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▶ To cite this version:

Francisco Bejarano, Gang Zheng. Observability of Singular Systems with Commensurate Time-Delays and Neutral terms. Automatica, Elsevier, 2017, 85, pp.462-467. <10.1016/j.automatica.2017.08.001>. <hal-01649585>

HAL Id: hal-01649585
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Submitted on 27 Nov 2017

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Observability of Singular Systems with Commensurate Time-Delays and Neutral terms[☆]

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Abstract

This paper deals with the observability problem of a sort of singular systems with commensurate time-delays in the trajectories of the system, in the time derivative of the trajectories (neutral terms), and in the output system. By using a recursive algorithm, sufficient conditions (easy testable) are proposed for guaranteeing the backward and the algebraic observability of the system. This condition implies that the trajectories of the system can be reconstructed by using the actual and past values of the system output.

Keywords: Singular systems, descriptor systems, systems with time-delays, observability

1. Introduction

The description of a variety of practical systems by means of singular systems, also called descriptive, implicit, or differential algebraic systems, has been shown to be useful since several decades ago as it is well explained in Campbell (1980). Such systems, as many others, may contain time-delay terms in the trajectory of the system, the input, and/or the system output. A compendium of new researching results for singular systems with time-delays has been recently published, Gu et al. (2013). A variety of definitions and necessary and sufficient conditions of observability can be found in Yip and Sincovec (1981); Cobb (1984); Hou and Müller (1999). Nevertheless, up to the authors' knowledge, there are few works dedicated to the study of the observability problem of singular systems with time delays, despite the increasing research on problems like solvability, stability, controllability (see, e.g. Cobb (2006)). In Perdon and Anderlucci (2006), an observer design is proposed for a general sort of discrete time singular systems with time-delays. For singular systems with one time-delay in the trajectories of the system, a condition guaranteeing the observability of the system is found in Wei (2013) (there, observability is interpreted as the reconstruction of the initial conditions). However such a condition seems to be difficult of checking. An observer design, based on a Lyapunov-Krasovsky functional, is proposed in Ezzine et al. (2013) for singular systems with a time-delay in the trajectories of the system (not in the system output nor in the time derivative of the trajectories of the system). Therefore, we may say that the observability problem of time-delay systems has not been tackled enough and certainly has not been completely solved.

The main motivation of this paper comes from the interest of tackling the observability problem of a general class of descriptor linear delay system with neutral terms, namely systems whose dynamics is governed by equations as these ones,

$$
\begin{aligned}
J\dot{x}(t) &= \sum_{i=1}^{k_f} F_i \dot{x}(t-ih) + \sum_{i=1}^{k_a} A_i x(t-ih) \\
y(t) &= \sum_{i=1}^{k_c} C_i x(t-ih)
\end{aligned}
$$

where the matrices J, F_i, A_i, and C_i are all constant and J could be a non square matrix, certainly it is assumed to be non invertible. The aim is to find out conditions under which the vector $x(t)$ may be reconstructed by using the trajectory of the system output $y(t)$. It is common to define the backward shift operator $\delta : x(t) \mapsto x(t-h)$ (see, e.g., Kamen (1991)), which allows for rewriting the above dynamic equations as

$$
\begin{aligned}
J\dot{x}(t) &= F(\delta)\dot{x}(t) + A(\delta)x(t) \\
y &= C(\delta)
\end{aligned}
$$

where, by definition, $F(\delta) = \sum_{i=0}^{k_f} F_i \delta^i$, $A(\delta) = \sum_{i=0}^{k_a} A_i \delta^i$, and $C(\delta) = \sum_{i=0}^{k_c} C_i \delta^i$. The definition $E(\delta) = J - F(\delta)$ yields the following compact representation of the previous system equations

$$
\begin{aligned}
E(\delta)\dot{x}(t) &= A(\delta)x(t) \\
y(t) &= C(\delta)x(t)
\end{aligned}
$$

The above compact representation allows for studying the system considering the elements of the matrices appearing in the system over the polynomial ring.

The following notation will be used along the paper. $\mathbb{R}$ is the field of real numbers, $\mathbb{N}$ is the ring of nonnegative integers.

