Triangulating stratified manifolds I: a reach comparison theorem

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Abstract

In this paper, we define the reach for submanifolds of Riemannian manifolds, in a way that is similar to the Euclidean case. Given a d-dimensional submanifold $\mathcal{S}$ of a smooth Riemannian manifold $\mathcal{M}$ and a point $p \in \mathcal{M}$ that is not too far from $\mathcal{S}$ we want to give bounds on local feature size of $\exp_p^{-1}(\mathcal{S})$. Here $\exp_p^{-1}$ is the inverse exponential map, a canonical map from the manifold to the tangent space. Bounds on the local feature size of $\exp_p^{-1}(\mathcal{S})$ can be reduced to giving bounds on the reach of $\exp_p^{-1}(\mathcal{B})$, where $\mathcal{B}$ is a geodesic ball, centred at c with radius equal to the reach of $\mathcal{S}$. Equivalently we can give bounds on the reach of $\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$, where now $\mathbf{B}_c$ is a ball in the tangent space $T_c\mathcal{M}$, with the same radius. To establish bounds on the reach of $\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$ we use bounds on the metric and on its derivative in Riemann normal coordinates.

This result is a first step towards answering the important question of how to triangulate stratified manifolds.

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1 Introduction and motivation

Motivation: Triangulating stratified manifolds

Triangulating of manifolds with boundary, submanifolds of Riemannian manifolds and stratified manifolds, is an important problem because of applications in dynamical systems, physics, and chemistry. Examples of stratified manifolds in applications include conformation spaces of molecules, such as discovered for cyclo-octane [26].

There already exists a vast literature on triangulating surfaces in $\mathbb{R}^3$, see for example [1, 6, 8, 11, 15], as well as results on the triangulation of general manifolds in arbitrary dimensional Euclidean space [5, 10]. The triangulation of solid objects with boundary and stratified solid objects in $\mathbb{R}^3$ has been the object of study in [13, 14, 12, 28, 29], and Chapter 15 of [11].

Algorithms have been proposed for separating the strata of stratified manifolds [4]; the resulting strata being manifolds with boundaries, see also [3]. The triangulation of Riemannian manifolds also has received significant interest in the last few years, see [7, 17].

The reach

The reach is a key concept used in triangulations and related problems, see for example the overviews [6, 11, 15], because it not only gives bounds on the local curvature, but gives more global information, such as how close different parts of the set $\mathcal{S}$ lie to one another.

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In Euclidean space, the reach of a subset $\mathcal{S}$ is defined as the minimal distance of a point on $\mathcal{S}$ to the medial axis (i.e. the set of points that have more than one closest point on $\mathcal{S}$).

The reach of subsets of Euclidean space was introduced by Federer [18]. The reach is denoted by rch. The same concept was called the condition number by Niyogi, Smale and Weinberger [27]. The local feature size, a local version of the reach and called the reach at a specific point by Federer, was introduced into the Computational geometry community by Amenta and Bern [2]. Remarkably, the authors of both [2] and [27] seem to have been unaware of the results of Federer.

In this paper, we'll generalize the reach to submanifolds of Euclidean space to submanifolds of arbitrary Riemannian manifolds. Defining and understanding such a generalization of the reach to this setting is essential to be able to leverage the results on triangulations in Euclidean space we mentioned above. We'll explain this in more detail now.

Let $\mathcal{M}$ be a Riemannian manifold. We denote by $T_p\mathcal{M}$ the tangent space at p and remind ourselves that the exponential map $\exp_p$ maps vectors emanating from p in $T_p\mathcal{M}$ to geodesics on $\mathcal{M}$ emanating from p with equal length, while also preserving the angles between these.

The reach of a submanifold of a Riemannian manifold can be defined in much the same way as in Euclidean space, roughly as the distance to the medial axis (i.e. the set of points that have more than one closest point on $\mathcal{S}$). In the manifold setting some technical extra condition on the reach, involving the geometry of $\mathcal{M}$ is also necessary, see Section 2.

