilon \|\tilde{z}_1^\epsilon\|_{L^2(\Omega)} = O(\epsilon)$ , then

$$439 \quad \|z_\epsilon - z_0\|_{L^2(\Omega)} \leq C\epsilon.$$

440 Since  $B_0$  is two orders smoothing, we have that

$$441 \quad \|B_0(z_\epsilon - z_0)\|_{H_0^2(\Omega)} \leq \|z_\epsilon - z_0\|_{L^2(\Omega)} \leq C\epsilon$$

442 The same proof holds for  $C_0^{\frac{1}{2}}$ .  $\square$

443 Now to derive the eigenvalue expansion, we will apply the Theorem of Osborn [23], which we state  
444 here for reader's convenience. Suppose  $X$  is a Hilbert space with inner product denoted by  $\langle \cdot, \cdot \rangle$   
445 and  $K_n : X \rightarrow X$  is a sequence of compact linear operators such that  $K_n$  converge in the operator  
446 norm to  $K$ . It then follows that the adjoint operators also converges in norm. Let  $\mu$  be a nonzero  
447 eigenvalue of  $K$  of algebraic multiplicity  $m$ . It is well known that for  $n$  large enough, there exist  $m$   
448 eigenvalues of  $K_n$ :  $\mu_1^n, \mu_2^n, \dots, \mu_m^n$  such that  $\mu_j^n \xrightarrow{j \rightarrow \infty} \mu$  pour tout  $j = 1, \dots, m$ . Let  $E$  be the spectral  
449 projection onto the generalized eigenspace of  $K$  corresponding to eigenvalue  $\mu$ . The space  $X$  can  
450 be decomposed in terms of the range and null space of  $E$  as  $X = R(E) \oplus N(E)$ . Then form the  
451 proof of Theorem 3 in [23], one can state the following theorem.

452 **THEOREM 4.7.** *Let  $\phi_1, \phi_2, \dots, \phi_m$  be a normalized basis for  $R(E)$ , and let  $\phi_1^*, \phi_2^*, \dots, \phi_m^*$  be the*  
453 *dual basis of  $R(E)$  such that  $\langle v, \phi_j^* \rangle = 0$  for all  $v \in N(E)$ . Then there exists a constant  $C$  such*  
454 *that :*

$$455 \quad (48) \quad \left| \mu - \frac{1}{m} \sum_{j=1}^m \mu_j^n - \frac{1}{m} \sum_{j=1}^m \langle (K - K_n)\phi_j, \phi_j^* \rangle \right| \leq C \| (K - K_n)|_{R(E)} \| \| (K^* - K_n^*)|_{R(E)^*} \|$$

456 In order to apply Theorem 4.7 and obtain explicit expression for the first order asymptotic  
457  $\sum_{j=1}^m \langle (K - K_n)\phi_j, \phi_j^* \rangle$ , one has to construct the basis  $\phi_j^*$ . Remark that in the case of selfadjoint  
458 operators,  $\phi_j^* = \phi_j$ , but this does not apply to our problem. One easily check that  $\phi_j^*$  are necessarily  
459 a basis of the generalized eigenspace of  $K^*$  associated with  $\bar{\mu}$ .

460 We now turn our attention to application of this theorem with  $K_n \equiv \mathcal{T}_\epsilon$  and  $K \equiv \mathcal{T}_0$  and  
461  $X \equiv H_0^2(\Omega) \times H_0^2(\Omega)$ . We already showed that  $\mathcal{T}_\epsilon$  converges to  $\mathcal{T}_0$  in the operator norm in  
462 Theorem 4.1. In order to simplify the calculations we define the inner product on  $H_0^2(\Omega) \times H_0^2(\Omega)$   
463 by:

$$464 \quad \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle := (A_0 u, w)_{H_0^2(\Omega)} + (v, z)_{H_0^2(\Omega)}.$$

465 Let  $\tau_0$  be a simple real eigenvalue of  $\mathcal{T}_0$ , then for  $\epsilon$  small enough, some eigenvalue  $\tau_\epsilon$  of  $\mathcal{T}_\epsilon$  is such  
466 that  $\tau_\epsilon \xrightarrow{\epsilon \rightarrow 0} \tau_0$ .

