Asymptotic expansion for vector-valued sequences of random variables with focus on Wiener chaos

Ciprian Tudor, Nakahiro Yoshida

Asymptotic expansion for vector-valued sequences of random variables with focus on Wiener chaos

Ciprian A. Tudor [1] * Nakahiro Yoshida [2,3,†]

[1] Laboratoire Paul Painlevé, Université de Lille 1
F-59655 Villeneuve d'Ascq, France.
tudor@math.univ-lille1.fr

[2]Graduate School of Mathematical Science, University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan
[3] Japan Science and Technology Agency CREST
nakahiro@ms.utokyo.ac.jp

December 8, 2017

Abstract

We develop the asymptotic expansion theory for vector-valued sequences $(F_N)_{N\geq 1}$ of random variables in terms of the convergence of the Stein-Malliavin matrix associated to the sequence F_N. Our approach combines the classical Fourier approach and the recent theory on Stein method and Malliavin calculus. We find the second order term of the asymptotic expansion of the density of F_N and we illustrate our results by several examples.

2010 AMS Classification Numbers: 62M09, 60F05, 62H12

Key Words and Phrases: Asymptotic expansion, Stein-Malliavin calculus, Quadratic variation, Fractional Brownian motion, Central limit theorem, Fourth Moment Theorem.

1 Introduction

The analysis of the convergence in distribution of sequences of random variables constitutes a fundamental direction in probability and statistics. In particular, the asympotic expan-

*. The author was partially supported by Project ECOS NCRS-CONICYT C15E05,MEC PAI80160046 and MATHAMSUD 16-MATH-03 SIDRE Project.

†The author was in part supported by Japan Science and Technology Agency CREST JPMJCR14D7; Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research No. 17H01702 (Scientific Research) and by a Cooperative Research Program of the Institute of Statistical Mathematics.

sion represents a technique that allows to give a precise approximation for the densities of probability distributions,

Our work concerns the convergence in law of random sequences to the Gaussian distribution. Although our context is more general, we will focus on examples based on elements in Wiener chaos. The characterization of the convergence in distribution of sequences of random variables belonging to a Wiener chaos of fixed order to the Gaussian law constitutes an important research direction in stochastic analysis in the last years. This research line was initiated by the seminal paper [11] where the authors proved the famous Fourth Moment Theorem. This result, which basically says that the convergence in law of a sequence of multiple stochastic integrals with unit variance is equivalent to the convergence of the sequence of the fourth moments to the fourth moment of the Gaussian distribution, has then been extended, refined, completed or applied to various situations. The reader may consult the recent monograph [8] for the basics of this theory.

In this work, we are also concerned with some refinements of the Fourt Moment Theorem for sequences of random variables in a Wiener chaos of fixed order. More concretely, we aim to find the second order expansion for the density. This is a natural extension of the Fourth Moment Theorem and of its ramifications. In order to explain this link, let us briefly describe the context. Let H be a real and separable Hilbert space, $(W(h), h \in H)$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{F}, P)$ and let I_q denote the qth multiple stochastic integral with respect to W. Let $Z \sim N(0,1)$. Denote by $d_{TV}(F,G)$ the total variation distance between the laws of the random variables F and G and let D be the Malliavin derivative with respect to W. The Fourth Moment Theorem from [11] can be stated as follows.

Theorem 1 *Fix $q \geq 1$. Consider a sequence $\{F_k = I_q(f_k), k \geq 1\}$ of random variables in the q-th Wiener chaos. Assume that*

$$\lim_{k\to\infty} \mathbf{E}F_k^2 = \lim_{k\to\infty} q!\|f_k\|^2_{H^{\otimes q}} = 1. \tag{1}$$

Then, the following statements are equivalent:

1. *The sequence of random variables $\{F_k = I_n(f_k), k \geq 1\}$ converges in distribution as $k \to \infty$ to the standard normal law $N(0,1)$.*
2. $\lim_{k\to\infty} \mathbf{E}[F_k^4] = 3$.
3. $\|DF_k\|^2_H$ *converges to q in $L^2(\Omega)$ as $k \to \infty$.*
4. $d_{TV}(F_k, Z) \to_{k\to\infty} 0$.

A more recent point of interest in the analysis of the convergence to the normal distribution of sequences of multiple integrals is represented by the analysis of the convergence of the sequence of densities. For every $N \geq 1$, denote by p_{F_N} the density of the random variable F_N (which exists due to a result in [14]). Since the total variation distance between F and G (or between the distributions of the random variables F and G) can be written as $d_{TV}(F,G) = \frac{1}{2}\int_{\mathbb{R}} |p_F(x) - p_G(x)|dx$, we can notice that point 4. in the Fourth Moment Theorem implies that

$$\|p_{F_N}(x) - p(x)\|_{L^1(\mathbb{R})} \to_{N\to\infty} 0$$

where $p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ denotes, throughout our work, the density of the standard normal distribution. The result has been improved in the work [7], where the authors showed that if

$$\overline{\lim_{N\to\infty}} \mathbf{E}\|DF_N\|_H^{-4-\varepsilon} < \infty \tag{2}$$

then the conditions 1.-4. in the Fourth Moment Theorem are equivalent to

$$\sup_{x\in\mathbb{R}} |p_{F_N}(x) - p(x)| \le c\sqrt{(\mathbf{E}F_N^4 - 3)}. \tag{3}$$

We also refer to [3], [4] for other references related to the density convergence on Wiener chaos.

Our purpose is to find the asymptotic expansion in the Fourth Moment Theorem (and, more generally, for sequences of random variables converging to the normal law). By finding the asymptotic expansion we mean to find a sequence $(p_N)_{N\geq 1}$ of the form $p_N(x) = p(x) + r_N\varphi(x)$ such that $r_N \to_{N\to\infty} 0$ which approximates the density p_{F_N} in the following sense

$$\sup_{x\in\mathbb{R}} |x^\alpha|\,|p_{F_N}(x) - p_N(x)| = o(r_N) \tag{4}$$

for any $\alpha \in \mathbb{Z}_+$. This improves the main result in [7] and it will also allow to generalize the Edgeworth -type expansion (see [9], [1]) for sequences on Wiener chaos to a larger class of measurable functions. The main idea to get the asymptotic expansion is to analyze the asymptotic behavior of the characteristic function of F_N and then to obtain the approximate density p_N by Fourier inversion of the dominant part of the characteristic function.

As mentioned before, we develop our asymptotic expansion theory for a general class of random variables, which are not necessarily multiple stochastic integrals, although our main examples are related to Wiener chaos. Our main assumption is that the vector-valued random sequence $\left(F_N, \gamma_N^{-1}(\Lambda(F_N) - C)\right)$ converges to a $d+d\times d$-dimensional Gaussian vector, where $(\gamma_N)_{N\geq 1}$ is a deterministic sequence converging to zero as $N\to\infty$, $\Lambda(F_N)$ is the so-called Stein-Malliavin matrix of F_N whose components are $\Lambda_{i,j} = \langle DF_N^{(i)}, D(-L)^{-1}F_N^{(j)}\rangle_H$ for $i,j = 1,d$ (L is the Ornstein-Uhlenbeck operator) and $C_{i,j} = \lim_{N\to\infty} \mathbf{E}F_N^{(i)}F_N^{(j)}$. In the one-dimensional case the assumption becomes

$$\left(F_N, \gamma_N^{-1}(\langle DF_N, D(-L)^{-1}F_N\rangle_H - 1)\right) \tag{5}$$

converges in distribution, as $N\to\infty$, to a two-dimensional centered Gaussian vector (Z_1, Z_2) with $\mathbf{E}Z_1^2 = \mathbf{E}Z_2^2 = 1$ and $\mathbf{E}Z_1Z_2 = \rho \in (-1,1)$.

The condition (5) is related to what is usually assumed in the asymptotic expansion theory for martingales, which has been developed in the last decades and it is based on the so-called Fourier approach. Let us refer, among many others, to [6], [17], [13], [15], [16] for few important works on the martingale approach in asymptotic expansion. Recall that if $(M_N)_{N\geq 1}$ denotes a sequence of random variables converging to $Z \sim N(0,1)$ such that M_N is the terminal value of a continuous martingale and if $\langle M\rangle_N$ is the martingale's bracket, then one assumes in e.g. [17]), [15] that

$$\langle M\rangle_N \to_{N\to\infty} 1 \text{ in probability}$$

and the following joint convergence holds true

$$\left(M_N, \frac{\langle M\rangle_N - 1}{\gamma_N}\right) \to^{(d)}_{N\to\infty} (Z_1, Z_2) \tag{6}$$

where (Z_1, Z_2) is a centered Gaussian vector as above. The notation "$\to^{(d)}$" stands for the convergence in distribution.

In our assumption (5), the role of the martingale bracket is played by $\langle DF_N, D(-L)^{-1}F_N\rangle_H$ which becomes $\frac{1}{q}\|DF_N\|_H^2$ when F_N is an element of the qth Wiener chaos. The choice is natural, suggested by the identity $\mathbf{E}F_N^2 = \mathbf{E}\langle F\rangle_N^2$ which playes the role of the Itô isometry and by the fact that $\frac{\|DF_N\|^2}{q} - 1$ converges to zero in $L^2(\Omega)$, due to the Fourth Moment Theorem.

We organized our paper as follows. In Section 2, we fix the general context of our work, the main asumptions and some notation. In Section 3, we analyze the asymptotic behavior of the characteristic function of a random sequence that satisfies our assumptions while in Section 4 we obtain the approximate demsity and the asymptotic behavior via Fourier inversion. In Section 5, we illustrate our theory by several examples related to Wiener chaos.

2 Asssumptions and notation

We will here present the basic assumptions and the notation utilised throughout our work.

Let H be a real and separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_H$ and consider $(W(h), h\in H)$ an isonormal process on the probability space $(\Omega, \mathcal{F}, P)$. In the sequel, D, L and δ represent the Malliavin derivative, the Ornstein-Uhlenbeck operator and the divergence integral with respect to W. See the Appendix for their definition and basic properties.

2.1 The main assumptions

Consider a sequence of centered random variables $(F_N)_{N\geq 1}$ in $\mathbb{R}^d$ of the form

$$F_N = \left(F_N^{(1)}, \ldots, F_N^{(d)}\right).$$

Denote by $\Lambda(F_N) = (\Lambda_{i,j})_{i,j=1,..,d}$ the Stein-Malliavin matrix with components

$$\Lambda_{i,j} = \langle DF_N^{(i)}, D(-L)^{-1}F_N^{(j)}\rangle_H$$

for every $i, j = 1, .., d$.

We consider the following assumptions:

[C1] There exists a symmetric (strictly) positive definite $d \times d$ matrix C and a deterministic sequence $(\gamma_N)_{N\geq 1}$ converging to zero as $N \to \infty$ such that $\big(F_N, \gamma_N^{-1}(\Lambda(F_N) - C)\big)$ converges in distribution to a $d + d \times d$ Gaussian random vector (Z_1, Z_2) such that $Z_1 = (Z_1^{(1)}, \dots, Z_1^{(d)})$, $Z_2 = (Z_2^{(k,l)})_{k,l=1,..,d}$ with

$$\mathbf{E}Z_1^{(i)}Z_1^{(j)} = C_{i,j}, \text{ for every } i, j = 1, .., d.$$

[C2] For every $i = 1, .., d$, the sequence $(F_N^{(i)})_{N\geq 1}$ is bounded in $\mathbb{D}^{\ell+1,\infty} = \cap_{p>1}\mathbb{D}^{\ell+1,p}$ for some $l \geq d + 3$.

[C3] With γ_N, C from **[C1]**, for every $i, j = 1, d$ the sequence $\big(\gamma_N^{-1}\big(\Lambda_{i,j} - C_{i,j}\big)\big)_{N\in\mathbb{N}}$ is bounded in $\mathbb{D}^{\ell,\infty}$ for some $l \geq d + 3$.

Notice that the matrix $\Lambda(F_N)$ is not necessarily symmetric (except when all the components belong to the same Wiener chaos) since in general $\Lambda_{i,j} \neq \Lambda_{j,i}$. Nevertheless, we assumed in **[C1]** that it converges to the symmetric matrix C, in the sense that $\gamma_N^{-1}(\Lambda(F_N) - C\big)$ converges in law to normal random variable.