Suppose that we have bounds on the reach of a submanifold[1] $\mathcal{S}$ of a Riemannian manifold $\mathcal{M}$. We now want to consider a point $p \in \mathcal{M}$ that is not too far from $\mathcal{S}$. In fact, for the sake of the argument, we'll assume that $\mathcal{S}$ lies well inside a convex geodesic ball centred at p. This assumption guarantees that $\exp_p^{-1}(\mathcal{S})$ is well defined and we can talk about the (global) reach of $\exp_p^{-1}(\mathcal{S})$. Working with more local assumptions (such as the local feature size) should be possible, but makes the argument even more technical.

Now we would like to give bounds on the reach of $\exp_p^{-1}(\mathcal{S})$ in $T_p\mathcal{M}$. With bounds on the reach, we can use results on triangulations in Euclidean space to triangulate in $T_p\mathcal{M}$ and previous results on distortion Riemannian simplices under the exponential map to map the triangulation back to the manifold.

In preparation for the larger problem of triangulating Riemannian manifolds with boundary, submanifolds of a Riemannian manifold, and stratified Riemannian manifolds, this paper focuses on the essential prerequisite of understanding the reach or local feature size of $\exp_p^{-1}(\mathcal{S})$ in terms of the reach of the submanifold $\mathcal{S}$ and the geometry of the manifold $\mathcal{M}$ in which $\mathcal{S}$ is embedded. Along the way, we will give an introduction to some fundamental results in Riemannian geometry, such as (a simplified version of) the Toponogov comparison theorem.

Future work

As we mentioned above, this work is part of a larger project on the triangulation of several kinds of Riemannian manifolds with substructure, for which a significant amount of work needs to be done. This is also reflected in the title of this paper.

To be able to go from this result to the triangulation of stratified manifolds we also need to extend and/or generalize results on

[1] The result for a stratum would be similar.

- The triangulation of piecewise smooth manifolds, and stratified manifolds in three dimensional Euclidean space.
- The results on the triangulation of Riemannian manifolds to accommodate constraints.

This will be reported on in forthcoming papers.

2 The reach

Similar to the definition for manifolds embedded in Euclidean space we define the reach for C^∞ compact Riemannian (sub)manifolds $\mathcal{S} \subset \mathcal{M}$. We shall denote the normal space of $\mathcal{S}$ at a point x by $N_x\mathcal{S}$, and the bundle by $N\mathcal{M}$.

▶ **Definition 1** (The reach)**.** We let the medial axis $\mathrm{ax}_\mathcal{M}(\mathcal{S})$ be the set of points in $\mathcal{M}$ that do not have a unique closest point on $\mathcal{S}$, with respect to the Riemannian metric. We denote the projection of a point x in $\mathcal{M}$ on the closest point on $\mathcal{S}$ by $\pi_\mathcal{S}(x)$. The pre-reach $p\mathrm{rch}_\mathcal{M}(\mathcal{S})$ is then the shortest distance between $\mathrm{ax}_\mathcal{M}(\mathcal{S})$ and $\mathcal{S}$. We now define the reach $\mathrm{rch}_\mathcal{M}(\mathcal{S})$ to be

$$\min\{p\mathrm{rch}_\mathcal{M}(\mathcal{S}), \iota_\mathcal{M}\}, \tag{1}$$

where $\iota_\mathcal{M}$ is the injectivity radius of $\mathcal{M}$.

The injectivity radius is the largest radius r such that for any x, $\exp_x$ restricted to the open ball centred at x with radius r in $T_x\mathcal{M}$ is a diffeomorphism onto its image. Adding this bound to our definition of the reach is essential, because we would like to have that a tangent ball to $\mathcal{S}$ is indeed a topological ball.

▶ Remark. In the mathematics literature, the closure of the medial axis is often called the cut locus in this context, see [9, Section 2] or [25]. We have chosen to use the terminology and concept most common in the computational geometry community in view of future applications.

We now give the result, which is an extension of Theorem 4.8.8 of Federer [18] to the setting of Riemannian manifolds:

▶ **Theorem 2** (Tubular neighbourhood)**.** *Let $B_{N_p\mathcal{S}}(r)$, be the ball of radius r centred at p in the normal space $N_p\mathcal{S} \subset T_p\mathcal{M}$ of a C^∞ manifold with reach $\mathrm{rch}_\mathcal{M}(\mathcal{S})$, where $r < \mathrm{rch}_\mathcal{M}(\mathcal{S})$. For every point $x \in \exp_p(B_{N_p\mathcal{S}}(r))$, we have $\pi_\mathcal{S}(x) = p$.*

The proof of Theorem 2 is not so difficult and mainly follows Hirsch [21, Section 4.5] and Spivak [30, Appendix I of Chapter 9] with some variations.

