Bounded Backstepping through a Dynamic Extension with Delay

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***Abstract*— We provide a bounded backstepping result that ensures global asymptotic convergence for a broad class of partially linear systems with an arbitrarily large number of integrators. We use one artificial delay, and we assume that the nonlinear subsystems satisfy a converging-input-converging-state assumption. When the nonlinear subsystem is control affine with the state of the first integrator as the control, we provide sufficient conditions for our converging-input-converging-state assumption to hold. Our example illustrates the novelty and utility of our main result.**

***Index Terms*— Backstepping, delays, stabilization**

I. INTRODUCTION

This work continues our team's search (begun in [7], [8], [9], [10], [12], and [13]) for new backstepping results that can help overcome the obstacles to using standard backstepping; see [5] for traditional backstepping. Classical backstepping synthesizes globally asymptotically stabilizing feedbacks, by recursively finding globally asymptotically stabilizing controls and Lyapunov functions for subsystems; see [6] for more recent backstepping that includes nonlinearities and uncertainties, and see [1], [2], and [3] for applications of backstepping for adaptive, aerospace, and robotic systems. However, there are important cases that call for backstepping where existing backstepping methods do not apply, e.g., for general nonlinear subsystems having bounds on the allowable control magnitudes, which produce challenges that we help address here.

We focus on systems having the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \mathcal{F}(t, x(t), z_1(t)) \\ \dot{z}_i(t) & = & z_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{z}_k(t) & = & u(t) + \sum\limits_{j=1}^{k} v_j z_j(t) \end{array} \right. \tag{1}$$

with a scalar control u and any number $k \geq 2$ of integrators. Here, x is valued in $\mathbb{R}^n$ for any n, the v_j's are constant real numbers, and the nonlinear subsystem will enjoy a converging-input-converging-state condition that we define and discuss below. We write our closed loop controls as $u(t)$, but they will be feedbacks that depend on t through their dependence on states of an enlarged system.

Mazenc is with EPI DISCO INRIA-Saclay, Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS, CentraleSupélec, Université Paris-Sud, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France, `Frederic.MAZENC@lss.supelec.fr`.

Malisoff is with the Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA, `malisoff@lsu.edu`.

Burlion is with ONERA, The French Aerospace Lab, 31055 Toulouse, France, `lburlion@onera.fr`.

Gibert is with Airbus Operations, S.A.S, Toulouse, France. `victor.gibert@airbus.com`.

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While we assume that the current state of the system can be measured, we also use a delay in the state values in our feedback since this artificial delay is needed to design our bounded control. Our work [10] used the converging-input-converging-state assumption and an artificial delay, but a significant improvement in this note in comparison with [10] is that here we allow an arbitrarily large number k of integrators, while [10] only allowed one integrator. Our bounded backstepping work [12] also allowed an arbitrarily large number of integrators, but the present work produces a globally bounded control for (1) while the controls for the original systems in [12] were not bounded. In addition, while [12] required k artificial delays and did not use dynamic extensions, here we only require one artificial delay, which may make our feedback simpler to apply.

Our works [7], [8], [9], and [13] did not use converging-input-converging-state conditions or artificial delays. Moreover, our work differs from [8] (which was based on a forwarding method for one integrator case), [7] (which was limited to one integrator), [13] (which gives unbounded controls), and [16] and [17] (which employ Lie derivatives without satisfying input constraints). Hence, our novel combination of converging-input-converging-state conditions, artificial delays, and bounded controllers for (1) in this work is useful.