In the case of random variables in Wiener chaos, from the main result in [12] (see Theorem 4 later in the paper), we can prove that, under the convergence in law of $F_N^{(i)}$, $i = 1, .., d$ to the Gaussian law, the quantities $\Lambda_{i,j} - C_{i,j}, \Lambda_{j,i} - C_{i,j}$ converges to zero in $L^2(\Omega)$. Thus condition **[C1]** intuitively means that these two terms have the same rate of convergence to zero.

2.2 The truncation

Let us introduce the truncation sequence $(\Psi_N)_{N\geq 1}$ which will be widely used throughout our work. Its use allows to avoid the condition on the existence of negative moments for the Malliavin derivative of F_N (2).

We consider the function $\Psi_N \in \mathbb{D}^{\ell,p}$ for $p > 1$ and $\ell \geq d + 3$ such that

$$\Psi_N = \psi\left(\left(K\frac{|\Lambda(F_N) - C|_{d\times d}}{\gamma_N^{\delta}}\right)^2\right) \tag{7}$$

with some $K, 0 < \delta < 1$, where $\psi \in C^{\infty}(\mathbb{R}; [0, 1])$ is such that $\psi(x) = 1$ is $|x| \leq \frac{1}{2}$ and $\psi(x) = 0$ if $|x| \geq 1$. The square in the argument pf ψ in order to have Ψ_N differentiable in the Mlliavin sense.

From this definition, clearly we have

1. $0 \leq \Psi_N \leq 1$ for every $N \geq 1$.

2. $\Psi_N \to_{N\to\infty} 1$ in probability.

A crucial fact in our computations is that on the set $\{\Psi_N > 0\}$ we have

$$|\Lambda(F_N) - C|_{d\times d} \leq \frac{1}{K}\gamma_N^{\delta}.$$

This also implies (see e.g. [17]) that there exists two positive constants c_1, c_2 such that for N large enough

$$c_1 \leq \det \Lambda(F_N) \leq c_2. \tag{8}$$

2.3 Notation

For every $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}_+$ we use the notation

$$\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}....x_d^{\alpha_d} \text{ and } \frac{\partial^{\boldsymbol{\alpha}}}{\partial x^{\boldsymbol{\alpha}}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}....\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$$

If $x, y \in \mathbb{R}^d$, we denote by $\langle x, y\rangle_d := \langle x, y\rangle$ their scalar product in $\mathbb{R}^d$. By $|x|_d := |x|$ we will denote the Euclidean norm of x.

The following will be fixed in the sequel:

- The sequence $(F_N)_{N\geq 1}$ defined in Section 2.1.
- The truncation sequence $(\Psi_N)_{N\geq 1}$ is as in Section 2.2.
- $o_\lambda(\gamma_N)$ denotes a deteministic sequence that depends on λ such that $\frac{o_\lambda(\gamma_N)}{\gamma_N} \to_{N\to\infty} 0$. By $o(\gamma_N)$ we refer as usual to a sequence such that $\frac{o(\gamma_N)}{\gamma_N} \to_N 0$.
- By $\Phi(\lambda) = e^{-\frac{\lambda^2}{2}}$ we denote the characteristic function of $Z \sim N(0,1)$ and by $p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ its density.
- By $c, C, C_1, ...$ we denote generic strictly positive constants that may vary from line to line and by $\langle\cdot,\cdot\rangle$ we denote the Euclidean scalar product $\mathbb{R}^d$. On the other hand, we will keep the subscript for the inner product in H, denoted by $\langle\cdot,\cdot\rangle_H$.
- We will use bold letters to indicate vectors in $\mathbb{R}^d$ when we need to differentiate them from their one-dimensional components.

3 Asymptotic behavior of the characteristic function

The first step in finding the asymptotic expansion for the sequence $(F_N)_{N\geq 1}$ which satisfies **[C1]** - **[C3]** is to analyze the asymptotic behavior as $N \to \infty$ of the characteristic function of F_N. In order to avoid troubles related to the integrability over the whole real line, we will work under truncation, in the sense that we will always keep the multiplicative factor Ψ_N given by (7).

Let $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_d) \in \mathbb{R}^d$ and $\theta \in [0,1]$ and let us consider the truncated interpolation

$$\varphi_N^{\Psi}(\theta, \boldsymbol{\lambda}) = \mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \right). \tag{9}$$

Notice that $\varphi_N(1, \boldsymbol{\lambda}) = \mathbf{E}(\Psi_N e^{i\boldsymbol{\lambda}, F_N\rangle})$, the "truncated" characteristic function of F_N, while $\varphi_N(0, \boldsymbol{\lambda}) = (\mathbf{E}\Psi_N)e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}}$, the "truncated"characteristic function of the limit in law of F_N. Therefore, it is useful to analyze the behavior of the derivative of $\varphi_N^{\Psi}(\theta, \boldsymbol{\lambda})$ with respect to the variable θ.

We have the following result.

Proposition 1 *Assume* **[C1]** - **[C3]** *and suppose that*

$$\mathbf{E}\left(Z_2^{(k,l)} | Z_1 \right) = \sum_{a=1}^{d} \rho_{k,l}^a Z_1^{(a)} \tag{10}$$

for every $k, l = 1, .., d$, where (Z_1, Z_2) is the Gaussian random vector from **[C1]**. *Then we have the convergence*

$$\gamma_N^{-1} \frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \boldsymbol{\lambda}) \to_{N\to\infty} -\mathtt{i}\theta^2 \left(\sum_{i,j,k,a=1}^{d} \lambda_j \lambda_k \lambda_i \rho_{j,k}^a C_{ia} \right) e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}}. \tag{11}$$

Proof: By differentiating (9) with respect to θ and using the formula $F_N^{(i)} = \delta D(-L)^{-1} F_N^{(i)}$ for every $i = 1, .., d$, we can write

$$\begin{aligned}
\frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \boldsymbol{\lambda}) &= \mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \left(i\langle \boldsymbol{\lambda}, F_N\rangle + \theta \boldsymbol{\lambda}^T C \boldsymbol{\lambda} \right) \right) \\
&= \mathtt{i}\mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \sum_{j=1}^{d} \lambda_j \delta D(-L)^{-1} F_N^{(j)} \right) \\
&\quad + \theta \sum_{j,k=1}^{d} \lambda_j \lambda_k C_{j,k} \mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \right)
\end{aligned}$$

and by the duality relationship (65)

$$\begin{aligned}\frac{\partial}{\partial\theta}\varphi_N^{\Psi}(\theta,\boldsymbol{\lambda}) &= \mathtt{i}\sum_{j=1}^{d}\lambda_j\mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^TC\boldsymbol{\lambda}}{2}}\mathtt{i}\theta\langle D(-L)^{-1}F_N^{(j)},D\langle\boldsymbol{\lambda},F_N\rangle\rangle_H\right)\\ &\quad+\mathtt{i}\sum_{j=1}^{d}\lambda_j\mathbf{E}\left(e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^TC\boldsymbol{\lambda}}{2}}\langle D\Psi_N,D(-L)^{-1}F_N^{(j)}\rangle_H\right)\\ &\quad+\theta\sum_{j,k=1}^{d}\lambda_j\lambda_kC_{j,k}\mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^TC\boldsymbol{\lambda}}{2}}\right).\end{aligned}$$

We obtain, since $D\langle\boldsymbol{\lambda},F_N\rangle=\sum_{k=1}^{d}\lambda_kDF_N^{(k)}$,

$$\begin{aligned}\frac{\partial}{\partial\theta}\varphi_N^{\Psi}(\theta,\boldsymbol{\lambda}) &= -\theta\sum_{j,k=1}^{d}\lambda_j\lambda_k\mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^TC\boldsymbol{\lambda}}{2}}\left(\langle DF_N^{(k)},D(-L)^{-1}F_N^{(j)}\rangle_H-C_{j,k}\right)\right)\\ &\quad+\mathtt{i}\sum_{j=1}^{d}\lambda_j\mathbf{E}\left(e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^TC\boldsymbol{\lambda}}{2}}\langle D\Psi_N,D(-L)^{-1}F_N^{(j)}\rangle_H\right)\\ &:= A_N(\theta,\boldsymbol{\lambda})+B_N(\theta,\boldsymbol{\lambda}). \qquad (12)\end{aligned}$$

Using the definition of the truncation function Ψ_N, we can prove that

$$\gamma_N^{-1}B_N(\theta,\boldsymbol{\lambda})=\gamma_N^{-1}\sum_{j=1}^{d}\mathbf{E}\left(e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^TC\boldsymbol{\lambda}}{2}}\langle D\Psi_N,D(-L)^{-1}F_N^{(j)}\rangle_H\right)\to_{N\to\infty}0. \quad (13)$$

Indeed, by the chain rule (66)

$$D\Psi_N=\psi'\left(\left(K\frac{|\Lambda(F_N)-C|_{d\times d}}{\gamma_N^{\delta}}\right)^2\right)D\left(\frac{|\Lambda(F_N)-C|_{d\times d}}{\gamma_N^{\delta}}\right)^2$$

and by using **[C2]**-**[C3]** and (7), we get for every $p>1$

$$\begin{aligned}\gamma_N^{-1}|B_N(\theta,\boldsymbol{\lambda})| &\le c\gamma_N^{-1}\mathbf{E}\left|\langle D\Psi_N,D(-L)^{-1}F_N^{(j)}\rangle_H\right|\\ &\le c\gamma_N^{-1}P\left(|\Lambda(F_N)-C|_{d\times d}\ge\frac{1}{2}\gamma_N^{\delta}\right)^{\frac{1}{2}}\\ &\le c\mathbf{E}\left(|\Lambda(F_N)-C|_{d\times d}\right)^p\gamma_N^{-1-\delta p}\le c\gamma_N^{p(1-\delta)-1}\end{aligned}$$

where we used again **[C2]**. Since $\delta<1$ and p is arbitrarily large, we obtain the convergence (13).

On the other hand, by assumption **[C1]**, we have the convergence

$$\gamma_N^{-1} A_N(\theta, \boldsymbol{\lambda}) \to_{N\to\infty} -\theta \sum_{j,k=1}^{d} \lambda_j \lambda_k \mathbf{E}\left(e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} Z_2^{(k,l)} \right). \tag{14}$$

Using our assumption (10), we can express the above limit in a more explicit way. We have

$$\begin{aligned}
\mathbf{E}\left(e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} Z_2^{(j,k)} \right) &= \mathbf{E}\left(e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \mathbf{E}(Z_2^{(j,k)}|Z_1) \right) \\
&= \mathbf{E}\left(e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \sum_{a=1}^{d} \rho_{j,k}^{a} Z_1^{(a)} \right) \qquad (15)
\end{aligned}$$

and, moreover, for every $a = 1,..,d$, from the differential equation verified by the characteristic function of the normal distribution,

$$\begin{aligned}
&\mathbf{E}\left(e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} Z_1^{(a)} \right) = e^{-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \mathbf{E}\left(e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle} Z_1^{(a)} \right) \\
&= e^{-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \frac{-1}{\mathtt{i}\theta} (\sum_{i=1}^{d} \lambda_i C_{ia}) \theta^2 e^{-\theta^2 \frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \\
&= e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \frac{-\theta}{\mathtt{i}} (\sum_{i=1}^{d} \lambda_i C_{ia}) = \mathtt{i}\theta e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \sum_{i=1}^{d} \lambda_i C_{ia}. \qquad (16)
\end{aligned}$$

Thus, by replacing (15) and (16) in (14), we obtain

$$\gamma_N^{-1} A_N(\theta, \boldsymbol{\lambda}) \to_{N\to\infty} -\mathtt{i}\theta^2 \left(\sum_{i,j,k,a=1}^{d} \lambda_j \lambda_k \lambda_i \rho_{j,k}^{a} C_{ia} \right) e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}}. \tag{17}$$

Relation (17), together with with (13) and (12) will imply the conclusion. ■

We will also need to analyze the asymptotic behavior of the partial derivatives with respect to the variable $\boldsymbol{\lambda}$ of (9). Let us first introduce some notation borrowed from [16].