[☆]F.J. Bejarano acknowledges the financial support by Proyecto SIP 20171673, Instituto Politécnico Nacional. Gang Zheng acknowledges the financial support by Ministry of Higher Education and Research Nord-Pas de Calais Regional Council, by FEDER through the Contrat de Projets Etat Region (CPER), and by ARCIR Project ESTIREZ Nord-Pas de Calais Regional Council.

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$\mathbb{R}[\delta]$ is the polynomial ring over the real field $\mathbb{R}$. I_n is the identity matrix of dimension n by n. Since hereinafter mostly matrices with terms in the polynomial ring $\mathbb{R}[\delta]$ will be used, instead of using the symbol (δ) in front of a matrix to indicate that the latter has terms in $\mathbb{R}[\delta]$, we will use the following notation. Given the ring $\mathfrak{R} = \mathbb{R}[\delta]$, $\mathfrak{R}^n$ denotes the module of column vectors with n terms in $\mathfrak{R}$ and $\mathfrak{R}^{1\times n}$ is the module of row vectors. $\mathfrak{R}^{r\times s}$ is meant for the set of matrices of dimension r by s, all of whose entries are in $\mathfrak{R}$. A square matrix M whose terms belong to $\mathfrak{R}$ is called unimodular if its determinant is a nonzero constant. A matrix $M \in \mathfrak{R}^{r\times s}$ is called left invertible if there exists a matrix $M^+ \in R^{s\times r}$ such that $M^+M = I_s$. For a matrix F (with terms in $\mathfrak{R}$), $\operatorname{rank} F$ denotes the rank of F over $\mathfrak{R}$. The degree of a polynomial $p(\delta) \in \mathbb{R}[\delta]$ is denoted by $\deg p(\delta)$. For a matrix F, with terms in $\mathfrak{R}$, $\deg F$ denotes the greatest degree of all entries of F. By $\operatorname{Inv}_s F$ we denote the set of invariant factors (or invariant polynomials) of the matrix F (Gohberg et al. (2009)). The limit from below of a time valued function is denoted as $f(t_-)$.

2. Formulation of the problem

Hence, we will consider the sort of systems that can be represented by the following equations

$$E\dot{x}(t) = Ax(t) \tag{1a}$$
$$y(t) = Cx(t) \tag{1b}$$

where, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. The dimension of the matrices is as follows, $E \in \mathfrak{R}^{\bar{n}\times n}$, $A \in \mathfrak{R}^{\bar{n}\times n}$, $C \in \mathfrak{R}^{p\times n}$ ($\mathfrak{R} = \mathbb{R}[\delta]$). According to the notation defined at the introduction, we use δ as the shift backward operator, i.e., $\delta : x(t) \mapsto x(t-h)$, where h is a real positive number. We assume that there exists a solution of (1a) (which might be not unique) and that every solution of (1a) is piecewise differentiable.

The following definitions are taking as the starting point for the observability analysis that will be done further.

Definition 1. The system (1) is called backward observable (**BO**) on $[t_1, t_2]$ if, and only if, for each $\tau \in [t_1, t_2]$ there exist $\bar{t}_1$ and $\bar{t}_2 \leq \tau$ such that $y(t) = 0$ for all $t \in [\bar{t}_1, \bar{t}_2]$ implies $x(\tau_-) = 0$.

The previous definition is somewhat different to definitions given in Delfour and Mitter (1972) and Lee and Olbrot (1981). The main difference has to do with the fact that backward observability considers only the previous values of the system output. In that sense, backward observability is related more with the final observability given in Lee and Olbrot (1981). In fact, final observability implies backward observability.

Definition 2. The system (1) is algebraically observable (**AO**) if it exists a time t_1 such that $x(t)$ can be expressed for all $t \geq t_1$ by a formula of the type

$$x(t) = \beta_0 y(t) + \beta_1 \dot{y}(t) + \cdots + \beta_l y^{(l)}(t), \tag{2}$$

for some non negative integer l, where $\beta_i \in \mathfrak{R}^{n\times p}$ ($i = 0, 1, \ldots, l$), provided that the system output $y(t)$ is a smooth function.