467 Let  $U_0 = \begin{pmatrix} u_0 \\ \lambda_0 C_0^{\frac{1}{2}} u_0 \end{pmatrix}$  be an eigenvector of  $\mathcal{T}_0$  associated with  $\tau_0$ . Using the expression of  $\mathcal{T}_0$  one

468 easily observes that  $\tilde{U}_0^* = \begin{pmatrix} u_0 \\ -\lambda_0 C_0^{\frac{1}{2}} u_0 \end{pmatrix}$  is an eigenvector of  $\mathcal{T}_0^*$  associated with  $\tau_0$ . Then this  
469 eigenvector is proportional to the dual basis of  $U_0$  if and only if

$$470 \quad (49) \quad -\beta_0 := \langle U_0, \tilde{U}_0^* \rangle = (A_0 u_0, u_0) - \lambda_0^2 (C_0 u_0, u_0) \neq 0.$$

We remark that since  $\tau_0$  is assumed to be a simple eigenvalue (i.e. also with geometrical multiplicity equals 1), then (49) holds. We then can define the dual vector as

$$U_0^* = \frac{-1}{\beta_0} \tilde{U}_0^*$$

471 and apply Theorem 4.7 to get that

$$472 \quad (50) \quad \left| \tau_0 - \tau_\epsilon - \langle (\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0, U_0^* \rangle \right| \leq C \| (\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0 \|_{H_0^2(\Omega)} \| (\mathcal{T}_0^* - \mathcal{T}_\epsilon^*)U_0^* \|_{H_0^2(\Omega)}.$$

473 We are now in position to prove the main result of Theorem of this paper. We refer to Theorem  
474 4.11 for an extension to the case of transmission eigenvalues with higher multiplicities.

475 THEOREM 4.8. Assume that  $n_0$  and  $n_1$  are in  $C^4(\overline{\Omega})$ . Let  $\lambda_0$  be a simple real transmission  
 476 eigenvalue corresponding to  $n_0$  and let  $u_0 \in H_0^2(\Omega)$  be the corresponding eigenvector. This implies  
 477 in particular that (49) holds. Further assume that  $u_0$  and  $A_0^{-1}u_0$  are in  $C^6(\overline{\Omega})$ . Then, for  $\epsilon > 0$   
 478 small enough, there exists a transmission eigenvalue  $\lambda_\epsilon$  corresponding to  $n_\epsilon$  such that

$$479 \quad (51) \quad \frac{1}{\lambda_\epsilon} - \frac{1}{\lambda_0} = -\frac{\epsilon}{\beta_0 \lambda_0} \int_{\Gamma} \frac{n_0 - n_1}{(1 - n_0)^2} |\Delta u_0|^2 ds(x) + O(\epsilon^{\frac{3}{2}}).$$

480 *Proof.* Using estimate (50) with  $\lambda_0 = \frac{1}{\tau_0}$ , we have

$$481 \quad (52) \quad \left| \frac{1}{\lambda_0} - \frac{1}{\lambda_\epsilon} - \langle (\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0, U_0^* \rangle \right| \leq C \|(\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0\|_{H_0^2(\Omega)} \|(\mathcal{T}_0^* - \mathcal{T}_\epsilon^*)U_0^*\|_{H_0^2(\Omega)}$$

482 From the definition of (9) of  $\mathcal{T}_\epsilon$ , we have

$$\begin{aligned} 483 \quad \mathcal{T}_\epsilon U_0 &= \begin{pmatrix} -A_\epsilon^{-1} B_\epsilon u_0 - \lambda_0 A_\epsilon^{-1} C_\epsilon^{\frac{1}{2}} C_0^{\frac{1}{2}} u_0 \\ C_\epsilon^{\frac{1}{2}} u_0 \end{pmatrix} \\ 484 &= \begin{pmatrix} -A_0^{-1} B_\epsilon u_0 - \lambda_0 A_0^{-1} C_\epsilon^{\frac{1}{2}} C_0^{\frac{1}{2}} u_0 \\ C_\epsilon^{\frac{1}{2}} u_0 \end{pmatrix} + \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_0 u_0 + \lambda_0 C_0 u_0) \\ 0 \end{pmatrix} \\ 485 &+ \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_\epsilon - B_0)u_0 - \lambda_0(A_\epsilon^{-1} - A_0^{-1})(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}} u_0 \\ 0 \end{pmatrix} \\ 486 \end{aligned}$$