We use standard notation and definitions. We omit arguments of functions when they are clear, and the dimensions of our Euclidean spaces are arbitrary. We use $|\cdot|$ to denote the usual Euclidean norm and the induced matrix norm, and $|\phi|_\infty$ (resp., $|\phi|_{\mathcal{I}}$) is the essential supremum (resp., supremum over any interval $\mathcal{I}$) for bounded measurable functions; ϕ. Set $\mathbb{N} = \{1, 2, \dots\}$. For a fixed constant $T > 0$, let $C_{\rm in}$ denote the set of all continuous functions $\phi : [-T, 0] \to \mathbb{R}^a$, which we call the set of all *initial functions*. We define $\Xi_t \in C_{\rm in}$ and $\dot{\Xi}_t \in C_{\rm in}$ by $\Xi_t(s) = \Xi(t+s)$ and $\dot{\Xi}_t(s) = \Xi'(t+s)$ for all choices of Ξ, $s \leq 0$, and $t \geq 0$ for which the equalities are defined. We assume for simplicity that the initial times for our solutions are $t_0 = 0$ and that the initial functions are constant at time 0 (so the states are constant on $[-T, 0]$, where T will be the artificial delay). We denote by $f^{(i)}(t)$ the ith derivative of a function $f : [0, +\infty) \to \mathbb{R}$ with $f^{(0)}(t) = f(t)$, and $\sigma_r : \mathbb{R} \to [-r, r]$ is the standard saturation that is defined for all constants $r > 0$ by $\sigma_r(s) = s$ for all $s \in [-r, r]$ and $\sigma_r(s) = r\text{sign}(s)$ otherwise.

II. MAIN RESULT

To state our theorem, we require the following two lemmas, the first of which follows from elementary calculations that we omit:

Lemma 1: Let $q > 0$ and $T > 0$ be given constants, and $\mu_0 : [-T, +\infty) \to \mathbb{R}$ be any continuous function, and set

$$\begin{aligned} &\zeta(t) = \textstyle\int_{t-T}^{t} e^{q(\ell-t)} Q(t,\ell,\ell+T)\mu_0(\ell)\mathrm{d}\ell, \\ &\text{and } \Omega_j(t) = \zeta^{(j-1)}(t) \\ &\text{and } \mu_i(t) = \tfrac{1}{T}\textstyle\int_{t-T}^{t} e^{q(\ell-t)} \tfrac{(t-\ell)^{i-1}}{(i-1)!}\mu_0(\ell)\mathrm{d}\ell \end{aligned} \tag{2}$$

for all $j \in \{1, ..., k+1\}$ and $i \in \mathbb{N}$, where $Q(t,a,b) = (t-a)^{k-1}(t-b)^{k-1}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k \geq 2$. Then there are constants $c_{i,j}(q,T) \in \mathbb{R}$ for all $i \in \{1, ..., 2k-1\}$ and all $j \in \{1, ..., k\}$, and constants $g_i(q,T) \in \mathbb{R}$ for all $i \in \{-1, 0, ..., 2k-1\}$, such that

$$\begin{aligned} &\Omega_j(t) = \sum_{i=1}^{2k-1} c_{i,j}(q,T)\mu_i(t) \text{ and} \\ &\Omega_{k+1}(t) = \sum_{j=0}^{2k-1} g_j(q,T)\mu_j(t) + g_{-1}(q,T)\mu_0(t-T) \end{aligned} \tag{3}$$

hold for all $t \geq 0$. □

Lemma 1 follows by first using the Binomial Theorem to write $Q(t,\ell,\ell+T)$ as a linear combination of the terms $(t-\ell)^r$ with $k-1 \leq r \leq 2k-2$. In the next lemma (which can be deduced from [15] on linear systems), we say that a linear system is *not exponentially unstable* provided its poles are all in the closed left-half plane:

Lemma 2: Let $k \geq 2$ be an integer and $v = (v_1, \ldots, v_k)$ be any vector of k real constants such that

$$\left\{ \begin{array}{rcl} \dot{z}_i(t) &=& z_{i+1}(t), \;\; i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) &=& u + \sum\limits_{i=1}^{k} v_i z_i \end{array} \right. \tag{4}$$

is not exponentially unstable when $u = 0$. Then there is a bounded locally Lipschitz function ϑ such that (4), in closed loop with $u = \vartheta(Z)$ where $Z = (z_1, \ldots, z_k)^\top$, is globally asymptotically and locally exponentially stable to 0. □

In addition to the notation from the preceding two lemmas, our theorem will also use the nonzero constant

$$b_T = \textstyle\int_{-T}^{0} e^{q\ell}\ell^{k-1}(\ell+T)^{k-1}\mathrm{d}\ell \tag{5}$$

and the following assumption:

Assumption 1: (i) The function $\mathcal{F}$ in (1) is continuous in t and globally Lipschitz in (x, z_1) and satisfies $\mathcal{F}(t,0,0) = 0$ for all $t \geq 0$. (ii) There is a globally Lipschitz bounded function $\omega : \mathbb{R}^n \to [-\bar{\omega}, \bar{\omega}]$ having some constant bound $\bar{\omega} > 0$ such that $\omega(0) = 0$, and constants $T > 0$ and $q > 0$, such that for each continuous function $\delta : [0, +\infty) \to \mathbb{R}$ that exponentially converges to 0, the following is true: All solutions $\xi : [0, +\infty) \to \mathbb{R}^n$ of the system

$$\begin{aligned} &\dot{\xi}(t) = \\ &\mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \tfrac{e^{q(\ell-t)}Q(t,\ell,\ell+T)\omega(\xi(\ell))}{b_T}\mathrm{d}\ell + \delta(t)\right) \end{aligned} \tag{6}$$

satisfy $\lim_{t\to+\infty} \xi(t) = 0$, where $Q(t,a,b) = (t-a)^{k-1}(t-b)^{k-1}$ and $k \geq 2$ is from (1). □

We refer to part (ii) of Assumption 1 as our converging-input-converging-state assumption; see Section III for sufficient conditions. The system (6) differs from the nonlinear subsystem of (1) because the third argument of $\mathcal{F}$ in (1) has been replaced by the sum of $\delta(t)$ and an integral. In terms of the Jordan matrix

$$J_{2k-1} = \left[\begin{array}{ccccc} -q & 1 & 0 & \ldots & 0 \\ 0 & -q & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -q & 1 \\ 0 & \ldots & \ldots & 0 & -q \end{array} \right] \in \mathbb{R}^{(2k-1)\times(2k-1)},$$

our main result is as follows, where the forward completeness of the closed loop system follows from the boundedness of ω and the control and the global Lipschitzness of $\mathcal{F}$ and ω:

Theorem 1: Let the constants $q > 0$, $k \geq 2$, and $T > 0$ and the functions $\mathcal{F}$ and ω be such that Assumption 1 holds, where $k \in \mathbb{N}$. Let ϑ and v satisfy the requirements from Lemma 2. Consider the augmented (x, Z, Y) system, consisting of (1) and the $2k-1$ dimensional system

$$\dot{Y}(t) = J_{2k-1}Y(t) + \tfrac{e_{2k-1}}{T}\tfrac{\omega(x(t))}{b_T} \tag{7}$$

where $e_{2k-1} = (0, 0, \ldots, 1)^\top \in \mathbb{R}^{2k-1}$ is the $(2k-1)$-st standard basis vector, in closed loop with the control

$$\begin{aligned} &u(Z(t), Y_t, x_t) = \sigma_{\bar{a}}\left(\mathcal{M}(Y_t)\right) \\ &+ g_0(q,T)\tfrac{\omega(x(t))}{b_T} + g_{-1}(q,T)\tfrac{\omega(x(t-T))}{b_T} + \vartheta(Z_\star(t)) \end{aligned} \tag{8}$$

with the saturation level

$$\bar{a} = \left| \sum_{j=1}^{k} v_j \mathcal{C}_j(q,T) - \mathcal{G}(q,T) \right| e^{|J_{2k-1}|T}\tfrac{\bar{\omega}}{b_T}, \tag{9}$$

where

$$\mathcal{M}(Y_t) = \left(\mathcal{G}(q,T) - \sum_{j=1}^{k} v_j \mathcal{C}_j(q,T) \right) \Psi(Y_t) \tag{10}$$

and $\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T)$ and

$$Z_\star(t) = \left[\begin{array}{c} z_1(t) - \mathcal{C}_1(q,T)\Psi(Y_t) \\ \vdots \\ z_k(t) - \mathcal{C}_k(q,T)\Psi(Y_t) \end{array} \right], \tag{11}$$

and $\mathcal{G}(q,T) = [g_{2k-1}(q,T) \;......\; g_1(q,T)]$ and

$$\mathcal{C}_j(q,T) = [c_{2k-1,j}(q,T) \;......\; c_{1,j}(q,T)], \; 1 \leq j \leq k \tag{12}$$

and where the constants $c_{i,j}$ and g_i satisfy the requirements from Lemma 1 for the function $\mu_0(t) = \omega(x(t))/b_T$. Then all maximal solutions $(x, Z, Y)(t)$ of the augmented (x, Z, Y) system satisfy $\lim_{t\to+\infty}(x, Z, Y)(t) = 0$. □