The backward observability is more related with the map between the trajectories of the system and the system output, whereas a more explicit relationship is given by the algebraic observability. Furthermore, by (2), it is clear that AO implies BO. However, as we will see in the next example, in general, BO does not imply AO.

Example 1. Let us see a system that is backward observable, but is not algebraically observable.

$$\begin{aligned}\dot{x}(t) &= x(t-h)\\ y(t) &= x(t-h)\end{aligned}$$

There, if $y(\xi) = x(\xi - h) = 0$ on the interval $[0, \gamma h]$ ($\gamma \geq 2$) then, $x(\xi) = 0$ on $[-h, (\gamma-1)h]$ and $x(\xi)$ is constant on $[0, \gamma h]$. Therefore, $x(\xi) = 0$ on $[(\gamma-1)h, \gamma h]$. Therefore, we can say that $x(t)$ is BO on $[\gamma h, \gamma h + \bar{t}]$ for any $\bar{t} > 0$.

However, as it is possible to verify, $x(t)$ cannot be expressed as in (2). Indeed, for the initial condition $x(t) = \begin{cases} 0, & t \in [-h, 0) \\ 1, & t = 0 \end{cases}$, we have

$$x(t) = \begin{cases} 1, & t \in [0, h] \\ t - h + 1, & t \in [h, 2h] \\ \frac{t^2}{2} + (1-h)t + 1 - h, & t \in [2h, 3h] \end{cases}$$

By the previous equation we can see that it is not possible to have an expression for $x(t)$ like that of (2).

3. Like Silverman-Molinari algorithm

The technique to be used is based on the approach followed in Bejarano et al. (2013) and Bejarano and Zheng (2014). The condition guaranteeing the observability will be checked by means of a matrix denoted by N_{k^*} that will be defined further. As for, let us select a unimodular matrix S_0 so that the following equation is obtained,

$$S_0 \left[\begin{array}{c|c} -I_{\bar{n}} & E \end{array}\right] = \left[\begin{array}{c|c} J_0 & R_0 \\ H_0 & 0 \end{array}\right] \text{ such that } R_0 \in \mathfrak{R}^{\beta_0\times n} \tag{3}$$

where $\beta_0 = \operatorname{rank}(E)$.

Now, let us define $\Delta_0 = C$. For the k-th step ($k \geq 1$) the matrices Δ_k, N_k and H_k are generated by using the following general procedure. The matrix Δ_k is defined as follows,

$$\Delta_k = H_{k-1}A \tag{4}$$

The matrix N_k is formed by the concatenation of matrices Δ_0 to Δ_k ($k \geq 1$), that is,

$$N_k = \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \vdots \\ \Delta_k \end{bmatrix} \tag{5}$$

For the construction of the matrix H_k, we require to select a unimodular matrix S_k so that

$$S_k \left[\begin{array}{c|c} -I_{\bar{n}} & E \\ \hline 0 & N_k \end{array}\right] = \left[\begin{array}{c|c} J_k & R_k \\ \hline H_k & 0 \end{array}\right] \text{ such that } R_k \in \mathfrak{R}^{\beta_k\times n} \tag{6}$$

where $\beta_k = \operatorname{rank}\begin{bmatrix} E \\ N_k \end{bmatrix}$.

Proposition 1. *The matrix E belongs to the null space of H_0. Moreover, for $k \geq 1$, there exists a matrix Γ_k (which is a submatrix of S_k) that satisfies the equation*

$$H_k E = \Gamma_k N_k \tag{7}$$

Furthermore, if $HE = \Gamma N_k$, for some matrices H and Γ, then $\operatorname{rank} HE \leq \operatorname{rank} H_k E$.

Proof. The existence of such a matrix Γ_k is straightforwardly verified from (6). Indeed, let us give a partition of S_k as $S_k = \begin{bmatrix} S_{k,1} & S_{k,2} \\ S_{k,3} & S_{k,4} \end{bmatrix}$, where $S_{k,3}(-I) = H_k$. Then, readily we obtain that $-H_k E + S_{k,4} N_k = 0$.