487 Using the definition (10) of  $\mathcal{T}_0$ , we obtain

$$\begin{aligned} 488 \quad (\mathcal{T}_\epsilon - \mathcal{T}_0)U_0 &= \begin{pmatrix} -A_0^{-1}(B_\epsilon - B_0)u_0 - \lambda_0 A_0^{-1}(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}} u_0 \\ (C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u_0 \end{pmatrix} \\ 489 &+ \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_0 u_0 + \lambda_0 C_0 u_0) \\ 0 \end{pmatrix} \\ 490 &+ \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_\epsilon - B_0)u_0 - \lambda_0(A_\epsilon^{-1} - A_0^{-1})(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}} u_0 \\ 0 \end{pmatrix} \\ 491 \end{aligned}$$

492 From (33) and (36), we have

$$493 \quad (53) \quad \|(\mathcal{T}_\epsilon - \mathcal{T}_0)U_0\|_{H_0^2(\Omega)} = O(\epsilon^{\frac{1}{2}}).$$

494 On the other hand, we have

$$\begin{aligned} 495 \quad (\mathcal{T}_\epsilon^* - \mathcal{T}_0^*)U_0^* &= -\frac{1}{\beta_0} \begin{pmatrix} -(B_\epsilon - B_0)A_0^{-1}u_0 - B_0(A_\epsilon^{-1} - A_0^{-1})u_0 - \lambda_0(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}} u_0 \\ -(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})A_0^{-1}u_0 - C_0^{\frac{1}{2}}(A_\epsilon^{-1} - A_0^{-1})u_0 \end{pmatrix} \\ 496 &- \frac{1}{\beta_0} \begin{pmatrix} -(B_\epsilon - B_0)(A_\epsilon^{-1} - A_0^{-1})u_0 \\ -(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})(A_\epsilon^{-1} - A_0^{-1})u_0 \end{pmatrix} \\ 497 \end{aligned}$$

498 From estimates (33) and (47), we obtain

$$499 \quad (54) \quad \|(\mathcal{T}_\epsilon^* - \mathcal{T}_0^*)U_0^*\|_{H_0^2(\Omega)} = O(\epsilon).$$

500 Next, (52) implies

$$501 \quad \frac{1}{\lambda_\epsilon} - \frac{1}{\lambda_0} = \langle (\mathcal{T}_\epsilon - \mathcal{T}_0)U_0, U_0^* \rangle + O(\epsilon^{\frac{3}{2}}).$$



502 Using the expression of  $U_0^*$  we see that

$$\begin{aligned}
503 \quad \langle (\mathcal{T}_\epsilon - \mathcal{T}_0)U_0, U_0^* \rangle &= \frac{1}{\beta_0}((B_\epsilon - B_0)u_0, u_0) + \frac{1}{\beta_0}\lambda_0((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}}u_0, u_0) \\
504 \quad &+ \frac{1}{\beta_0}(A_0((A_\epsilon^{-1} - A_0^{-1})(B_0u_0 + \lambda_0C_0u_0), u_0) \\
505 \quad &+ \frac{1}{\beta_0}((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u_0, \lambda_0C_0^{\frac{1}{2}}u_0) + O(\epsilon^{\frac{3}{2}}) \\
506
\end{aligned}$$

507 Recall that, by definition of  $u_0$

$$508 \quad (55) \quad A_0u_0 + \lambda_0B_0u_0 + \lambda_0^2C_0u_0 = 0.$$

509 Since  $C_0^{\frac{1}{2}}$  is self-adjoint, we have

$$\begin{aligned}
510 \quad \langle (\mathcal{T}_\epsilon - \mathcal{T}_0)U_0, U_0^* \rangle &= \frac{1}{\beta_0}((B_\epsilon - B_0)u_0, u_0) + \frac{2}{\beta_0}\lambda_0((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}}u_0, u_0) \\
511 \quad &- \frac{1}{\beta_0}\frac{1}{\lambda_0}((A_\epsilon^{-1} - A_0^{-1})A_0u_0, A_0u_0) + O(\epsilon^{\frac{3}{2}}). \\
512
\end{aligned}$$

513 We then deduce

$$\begin{aligned}
514 \quad \frac{1}{\lambda_\epsilon} - \frac{1}{\lambda_0} &= \frac{1}{\beta_0}((B_\epsilon - B_0)u_0, u_0) + \frac{2}{\beta_0}\lambda_0((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})C_0^{\frac{1}{2}}u_0, u_0) \\
515 \quad (56) \quad &- \frac{1}{\beta_0}\frac{1}{\lambda_0}((A_\epsilon^{-1} - A_0^{-1})A_0u_0, A_0u_0) + O(\epsilon^{\frac{3}{2}}). \\
516
\end{aligned}$$

517 In order to conclude, we use the results of the two lemmas below that treat the asymptotic for  
518 each term in (56).