We next provide sufficient conditions for our converging-input-converging-state condition from Assumption 1 to hold, and then we prove Theorem 1 in Section IV.

III. Checking Assumption 1

We provide sufficient conditions for our converging-input-converging-state conditions on (6) to hold, based on Lyapunov functions, using

$$\dot{q}(t) = \mathcal{F}(t, q(t), \omega(q(t))), \tag{13}$$

where $\mathcal{F}$ is from (1), and using the standard definitions of positive definiteness, properness, and $\mathcal{K}_\infty$ [6]. In the next assumption, V_t and V_x are the partial derivative with respect to t and the gradient with respect to x, respectively, and uniform global Lipschitzness in x means that the global Lipschitz constants can be chosen independently of t:

Assumption 2: There exist functions $f : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ that are uniformly globally Lipschitz in x and continuous on $[0,+\infty) \times \mathbb{R}^n$, such that

$$\mathcal{F}(t,x,p) = f(t,x) + g(t,x)p \tag{14}$$

holds for all $t \geq 0$, $x \in \mathbb{R}^n$, and $p \in \mathbb{R}$. Also, there exist a C^1 uniformly proper and positive definite function $V : [0,+\infty) \times \mathbb{R}^n \to [0,+\infty)$; a uniformly continuous positive definite function $W : \mathbb{R}^n \to [0,+\infty)$; positive constants r_0, r_1, and r_3; and a constant $r_2 \geq 0$ such that for all $(t,x) \in [0,+\infty) \times \mathbb{R}^n$, we have

$$\begin{aligned} &V_t(t,x) + V_x(t,x)\big(f(t,x) + g(t,x)\omega(x)\big) \leq -W(x),\\ &|V_x(t,x)g(t,x)| \leq r_0\sqrt{W(x)},\ |\omega(x)| \leq r_1\sqrt{W(x)},\\ &|f(t,x)| \leq r_2\sqrt{W(x)},\ \text{ and }\ |g(t,x)| \leq r_3, \end{aligned} \tag{15}$$

where $\omega : \mathbb{R}^n \to \mathbb{R}$ is bounded and satisfies $\omega(0) = 0$, and there is a global Lipschitz constant $C > 0$ for ω on $\mathbb{R}^n$. □

See [11] for conditions under which Assumption 2 holds.

Proposition 1: If Assumption 2 holds, then for all constants $T > 0$ such that

$$4(Tr_0C)^2\left[2r_2^2 + \frac{5}{2}\left(\frac{r_1r_3T^{2k-1}}{|b_T|}\right)^2\right] < 1, \tag{16}$$

and for all integers $k \geq 2$, Assumption 1 is satisfied. □

Proof: (Sketch.) The proof has a similar structure to the proof of the sufficient conditions from [10], so we only sketch the proof. Fix any continuous function $\delta : [0,+\infty) \to \mathbb{R}$ that exponentially converges to 0. By (5), we have

$$\int_{t-T}^{t} \frac{e^{q(\ell-t)}Q(t,\ell,\ell+T)}{b_T}\mathrm{d}\ell = 1 \tag{17}$$

for all $t \geq T$. Hence, along all solutions $x(t)$ of (6), we get

$$\begin{aligned} &\dot{V}(t) \leq -W(x(t)) + r_0\sqrt{W(x(t))}\\ &\times\left(\sup_{\ell\in[t-T,t]}|\omega(x(\ell)) - \omega(x(t))| + |\delta(t)|\right) \end{aligned} \tag{18}$$