Now, since the rank of the matrix $\begin{bmatrix} H_k & \Gamma_k \end{bmatrix}$ is equal to the dimension of the left null space of the matrix $\begin{bmatrix} E \\ N_k \end{bmatrix}$, then, among all the matrices Γ and H satisfying the identity $HE = \Gamma N_k$, the matrices H_k and Γ_k are such that the rank of $H_k E$ is maximal. $\square$

Theorem 1. *There exists a positive integer k^* with the following properties:*

1. k^* and the set $\operatorname{Inv_s}(N_{k^*})$ are independent of the choices of the matrices $\{S_i\}$, for $i = 0, 1, 2, \ldots, k^*$,
2. $\operatorname{Inv_s}(N_{k^*+i}) = \operatorname{Inv_s}(N_{k^*})$, for all $i \geq 0$.
3. k^* is the least positive integer for which the identity $\operatorname{Inv_s} N_{k^*+1} = \operatorname{Inv_s} N_{k^*}$ is satisfied,

Proof. 1. We will prove it by using induction. Let S_0 and $\bar{S}_0$ two matrices used in (3) to generate H_0 and $\bar{H}_0$, respectively. We obtain the equation

$$\begin{bmatrix} J_0 & R_0 \\ H_0 & 0 \end{bmatrix} = \underbrace{S_0 \bar{S}_0^{-1}}_{S_0^*} \begin{bmatrix} \bar{J}_0 & \bar{R}_0 \\ \bar{H}_0 & 0 \end{bmatrix}$$

Hence, since R_0 and $\bar{R}_0$ have both the same rank, which is equal to their number of rows, it is easy to verify that S_0^* is a block triangular matrix, which in turn implies that there exists a unimodular matrix Θ_0 such that $H_0 = \Theta_0 \bar{H}_0$. Thus, the previous equation and (4) imply that $\Delta_1 = \Theta_0 \bar{\Delta}_1$.

Now, let us assume that for $i \geq 1$ there exists a unimodular matrix Ω_i such that $N_i = \Omega_i \bar{N}_i$. Then, in view of (6), we obtain the equation

$$\begin{bmatrix} J_i & R_i \\ H_i & 0 \end{bmatrix} = S_i \begin{bmatrix} I & 0 \\ 0 & \Omega_i \end{bmatrix} \bar{S}_i^{-1} \begin{bmatrix} \bar{J}_i & \bar{R}_i \\ \bar{H}_i & 0 \end{bmatrix}$$

Hence $H_i = \Theta_i \bar{H}_i$ for a unimodular matrix Θ_i, which in turn implies that $\Delta_{i+1} = \Theta_i \bar{\Delta}_{i+1}$. Thereby, there exists a unimodular matrix $\Omega_{i+1} = \operatorname{diag}\{\Omega_i, \Theta_i\}$ such that $N_{i+1} = \Omega_{i+1} \bar{N}_{i+1}$.

2. Let us define $\mathcal{N}_k$ as the $\mathfrak{R}$-module generated by the rows of N_k. Thus, we obtain the following ascending chain

$$\mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_k \subset \mathcal{N}_{k+1} \subset \cdots \tag{8}$$

Defined as above, every $\mathcal{N}_k$ is a submodule of $\mathfrak{R}^{1\times n}$, which is a free Noetherian module (see e.g. Proposition 6.5 in Atiyah and Macdonald (1994)). Thereby, the chain in (8) is stationary, that is, there exists a least positive integer, let say k^*, such that $\mathcal{N}_{k^*+i} = \mathcal{N}_{k^*}$, for any $i \geq 0$. Thereafter, we deduce that N_{k^*} and N_{k^*+i} have both the same invariant factors.

3. Let us take for granted that $\operatorname{Inv_s}(N_{k+1}) = \operatorname{Inv_s}(N_k)$ for some $k \geq 1$. Now, let us assume that $\operatorname{Inv_s}(N_{k+j+1}) = \operatorname{Inv_s}(N_{k+j})$ for a non negative integer $j \geq 0$. Thus, we have that $\mathcal{N}_{k+j+1} = \mathcal{N}_{k+j}$. Therefore, $\Delta_{k+j+1} = X_{k+j+1} N_{k+j}$, for some matrix X_{k+j+1}. In view of this, we obtain the following equation