We start with Lemma 19 of Spivak [30, Chapter 9]:

▶ **Lemma 3.** *Let X be a compact metric space and $X_0 \subset X$ a closed subset. Let $f : X \to Y$ be a local homeomorphism such that $f|_{X_0}$ is injective. Then there exists a neighbourhood U of X_0 such that $f|_U$ is injective.*

With this lemma we can prove an embedding result, for which we have to make the following definition:

▶ **Definition 4.** Let NB_ϵ denote the ϵ-neighbourhood in $N\mathcal{S}$ of $\mathcal{S}$, that is the neighbourhood of all points closer than ϵ to $\mathcal{S}$, where we identify $\mathcal{S} \subset N\mathcal{S}$ via the zero section. Moreover write $f : NB_\epsilon \to \mathcal{M}$, for the map defined by sending $\mathcal{S} \subset N\mathcal{S}$ to $\mathcal{S}$ and fiberwise sending $N_p\mathcal{S}$ to $\exp_p(N_p\mathcal{S})$.

▶ **Theorem 5.** *There exists an $\epsilon > 0$ such that $f : NB_\epsilon \to \mathcal{M}$ is a (global) homeomorphism onto its image, that is an embedding.*

The proof in the appendix combines arguments from Section 4.5 of [21] with Appendix I of Chapter 9 of [30], where small variations of this statement can be found.

We now define $N\mathcal{S}(r)$ fiberwise as those points in $N_p\mathcal{S}$ that are at a distance r from p. We refer to r as the radius of the tubular neighbourhood. For any smooth manifold $\mathcal{S}$ embedded in $\mathcal{M}$ and $x \in \mathcal{M}$, we know that the geodesic from x to $\pi_\mathcal{S}(x)$ is normal to $\mathcal{S}$ at $\pi_\mathcal{S}(x)$, as a direct consequence of the Gauss lemma (the Gauss lemma that refers to the exponential map). It follows that, for all ϵ such that $f : NB_\epsilon \to \mathcal{M}$ is a homeomorphism, each point in $f(N\mathcal{S}(r))$ is a distance r from $\mathcal{M}$, for all $0 < r < \epsilon$.

Because $\mathcal{S}$ is compact, so is $N\mathcal{S}(r)$ as is the closed ϵ-neighbourhood $\overline{NB}_\epsilon$. Clearly f is continuous. In general, a continuous bijection from a compact to a Hausdorff space is a homeomorphism. This means that the map f from both $N\mathcal{S}(r)$ and the closed r-neighbourhood $\overline{NB}_r$ to their images are homeomorphisms, that is embeddings, if and only if f from both $N\mathcal{S}(r)$ and $\overline{NB}_r$ are injective.

Because $\overline{NB}_r$ is closed and $N\mathcal{S}(r)$ is the boundary of $\overline{NB}_r$, we have that if f from $N\mathcal{S}(r)$ and $\overline{NB}_r$ to their respective images are a homeomorphisms, that is embeddings, there is an $\eta > 0$ such that the same holds for $N\mathcal{S}(r+\eta)$ and $\overline{NB}_{r+\eta}$. This means that we can make the neighbourhood larger, until a critical radius r' where $N\mathcal{S}(r')$ no longer gets embedded by f. Here we assume that $r' < \iota_\mathcal{M}$. Moreover, as we have just seen this is equivalent to the map not being injective, so at least two points x and y are mapped to the same image. Because we assume that $r' < \iota_\mathcal{M}$, we cannot have that $x, y \in N_q\mathcal{S}$ for some $q \in Su$, if it were, then $\iota_\mathcal{M} = r'$. This means that the image of x, which is also the image of y, under f does not have a unique closest point on the manifold.