519 Applying Lemma 4.9 with  $u = u_0$  and  $\phi = u_0$  we have

$$520 \quad (57) \quad ((B_\epsilon - B_0)u_0, u_0) \leq C\epsilon^{\frac{3}{2}}$$

521 and with  $u = u_0$  and  $\phi = C_0^{\frac{1}{2}}u_0$ , we obtain

$$522 \quad (58) \quad ((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u_0, C_0^{\frac{1}{2}}u_0) \leq C\epsilon^2.$$

523

524 Applying Lemma 4.10 with  $u = A_0u_0$  (therefore  $z_0 = u_0$ ) and  $\phi = u_0$ , we get, using the fact  
525 that  $A_0$  is selfadjoint,

$$526 \quad ((A_\epsilon^{-1} - A_0^{-1})A_0u_0, A_0u_0) = \epsilon \int_0^L \frac{n_0 - n_1}{(1 - n_0)^2} \Delta u_0(\varphi(s, 0)) \Delta u_0(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}) \\ 527$$

528 We finally obtain

$$529 \quad (59) \quad \frac{1}{\lambda_\epsilon} - \frac{1}{\lambda_0} = -\frac{\epsilon}{\beta_0\lambda_0} \int_0^L \frac{n_0 - n_1}{(1 - n_0)^2} |\Delta u_0(\varphi(s, 0))|^2 ds + O(\epsilon^{\frac{3}{2}}) \quad \square$$

530 which corresponds with the formula announced in the theorem and concludes the proof.

531 **LEMMA 4.9.** *Under the assumptions of Theorem 4.8 one has*

$$532 \quad ((B_\epsilon - B_0)u, \phi) \leq C\epsilon^{\frac{3}{2}}\|\phi\|_{H^2(\Omega)} \quad \text{and} \quad ((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u, \phi) \leq C\epsilon^2\|\phi\|_{H^2(\Omega)}$$

533 for some  $C$  independent of  $\epsilon$  and  $\phi$ .

534 *Proof.* Since

$$535 \quad ((B_\epsilon - B_0)u, \phi) = \int_{\Omega_\epsilon} \left( \frac{1}{1 - n_1} - \frac{1}{1 - n_0} \right) (u\Delta\phi + \Delta u\phi) dx,$$

536 Using the local coordinates in  $\Omega_\epsilon$ , we obtain

$$\begin{aligned}
537 \quad \int_{\Omega_\epsilon} \left( \frac{1}{1-n_1} - \frac{1}{1-n_0} \right) \Delta u \phi dx &= \int_0^L \int_0^1 m \Delta u(\varphi(s, \epsilon\xi)) \tilde{\phi}(s, \epsilon\xi) \epsilon(1 + \xi\epsilon\kappa) ds d\xi \\
538 \quad &= \int_0^L \int_0^1 m \Delta u(\varphi(s, 0)) \left( \int_0^{\epsilon\xi} \frac{\partial^2 \tilde{\phi}}{\partial \eta^2}(s, \eta) d\eta \right) \epsilon(1 + \xi\epsilon\kappa) ds d\xi \\
539 \quad &\leq C\epsilon^{\frac{3}{2}} \|\phi\|_{H^2(\Omega)}
\end{aligned}$$

541 Hence  $((B_\epsilon - B_0)u, \phi) \leq C\epsilon^{\frac{3}{2}} \|\phi\|_{H^2(\Omega)}$ . Similarly, we compute the asymptotic formula of

$$\begin{aligned}
542 \quad ((C_\epsilon - C_0)u, \phi) &= \int_0^L \int_0^1 \frac{n_0}{1-n_0} u(\varphi(s, \epsilon\xi)) \phi(\varphi(s, \epsilon\xi)) \epsilon(1 + \xi\epsilon\kappa) ds d\xi \\
543 \quad &= \int_0^L \int_0^1 \frac{n_0}{1-n_0} \left( \int_0^{\epsilon\xi} \frac{\partial \tilde{u}}{\partial \eta}(s, \eta) d\eta \int_0^{\epsilon\xi} \frac{\partial \tilde{\phi}}{\partial \eta}(s, \eta) d\eta \right) \epsilon(1 + \xi\epsilon\kappa) ds d\xi \\
544 \quad &\leq C\epsilon^2 \|\phi\|_{H^2(\Omega)}
\end{aligned}$$