$$\underbrace{\begin{bmatrix} S_{k+j} & 0 \\ \begin{bmatrix} 0 & -X_{k+j} \end{bmatrix} & I \end{bmatrix}}_{S_{k+j+1}} \begin{bmatrix} -I & E \\ 0 & N_{k+j} \\ 0 & \Delta_{k+j+1} \end{bmatrix} = \begin{bmatrix} J_{k+j} & R_{k+j} \\ H_{k+j} & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, with this selection of S_{k+j+1}, we obtain by (6) that $H_{k+j+1} = \begin{bmatrix} H_{k+j} \\ 0 \end{bmatrix}$, which in turn implies that $\Delta_{k+j+2} = \begin{bmatrix} \Delta_{k+j+1} \\ 0 \end{bmatrix}$. Thence, $N_{k+j+2} = \begin{bmatrix} N_{k+j+1} \\ 0 \end{bmatrix}$, which means that the rows of N_{k+j+1} and the rows N_{k+j+2} generate the same module, and therefore, both matrices have the same invariant factors.

We have just proved by induction that if $\operatorname{Inv_s}(N_{k+1}) = \operatorname{Inv_s}(N_k)$ then $\operatorname{Inv_s}(N_{k+i}) = \operatorname{Inv_s}(N_k)$ for all $i \geq 1$. Therefore, since k^* is the least positive integer for which $\operatorname{Inv_s}(N_{k^*+i}) = \operatorname{Inv_s}(N_{k^*})$ for all $i \geq 1$, then k^* is also the least positive integer that satisfies $\operatorname{Inv_s}(N_{k^*+1}) = \operatorname{Inv_s}(N_{k^*})$. $\square$

4. Main Results

4.1. Sufficient conditions of observability

In this section we propose a sufficient condition guaranteeing the backward observability of the system on an interval $[t^*, \infty)$, and we show that such a condition is also a sufficient for guaranteeing the algebraic observability. The main results are based on the following lemma.

Lemma 1. *Let t^* be defined as $t^* = \alpha_{k^*} h$ ($\alpha_{k^*} \triangleq \max_{1 \leq i \leq k^*-1} \deg H_i$). For any $T > t^*$, the identity $y(t) = 0$ for all $t \in [0, T]$ implies $N_{k^*} x(t) = 0$ for all $t \in [t^*, T)$.*

Proof. Let us take for granted that $y(t) = 0$ for all $t \in [0, T]$. Since, by definition, $\Delta_0 = C$, then $\Delta_0 x(t) = 0$ for all $t \in [0, T]$. Moreover, since $H_0 E = 0$, then $\Delta_1 x(t) = H_0 A x(t) = 0$ for all $t \in [\alpha_1 h, T)$, where $\alpha_1 = \deg H_0$. Now, let us suppose that $N_{k-1} x(t) = 0$, for all $t \in [\alpha_{k-1} h, T)$ $(\alpha_{k-1} \triangleq \max_{1 \leq i \leq k-1} \deg H_i)$. Then $\frac{d}{dt}(\Gamma_{k-1} N_{k-1}) x(t) = (H_{k-1} A) x(t) = 0$, that is, $\Delta_k x(t) = 0$, which in turn implies that $\Delta_k x(t) = 0$, for all $t \in [\alpha_k h, T)$ $(\alpha_k \triangleq \max_{1 \leq i \leq k} \deg H_i)$. Therefore, we conclude that $N_k x(t) = 0$, for all $t \in [\alpha_k h, T)$ $(\alpha_k \triangleq \max_{1 \leq i \leq k} \deg H_i)$, for all $k \geq 1$. $\square$

Theorem 2. *The system is* ***BO*** *on* $[t_1^*, T)$ *(for any* $T > t_1^*$*) provided that* $\mathrm{Inv}_s N_{k^*}$ *has* n *elements and* $\mathrm{Inv}_s N_{k^*} \subset \mathbb{R}$*, where* $t_1^* = (\alpha_{k^*} + \alpha_n) h$*,* $(\alpha_n = \deg N_{k^*}^{+})$*.*

Proof. If N_{k^*} has n real non zero invariant factors, then the equation $N_{k^*} x(t) = 0$ for all $t \in [t^*, \tau)$ implies that $x(t) = 0$ for all $t \in [t_1^*, \tau)$, from which we deduce that $x(\tau_-) = 0$. Therefore, the proof of this theorem follows by Lemma 1. □

Conditions given in the previous theorem guarantee also the algebraic observability of the system.