We now have the following:

▶ **Lemma 6.** *The reach r is the smallest radius, with $r \leq \iota_\mathcal{M}$, such that f restricted to $N\mathcal{S}(r)$ and $\overline{NB}_r$ are no longer homeomorphisms to their images. For all $r \leq \mathrm{rch}_\mathcal{M}(\mathcal{S})$ and all points in $f(N\mathcal{S}(r))$ are a distance r from $\mathcal{S}$.*

In particular we have proven Theorem 2.

Note that Theorem 2 immediately yields,

▶ **Corollary 7.** *Let $\mathcal{S}$ be a submanifold $\mathcal{M}$ and $p \in \mathcal{S}$. Any open ball $\mathcal{B}$ that is tangent to $\mathcal{S}$ at p and whose radius r satisfies $r \leq \mathrm{rch}_\mathcal{M}(\mathcal{S})$ does not intersect $\mathcal{M}$.*

Proof. Let $r < \mathrm{rch}_\mathcal{M}(\mathcal{S})$. Suppose that the intersection of $\mathcal{M}$ and the open ball is not empty, then the $\pi_\mathcal{S}(c) \neq p$ contradicting Theorem 2. The result for $r = \mathrm{rch}_\mathcal{M}(\mathcal{S})$ now follows by taking the limit. ◀

3 Bounds on the reach

As mentioned, the main goal of this paper is to understand the reach or local feature size of $\exp_p^{-1}(\mathcal{S})$ in terms of the reach of the submanifold $\mathcal{S}$ and the geometry of the manifold $\mathcal{M}$ in which $\mathcal{S}$ is embedded.

3.1 Approach

The lower bounds on the reach of $\exp_p^{-1}(\mathcal{S})$, in terms of the reach of $\mathcal{S}$ and geometric properties of the manifold, follow by considering tangent balls to $\mathcal{S}$ that do not contain points of $\mathcal{S}$, from various viewpoints, namely from the manifold and from various tangent spaces. The argument will be explained in detail using Figure 1.

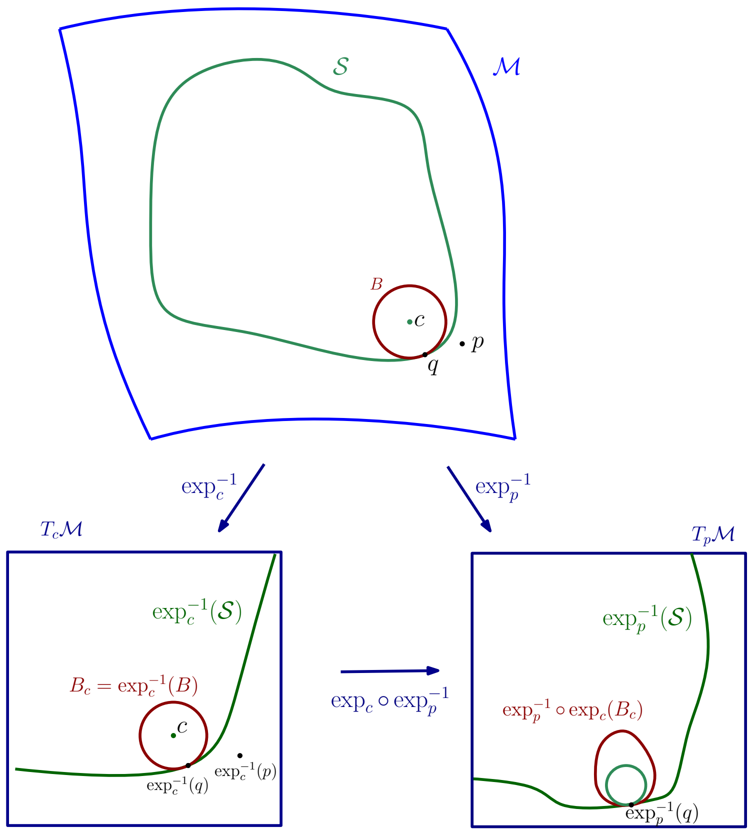

Figure 1 An overview of the approach.

Let $\mathcal{B}$ be a geodesic ball in $\mathcal{M}$ centered at c, tangent to $\mathcal{S}$ at a point q and with radius the reach of $\mathcal{S}$. As in Euclidean space and by Corollary 7, $\mathcal{B}$ has an empty intersection with $\mathcal{S}$. $\mathcal{B}$ is indicated in red in the top of Figure 1.