for all $t \geq T$ and

$$\begin{aligned} &|\dot{x}(t)| \leq r_2\sqrt{W(x(t))}+\\ &r_3\left(\frac{T^{2(k-1)}}{|b_T|}\int_{t-T}^{t}|\omega(x(\ell))|\mathrm{d}\ell + |\delta(t)|\right) \end{aligned} \tag{19}$$

along all solutions of (6). It follows from the Young and Jensen's inequalities that along all solutions of (6), we have

$$\begin{aligned} &|\dot{x}(t)|^2 \leq 2r_2^2W(x(t))+\\ &2r_3^2\left(5|\delta(t)|^2 + \frac{5}{4}\frac{T^{4(k-1)}Tr_1^2}{b_T^2}\int_{t-T}^{t}W(x(\ell))\mathrm{d}\ell\right). \end{aligned} \tag{20}$$

Hence, Jensen's and Young's inequalities also give

$$\begin{aligned} \dot{V}(t) &\leq -\tfrac{1}{2}W(x(t)) + r_0^2\Bigg(|\delta(t)|^2+\\ &\qquad C^2\sup_{\ell\in[t-T,t]}|x(\ell) - x(t)|^2\Bigg)\\ &\leq -\tfrac{1}{2}W(x(t)) + r_0^2\Bigg(|\delta(t)|^2\\ &\qquad + C^2T\textstyle\int_{t-T}^{t}|\dot{x}(\ell)|^2\mathrm{d}\ell\Bigg)\\ &\leq -\tfrac{1}{2}W(x(t)) + r_0^2\Bigg(|\delta(t)|^2\\ &\qquad + C^2T\Bigg[2r_2^2\textstyle\int_{t-T}^{t}W(x(\ell))\mathrm{d}\ell\\ &\qquad + 10r_3^2T|\delta|^2_{[t-T,t]}\\ &\qquad + \tfrac{5}{2}r_3^2\tfrac{T^{4(k-1)}T^2}{b_T^2}r_1^2\textstyle\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell\Bigg]\Bigg)\\ &\leq -\tfrac{1}{2}W(x(t)) + \mathcal{N}_1\textstyle\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell\\ &\qquad + \mathcal{N}_2|\delta|^2_{[t-T,t]} \end{aligned} \tag{21}$$

along all solutions of (6) for all $t \geq T$, where

$$\mathcal{N}_1 = T(r_0C)^2\left(2r_2^2 + \tfrac{5}{2}\left(\tfrac{r_1r_3T^{2k-1}}{|b_T|}\right)^2\right) \tag{22}$$

and $\mathcal{N}_2 = 10(r_0r_3TC)^2 + r_0^2$. Then our condition (16) provides a constant $\lambda > 1$ such that $2T\mathcal{N}_1\lambda < 1/2$, so

$$V_1(t,x_t) = V(t,x(t)) + \lambda\mathcal{N}_1\textstyle\int_{t-2T}^{t}\int_{s}^{t}W(x(\ell))\mathrm{d}\ell\mathrm{d}s \tag{23}$$

satisfies the following along all solutions of (6):

$$\dot{V}_1 \leq -\big\{\tfrac{1}{2} - 2T\mathcal{N}_1\lambda\big\}W(x(t)) + \mathcal{N}_2|\delta|^2_{[t-T,t]} \tag{24}$$

for all $t \geq 2T$. The result now follows from our choice of λ and applying Barbalat's Lemma to $W(x(t))$. ■

IV. Proof of Theorem 1

Theorem 1 will follow from three more lemmas, which we state next. The following follows from [14, Appendix A.3]:

Lemma 3: For the Jordan matrix J_{2k-1}, the equality

$$e^{J_{2k-1}t} = e^{-qt}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{2(k-1)}}{(2(k-1))!}\\ 0 & 1 & t & \dots & \frac{t^{2k-3}}{(2k-3)!}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 1 & t\\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{25}$$

holds for all $t \in \mathbb{R}$ and integers $k \geq 2$. □

Lemma 4: Let $\mu_0 : [-T,+\infty) \to [-\bar{\mu},\bar{\mu}]$ be any continuous function having a bound $\bar{\mu}$. Then the functions μ_i from (2) in Lemma 1, and the functions $\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T)$ for all solutions Y of

$$\dot{Y}(t) = J_{2k-1}Y(t) + \tfrac{e_{2k-1}}{T}\mu_0(t), \tag{26}$$

are such that for all $t \geq T$, we have

$$\nu_{2k-1}(t) = \Psi(Y_t) \ \text{ and }\ |\Psi(Y_t)| \leq e^{|J_{2k-1}|T}\bar{\mu}, \tag{27}$$

where $\nu_{2k-1}(t) = (\mu_{2k-1}(t),\dots,\mu_1(t))^\top$ for all $t \geq T$. □