Theorem 3. *The system is* ***AO*** *if* $\mathrm{Inv}_s N_{k^*}$ *has* n *elements and* $\mathrm{Inv}_s N_{k^*} \subset \mathbb{R}$.

Proof. Since by definition, $\Delta_0 = C$, then $y(t) = \Delta_0 x(t)$ for all $t \in [0, \infty)$. Moreover, since $H_0 E = 0$, then $\Delta_1 x(t) = H_0 A x(t) = 0$ for all $t \geq \alpha_1 h$, where $\alpha_1 = \deg H_0$.Then, we obtain the equation

$$N_1 x(t) = \begin{bmatrix} y(t) \\ 0 \end{bmatrix} =: \bar{y}(t)$$

where the number of elements of the zero vector given above is equal to the number of rows of H_0. Now, since $\Gamma_1 N_1 = H_1 E$, then

$$\frac{d}{dt}(\Gamma_1 N_1) x(t) = \frac{d}{dt}(H_1 E) x(t) = (H_1 A) x(t) = \Delta_2 x(t).$$

for all $t \geq \alpha_2 h$ ($\alpha_2 \triangleq \max_{0 \leq i \leq 1} \deg H_i$). Thus, since $y(t)$ is assumed to be smooth, we obtain the equation,

$$N_2 x(t) = \begin{bmatrix} I & 0 \\ 0 & \Gamma_1 \end{bmatrix} \begin{bmatrix} \bar{y}(t) \\ \dot{\bar{y}}(t) \end{bmatrix} = F_2 \begin{bmatrix} \bar{y}(t) \\ \dot{\bar{y}}(t) \end{bmatrix} \tag{9}$$

for all $t \geq \alpha_2 h$ ($\alpha_2 \triangleq \max_{0 \leq i \leq 1} \deg H_i$). Again, since $\Gamma_2 N_2 = H_2 E$, then, for all $t \geq \alpha_3 h$ ($\alpha_3 \triangleq \max_{0 \leq i \leq 2} \deg H_i$), we obtain the equation

$$\frac{d}{dt}(\Gamma_2 N_2) x(t) = \frac{d}{dt}(H_2 E) x(t) = (H_2 A) x(t) = \Delta_3 x(t).$$

After taking into account (5) and (9), the previous equation implies that

$$N_3 x(t) = \begin{bmatrix} F_2 & 0 \\ 0 & \Gamma_2 F_2 \end{bmatrix} \begin{bmatrix} \bar{y}(t) \\ \dot{\bar{y}}(t) \\ \ddot{\bar{y}}(t) \end{bmatrix} = F_3 \begin{bmatrix} \bar{y}(t) \\ \dot{\bar{y}}(t) \\ \ddot{\bar{y}}(t) \end{bmatrix} \tag{10}$$

for all $t \geq \alpha_3 h$ ($\alpha_3 \triangleq \max_{0 \leq i \leq 2} \deg H_i$), with the matrix F_3 is implicitly defined.

Iteration of the procedure explained above yields the following equation

$$N_{k^*} x(t) = \begin{bmatrix} F_{k*-1} & 0 \\ 0 & \Gamma_{k^*-1} F_{k*-1} \end{bmatrix} \begin{bmatrix} \bar{y}(t) \\ \dot{\bar{y}}(t) \\ \vdots \\ \bar{y}^{(k^*-1)}(t) \end{bmatrix} = F_{k^*} \begin{bmatrix} \bar{y}(t) \\ \dot{\bar{y}}(t) \\ \vdots \\ \bar{y}^{(k^*-1)}(t) \end{bmatrix} \tag{11}$$