The exponential map $\exp_c$ gives coordinates on a neighbourhood of the manifold, as does $\exp_p$. We will use the inverse of the exponential maps $\exp_c$ and $\exp_p$, and write $\mathbf{B}_c = \exp_c^{-1} \mathcal{B}$ and $\mathcal{B}_p = \exp_p^{-1} \mathcal{B}$ for the images (in $T_c\mathcal{M}$ and in $T_p\mathcal{M}$ respectively) of the geodesic ball $\mathcal{B}$. Note that $\mathbf{B}_c$ is an Euclidean ball which implies that its reach is equal to its radius, which is also the (geodesic) radius of $\mathcal{B}$. See bottom left part of Figure 1. Note also that $\mathbf{B}_c$ is tangent to $\mathcal{S}_c = \exp_c^{-1}(\mathcal{S})$ and that $\mathcal{B}_p$ is tangent to $\mathcal{S}_p = \exp_p^{-1}(\mathcal{S})$. The composition $\exp_p^{-1} \circ \exp_c$ gives a transformation between the two coordinate neighbourhoods, as indicated by the arrow from the bottom left to right in Figure 1.

Thanks to the Toponogov comparison theorem and a higher order variant, we have bounds on the metric as well as on the derivatives of the metric, both expressed in the coordinates induced by the exponential maps $\exp_p$ and $\exp_c$. The bounds on the metric and its derivatives can then be used to give bounds on the first and second order derivatives of the transformation $\exp_p^{-1} \circ \exp_c$, the bottom arrow in Figure 1. Thanks to a result by Federer [18] one can find a bound on the reach after the transformation, based on the reach of the original. This gives bounds on the reach (in $T_p\mathcal{M}$) of $\mathcal{B}_p$, that is on the reach of $\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$. This is indicated in the bottom right of Figure 1, by the green Euclidean ball inside the red deformed ball.

Consider the Euclidean ball $\mathbf{b}_p \subseteq \mathcal{B}_p \subset T_p\mathcal{M}$ with radius $\mathrm{rch}(\mathcal{B}_p)$ that is tangent to $\partial\mathcal{B}_p$ at $\exp_p^{-1}(q)$, where ∂ indicates the boundary. The Euclidean ball $\mathbf{b}_p$ is also tangent to $\mathcal{S}_p$ at $\exp_p^{-1}(q)$ and, since $\mathcal{B}$ is empty of points of $\mathcal{S}$, $\mathcal{B}_p$ does not intersect $\mathcal{S}_p$ nor does $\mathbf{b}_p$.

The reach of $\mathcal{S}_p$ is lower bounded by the minimum radius of any such Euclidean ball $\mathbf{b}_p$, that is

$$\mathrm{rch}(\mathcal{S}_p) \geq \min_{q\in\mathcal{S}} \min_{\mathcal{B}} \mathrm{rch}(\mathcal{B}_p) = \min_{q\in\mathcal{S}} \min_{\mathcal{B}} \mathrm{radius}(\mathbf{b}_p),$$

where the minimum over $\mathcal{B}$ is a minimum over geodesic balls in $\mathcal{M}$ of radius $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ that are tangent to $\mathcal{S}$ at q. For the local feature size one takes q fixed instead of minimizing over $\mathcal{S}$.

Overview

The outline of this section is as follows:

- In section 3.2 we focus on Riemann normal coordinate systems. Thanks to standard comparison theorems such as the Toponogov comparison theorem [20, 23], we are able to give bounds on the metric in Riemann normal coordinates. The Riemann normal coordinates of a neighbourhood of p are those coordinates that are found by lifting the metric to the tangent space at p via the exponential map. The work by Kaul [24] provides us bounds on the Christoffel symbols in the Riemann normal coordinate neighbourhood, and thus indirectly bounds on the derivative of the metric.
- In Section 3.3 we'll study the coordinates transformation $\exp_c^{-1} \circ \exp_p^{-1}$, the bottom arrow in Figure 1. In Section 3.3.1, we first see how we can go from bounds on the metric in the Riemann normal coordinates to bounds on the coordinate transformation $\exp_c^{-1} \circ \exp_p^{-1}$. In Section 3.3.2, we'll be applying a result by Federer [18], that will yields the reach.