For every $u, z \in \mathbb{R}^d, \boldsymbol{\alpha} \in \mathbb{Z}_+^d$ and for every symmetric $d \times d$ matrix C, we will denote by

$$P_{\boldsymbol{\alpha}}(u, z, C) = e^{-\mathtt{i}\langle u, z\rangle_d - \frac{1}{2}u^T C u} (-\mathtt{i})^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}}}{\partial u^{\boldsymbol{\alpha}}} e^{\mathtt{i}\langle u, z\rangle_d + \frac{1}{2}u^T C u} \tag{18}$$

where $|\boldsymbol{\alpha}| := \alpha_1 + ... + \alpha_d$ if $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_d) \in \mathbb{Z}_+^d$. Then clearly,

$$(-\mathtt{i})^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}}}{\partial u^{\boldsymbol{\alpha}}} e^{\mathtt{i}\langle u, z\rangle_d + \frac{1}{2}u^T C u} = e^{\mathtt{i}\langle u, z\rangle_d + \frac{1}{2}u^T C u} P_{\boldsymbol{\alpha}}(u, z, C). \tag{19}$$

In the next lemma we recall the explicit expression of the polynomial $P_{\boldsymbol{\alpha}}$ together on some estimates on this polynomial. We refer to Lemma 1 in [16] for the proof.

Lemma 1 *If $P_{\boldsymbol{\alpha}}$ is given by (18), then for every $u, z \in \mathbb{R}^d$ and evey $d \times d$ matrix C, we have*

$$P_{\boldsymbol{\alpha}}(u, z, C) = \sum_{0 \le \beta \le \boldsymbol{\alpha}} C_{\boldsymbol{\alpha}}^{\beta} (-\mathtt{i})^{\beta} z^{\boldsymbol{\alpha}-\beta} S_{\beta}(u, C)$$

with

$$S_{\beta}(u, C) = e^{-\frac{1}{2}u^T C u} \frac{\partial^{\beta}}{\partial u^{\beta}} e^{\frac{1}{2}u^T C u}.$$

Moreover, we have the estimate

$$|S_{\beta}(u, C)| \le \sum_{j=0}^{|\beta|} c_j^{|\beta|} |u|^j |C|^{(j+|\beta|)/2}$$

for every $u \in \mathbb{R}^d$ and every $d \times d$ matrix C.

Let us now state and prove the asymptotic behavior of the partial derivatives of the truncated interpolation φ_N^{Ψ}.

Proposition 2 *Assume* **[C1]** -**[C3]** *and (10). Let φ_N^{Ψ} be given by (9). Then for every $\boldsymbol{\alpha} \in \mathbb{Z}_+^d$, we have*

$$\gamma_N^{-1} \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} \frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \boldsymbol{\lambda}) \to_{N \to \infty} \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} \left[-\mathtt{i}\theta^2 \left(\sum_{i,j,k,a=1}^{d} \lambda_j \lambda_k \lambda_i \rho_{j,k}^{a} C_{ia} \right) e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \right].$$

Proof: If $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}_+^d$, then by (12) we have

$$\frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} \frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \boldsymbol{\lambda}) = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} A_N(\theta, \boldsymbol{\lambda}) + \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} B_N(\theta, \boldsymbol{\lambda})$$

with A_N, B_N defined in (12). Now, since by (19)

$$\frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\alpha}} e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} = \mathtt{i}^{|\boldsymbol{\alpha}|} P_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \theta F_N, -(1-\theta^2)C) e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}}$$

we can write

$$\begin{aligned}
&\frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\alpha}} \frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \boldsymbol{\lambda}) \\
= \quad & -\theta \sum_{j,k=1}^{d} \lambda_j \lambda_k \mathbf{E} \left(\Psi_N e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \mathtt{i}^{\boldsymbol{\alpha}} P_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \theta F_N, -(1-\theta^2), C) \left(\langle DF_N^{(k)}, D(-L)^{-1} F_N^{(j)} \rangle_H - C_{j,k} \right) \right) \\
& + \mathtt{i} \sum_{j=1}^{d} \lambda_j \mathbf{E} \left(\Psi_N e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, F_N\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \mathtt{i}^{\boldsymbol{\alpha}} P_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \theta F_N, -(1-\theta^2), C) \langle D\Psi_N, D(-L)^{-1} F_N^{(j)} \rangle_H \right) \qquad (20)
\end{aligned}$$

As in the proof of (13), the second summand above multiplied by γ_N^{-1} converges to zero as $N \to \infty$. Concerning the first summand in (20), the hypothesis **[C1]** implies that it converges as $N \to \infty$, to

$$
\begin{aligned}
& -\theta \sum_{j,k=1}^{d} \lambda_j \lambda_k \mathbf{E}\left(\mathtt{i}^{\boldsymbol{\alpha}} P_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \theta Z_1, -(1-\theta^2)C) e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} Z_2^{(k,l)} \right) \\
= \;& -\theta \sum_{j,k=1}^{d} \lambda_j \lambda_k \mathbf{E}\left(\left(\frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\alpha}} e^{\mathtt{i}\theta\langle \boldsymbol{\lambda}, Z_1\rangle - (1-\theta^2)\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \right) Z_2^{(k,l)} \right) \\
= \;& \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} \left[-\mathtt{i}\theta^2 \left(\sum_{i,j,k,a=1}^{d} \lambda_j \lambda_k \lambda_i \rho_{j,k}^{a} C_{ia} \right) e^{-\frac{\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}{2}} \right].
\end{aligned}
$$

Then the conclusion is obtained. ■

Remark 1 *If $d = 1$, relation (11) becomes*

$$
\gamma_N^{-1} \frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \lambda) \to_{N \to \infty} -\mathtt{i}\theta^2 \lambda^3 \rho e^{-\frac{\lambda^2}{2}}
$$

where $\rho = \mathbf{E}(Z_2/Z_1)$.

The following integration by parts formula plays an important role in the sequel. Its proof is a slightly adaptation of the proof of Proposition 1 in [16]. For its proof the nondegeneracy (8) is needed.

Lemma 2 *Assume* **[C2]** *-***[C3]***. Consider a function $f \in C_p^l(\mathbb{R}^d)$ with $l \geq 0$ and let G be a random variable in $\mathbb{D}^{l,\infty}$. Then it holds*

$$
\mathbf{E}\left(\frac{\partial^l}{\partial x^l} f(F_N) G \Psi_N \right) = \mathbf{E}\left(f(F_N) H_l(G\Psi_N) \right) \tag{21}
$$

where the random variable $H_l(G\Psi_N)$ belongs to $L^p(\Omega)$ for every $p \geq 1$.

We will use the integration by parts formula in the above Lemma 2 to show that $\frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} \frac{\partial}{\partial \theta} \varphi_N^{\Psi}(\theta, \boldsymbol{\lambda})$ is dominated by an integrable function uniformly with respect to $(\theta, \boldsymbol{\lambda}) \in [0,1] \times \mathbb{R}^d$.

Lemma 3 *Assume* **[C2]***-***[C3]***. For every $\boldsymbol{\alpha} \in \mathbb{Z}_+^d$, there exists a positive constant $C_{\boldsymbol{\alpha}}$ such that*

$$
\gamma_N^{-1} \left| \partial_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} \frac{\partial}{\partial \theta} \varphi_{F_N}^{\Psi}(\theta, \boldsymbol{\lambda}) \right| \leq C_{\boldsymbol{\alpha}} (1 + |\boldsymbol{\lambda}|)^{-\ell+2} \tag{22}
$$

for all $\boldsymbol{\lambda} \in \mathbb{R}^d$, $\theta \in [0,1]$ and $N \geq 1$.

Proof: First we consider the case $\boldsymbol{\alpha} = 0 \in \mathbb{Z}_+^d$. Consider the function $f(x) = e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},x\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}$ for $x \in \mathbb{R}^d$ with $\frac{\partial^{\boldsymbol{\alpha}}}{\partial x^{\boldsymbol{\alpha}}} f(x) = f(x)(i\boldsymbol{\lambda})^{\boldsymbol{\alpha}}$.

For $\theta \in (0,1]$, by using (12) and the integration by parts formula (21) to the function f ℓ-times, we obtain

$$
\begin{aligned}
& (\mathtt{i}\theta\boldsymbol{\lambda})^\ell \frac{\partial}{\partial\theta}\varphi_{F_N}^{\Psi}(\theta,\boldsymbol{\lambda}) \\
= & -\theta(\mathtt{i}\theta\boldsymbol{\lambda})^\ell \sum_{j,k=1}^d \lambda_j\lambda_k \mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}\left(\langle DF_N^{(k)}, D(-L)^{-1}F_N^{(j)}\rangle_H - C_{j,k}\right)\right) \\
& +\mathtt{i}(\mathtt{i}\theta\boldsymbol{\lambda})^\ell \sum_{j=1}^d \lambda_j \mathbf{E}\left(e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}\langle D\Psi_N, D(-L)^{-1}F_N^{(j)}\rangle_H\right) \\
= & -\theta\sum_{j,k=1}^d \lambda_j\lambda_k \mathbf{E}\left(\Psi_N e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}} H_\ell\Big(\big(q^{-1}\langle DF_N, DF_N\rangle_H - 1\big)\Psi_N\Big)\right) \\
& +\sum_{j=1}^d \lambda_j \mathbf{E}\left(e^{\mathtt{i}\theta\langle\boldsymbol{\lambda},F_N\rangle-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}} H_\ell\Big(\langle i\boldsymbol{\lambda}(-q)^{-1}DF_N, D\Psi_N\rangle_H\Big)\right)
\end{aligned}
$$

We use this estimate for $\theta \geq 1/2$. For $\theta \in [0,1/2]$, the inequality is trivial thanks to the exponential function $\exp(-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2})$. Thus we obtained the desired estimate when $\boldsymbol{\alpha} = 0$. For $\boldsymbol{\alpha} \in \mathbb{Z}_+^d, \boldsymbol{\alpha} \neq (0,...0)$, we can follow a similar argument with the estimate

$$
\sup_{\theta,\boldsymbol{\lambda}} \big((1-\theta^2)(1+\boldsymbol{\lambda}^2)\big)^m \exp\big(-(1-\theta^2)\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}\big) \quad < \quad \infty
$$

for every $m \in \mathbb{Z}_+$. Thus we obtained the result. ■

Let us denote by φ_N the sum of the characteristic function of the law $N(0,C)$ and of the integral over $[0,1]$ with respect to θ of the limit in (11), i.e.

$$
\varphi_N(\boldsymbol{\lambda}) = e^{-\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}} - \mathtt{i}\gamma_N\frac{1}{3}\left(\sum_{i,j,k,a=1}^d \lambda_j\lambda_k\lambda_i\rho_{j,k}^a C_{ia}\right)e^{-\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}. \tag{23}
$$

The next result shows that the difference between the truncated characteristic function of F_N and the characteristic function of the weak limit of the sequence F_N can be approximated by φ_N.

Proposition 3 *Suppose that* **[C1]**-**[C3]** *are fulfilled for $\ell = d+3$. For every $N \geq 1$, let φ_N be given by (23). Then*

$$
\gamma_N^{-1}\int_{\mathbb{R}^d}\left|\partial_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\Big(\mathbf{E}\big[e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\Psi_N\big] - \mathbf{E}[\Psi_N]\varphi_N(\boldsymbol{\lambda})\Big)\right|d\boldsymbol{\lambda} \quad \to_{N\to\infty} \quad 0
$$

for every $\boldsymbol{\alpha} \in \mathbb{Z}_+^d$.