546 Using the square root Lemma in [24] and the fact that  $C_\epsilon^n$  converges to  $C_0^n$  at the same order  
547  $O(\epsilon^2)$ , we conclude that  $C_\epsilon^{\frac{1}{2}}$  converges to  $C_0^{\frac{1}{2}}$  at the same order  $O(\epsilon^2)$ . Thus we have

$$548 \quad ((C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u, \phi) \leq C\epsilon^2 \|\phi\|_{H^2(\Omega)} \quad \square$$

549 **LEMMA 4.10.** *Under the assumptions of Theorem 4.8 one has for any  $\phi \in H_0^2(\Omega) \cap C^4(\bar{\Omega})$*

$$550 \quad (A_0(A_\epsilon^{-1} - A_0^{-1})u, \phi) = \epsilon \int_0^L \frac{n_0 - n_1}{(1-n_0)^2} \Delta z_0(\varphi(s, 0)) \Delta \phi(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}})$$

552 where  $z_0 := A_0^{-1}u_0$ .

553 *Proof.* With  $z_\epsilon := A_\epsilon^{-1}u_0$ ,

$$\begin{aligned}
554 \quad \int_{\Omega} \frac{1}{1-n_0} \Delta(z_\epsilon - z_0) \Delta \phi dx &= \int_{\Omega} \left( \frac{1}{1-n_0} - \frac{1}{1-n_\epsilon} \right) \Delta z_\epsilon \Delta \phi dx = \int_{\Omega_\epsilon} m \Delta z_\epsilon \Delta \phi dx \\
555 \quad &= \int_{\Omega_\epsilon} m \Delta(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon) \Delta \phi dx + \int_{\Omega_\epsilon} m \Delta(z_0 + \epsilon \tilde{z}_1^\epsilon) \Delta \phi dx \\
556 \quad &
\end{aligned}$$

557 Applying Lemma 4.5 we obtain

$$\begin{aligned}
558 \quad \int_{\Omega} \frac{1}{1-n_0} \Delta(z_\epsilon - z_0) \Delta \phi dx &= \int_{\Omega_\epsilon} m \Delta(z_0 + \epsilon \tilde{z}_1^\epsilon) \Delta \phi dx + O(\epsilon^{\frac{3}{2}}) \\
559 \quad &= \int_{\Omega_\epsilon} m \Delta z_0 \Delta \phi dx - \epsilon \int_{\Omega_\epsilon} m \Delta \psi \Delta \phi dx + O(\epsilon^{\frac{3}{2}}). \\
560 \quad &
\end{aligned}$$

561 Making use of the local coordinates we show

$$\begin{aligned}
562 \quad \int_{\Omega_\epsilon} m \Delta z_0 \Delta \phi dx &= \int_0^L \int_0^1 m \Delta z_0(\varphi(s, \epsilon\xi)) \Delta \phi(\varphi(s, \epsilon\xi)) \epsilon(1 + \xi\epsilon\kappa) ds d\xi \\
563 \quad &= \epsilon \int_0^L m \Delta z_0(\varphi(s, 0)) \Delta \phi(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}). \\
564 \quad &
\end{aligned}$$

565 For the second term

$$566 \quad \epsilon \int_{\Omega_\epsilon} m \Delta \psi \Delta \phi dx = \epsilon \int_{\Omega_\epsilon} \psi \Delta m \Delta \phi dx - \epsilon \int_{\Gamma} m \frac{\partial \psi}{\partial \nu} \Delta \phi ds(x) + \epsilon \int_{\Gamma} m \psi \frac{\partial \Delta \phi}{\partial \nu} ds(x)$$