Proof: By integrating (26), we deduce that

$$\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T) = \varrho(t), \qquad (28)$$

$$\text{where} \quad \varrho(t) = \int_{t-T}^{t} e^{J_{2k-1}(t-\ell)} \frac{e_{2k-1}}{T}\mu_0(\ell)\mathrm{d}\ell \qquad (29)$$

for all $t \geq T$. On the other hand, using (25), we obtain

$$\varrho(t) = \frac{1}{T}\int_{t-T}^{t} e^{-q(t-\ell)} \begin{bmatrix} \frac{(t-\ell)^{2(k-1)}}{(2(k-1))!} \\ \vdots \\ t-\ell \\ 1 \end{bmatrix} \mu_0(\ell)d\ell = \nu_{2k-1}(t), \qquad (30)$$

which proves the first conclusion of the lemma. The second conclusion of the lemma follows from (28)-(29). ■

Lemma 5: Let $\mu_0 : [-T, +\infty) \rightarrow [-\bar{\mu}, \bar{\mu}]$ be any continuous function having a bound $\bar{\mu}$, and let the constants v_i and the function ϑ satisfy the requirements from Lemma 2. Consider the linear system

$$\left\{ \begin{array}{rcl} \dot{z}_i(t) & = & z_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) & = & u(t) + \sum_{j=1}^{k} v_j z_j(t) \end{array} \right. \qquad (31)$$

in closed loop with the control

$$\begin{array}{l} u(Z(t), Y_t, x_t) = \sigma_{\bar{a}}\left(\mathcal{M}(Y_t)\right) \\ + g_0(q,T)\mu_0(t) + g_{-1}(q,T)\mu_0(t-T) + \vartheta(Z_\star(t)) \end{array} \qquad (32)$$

with the saturation level $\bar{a}$ for $\sigma_{\bar{a}}$ defined by

$$\bar{a} = \left| \sum_{j=1}^{k} v_j \mathcal{C}_j(q,T) - \mathcal{G}(q,T) \right| e^{|J_{2k-1}|T}\bar{\mu} \qquad (33)$$

and where Y satisfies (26) and $\mathcal{M}$, Ψ, Z_*, $\mathcal{G}$, and the $\mathcal{C}_j$'s and g_j's are defined as in Theorem 1. Then the dynamics for the vector $\tilde{Z}(t) = (\tilde{z}_1(t), ..., \tilde{z}_k(t))$ are globally asymptotically and locally exponentially stable to the origin, where $\tilde{z}_i(t) = z_i(t) - \Omega_i(t)$ for $i = 1, 2, \ldots, k$ and the Ω_i's are defined in (2) in Lemma 1. □

Proof: The fact that $\dot{\Omega}_i = \Omega_{i+1}$ for all $i \in \{1, 2, \ldots, k\}$ and the structure of the dynamics (31) allow us to conclude that the dynamics for the functions $\tilde{z}_i(t) = z_i(t) - \Omega_i(t)$ are

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & u(t) - \Omega_{k+1}(t) \\ & & + \sum_{j=1}^{k} v_j[\tilde{z}_j(t) + \Omega_j(t)] \end{array} \right. . \qquad (34)$$