Proof: If $\boldsymbol{\alpha} \in \mathbb{Z}_+^d$, using

$$\varphi_N^{\Psi}(1,\boldsymbol{\lambda}) = \mathbf{E}\big[e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\Psi_N\big] \text{ and } \varphi_N^{\Psi}(0,\boldsymbol{\lambda}) = e^{-\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}$$

we can write

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left|\gamma_N^{-1}\partial_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\Big(\mathbf{E}\big[e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\Psi_N\big]-\mathbf{E}[\Psi_N]\varphi_N(\boldsymbol{\lambda})\Big)\right|d\boldsymbol{\lambda}\\
= & \int_{\mathbb{R}}\left|\partial_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\int_0^1\left(\gamma_N^{-1}\partial_\theta\varphi_N^{\Psi}(\theta,\boldsymbol{\lambda})-(-\mathtt{i})\theta^2\mathbf{E}[\Psi_N]\left(\sum_{i,j,k,a=1}^{d}\lambda_j\lambda_k\lambda_i\rho_{j,k}^a C_{ia}\right)e^{-\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}\right)d\theta\right|d\boldsymbol{\lambda}\\
\leq & \int_{\mathbb{R}^d}\int_0^1\left|\gamma_N^{-1}\partial_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\partial_\theta\varphi_N^{\Psi}(\theta,\boldsymbol{\lambda})-\partial_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\big((-\mathtt{i})\theta^2\left(\sum_{i,j,k,a=1}^{d}\lambda_j\lambda_k\lambda_i\rho_{j,k}^a C_{ia}\right)e^{-\frac{\boldsymbol{\lambda}^T C\boldsymbol{\lambda}}{2}}\big)\right|d\theta d\boldsymbol{\lambda}\\
&+C\left(\|1-\Psi_N\|_2^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The last quantity converges to zero as $N\to\infty$ by the choice of the truncation function and also Lemma 3 and the dominated convergence theorem since the integrand is bounded uniformly in (θ,N). ■

Let φ_{F_N} be the characteristic function of the random vector F_N, i.e. $\varphi_{F_N}(\boldsymbol{\lambda}) = \mathbf{E}e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}$ for $\boldsymbol{\lambda}\in\mathbb{R}^d$. If φ_N is given by (23), we have

$$\begin{aligned}
&\varphi_{F_N}(\lambda)\\
= & \varphi_N(\boldsymbol{\lambda})\mathbf{E}\Psi_N+\mathbf{E}\left(\Psi_N e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\right)-\mathbf{E}(\Psi_N)\varphi_N(\boldsymbol{\lambda})+\mathbf{E}\left((1-\Psi_N)e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\right)\\
= & \varphi_N(\boldsymbol{\lambda})+\varphi_N(\boldsymbol{\lambda})(\mathbf{E}\Psi_N-1)+\mathbf{E}\left(\Psi_N e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\right)-\mathbf{E}(\Psi_N)\varphi_N(\boldsymbol{\lambda})+\mathbf{E}\left((1-\Psi_N)e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\right)
\end{aligned}$$

Proposition 3 shows that the difference $\mathbf{E}\left(\Psi_N e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}\right)-\mathbf{E}(\Psi_N)\varphi_N(\boldsymbol{\lambda})$ while from the definition of the truncation function Ψ_N we see that the term $\mathbf{E}((1-\Psi_N)e^{\mathtt{i}\langle\boldsymbol{\lambda},F_N\rangle}+\varphi_N(\boldsymbol{\lambda})(\mathbf{E}\Psi_N-1)$ is also small. Consequently, $\varphi_{F_N}(\boldsymbol{\lambda})$ can be approximated by φ_N. Actually, we have

$$\varphi_{F_N}(\boldsymbol{\lambda}) = \varphi_N(\boldsymbol{\lambda})+s_N(\boldsymbol{\lambda}) \tag{24}$$

where $s_N(\boldsymbol{\lambda})$ is a "small term"' such that $\gamma_N^{-1}s_N(\boldsymbol{\lambda})$ and its derivatives are dominated by the right-hand side of (20) and it that satisfies

$$s_N(\boldsymbol{\lambda}) \leq C\|1-\Psi_N\|_p+o_{\boldsymbol{\lambda}}(\gamma_N) \tag{25}$$

for every $p\geq 1$, where $o_{\boldsymbol{\lambda}}(\gamma_N)$ denotes a deteministic sequence that depends on λ such that $\frac{o_{\boldsymbol{\lambda}}(\gamma_N)}{\gamma_N}\to_{N\to\infty}0$.

Therefore φ_N is the dominant part in the asymptotic expansion of φ_{F_N}. By inverting φ_N, we get the approximate density.

4 The approximate density

In this paragraph, our purpose is to find the first and second-order term in the asymptotic expansion of the density of F_N. We will assume throughout this section **[C1]**-**[C3]** are fulfilled for $\ell = d+3$. Let us denote by $\varphi^{\Psi}_{F_N}$ the truncated characteristic function of F_N, i.e.

$$\varphi^{\Psi}_{F_N}(\boldsymbol{\lambda}) = \mathbf{E}\left(\Psi_N e^{\mathtt{i}\langle \boldsymbol{\lambda}, F_N\rangle}\right) = \varphi^{\Psi}_N(1,\boldsymbol{\lambda}) \tag{26}$$

with $\varphi^{\Psi}_N(1,\boldsymbol{\lambda})$ from (9).

We define the approximate density of F_N via the Fourier inversion of the dominant part of $p^{\Psi}_{F_N}$ defined by (26).

Definition 1 *For every $N \geq 1$, we define the approximate density p_N by*

$$p_N(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}}^{d} e^{-\mathtt{i}\langle \boldsymbol{\lambda}, x\rangle}\varphi_N(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \qquad x \in \mathbb{R}^d \tag{27}$$

where φ_N is given by (23).

Let us first notice that in the case $d=1$, we can obtain an explicit expression for p_N in the next result. Note that for $d=1$, assuming $C_{1,1}=1$ and $\mathbf{E}Z_1Z_2=\rho$, we have from (23)

$$\varphi_N(\lambda) = e^{-\frac{\lambda^2}{2}} - \mathtt{i}\frac{1}{3}\lambda^3\rho e^{-\frac{\lambda^2}{2}}, \text{ for every } \lambda \in \mathbb{R}. \tag{28}$$

Recall that $p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ denotes the density of the standard normal law. By H_k we denote the Hermite polynomial of degree $k \geq 0$ given by $H_0(x)=1$ and for $k \geq 1$

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}}\frac{d^k}{dx^k}\left(e^{-\frac{x^2}{2}}\right).$$

Proposition 4 *If p_N is given by (27) then for every $x \in \mathbb{R}$,*

$$p_N(x) = p(x) - \frac{\rho}{3}\gamma_N H_3(x)p(x).$$

Proof: This follows from (28) and the formula

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathtt{i}\lambda x}(\mathtt{i}\lambda)^k\Phi(\lambda)d\lambda = H_k(x)p(x) \tag{29}$$

for every $k \geq 0$ integer. Recall the notation $\Phi(\lambda) := e^{-\frac{\lambda^2}{2}}$. ■

We prefer to work with the truncated density of F_N, which can be easier controlled.

Definition 2 *The local (or truncated) density of the random variable F_N is defined as the inverse Fourier transform of the truncated characteristic function*

$$p^{\Psi}_{F_N}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-\mathtt{i}\langle \boldsymbol{\lambda}, x\rangle}\varphi^{\Psi}_{F_N}(\boldsymbol{\lambda})d\boldsymbol{\lambda}$$

for $x \in \mathbb{R}^d$.

The local density is well-defined since obviously the truncated characteristic function $\varphi^{\Psi}_{F_N}$ is integrable over $\mathbb{R}^d$.

Our purpose is to show that the local density $p^{\Psi}_{F_N}$ is well-approximated by the approximate density p_N given by (27) in the sense that for every $\boldsymbol{\alpha} \in \mathbb{Z}^d_+$

$$\sup_{x\in\mathbb{R}^d}\left|x^{\boldsymbol{\alpha}}\left(p^{\Psi}_{F_N}(x) - p_N(x)\right)\right|.$$

is of order $o(\gamma_N)$. This will be showed in the next result, based on the asymptotic behavior of the characteristic function of F_N.

Theorem 2 *For every $p \geq 1$, and for every integer $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_d) \in \mathbb{Z}^d_+$*

$$\sup_{x\in\mathbb{R}^d}\left|x^{\boldsymbol{\alpha}}\left(p^{\Psi}_{F_N}(x) - p_N(x)\right)\right| \leq c\|1-\Psi_N\|_p + o(\gamma_N) = o(\gamma_N). \tag{30}$$

Proof: We have, by integrating by parts

$$\begin{aligned}
x^{\boldsymbol{\alpha}}\left(p^{\Psi}_{F_N}(x) - p_N(x)\right) &= x^{\boldsymbol{\alpha}}\frac{1}{(2\pi)^d}\int_{\mathbb{R}} e^{-\mathtt{i}\langle \boldsymbol{\lambda}, x\rangle}\left(\varphi^{\psi}_{F_N}(\boldsymbol{\lambda} - \varphi_N(\lambda)\right)d\boldsymbol{\lambda} \\
&= \frac{1}{(2\pi)^d(-\mathtt{i})^{|\boldsymbol{\alpha}|}}\int_{\mathbb{R}^d} e^{-\mathtt{i}\langle \boldsymbol{\lambda}, x\rangle}\frac{\partial^{\boldsymbol{\alpha}}}{\partial\boldsymbol{\lambda}^{\boldsymbol{\alpha}}}\left(\varphi^{\psi}_{F_N}(\boldsymbol{\lambda}) - \varphi_N(\boldsymbol{\lambda})\right)d\boldsymbol{\lambda}
\end{aligned}$$

and then, by using (24) and Lemma 3 we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d}\left|x^{\boldsymbol{\alpha}}\left(p^{\Psi}_{F_N}(x) - p_N(x)\right)\right| &\leq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left|\frac{d^{\boldsymbol{\alpha}}}{d\boldsymbol{\lambda}^{\alpha}}\left(\varphi^{\psi}_{F_N}(\lambda) - \varphi_N(\boldsymbol{\lambda})\right)\right|d\boldsymbol{\lambda} \\
&\leq c\int_{\mathbb{R}^d}(1+|\boldsymbol{\lambda}|)^{-\ell+2}d\boldsymbol{\lambda}\left(o(\gamma_N) + c\|1-\Psi_N\|_p\right) \\
&\leq o(\gamma_N) + c\|1-\Psi_N\|_p = o(\gamma_N)
\end{aligned}$$

where we use $\ell - 2 > d$ since $\ell \geq d+3$ in **[C2]**. The fact that $c\|1-\Psi_N\|_p = o(\gamma_N)$ follows from the choice of our truncation function Ψ_N. ■

Let us make some comments on the above result.

Remark 2
- *Theorem 2 extends the inequality (3) when F_N belongs to the Wiener chaos of order $q \geq$ for every $N \geq 1$. It is known that for every $N \geq 1$, we have that $\gamma_N^2 = Var(\frac{1}{q}\|DF_N\|^2_H)$ behaves as $\mathbf{E}F_N^4 - 3$, more precisely (see Section 5.2.2 in [8])*

$$\gamma_N^2 \leq \frac{q-1}{3q}(\mathbf{E}F_N^4 - 3) \leq (q-1)\gamma_N^2.$$

It also extends some results in [3] (in particular Theorem 6.2 and Corollary 6. 6).

- *The appearance of truncation function Ψ in the right-hand side of (30) replaces the condition (2) on the existence of the negative moments of the Malliavin derivative.*

As a consequence of Theorem 2 we will obtain the asymptotic expansion for the expectation of measurable functionals of F_N.

Theorem 3 *Let p_N be given by (27) and fix $M \geq 0, \gamma > 0$. Then*

$$\sup_{f \in \mathcal{E}(M,\gamma)} \left| \mathbf{E} f(F_n) - \int_{\mathbb{R}^d} f(x) p_N(x) dx \right| = o(\gamma_N) \tag{31}$$

where $\mathcal{E}(M,\gamma)$ is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $|f(x)| \leq M(1 + |x|^\gamma)$ for every $x \in \mathbb{R}^d$.

Proof: For any measurable function f that satisfies the assumptions in the statement, we can write

$$\begin{aligned} \mathbf{E} f(F_n) - \int_{\mathbb{R}^d} f(x) p_N(x) dx &= \mathbf{E} f(F_N) - \mathbf{E} f(F_N) \Psi_N + \mathbf{E} f(F_N) \Psi_N - \int_{\mathbb{R}^d} f(x) p_N(x) dx \\ &= \mathbf{E} f(F_N)(1 - \Psi_N) + \int_{\mathbb{R}^d} f(x) \left[p^{\psi}_{F_N}(x) - p_N(x) \right] dx \end{aligned}$$

with $p^{\psi}_{F_N}$ defined in (26). Then for every $p > 1$ and for $\alpha \geq 0$ large enough (it may depend on M, γ) with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, we have, by using Hölder's inequality,

$$\begin{aligned} & \left| \mathbf{E} f(F_n) - \int_{\mathbb{R}^d} f(x) p_N(x) dx \right| \\ \leq & \left(\mathbf{E} \left| f(F_N) \right|^{p'} \right)^{\frac{1}{p'}} \|1 - \Psi_N\|_p + \int_{\mathbb{R}^d} |f(x)| \left(\frac{1}{1 + |x|^2} \right)^{\frac{\alpha}{2}} dx \times \sup_{x \in \mathbb{R}^d} \left(1 + |x|^2 \right)^{\frac{\alpha}{2}} \left[p^{\psi}_{F_N}(x) - p_N(x) \right] \\ \leq & \left(\mathbf{E} \left| f(F_N) \right|^{p'} \right)^{\frac{1}{p'}} \|1 - \Psi_N\|_p + (c \|1 - \Psi_N\|_p + o(\gamma_N)) \\ \leq & \; C \|1 - \Psi_N\|_p + o(\gamma_N) = o(\gamma_N). \end{aligned}$$

where we used (30). So the desired conclusion is obtained. ■

We insert some remarks related to the above result.