567

568 Or  $\psi|_{\Gamma} = 0$  and  $\frac{\partial\psi}{\partial\nu}|_{\Gamma} = \left(\frac{1-n_1}{1-n_0} - 1\right)\Delta z_0(\varphi(s, 0))$ . Then we have

$$\begin{aligned}
569 \quad \epsilon \int_{\Omega_\epsilon} m \Delta \psi \Delta \phi dx &= \epsilon \int_{\Omega_\epsilon} \psi \Delta m \Delta \phi dx - \epsilon \int_{\Gamma} m \frac{\partial \psi}{\partial \eta} \Delta \phi ds(x) \\
570 &= \epsilon \int_0^L \int_0^1 \tilde{\psi}(s, \xi) \Delta m \Delta \phi(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi \\
571 &- \epsilon \int_0^L m \left(\frac{1-n_1}{1-n_0} - 1\right) \Delta z_0(\varphi(s, 0)) \Delta \phi(\varphi(s, \epsilon \xi)) ds \\
572 &= \epsilon \int_0^L m \left(\frac{1-n_1}{1-n_0} - 1\right) \Delta z_0(\varphi(s, 0)) \Delta \phi(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}) \\
573
\end{aligned}$$

574 Consequently

$$\begin{aligned}
575 \quad \int_{\Omega} \frac{1}{1-n_0} \Delta(z_\epsilon - z_0) \Delta \phi dx &= \epsilon \int_0^L \frac{n_0 - n_1}{(1-n_0)^2} \Delta z_0(\varphi(s, 0)) \Delta \phi(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}) \\
576
\end{aligned}$$

577 which implies

$$\begin{aligned}
578 \quad (A_0(A_\epsilon^{-1} - A_0^{-1})u, \phi) &= \epsilon \int_0^L \frac{n_0 - n_1}{(1-n_0)^2} \Delta z_0(\varphi(s, 0)) \Delta \phi(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}) \\
579
\end{aligned}$$

580 and concludes the proof.  $\square$

581 We now indicate a possible extension to the case where the eigenvalue  $\tau_0$  is not simple. We need  
582 in that case to assume that the geometric multiplicity  $m$  coincides with the algebraic multiplicity  
583 so that a basis of  $R(E)$  is formed by eigenvectors of  $\mathcal{T}_0$  that we denote by  $U_0^j = \begin{pmatrix} u_0^j \\ \lambda_0 C_0^{\frac{1}{2}} u_0^j \end{pmatrix}$ . A

584 basis of  $R(E)^*$  is then formed by  $\tilde{U}_0^{j*} := \begin{pmatrix} u_0^j \\ -\lambda_0 C_0^{\frac{1}{2}} u_0^j \end{pmatrix}$ . If we assume that

$$\begin{aligned}
585 \quad (60) \quad -\beta_0^j &:= \langle U_0^j, \tilde{U}_0^{j*} \rangle = (A_0 u_0^j, u_0^j) - \lambda_0^2 (C_0 u_0^j, u_0^j) \neq 0,
\end{aligned}$$

then we can define the dual basis as

$$U_0^{j*} = \frac{-1}{\beta_0^j} \tilde{U}_0^{j*}.$$

Notice that the assumption on  $\beta_0^j$  is not guaranteed in general. Making this assumption makes the expression of the dual basis easier to express and allows us to follow the same calculations as above to express the leading term in

$$\langle (\mathcal{T}_0 - \mathcal{T}_\epsilon) U_0^j, U_0^{j*} \rangle.$$

586 We then obtain the following result as a consequence of the application of Theorem 4.7.

587 **THEOREM 4.11.** *Assume that  $n_0$  and  $n_1$  are in  $C^4(\bar{\Omega})$ . Let  $\lambda_0$  be a real transmission eigenvalue  
588 corresponding to  $n_0$  such that the associated eigenspace is formed only with eigenvectors  $u_0^j \in$   
589  $H_0^2(\Omega)$ ,  $j = 1, \dots, m$ . Assume in addition that  $\beta_0^j$  defined by (60) does not vanish and that  $u_0^j$  and  
590  $A_0^{-1} u_0^j$  are in  $C^6(\bar{\Omega})$ . Then, for  $\epsilon > 0$  small enough, there exists  $m$  transmission eigenvalues  $\lambda_\epsilon^j$   
591 corresponding to  $n_\epsilon$  such that*

$$\begin{aligned}
592 \quad (61) \quad \frac{1}{m} \sum_{i=1}^m \frac{1}{\lambda_\epsilon^i} - \frac{1}{\lambda_0} &= -\frac{\epsilon}{\lambda_0} \frac{1}{m} \sum_{i=1}^m \frac{1}{\beta_0^i} \int_{\Gamma} \frac{n_0 - n_1}{(1-n_0)^2} |\Delta u_0^i|^2 ds(x) + O(\epsilon^{\frac{3}{2}}).
\end{aligned}$$

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