Using our conclusion from Lemma 4 that

$$\nu_{2k-1}(t) = \Psi(Y_t) \qquad (35)$$

where $\nu_{2k-1}(t) = (\mu_{2k-1}(t), \ldots, \mu_1(t))^\top$ as before, it follows from (3) that

$$\begin{array}{rcl} \dot{\tilde{z}}_k(t) & = & u(t) - \mathcal{G}(q,T)\nu_{2k-1}(t) - g_0(q,T)\mu_0(t) \\ & & - g_{-1}(q,T)\mu_0(t-T) \\ & & + \sum_{j=1}^{k} v_j \tilde{z}_j(t) + \sum_{j=1}^{k} v_j \mathcal{C}_j(q,T)\nu_{2k-1}. \end{array} \qquad (36)$$

Hence (35) and our choice

$$\dot{Y}(t) = J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\mu_0(t) \qquad (37)$$

of our dynamic extension imply that for all $t \geq T$, we have

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & u(t) + \sum_{j=1}^{k} v_j \tilde{z}_j(t) - g_0(q,T)\mu_0(t) \\ & & - g_{-1}(q,T)\mu_0(t-T) + \bar{g}\Psi(Y_t) \end{array} \right. \qquad (38)$$

where

$$\bar{g} = \sum_{j=1}^{k} v_j \mathcal{C}_j(q,T) - \mathcal{G}(q,T). \qquad (39)$$

Next note that since Lemma 1 gives

$$\Omega_j = \mathcal{C}_j(q,T)\nu_{2k-1} \ \text{ for } 1 \leq j \leq k, \qquad (40)$$

it follows that for all $i \in \{1, \ldots, k\}$, we have

$$\tilde{z}_i(t) = z_i(t) - \Omega_i(t) = z_i(t) - \mathcal{C}_i(q,T)\nu_{2k-1}(t) \ . \qquad (41)$$

Thus, (35) gives $\tilde{z}_i(t) = z_i(t) - \mathcal{C}_i(q,T)\Psi(Y_t)$ for all $t \geq T$, so $Z_\star(t) = \tilde{Z}(t)$ for all $t \geq T$. Therefore, our choice (32) of the control gives

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & \sum_{j=1}^{k} v_j \tilde{z}_j(t) + \sigma_{\bar{a}}\left(-\bar{g}\Psi(Y_t)\right) \\ & & + \bar{g}\Psi(Y_t) + \vartheta(Z_\star(t)) \ . \end{array} \right. \qquad (42)$$

According to (27), we have $|\bar{g}\Psi(Y_t)| \leq |\bar{g}|\, e^{|J_{2k-1}|T}\overline{\mu}$ for all $t \geq T$. From the definition of $\sigma_{\bar{a}}$, it follows that for all $t \geq T$, we have

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & \vartheta(\tilde{Z}(t)) + \sum_{i=1}^{k} v_i \tilde{z}_i \end{array} \right. \qquad (43)$$

so the lemma follows from our choice of ϑ in Lemma 2. ■

We now combine the preceding lemmas to prove Theorem 1. First notice that the closed loop system is as follows:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \mathcal{F}(t, x(t), z_1(t)) \\ \dot{z}_i(t) & = & z_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) & = & u(Z(t), Y_t, x_t) + \sum_{j=1}^{k} v_j z_j(t) \\ \dot{Y}(t) & = & J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\frac{\omega(x(t))}{b_T} \end{array} \right. \qquad (44)$$

which is forward complete by our global Lipschitzness assumptions. Using the preceding lemma, we deduce that

$$\lim_{t \rightarrow +\infty} |z_i(t) - \Omega_i(t)| = 0 \qquad (45)$$

for all $i = 1$ to k, and $z_1 - \Omega_1$ exponentially converges to 0.

Next notice that $\zeta = \Omega_1$ in Lemma 1, and therefore that

$$\dot{x}(t) \ = \ \mathcal{F}(t, x(t), \zeta(t) + z_1(t) - \Omega_1(t)) \qquad (46)$$

when we choose the bounded function $\mu_0(t) = \omega(x(t))/b_T$

to obtain

$$\zeta(t) = \int_{t-T}^{t} e^{q(\ell-t)} Q_1(t,\ell,\ell+T) \frac{\omega(x(\ell))}{b_T} d\ell. \qquad (47)$$

Hence, we can use Assumption 1 with $\delta = z_1 - \Omega_1$ to conclude that

$$\lim_{t\to+\infty} |x(t)| = 0 \qquad (48)$$

and therefore that for all $i \in \{1, 2, \dots, k\}$, we have

$$\lim_{t\to+\infty} \Omega_i(t) = 0, \qquad (49)$$

since $\omega(0) = 0$ and ω is continuous at 0. Therefore, Theorem 1 follows by combining (45) with (48)-(49).