Remark 3
- *Theorem 3 is related to Theorem 9.3.1 in [8] (which reproduces the main result of [9]). Notice that we do not assume the differentiability of f and our result includes the uniform control in the case of the Kolmogorov distance.*
- *In particular, the asymptotic expansion (31) holds for any bounded function f and for polynomial functions, but the class of examples is obviously larger.*

In the case of sequences in a Wiener chaos of fixed order, we have the following result, which follows from the above results and from the hypercontractivity property (64).

Corollary 1 *Assume* $(F_N)_{N\geq 1} = (I_q(f_N))_{N\geq 1}$ *(with* $f_N \in H^{\otimes q}$ *for every* $N \geq 1$*) be a random sequence in the* q *th Wiener chaos* $(q \geq 2)$*. Assume* $\mathbf{E}F_N^2 = 1$ *for every* $N \geq 1$ *and suppose that the condition* **[C1]** *holds with* $\gamma_N = Var\left(\frac{\|DF_N\|_H^2}{q}\right)$ *i.e.*

$$\left(F_N, \gamma_N^{-1}\left(\frac{\|DF_N\|_H^2}{q} - 1\right)\right) \to^{(d)}_{N\to\infty} (Z_1, Z_2)$$

where (Z_1, Z_2) *is a centered Gaussian vector with* $\mathbf{E}Z_1^2 = \mathbf{E}Z_2^2 = 1$ *and* $\mathbf{E}Z_1Z_2 = \rho \in (0,1)$*. Then we have the asymptotic expansions (30) and (31).*

Proof: The results follows from Theorems 2 and 3 by noticing that (64) ensures that the conditions **[C2]** and **[C3]** are satisfied. ∎

5 Examples

In the last part of our work, we will present three examples in order two illustrate our theory. The first example concerns a one-dimensional sequence of random variables in the second Wiener chaos and it is inspired from [8], Chapter 9. The second example is a multidimensional one and it involves quadratic functions of two correlated fractional Brownian motions while the last example involves a sequence in a finite sum of Wiener chaoses.

5.1 A one-dimensional in a Wiener chaos of a fixed order: Quadratic variations of stationary Gaussian sequences

We consider a centered stationary Gaussian sequence $(X_k)_{k\in\mathbb{Z}}$ with $\mathbf{E}X_k^2 = 1$ for every $k \in \mathbb{Z}$. We set

$$\rho(l) := \mathbf{E}X_0X_l \text{ for every } l \in \mathbb{Z}$$

so $\rho(0) = 1$ and $|\rho(l)| \leq 1$ for every $l \in \mathbb{Z}$.

Without loss of generality and in order to fit to our context, we assume that $X_k = W(h_k)$ with $\|h_k\|_H = 1$ for every $k \in \mathbb{Z}$, where $(W(h), h \in H)$ is an isonormal process as defined in Section 6. Define the quadratic variation sequence

$$V_N = \frac{1}{\sqrt{N}} \sum_{k=0}^{N} \left(X_k^2 - 1\right), \quad N \geq 1. \tag{32}$$

and let $v_N := \mathbf{E}V_N^2$. Assume $\rho \in \ell^2(\mathbb{Z})$ (the Banach space of 2-summable functions endowed with the norm $\|\rho\|_{\ell^2(Z)}^2 = \sum_{k\in\mathbb{Z}} \rho(k)^2$). In this case, we know from [8] that the deterministic sequence v_N converges to the constant $2\sum_{k\in\mathbb{Z}}\rho(k)^2 > 0$ and consequently it plays no role in the sequel.

Define the renormalized sequence

$$F_N = \frac{V_N}{\sqrt{v_N}} = I_2\left(\frac{1}{\sqrt{Nv_N}}\sum_{k=0}^{N} h_k^{\otimes 2}\right) \text{ for every } N \geq 1. \tag{33}$$

Obviously $\mathbf{E}F_N = 0$ and $\mathbf{E}F_N^2 = 1$ for every $N \geq 1$. It is well known (see [2], see also Chapter 7 in [8]) that, as $N \to \infty$

$$F_N \to^{(d)} Z \sim N(0,1).$$

Let us see that the sequence $(F_N)_{N\geq 1}$ satisfies the conditions conditions **[C1]**-**[C3]** with

$$\gamma_N^2 = Var\left(\frac{1}{2}\|DF_N\|_H^2\right). \tag{34}$$

Actually, this follows from the computations contained in Section 5 of [8], but we recall the main ideas because they are needed later. Notice that $\langle DF_N, D(-L)^{-1}F_N\rangle_H = \frac{1}{2}\|DF_N\|_H^2$ since F_N belongs to the second Wiener chaos.

To prove **[C1]**, we need to assume

$$\rho \in \ell^{\frac{4}{3}}(\mathbb{Z}). \tag{35}$$

In this case γ_N^2 (which coincides with the fourth cumulant of F_N, denoted $k_4(F_N)$) behaves as $c\frac{1}{N}$ for N large, see Theorem 7.3.3 in [8]. The proof of Theorem 9.5.1 in [8] implies that, as $N \to \infty$,

$$\left(F_N, \gamma_N^{-1}\left(\frac{\|DF_N\|_H^2}{2} - 1\right)\right) = \left(F_N, \gamma_N^{-1}(\langle DF_N, D(-L)^{-1}F_N\rangle_H - 1\right) \to^{(d)} (Z_1, Z_2)$$

where (Z_1, Z_2) is a correlated centered Gaussian vector with $\mathbf{E}Z_1^2 = \mathbf{E}Z_2^2 = 1$. Therefore condition **[C1]** is fullfiled.

Conditions **[C2]** and **[C3]** are satisfied since F_N belongs to the second Wiener chaos and one can use the hypercontractivity property of multiple integrals (64). Consequently Theorems 3 and 2 can be then applied to the sequence (32).

Remark 4 *Let $(B_t^H)_{t\in\mathbb{R}}$ be a (two-sided) fractional Brownian motion with Hurst parameter $H \in (0,1)$. We recall that B^H is a centered Gaussian process with covariance function*

$$\mathbf{E}B_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

for every $s,t \in \mathbb{R}$. Let $X_k = B_{k+1}^H - B_k^H$ for every $k \in \mathbb{Z}$. In this case $\rho_H(k) := \mathbf{E}X_0X_k$ is given by

$$\rho_H(k) = \frac{1}{2}\left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\right), \quad \forall k \in \mathbb{Z}. \tag{36}$$

Since $\rho_H(k)$ behaves as $H(2H-1)|k|^{2H-2}$ for $|k|$ large enough, condition (35) is fullfiled for $0 < H < \frac{5}{8}$. Therefore Theorems 2 and 3 can be applied for $H \in (0, \frac{5}{8})$.

5.2 A multidimensional: quadratic variations of correlated fractional Brownian motions

For every $H \in (0,1)$, denote by

$$f_t^H(u) = d(H)\left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right), \quad t \in \mathbb{R}$$

with $d(H)$ a positive constant that ensures that $\int_{\mathbb{R}} f_t^H(u)^2 du = |t|^2$ for every $t \in \mathbb{R}$. For every $k \in \mathbb{Z}$, let

$$L_k^H = f_{k+1}^H - f_{k+1}^H. \tag{37}$$

Consider $(W_t)_{t\in\mathbb{R}}$ a Wiener process on the whole real line and define the two fractional Brownian motions

$$B_t^{H_i} = \int_{\mathbb{R}} f_t^{H_i}(s) dW_s, \quad i = 1, 2.$$

The fBms B^{H_1} and B^{H_2} are correlated. We actually have (see [5], Section 4), for every $k, l \in \mathbb{Z}$,

$$\begin{aligned} & \mathbf{E}(B_{k+1}^{H_1} - B_k^{H_1})(B_{l+1}^{H_2} - B_l^{H_2}) = \langle f_{k+1}^{H_1} - f_k^{H_1}, f_{l+1}^{H_2} - f_l^{H_2} \rangle = \langle L_k^{H_1}, L_l^{H_2} \rangle \\ = \; & D(H_1, H_2)\rho_{\frac{H_1+H_2}{2}}(k-l) \end{aligned} \tag{38}$$

where $D(H_1, H_2)$ is an explicit constant such that $D(H_i, H_i) = 1$ for $i = 1, 2$.

Assume $H_1, H_2 \in \left(0, \frac{3}{4}\right)$. Define the quadratic variations, for $i = 1, 2$ and for $N \geq 1$

$$V_N^{(i)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\left(B_{k+1}^{H_i} - B_k^{H_i} \right)^2 - 1 \right]$$

and

$$F_N^{(i)} = \frac{V_N^{(i)}}{\sqrt{v_n^{(i)}}} \tag{39}$$

with $v_N^{(i)} = \mathbf{E}(V_N^{(i)})^2$, $i = 1, 2$. Recall that $v_N^{(i)} \to_{N\to\infty} c_{v_i} := 2\sum_{k\in\mathbb{Z}} \rho_{H_i}(k)^2$ for $i = 1, 2$.

Let us show that the sequence $F_N = (F_N^{(1)}, F_N^{(2)})$ satisfies the conditions **[C1]**-**[C3]** with $\gamma_N = N^{-\frac{1}{2}}$. We will focus on **[C1]** (the assumptions **[C2]** and **[C3]** can be easily checked since we deal with elements of the second Wiener chaos and we can apply the hypercontractivity (64)).

From the previous paragraph we know that for $i = 1, 2$, the random vectors

$$\left(F_N^{(i)}, \sqrt{N}\langle DF_N^{(i)}, D(-L)^{-1}F_N^{(i)} \rangle_H - 1 \right) \tag{40}$$

converge in distribution, when $N \to \infty$, to some centered Gaussian vectors $(Z_i^{(1)}, Z_i^{(2)})$ with $\mathbf{E}(Z_i^{(1)})^2 = C_{i,i}$, $i = 1, 2$ and with nontrivial correlation.

The main tool to prove the the two-dimensional sequence $(F_N^{(1)}, F_N^{(2)})$ satisfies **[C1]** is the multidimensional version of the Fourth Moment Theorem. Let us recall this result which says that for sequences of random vectors on Wiener chaos, the componentwise convergence to the Gaussian law implies the joint convergence (see [12] or Theorem 6.2.3 in [8]).

Theorem 4 *Let $d \geq 2$ and $q_1, ..., q_d \geq 1$ integers. Consider the random vector*

$$F_N = \left(F_N^{(1)}, \ldots, F_N^{(d)}\right) = \left(I_{q_1}(f_N^{(1)}), \ldots, I_{q_d}(f_N^{(d)})\right)$$

with $f_N^{(i)} \in H^{\otimes q_i}$, $i = 1, .., d$. Assume that

$$\mathbf{E}F_N^{(i)} F_N^{(j)} \to_{N\to\infty} C(i,j) \text{ for every } i, j = 1, .., d. \tag{41}$$

Then the following two conditions are equivalent:

- *F_N converges in law, as $N \to \infty$, to a d dimensional centered Gaussian vector with covariance matrix C.*
- *For every $i = 1, .., d$, $F_N^{(i)}$ converges in law, as $N \to \infty$, to $N(0, C(i,i))$.*

We will need the two auxiliary lemmas below. The first concerns the asymptotic covariance of $F_N^{(1)}$ and $F_N^{(2)}$.