Algorithm 1 Checking the observability condition

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procedure MOLINARI(E, A, C) ▷ Finding N_{k^*} and its invariant factors.
  $N \leftarrow C$
  $\rho \leftarrow \mathrm{rank}(E)$
  $S \leftarrow$ a unimodular matrix so that $SE = \begin{bmatrix} R \\ 0 \end{bmatrix}$ and $\mathrm{rank}(R) = \rho$
  $S_2 \leftarrow$ a matriz obtained by splitting $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ so that $S_1 \in \mathfrak{R}^{\rho \times \bar{n}}$
  $H \leftarrow -S_2$
  $inv1 \leftarrow 0$
  $inv2 \leftarrow$ a vector with the invariant factors of N
  while $inv1 \neq inv2$ **do** ▷ The algorithm stops when the invariant factors do not change respect to the previous iteration.
    $inv1 \leftarrow inv2$
    $\Delta \leftarrow HA$
    $N_k \leftarrow \begin{bmatrix} N \\ \Delta \end{bmatrix}$
    $\rho \leftarrow \mathrm{rank}\left(\begin{bmatrix} E \\ N_k \end{bmatrix}\right)$
    $S \leftarrow$ a matrix so that $S \begin{bmatrix} E \\ N_k \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}$ and $\mathrm{rank}\, R = \rho$ ▷ S must be unimodular.
    $S_2 \leftarrow$ a matrix obtained by spliting $S = \begin{bmatrix} S_1 & S_3 \\ S_2 & S_4 \end{bmatrix}$ so that $S_1 \in \mathfrak{R}^{\rho \times \bar{n}}$
    $H \leftarrow -S_2$
    $N \leftarrow N_k$
    $inv2 \leftarrow$ a vector with its invariant factors.
  end while
  return N and $inv2$ ▷ The matrix N_{k^*} and a vector with the invariant factors of it.
end procedure
if $inv2 \in \mathbb{R}^n$ **then**
  The system is BO and AO
end if

---

for all $t \geq \alpha_{k^*} h$ ($\alpha_{k^*} \triangleq \max_{1 \leq i \leq k^*-1} \deg H_i$). Then, since N_{k^*} has n invariant factors all of them constant, then it has a left inverse matrix $N_{k^*}^{+}$ with terms within $\mathfrak{R}$ such that $N_{k^*}^{+} N_{k^*} = I$. Therefore, by (11) we can obtain an equation like (2). □

4.2. Algorithms with pseudo-codes

A computer program can be easily done with the proposed algorithm to check the observability and express the vector $x(t)$ in terms of $y(t)$ and some of its time derivatives. We give in Algorithm 1 the pseudo-code that allows for checking whether the sufficient condition of Theorems 2 and 3 is achieved.

The pseudo-code given in Algorithm 2 helps to get the matrix F_{k^*} defined in (11), which allows for obtaining an expression like (2). The matrices required to follow this algorithm can be obtained from Algorithm 1.

Algorithm 2 Algebraic expression for $x(t)$

procedure MATRIX $F_k^*(E, H_0, \ldots, H_{k^*}, N_1, \ldots, N_{k^*})$ $\triangleright$ Finding N_{k^*} and its invariant factors.
  $r \leftarrow$ number of rows of H_0
  $F \leftarrow$ the identity matrix of dimension r by r.
  for k:=1 **to** k^*-1 **do**
    $\Gamma \leftarrow$ a matrix chosen so that $\Gamma N_k = H_k E$
    $F_n \leftarrow \begin{bmatrix} F & 0 \\ 0 & \Gamma F \end{bmatrix}$
    $F \leftarrow F_n$
  end for
  return F $\triangleright$ The matrix F_{k^*}.
  $N_p \leftarrow$ a matrix so that $N_p N_{k^*} = I$
  $J \leftarrow N_p F$
  for i:=0 **to** k^*-1 **do**
    $\beta_i \leftarrow$ the submatrix of J by taking all the rows from the column $ip+1$ to the column $ip+p$.
  end for
end procedure
return We obtain the identity $x(t) = \beta_0 \bar{y}(t) + \beta_1 \dot{\bar{y}}(t) + \cdots + \beta_{k^*-1} \bar{y}^{(k^*-1)}(t)$