V. Illustration

We apply our method to the three-dimensional dynamics

$$\begin{cases} \dot{x}(t) &=& \frac{|x(t)|}{1+|x(t)|} + z_1(t) \\ \dot{z}_1(t) &=& z_2(t) \\ \dot{z}_2(t) &=& u(t) \end{cases} \qquad (50)$$

which is outside the scope of classical backstepping, as the right side of $\dot{x}(t)$ is not differentiable. With our earlier notation, we choose $k = 2$, $n = 1$, $q = 1$, and

$$\omega(x) = -\frac{|x|}{1+|x|} - 2\frac{x}{1+|x|} \ . \qquad (51)$$

We find a constant $T > 0$ such that Assumption 1 holds with

$$\mathcal{F}(t, x, z_1) = \frac{|x|}{1+|x|} + z_1 \ . \qquad (52)$$

Since (51)-(52) are globally Lipschitz functions in the states and $\mathcal{F}$ is affine in z_1 and ω is bounded, it suffices to produce constants r_i for $i = 0, 1, 2, 3$ and functions V and W that satisfy Assumption 2 with $f(t, x) = x/(1 + |x|)$ and $g(t, x) = 1$, and then we choose T such that (16) holds.

To this end, we check that Assumption 2 holds for

$$V(t, x) = \int_0^x \sigma_1(\ell)\mathrm{d}\ell \ \text{ and } \ W(x) = \frac{2\sigma_1(x)x}{1+|x|}. \qquad (53)$$

We have

$$f(t, x) + g(t, x)\omega(x) = -\frac{2x}{1+|x|} \qquad (54)$$

so our conditions on the r_i's from (15) will hold if

$$\begin{array}{l} |\sigma_1(x)| \le r_0\sqrt{\frac{2\sigma_1(x)x}{1+|x|}}, \ \frac{3|x|}{1+|x|} \le r_1\sqrt{\frac{2\sigma_1(x)x}{1+|x|}}, \\ \frac{|x|}{1+|x|} \le r_2\sqrt{\frac{2\sigma_1(x)x}{1+|x|}}, \ \text{ and } \ 1 \le r_3. \end{array} \qquad (55)$$

By separately considering points where $x \in [-1, 1]$ or $x \notin [-1, 1]$, we can readily check that Assumption 2 is satisfied using $C = 3$, $r_0 = 1$, $r_1 = \frac{3}{\sqrt{2}}$, $r_2 = 1$, and $r_3 = 1$. Moreover, for the case where $k = 2$ and $q = 1$, our constant b_T from (5) that is used in our condition (16) on T in Proposition 1 is $b_T = 2 - T - e^{-T}(2 + T)$, so our requirement (16) on $T > 0$ is that

$$\begin{array}{l} 1 > 4(Tr_0C)^2 \left[2r_2^2 + \frac{5}{2}\left(\frac{r_1 r_3 T^3}{|b_T|}\right)^2\right] \\ = 4(3T)^2 \left[2 + \frac{5}{2}\left(\frac{3T^3}{\sqrt{2}|2-T-e^{-T}(2+T)|}\right)^2\right] \end{array} \qquad (56)$$

and we can use Mathematica [18] to check that the right side of (56) is 0.912536 for $T = 0.11$. Hence, Assumption 1 holds with $T = 0.11$, so the desired dynamic controller is provided by Theorem 1.

VI. Conclusions

Our new bounded backstepping technique applies to a large class of partially linear systems with arbitrarily large numbers of integrators, when the nonlinear subsystems satisfy a converging-input-converging-state assumption. For many cases, our Lyapunov functions provide sufficient conditions for our converging-input-converging-state assumption to hold. Our theorem provides bounded controllers for the original system. We plan to combine our methods with the time delay methods in [4] to also allow measurement delays.

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