Lemma 4 *Let $F_N^{(1)}, F_N^{(2)}$ be defined by (39). Assume $H_1, H_2 \in \left(0, \frac{3}{4}\right)$. Then for every $H_1, H_2 \in (0, 1)$,*

$$\mathbf{E}F_N^{(1)} F_N^{(2)} \to_{N\to\infty} C_{1,2}$$

with

$$C_{1,2} = \frac{2D^2(H_1, H_2)}{\sqrt{c_{v_1} c_{v_2}}} \sum_{l\in\mathbb{Z}} \rho_{\frac{H_1+H_2}{2}}(l)^2. \tag{42}$$

Consequently,

$$\left(F_N^{(1)}, F_N^{(2)}\right) \to^{(d)}_{N\to\infty} (Z_1^{(1)}, Z_1^{(2)}) \tag{43}$$

where $(Z_1^{(1)}, Z_1^{(2)})$ is a Gaussian vector with $\mathbf{E}Z_1^{(1)} = \mathbf{E}Z_1^{(2)} = 0$ and

$$\mathbf{E}(Z_1^{(1)})^2 = C_{1,1} = 1, \;\; \mathbf{E}(Z_1^{(2)})^2 = C_{2,2} > 0, \;\; \mathbf{E}(Z_1^{(1)} Z_1^{(2)}) = C_{1,2}$$

with $C_{1,2}$ given by (42). Recall that we denoted by " $\to^{(d)}$ " the convergence in distribution.

Proof: By the chaos expression of $F_N^{(i)}$ (33) and by using the isometry (58)

$$\begin{aligned}
\mathbf{E}F_N^{(1)}F_N^{(2)} &= \frac{1}{N\sqrt{v_N^{(1)}v_N^{(2)}}}\mathbf{E}I_2\left(\sum_{i=0}^{N-1}(L_i^{H_1})^{\otimes 2}\right)I_2\left(\sum_{j=0}^{N-1}(L_j^{H_2})^{\otimes 2}\right)\\
&= \frac{1}{N\sqrt{v_N^{(1)}v_N^{(2)}}}2\sum_{i,j=0}^{N-1}\langle L_i^{H_1}, L_j^{H_2}\rangle^2
\end{aligned}$$

with L^H from (37). From the identity (38)

$$\begin{aligned}
\mathbf{E}F_N^{(1)}F_N^{(2)} &= \frac{1}{N\sqrt{v_N^{(1)}v_N^{(2)}}}2D^2(H_1,H_2)\sum_{i,j=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(i-j)^2\\
&= \frac{2}{\sqrt{v_N^{(1)}v_N^{(2)}}}\sum_{l=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(l)^2(1-\frac{l}{N})\\
&\to_{N\to\infty} \frac{2D^2(H_1,H_2)}{\sqrt{c_{v_1}c_{v_2}}}\sum_{l\in\mathbb{Z}}\rho_{\frac{H_1+H_2}{2}}(l)^2.
\end{aligned}$$

The sum $\sum_{l\in\mathbb{Z}}\rho_{\frac{H_1+H_2}{2}}(l)^2$ is finite since $H_1+H_2<\frac{3}{2}$. The limit (43) will then follow from Theorem 4. ■

We will also need to control the speed of convergence of the covariance function of $F_N^{(1)}$ and $F_N^{(2)}$. In this case we need to impose an additional restriction on the Hurst parameters.

Lemma 5 *Let $F_N^{(1)}, F_N^{(2)}$ be defined by (39) and let $C_{1,2}$ be given by (42). Assume $H_1, H_2 \in \left(0, \frac{5}{8}\right)$. Then*

$$\sqrt{N}(\mathbf{E}F_N^{(1)}F_N^{(2)} - C_{1,2}) \to_{N\to\infty} 0.$$

Proof: For every $N \geq 1$ we can write

$$\begin{aligned}
&\sqrt{N}(\mathbf{E}F_N^{(1)}F_N^{(2)} - C_{1,2})\\
=\ & 2D^2(H_1,H_2)\sqrt{N}\left[\frac{1}{\sqrt{v_N^{(1)}v_N^{(2)}}}\sum_{l=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(l)^2(1-\frac{l}{N}) - \frac{1}{\sqrt{c_{v_1}c_{v_2}}}\sum_{l\in\mathbb{Z}}\rho_{\frac{H_1+H_2}{2}}(l)^2\right].
\end{aligned}$$

Thus, for N large enough, using that $v_N^{(i)}$ converges to c_{v_i} for $i=1,2$, we can write

$$\begin{aligned}
&\sqrt{N}\left|\mathbf{E}F_N^{(1)}F_N^{(2)}-C_{1,2}\right| \\
\leq\ & C\sqrt{N}\sum_{l\geq N}\rho_{\frac{H_1+H_2}{2}}(l)^2+C\frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}l\times\rho_{\frac{H_1+H_2}{2}}(l)^2+C\sqrt{N}\left|\frac{1}{\sqrt{v_N^{(1)}v_N^{(2)}}}-\frac{1}{\sqrt{c_{v_1}c_{v_2}}}\right| \\
:=\ & T_{1,N}+T_{2,N}+T_{3,N}.
\end{aligned}$$

Since for l large, the function $\rho_H(l)$ behaves as $C\times|l|^{2H-2}$, we get, since $H_1,H_2<\frac{5}{8}$ (so $H_1+H_2<\frac{5}{4}$)

$$T_{1,N}\leq C\sqrt{N}N^{2H_1+2H_2-3}=CN^{2H_1+2H_2-\frac{5}{2}}\to_{N\to\infty}0$$

and

$$T_{2,N}\leq C\frac{1}{\sqrt{N}}N^{2H_1+2H_2-2}=CN^{2H_1+2H_2-\frac{5}{2}}\to_{N\to\infty}0.$$

Concerning the summand $T_{3,N}$, we have for N large enough

$$\begin{aligned}
T_{3,N}\ &\leq\ C\sqrt{N}\left|\frac{1}{\sqrt{v_N^{(1)}}}-\frac{1}{\sqrt{c_{v_1}}}\right|+C\sqrt{N}\left|\frac{1}{\sqrt{v_N^{(2)}}}-\frac{1}{\sqrt{c_{v_2}}}\right| \\
&\leq\ C\sqrt{N}\left(|v_N^{(1)}-c_{v_1}|+|v_N^{(2)}-c_{v_2}|\right) \\
&\leq\ C\sqrt{N}\left(\sum_{l\geq N}\rho_{H_1}^2(l)+\sum_{l\geq N}\rho_{H_2}^2(l)\right)\leq C(N^{4H_1-\frac{5}{2}}+N^{4H_2-\frac{5}{2}})\to_{N\to\infty}0
\end{aligned}$$

and the conclusion follows. ■

The following lemma is also important in order to check **[C1]**.

Lemma 6 *Assume $H_1<\frac{5}{8}$ and $H_2<\frac{5}{8}$. Let $F_N^{(1)},F_N^{(2)}$ be defined by (39) and let $C_{1,2}$ be given by (42). Then*

$$\sqrt{N}\left(\frac{1}{2}\langle DF_N^{(1)},DF_N^{(2)}\rangle_H-C_{1,2}\right)$$

converges in distribution, as N goes to infinity, to a Gaussian random variable.

Proof: Since $\mathbf{E}F_N^{(1)}F_N^{(2)}=\frac{1}{2}\langle DF_N^{(1)},DF_N^{(2)}\rangle_H$ and by Lemma 5

$$\sqrt{N}\left(\mathbf{E}\frac{1}{2}\langle DF_N^{(1)},DF_N^{(2)}\rangle_H-C_{1,2}\right)\to_{N\to\infty}0$$

it suffices to show that the sequence $(Y_N)_{N\geq 1}$ with

$$Y_N := \frac{1}{2}\sqrt{N}\left(\langle DF_N^{(1)}, DF_N^{(2)}\rangle_H - \mathbf{E}\langle DF_N^{(1)}, DF_N^{(2)}\rangle_H\right) \tag{44}$$

converges in distribution as N goes to infinity, to a Gaussian random variable. We will apply the Fourth Moment Theorem. In order to do it, we need to show that

$$\mathbf{E}Y_N^2 \to_{N\to\infty} C_0 > 0 \text{ and } Var(\|DY_N\|_H^2) \to_{N\to\infty} 0. \tag{45}$$

From (33) and the product formula for multiple integrals (60), we can write

$$\begin{aligned} Y_N &= \frac{1}{\sqrt{Nv_N^{(1)}v_N^{(2)}}} I_2\left(\sum_{k,l=0}^{N-1}(L_k^{H_1}\otimes L_l^{H_2})\langle L_k^{H_1}, L_l^{H_2}\rangle\right) \\ &= D(H_1,H_2)\frac{1}{\sqrt{Nv_N^{(1)}v_N^{(2)}}} I_2\left(\sum_{k,l=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)(L_k^{H_1}\otimes L_l^{H_2})\right) \end{aligned} \tag{46}$$

with L_k^H defined by (37). Consequently, since any four functions of two variable f,g,u,v we have

$$\langle f\tilde{\otimes}g, u\tilde{\otimes}v\rangle = \frac{1}{2}(\langle f,u\rangle\langle g,v\rangle + \langle f,v\rangle\langle g,u\rangle)$$

we will obtain

$$\begin{aligned} \mathbf{E}Y_N^2 &= D^2(H_1,H_2)\frac{1}{Nv_N^{(1)}v_N^{(2)}}\sum_{k,l,i,j=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)\langle L_k^{H_1}\tilde{\otimes}L_l^{H_2}, L_i^{H_1}\tilde{\otimes}L_j^{H_2}\rangle \\ &= D^2(H_1,H_2)\frac{1}{Nv_N^{(1)}v_N^{(2)}}\sum_{k,l,i,j=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j) \\ &\quad\times\frac{1}{2}\left[\rho_{H_1}(k-i)\rho_{H_2}(l-j) + D(H_1,H_2)^2\rho_{\frac{H_1+H_2}{2}}(i-l)\rho_{\frac{H_1+H_2}{2}}(k-j)\right]. \end{aligned} \tag{47}$$

For two sequences $(u(n), n\in\mathbb{Z})$ and $(v(n), n\in\mathbb{Z})$, we define their convolution by

$$(u*v)(j) = \sum_{n\in\mathbb{Z}} u(n)v(j-n).$$

We will need the Young 's inequality: if $s,p,q\geq 1$ with $\frac{1}{s}+1 = \frac{1}{p}+\frac{1}{q}$ then

$$\|u*v\|_{\ell^s(\mathbb{Z})} \leq \|u\|_{\ell^p(\mathbb{Z})}\|v\|_{\ell^q(\mathbb{Z})}. \tag{48}$$

By the proof of Theorem 7.3.3 in [8], we have

$$\frac{1}{N}\sum_{k,l,i,j=}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)\rho_{\frac{H_1+H_2}{2}}(i-l)\rho_{\frac{H_1+H_2}{2}}(k-j)\to_{N\to\infty}\langle\rho^{*3}_{\frac{H_1+H_2}{2}},\rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}$$

where $\langle\rho^{*3}_{\frac{H_1+H_2}{2}},\rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}<\infty$ when $H_1+H_2\leq\frac{5}{4}$ which implies that $\rho_{\frac{H_1+H_2}{2}}\in\ell^{\frac{4}{3}}(\mathbb{Z})$. The second summand in (47) can be handled as follows

$$\begin{aligned}
&\frac{1}{N}\sum_{k_1,k_2,k_3,k_4=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k_4-k_3)\rho_{\frac{H_1+H_2}{2}}(k_2-k_1)\rho_{H_1}(k_1-k_4)\rho_{H_2}(k_3-k_2)\\
=\;&\frac{1}{N}\sum_{k_1=0}^{N-1}\sum_{k_2,k_3,k_4=-k_1}^{N-1-k_1}\rho_{\frac{H_1+H_2}{2}}(k_4-k_3)\rho_{\frac{H_1+H_2}{2}}(k_2)\rho_{H_1}(k_4)\rho_{H_2}(k_3-k_2)\\
=\;&\frac{1}{N}\sum_{k_2,k_3,k_4\in\mathbb{Z}}\rho_{\frac{H_1+H_2}{2}}(k_4-k_3)\rho_{\frac{H_1+H_2}{2}}(k_2)\rho_{H_1}(k_4)\rho_{H_2}(k_3-k_2)\\
&\left[1\vee\left(1-\frac{\max(k_2,k_3,k_4)}{N}\right)-0\wedge\left(\frac{\min(k_2,k_3,k_4)}{N}\right)\right]1_{|k_2|<N,|k_3|<N,|k_4|<N}\\
&\to_{N\to\infty}\sum_{k_2,k_3,k_4\in\mathbb{Z}}\rho_{\frac{H_1+H_2}{2}}(k_4-k_3)\rho_{\frac{H_1+H_2}{2}}(k_2)\rho_{H_1}(k_4)\rho_{H_2}(k_3-k_2)\\
=\;&\langle\rho_{\frac{H_1+H_2}{2}}*\rho_{H_1}*\rho_{H_2},\rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}.
\end{aligned}\tag{49}$$