4.3. Numerical example

Example 2. Let us consider the following academic example

$$\begin{aligned}
\dot{x}_2(t) &= x_1(t) - \delta^2 x_3(t) + x_4(t) \\
\dot{x}_3(t) &= x_1(t) + \delta^2 x_3(t) - (1+\delta) x_4(t) \\
\dot{x}_4(t) &= \frac{1}{2} x_2(t) + \frac{1}{2} x_3(t) \\
0 &= x_1(t) - \frac{1}{2} x_2(t) + \frac{1}{2} x_3(t) - \delta x_4(t) \\
y_1(t) &= \frac{1}{2} x_2(t) + \frac{1}{2} x_3(t) \\
y_2(t) &= \delta x_4(t)
\end{aligned}$$

Then, the matrices of the system represented as in (1) are

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & -\delta^2 & 1 \\ 1 & 0 & \delta^2 & -(1+\delta) \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} & -\delta \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix}$$

To check whether the sufficient condition proposed in this paper is satisfied or not, we need to search for the value of the constant k^*, which consequently allows us for to obtain the matrix N_{k^*}. Thus, by using (3) and since $\Delta_1 = H_0 A$, we obtain that

$$H_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\delta \end{bmatrix}$$

Hence, the matrix $N_1 = \begin{bmatrix} C \\ \Delta_1 \end{bmatrix}$ has rank equal to $3 < 4 = n$. That is why we have to continue with the process. It is easy to verify that (6) and (4) yield

$$H_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \delta & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\delta \\ 1 & 0 & 0 & \frac{1}{2}\delta \\ 0 & \frac{1}{2}\delta & \frac{1}{2}\delta & 0 \end{bmatrix}$$

Thus, the matrix $N_2 = \begin{bmatrix} N_1 \\ \Delta_2 \end{bmatrix}$ has rank 4, but $\mathrm{Inv_s}(N_2) = \{1, 1, 1, \delta\}$. Therefore, the procedure cannot be stopped in this step. To construct the matrix N_3, first we obtain that

$$H_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \delta & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\delta & 0 \\ 0 & 0 & \frac{1}{2}\delta & 0 \\ \frac{1}{2}\delta & \frac{1}{2}\delta & 0 & 0 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\delta \\ 1 & 0 & 0 & \frac{1}{2}\delta \\ 0 & \frac{1}{2}\delta & \frac{1}{2}\delta & 0 \\ 0 & -\delta & \left(\delta - \frac{1}{2}\right)\delta & -1+\delta \\ 0 & \frac{1}{4}\delta & \frac{1}{4}\delta & 0 \\ \delta & 0 & 0 & \frac{1}{2}\delta^2 \end{bmatrix}$$

At this step, we obtain that $\mathrm{Inv_s}(N_3) = \{1, 1, 1, 1\}$. Thereby, we conclude that $\mathrm{Inv_s}(N_4) = \mathrm{Inv_s}(N_3)$ since $N_4 = \begin{bmatrix} N_3 \\ \Delta_4 \end{bmatrix}$. Thus, $k^* = 3$, and we stop the procedure here. According to Theorems 2 and 3, the system is BO and AO.

Following Algorithm 2, the terms of the vector $x(t)$ can be reconstructed by means of the following explicit formula,

$$\begin{aligned}
x_1(t) &= \frac{1}{2} y_2(t) + \dot{y}_1(t) \\
x_2(t) &= y_1(t) - \frac{1}{2} y_2(t) + \dot{y}_1(t) \\
x_3(t) &= y_1(t) + \frac{1}{2} y_2(t) - \dot{y}_1(t) \\
x_4(t) &= \delta^2 y_1(t) + \frac{1}{2}\left(\delta^2 - \frac{1}{2}\right) y_2(t) - \delta^2 \dot{y}_1(t) - \frac{1}{2} \dot{y}_2(t) + \ddot{y}_1(t)
\end{aligned}$$

Conclusions

We have found a sufficient condition to check the backward and the algebraic observability of a general class of linear systems with neutral terms. The obtained condition can be verified by checking the invariant factors of the matrix N_{k^*}. We have shown that N_{k^*} is obtained by a finite number of steps of the proposed algorithm. The calculation to get N_{k^*} can be carried out by any software able to work with polynomial matrices. For future work it would be interesting to extend the obtained results to other sort of systems, like networked systems (Wang et al. (2014),Wang (2014)).

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