Again notice that $\langle|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|,|\rho_{\frac{H_1+H_2}{2}}|\rangle_{\ell^2(\mathbb{Z})}<\infty$ since by Hölder's and Young's inequalities, by using $\rho_{\frac{H_1+H_2}{2}}\in\ell^{\frac{4}{3}}(\mathbb{Z})$,

$$\begin{aligned}
\langle|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|,|\rho_{\frac{H_1+H_2}{2}}|\rangle_{\ell^2(\mathbb{Z})}&\leq C\||\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|\|_{\ell^4(\mathbb{Z})}\\
&\leq C\||\rho_{\frac{H_1+H_2}{2}}|\|_{\ell^{\frac{4}{3}}(\mathbb{Z})}\times\||\rho_{H_1}|*|\rho_{H_2}|\|_{\ell^2(\mathbb{Z})}\\
&\leq\|\rho_{\frac{H_1+H_2}{2}}\|_{\ell^{\frac{4}{3}}(\mathbb{Z})}\times\||\rho_{H_1}|\|_{\ell^{\frac{4}{3}}(\mathbb{Z})}\times\||\rho_{H_2}|\|_{\ell^{\frac{4}{3}}(\mathbb{Z})}
\end{aligned}$$

where we applied twice (48). Thus

$$\mathbf{E}Y_N^2\to_{N\to\infty}C_0:=\frac{D^2(H_1,H_2)}{c_{v_1}c_{v_2}}\frac{1}{2}\left(\langle\rho_{\frac{H_1+H_2}{2}}*\rho_{H_1}*\rho_{H_2},\rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}+D^2(H_1,H_2)\langle\rho^{*3}_{\frac{H_1+H_2}{2}},\rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}\right)$$

and the first part of (45) is obtained. Next, we check the convergence of the Malliavin derivative in (45). From (46) and the product formula (60)

$$\|DY_N\|_H^2=\frac{4D(H_1,H_2)^2}{Nv_N^{(1)}v_N^{(2)}}I_2\left(\sum_{i,j,k.l=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)(L_k^{H_1}\tilde{\otimes}L_l^{H_2})\otimes_1(L_i^{H_1}\tilde{\otimes}L_j^{H_2})\right)$$

where the contraction $\otimes_1$ is defined in (63). Since for every $f,g,u,v \in L^2(\mathbb{R}^2)$, we have

$$(f\tilde{\otimes}g)\otimes_1(u\tilde{\otimes}v) = \frac{1}{4}\left(\langle f,u\rangle(g\otimes v)+\langle f,v\rangle(g\otimes u)+\langle g,u\rangle(f\otimes v)+\langle g,v\rangle(f\otimes u)\right)$$

we obtain, via (38)

$$\begin{aligned}\|DY_N\|_H^2 &= \frac{D(H_1,H_2)^2}{Nv_N^{(1)}v_N^{(2)}} I_2\Bigg(\sum_{i,j,k.l=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)\\ &\Big[\rho_{H_1}(k-i)(L_l^{H_2}\otimes L_j^{H_2})+D(H_1,H_2)\rho_{\frac{H_1+H_2}{2}}(k-j)(L_l^{H_2}\otimes L_i^{H_1})\\ &+D(H_1,H_2)\rho_{\frac{H_1+H_2}{2}}(l-i)(L_k^{H_1}\otimes L_j^{H_2})+\rho_{H_2}(l-j)(L_k^{H_1}\otimes L_i^{H_1})\Big]\Bigg)\end{aligned}$$

and by the isometry formula (58)

$$\begin{aligned}& Var(\|DY_N\|_H^2)\\ =\ & C_1\frac{1}{N^2(v_N^{(1)})^2(v_N^{(2)})^2}\sum_{i,j,k.l,i',j',k',l'=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)\rho_{\frac{H_1+H_2}{2}}(k'-l')\rho_{\frac{H_1+H_2}{2}}(i'-j')\\ & \times\rho_{\frac{H_1+H_2}{2}}(k-i)\rho_{\frac{H_1+H_2}{2}}(k'-i')\rho_{\frac{H_1+H_2}{2}}(l-l')\rho_{\frac{H_1+H_2}{2}}(j-j')\\ & +C_2\frac{1}{N^2(v_N^{(1)})^2(v_N^{(2)})^2}\sum_{i,j,k.l,i',j',k',l'=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)\rho_{\frac{H_1+H_2}{2}}(k'-l')\rho_{\frac{H_1+H_2}{2}}(i'-j')\\ & \times\rho_{H_1}(k-i)\rho_{H_1}(k'-i')\rho_{H_2}(l-l')\rho_{H_2}(j-j')\\ & +C_2\frac{1}{N^2(v_N^{(1)})^2(v_N^{(2)})^2}\sum_{i,j,k.l,i',j',k',l'=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(k-l)\rho_{\frac{H_1+H_2}{2}}(i-j)\rho_{\frac{H_1+H_2}{2}}(k'-l')\rho_{\frac{H_1+H_2}{2}}(i'-j')\\ & \times\rho_{H_1}(k-i)\rho_{\frac{H_1+H_2}{2}}(k'-j')\rho_{H_2}(l-l')\rho_{\frac{H_1+H_2}{2}}(j-i').\end{aligned}$$

We can write, using the convolution symbol

$$\begin{aligned}& Var(\|DY_N\|_H^2)\\ \leq\ & C\frac{1}{N^2}\sum_{l,j'=0}^{N-1}\Big[\left(|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|\right)(l-j')^2\\ & +\left(|\rho_{\frac{H_1+H_2}{2}}||\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|\right)(l-j')^2\\ & +\left(|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|\right)(l-j')\left(|\rho_{\frac{H_1+H_2}{2}}||\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|\right)(l-j')\Big]\\ \leq\ & C\frac{1}{N}\sum_{l=-(N-1)}^{N-1}g(l) \qquad (50)\end{aligned}$$

with

$$g(l)=\left(|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|\right)(l)^2+\left(|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|\right)(l)^2$$

$$+\left(|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|*|\rho_{\frac{H_1+H_2}{2}}|\right)(l)\left(|\rho_{\frac{H_1+H_2}{2}}||\rho_{\frac{H_1+H_2}{2}}|*|\rho_{H_1}|*|\rho_{H_2}|\right)(l).$$

Since for any two sequence $u,v\in\ell^2(\mathbb{Z})$ we have

$$\overline{\lim_{n\to\infty}}(|u|*|v|)(n)\leq\sqrt{\sum_{j\geq M}u(j)^2}\sqrt{\sum_{j\geq M}v(j)^2}$$

for every $M\geq 1$ (see [8], page 168), we obtain as in [8], by using again Young's inequality

$$g(N)\to_{N\to\infty}0.$$

The convergence of $Var(\|DY_N\|_H^2)$ to zero follows from (50) and Césaro theorem. ■

Let us go back to the sequence $F_N=(F_N^{(1)},F_N^{(2)})$ from (39).

Proposition 5 *The two-dimensional random sequence* $F_N=\left(F_N^{(1)},F_N^{(2)}\right)$ *defined by (39) satisfies condition* **[C1]** *with* $\gamma_N=N^{-\frac{1}{2}}$.

Proof: Since for every $N\geq 1$

$$\langle DF_N^{(1)},D(-L)^{-1}F_N^{(2)}\rangle=\langle DF_N^{(2)},D(-L)^{-1}F_N^{(1)}\rangle=\frac{1}{2}\langle DF_N^{(1)},DF_N^{(2)}\rangle \tag{51}$$

we need to show that the random vector

$$\left(F_N^{(1)},F_N^{(2)},\sqrt{N}\left(\frac{1}{2}\|DF_N^{(1)}\|_H^2-1\right),\sqrt{N}\left(\frac{1}{2}\|DF_N^{(2)}\|_H^2-1\right),\sqrt{N}(\frac{1}{2}\langle DF_N^{(1)},DF_N^{(2)}\rangle_H-C_{1,2})\right) \tag{52}$$

converges to a Gaussian vector.

All the components of the above vector are multiple integrals of order 2 or sum of a multiple integral of order 2 and a deterministic term tending to zero (the case of the last component). We will use the multidimensional Fourth Moment Theorem to prove the convergence of (52). From the convergence of the vectors (40), by using Lemmas 4 and 6, it suffices to check the condition (41) from the multidimensional Fourth Moment Theoren (Theorem 4), i.e. the covariances of the components of the vector (52) converge to constants as $N\to\infty$. The covariance of $F_N^{(1)}$ and $F_N^{(2)}$ converges to $C_{1,2}$ (42) due to Lemma 4. Concerning the other limits, we have from (33) (recall the the symbol $\sim$ means that the sides have the same limit as $N\to\infty$)

$$
\begin{aligned}
\mathbf{E}F_N^{(1)}\left(\frac{\|DF_N^{(2)}\|_H^2}{2}-1\right) &\sim C\mathbf{E}\frac{1}{N}I_2\left(\sum_{i=0}^{N-1}(L_k^{H_1})^{\otimes 2}\right)I_2\left(\sum_{j,k=0}^{N-1}\rho_{\frac{H_2+H_2}{2}}(j-k)(L_j^{H_1}\otimes L_k^{H_2})\right)\\
&= C\frac{1}{N}\sum_{i,j,k=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(j-k)\rho_{H_1}(i-j)\rho_{\frac{H_1+H_2}{2}}(i-k)\\
&\quad \to_{N\to\infty} \langle \rho_{\frac{H_1+H_2}{2}} * \rho_{H_1}, \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})} < \infty
\end{aligned}
$$

where the limit is obtained similarly as (49) and the last series is finite by using Young inequality and $\rho_{\frac{H_1+H_2}{2}}, \rho_{H_1}, \rho_{H_2} \in \ell^{\frac{4}{3}}(\mathbb{Z})$. Also, by symmetry

$$
\mathbf{E}F_N^{(2)}\left(\frac{\|DF_N^{(1)}\|_H^2}{2}-1\right) \to_{N\to\infty} \langle \rho_{\frac{H_1+H_2}{2}} * \rho_{H_2}, \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}.
$$

Next, denote by Y_N' the last component of the vector (52), i.e.

$$
Y_N' := \sqrt{N}(\frac{1}{2}\langle DF_N^{(1)}, DF_N^{(2)}\rangle_H - C_{1,2}).
$$

By Lemma 5

$$
\begin{aligned}
\mathbf{E}F_N^{(1)}Y_N' &\sim \mathbf{E}F_N^{(1)}Y_N \sim C\frac{1}{N}\mathbf{E}I_2\left(\sum_{i=0}^{N-1}(L_k^{H_1})^{\otimes 2}\right)I_2\left(\sum_{j,k=0}^{N-1}\rho_{\frac{H_1+H_2}{2}}(j-k)(\Lambda_j^{H_1}\otimes L_k^{H_2})\right)\\
&= \frac{1}{N}\sum_{i,j,k=0}^{N-1}\langle (L_k^{H_1})^{\otimes 2}, (L_j^{H_1}\tilde{\otimes}L_k^{H_2})\rangle\\
&= C\frac{1}{N}\sum_{i,j,k}\rho_{\frac{H_1+H_2}{2}}(j-k)\rho_{H_1}(i-j)\rho_{\frac{H_1+H_2}{2}}(i-k) \to_{N\to\infty} \langle \rho_{\frac{H_1+H_2}{2}} * \rho_{H_1}, \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}
\end{aligned}
$$

and similarly

$$
\mathbf{E}F_N^{(2)}Y_N' \to_{N\to\infty} C\langle \rho_{\frac{H_1+H_2}{2}} * \rho_{H_2}, \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}.
$$

Finally, we can also prove that

$$
\begin{aligned}
\mathbf{E}\left(\frac{\|DF_N^{(1)}\|_H^2}{2}-1\right)\left(\frac{\|DF_N^{(2)}\|_H^2}{2}-1\right) &\sim C\frac{1}{N}\sum_{i,j,k,l}\rho_{H_1}(i-j)\rho_{H_2}(k-l)\langle (L_i^{H_1}\tilde{\otimes}L_j^{H_1}), (L_k^{H_2}\tilde{\otimes}L_l^{H_2})\rangle\\
&= C\frac{1}{N}\sum_{i,j,k.l}\rho_{H_1}(i-j)\rho_{H_2}(k-l)\rho_{\frac{H_1+H_2}{2}}(k-i)\rho_{\frac{H_1+H_2}{2}}(l-j)\\
&\quad \to_{N\to\infty} \langle \rho_{H_1} * \rho_{\frac{H_1+H_2}{2}}, \rho_{H_2} * \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}
\end{aligned}
$$

and

$$\mathbf{E}\left(\frac{\|DF_N^{(1)}\|_H^2}{2}-1\right)Y_N' \sim \mathbf{E}\left(\frac{\|DF_N^{(1)}\|_H^2}{2}-1\right)Y_N \to_{N\to\infty} \langle \rho_{H_1} * \rho_{\frac{H_1+H_2}{2}}, \rho_{H_1} * \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}$$

$$\mathbf{E}\left(\frac{\|DF_N^{(2)}\|_H^2}{2}-1\right)Y_N' \sim \mathbf{E}\left(\frac{\|DF_N^{(2)}\|_H^2}{2}-1\right)Y_N \to_{N\to\infty} \langle \rho_{H_2} * \rho_{\frac{H_1+H_2}{2}}, \rho_{H_2} * \rho_{\frac{H_1+H_2}{2}}\rangle_{\ell^2(\mathbb{Z})}.$$

So, condition (41) is fullfilled and the conclusion follows from Theorem 4. ■

5.3 An example in a sum of finite Wiener chaoses

Consider a sequence $I_2(f_N)$ in a second Wiener chaos that satisfies condition **[C1]**- **[C3]** (take for example the sequence (33)). Consider a "perturbation" $G_N = I_q(g_N)$ with $q \geq 3$ and such that for $N \geq 1$

$$\|g_N\|_{H^{\otimes q}}^2 \leq C\frac{1}{N^{1+\beta}} \tag{53}$$

with some $\beta > 0$. In particular, (53) implies that $G_N \to_{N\to\infty} 0$ in $L^2(\Omega)$. Define,

$$F_N = I_2(f_N) + G_N, \qquad N \geq 1. \tag{54}$$

Clearly F_N verifies **[C2]** . Let us show that F_N satisfies **[C1]** with $\gamma_N = \sqrt{Var\left(\frac{1}{2}\|DI_2(f_N)\|_H^2\right)}$ (which behaves as $C\frac{1}{\sqrt{N}}$ for N large). We have, by the product formula (60)

$$\begin{aligned}\gamma_N^{-1}\left(\langle DF_N, D(-L)^{-1}F_N\rangle - 1\right) &= \gamma_N^{-1}\left(\frac{1}{2}\|DI_2(f_N)\|_H^2 - 1\right) + \frac{1}{q}\gamma_N^{-1}\|DG_N\|_H^2 \\ &\quad +(2+q)\gamma_N^{-1}\left(I_q(f_N \otimes_1 g_N) + (q-1)I_{q-1}(f \otimes_2 g)\right)\end{aligned}$$

Notice that, by (53)

$$\mathbf{E}\left(\gamma_N^{-1}\|DG_N\|_H^2\right) \leq CN^{-\frac{1}{2}}\|g_N\|_{H^{\otimes q}}^2 \leq CN^{-\beta-\frac{1}{2}} \to_{N\to\infty} 0. \tag{55}$$

Also, since from (63) $\|f_N \otimes_k g_N\|_{H^{\otimes(2+q-2k)}} \leq \|f_N\|_{H^{\otimes 2}}\|g_N\|_{H^{\otimes q}}$ for $k = 1, 2$

$$\mathbf{E}\left(\gamma_N^{-1}I_q(f_N \otimes_1 g_N)\right)^2 \leq CN\|f_N \otimes_1 g\|_{H^{\otimes q}}^2 \leq CN\|f_N\|_{H^{\otimes 2}}^2\|g_N\|_{H^{\otimes q}}^2 \leq CN^{-\beta} \tag{56}$$

and

$$\mathbf{E}\left(\gamma_N^{-1}I_{q-2}(f_N \otimes_2 g_N)\right)^2 \leq CN\|f_N \otimes_2 g\|_{H^{\otimes(q-2)}}^2 \leq CN\|f_N\|_{H^{\otimes 2}}^2\|g_N\|_{H^{\otimes q}}^2 \leq CN^{-\beta}. \tag{57}$$

Relations (55), (56) and (57), together with the fact that $\gamma_N^{-1}\left(\frac{1}{2}\|DI_2(f_N) - 1\right)$ converges to a Gaussian law as $N \to \infty$, implies **[C1]**. The same relations together with (64) will give **[C3]**.

6 Appendix: Elements from Malliavin calculus

We briefly describe the tools from the analysis on Wiener space that we will need in our work. For complete presentations, we refer to [10] or [8]. Consider H a real separable Hilbert space and $(W(h), h \in H)$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, which is a centered Gaussian family of random variables such that $\mathbf{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_H$. Denote by I_n the multiple stochastic integral with respect to B (see [10]). This mapping I_n is actually an isometry between the Hilbert space $H^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{H^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $h_n(W(h))$ where $h \in H, \|h\|_H = 1$ and h_n is the Hermite polynomial of degree $n \in \mathbb{N}$

$$h_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as follows: for m, n positive integers,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle \tilde{f}, \tilde{g}\rangle_{H^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{58}$$

It also holds that $I_n(f) = I_n(\tilde{f})$ where $\tilde{f}$ denotes the symmetrization of f.
We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{59}$$

where $f_n \in H^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n \geq 0} n I_n(f_n)$$

if F is given by (59) and it is such that $\sum_{n=1}^{\infty} n^2 n! \|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty$.

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(W(h_1), \ldots, W(h_n))$ (g is a smooth function with compact support and $h_i \in H$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(H)$.

We recall the product formula for multiple integrals. It is well-known that for $f \in H^{\odot n}$ and $g \in H^{\odot m}$

$$I_n(f)I_m(g) = \sum_{r=0}^{n\wedge m} r! \left(\begin{array}{c} n \\ r \end{array} \right) \left(\begin{array}{c} m \\ r \end{array} \right) I_{m+n-2r}(f \otimes_r g) \tag{60}$$

where $f \otimes_r g$ means the r-contraction of f and g. This contraction is defined in the following way. Consider $(e_j)_{j\geq 1}$ a complete orthonormal system in H and let $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ be two symmetric functions with $n, m \geq 1$. Then

$$f = \sum_{j_1,..,j_n \geq 1} \lambda_{j_1,..,j_n} e_{j_1} \otimes ... \otimes e_{j_n} \tag{61}$$

and

$$g = \sum_{k_1,..,k_m \geq 1} \beta_{k_1,..,k_m} e_{k_1} \otimes .. \otimes e_{k_m} \tag{62}$$

where the coefficients λ_i and β_j satisfy $\lambda_{j_{\sigma(1)},...j_{\sigma(n)}} = \lambda_{j_1,..,j_n}$ and $\beta_{k_{\pi(1)},...,k_{\pi(m)}} = \beta_{k_1,..,k_m}$ for every permutation σ of the set $\{1,...,n\}$ and for every permutation π of the set $\{1,..,m\}$. Actually $\lambda_{j_1,..,j_n} = \langle f, e_{j_1} \otimes ... \otimes e_{j_n} \rangle$ and $\beta_{k_1,..,k_m} = \langle g, e_{k_1} \otimes .. \otimes e_{k_m} \rangle$ in (61) and (62).

If $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ are symmetric given by (61), (62) respectively, then the contraction of order r of f and g is given by

$$\begin{aligned} f \otimes_r g \quad &= \sum_{i_1,..,i_r \geq 1} \sum_{j_1,..,j_{n-r} \geq 1} \sum_{k_1,..,k_{m-r} \geq 1} \lambda_{i_1,..,i_r,j_1,..,j_{n-r}} \beta_{i_1,..,i_r,k_1,..,k_{m-r}} \\ &\times \left(e_{j_1} \otimes .. \otimes e_{j_{n-r}}\right) \otimes \left(e_{k_1} \otimes .. \otimes e_{k_{m-r}}\right) \end{aligned} \tag{63}$$

for every $r = 0,..,m \wedge n$. In particular $f \otimes_0 g = f \otimes g$. Note that $f \otimes_r g$ belongs to $H^{\otimes(m+n-2r)}$ for every $r = 0,..,m \wedge n$ and it is not in general symmetric. We will denote by $f \tilde{\otimes}_r g$ the symmetrization of $f \otimes_r g$.

Another important property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^{n} I_k(f_k)$ with $f_k \in H^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p \left(\mathbf{E}F^2\right)^{\frac{p}{2}}. \tag{64}$$

for every $p \geq 2$.

We denote by D the Malliavin derivative operator that acts on smooth functionals of the form $F = g(W(\varphi_1), \ldots, W(\varphi_n))$ (here g is a smooth function with compact support and $\varphi_i \in H$ for $i = 1,..,n$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

The operator D can be extended to the space $\mathbb{D}^{1,2}$ since it is closable.

The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. Its domain ($Dom(\delta)$) coincides with the class of stochastic processes $u \in L^2(\Omega \times T)$ such that

$$|\mathbf{E}\langle DF, u\rangle| \leq c\|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$ and $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relationship

$$\mathbf{E}(F\delta(u)) = \mathbf{E}\langle DF, u\rangle_H. \tag{65}$$

The chain rule for the Malliavin derivative (see Proposition 1.2.4 in [10]) will be used several times. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and $F \in \mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \varphi'(F)DF. \tag{66}$$

References

[1] H. Biermé, A. Bonami, I. Nourdin and G. Peccati (2012): *Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants.* ALEA Lat. Am. J. Probab. Math. Stat. 9, no. 2, 473-500.

[2] P. Breuer and P. Major (1983): *Central limit theorem for non-linear functionals of Gaussian fields.* J. Mult. Anal. 13, 425-441.

[3] Y. Hu, F. Lu and D. Nualart (2014): *Convergence of densities of some functionals of Gaussian processes.* J. Funct. Anal. 266 (2), 814-875.

[4] Y. Hu, D. Nualart, S. Tindel, F. Xu (2015): *Density convergence in the Breuer-Major theorem for Gaussian stationary sequences.* Bernoulli 21 (4), 2336-2350.

[5] M. Maejima and C. A. Tudor (2012): *Selfsimilar processes with stationary increments in the second Wiener chaos.* Probability and Mathematical Statistics, 32, 167-186.

[6] P. A. Mykland (1992): *Asymptotic expansions and bootstrapping distributions for dependent variables: a martingale approach.* Ann. Statist. 20(2), 623-654.

[7] I. Nourdin and D. Nualart (2016): *Fisher information and the fourth moment theorem.* Ann. Inst. Henri Poincaré Probab. Stat. 52 (2), 849-867.

[8] I. Nourdin and G. Peccati (2012): Normal Approximations with Malliavin Calculus From Stein's Method to Universality. Cambridge University Press.

[9] I. Nourdin and G. Peccati (2009): *Stein's method and exact Berry-Esseen asymptotics for functionals of Gaussian fields.* Ann. Probab. 37, no. 6, 2231-2261.

[10] D. Nualart (2006): *Malliavin Calculus and Related Topics. Second Edition.* Springer.

[11] D. Nualart and G. Peccati (2005): *Central limit theorems for sequences of multiple stochastic integrals*, Ann. Probab. 33, 177-193.

[12] G. Peccati and C.A. Tudor (2004): *Gaussian limits for vector-valued multiple stochastic integrals.* Séminaire de Probabilités, XXXIV, 247-262.

[13] M. Podolskij and N. Yoshida (2016): *Edgeworth expansion for functionals of continuous diffusion processes.* Ann. Appl. Probab. 26 (6), 3415-3455.

[14] I. Shigekawa (1980): *Derivatives of Wiener functionals and absolute continuity of induced measures.* J. Math. Kyoto Univ. 20(2), 263-289.

[15] N. Yoshida (1997): *Malliavin calculus and asymptotic expansion for martingales.* Probab. Theory and Related Fields, 109. 301-342.

[16] N. Yoshida (2001): *Malliavin calculus and martingale expansion.* Bull. Sci, math., 125 (6-7), 431-456.

[17] N. Yoshida (2013): *Martingale expansion in mixed normal limit.* Stochastic Processes and Their Applications, 123 (3), 887-933.