3.2 Bounds on the metric and Kaul's bound on the Christoffel symbols

In this section we review the bounds on the metric and Kaul's bounds on the Christoffel symbols, see [24]. The expressions for these bounds have been simplified by Von Deylen [31, Section 6], at the cost of weakening the bounds. We shall make use of his simplification.

Here and throughout d denotes the dimension of $\mathcal{M}$ and we adopt the Einstein summation convention, that is, if an index occurs twice (once as an upper and once as a lower index) we sum over this index without writing a summation sign. We denote by $g_{ij}(x)$ the metric in this coordinate system at x and δ_{ij} is the standard Euclidean metric. The inverses of the metrics are denotes by $g^{ij}(x)$ and δ^{ij}, respectively. The norm with respect to the Riemannian metric is denoted by $|\cdot|_g$ while the norm with respect to the Euclidean metric is denoted by $|\cdot|_{\mathbb{E}}$. Distances on $\mathcal{M}$ will be denoted by $d_{\mathcal{M}}$. As before, ι_M is the injectivity radius.

The Christoffel symbols are

$$\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}),$$

where ∂_μ denotes the partial derivative with respect to the coordinate x^μ The Christoffel symbols are used to express the covariant derivative of a vector field v^κ in local coordinates,

$$\nabla_\nu v^\kappa = \partial_\nu v^\kappa + \Gamma^{\kappa}_{\mu\nu} v^\mu.$$

$R^{\sigma}_{\mu\nu\lambda}$ will denote the Riemann curvature tensor in local coordinates. We refer to Do Carmo [16] and Spivak [30], as some of the standard texts introducing these concepts.

We shall now assume that the curvature and its derivative are bounded, that is in any orthonormal coordinate system,

$$|R^{\sigma}_{\mu\nu\lambda}| \leq R_{\max} \tag{2}$$

$$|\nabla_\kappa R^{\sigma}_{\mu\nu\lambda}| \leq R^{\nabla}_{\max}. \tag{3}$$

We now consider the Riemann normal coordinates around p:

$$x : (x^1, \dots, x^d) \mapsto \exp_p(x^i E_i)$$

for some orthonormal basis E_i of $T_{\mathcal{M}}$.

We now have the following simplification of the Toponogov comparison theorem:

▶ **Lemma 8** (Lemma 6.8 of [31]). *If $d_{\mathcal{M}}(p,x) \leq r$, with $R_{max} r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then $|g_{ij}(x) - \delta_{ij}| \leq R_{max} r^2$.*

Moreover the result of Kaul [24] simplifies to:

▶ **Lemma 9** (Lemma 6.9 of [31]). *If $d_{\mathcal{M}}(p,x) \leq r$, with $R_{max} r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then*

$$|g_{\kappa\lambda}(x)\Gamma^{\kappa}_{\mu\nu}(x) v^\mu w^\nu u^\lambda| \leq 10 R_{max} r + 5 R^{\nabla}_{max} r^2,$$

for all $v, u, w \in T_x\mathcal{M}$ such that $|u|_g = |v|_g = |w|_g = 1$.

We now note that thanks to Lemma 8, we have that

$$\begin{aligned}
||v|_g^2 - |v|_{\mathbb{E}}^2| &= \left| g_{ij} v^i v^j - \delta_{ij} v^i v^j \right| \\
&\leq \sum_{ij} |g_{ij} - \delta_{ij}| |v^i v^j| \\
&\leq \sum_{ij} R_{\max} r^2 |v^i| |v^j| \\
&= R_{\max} r^2 \left(\sum_i 1 |v^i| \right)^2 \\
&\leq R_{\max} r^2 \left(\sum_i 1 \right) \left(\sum_i |v^i|^2 \right), \\
&= d R_{\max} r^2 |v|_{\mathbb{E}}^2,
\end{aligned}$$

where we used the Cauchy-Schwartz inequality, and made the summation explicit at several points for reasons of clarity. Combining this with Lemma 9, we see that

$$|g_{\kappa\lambda}(x)\Gamma^{\kappa}_{\mu\nu}(x)| \leq \frac{10 R_{\max} r + 5 R^{\nabla}_{\max} r^2}{(1 - R_{\max} d\, r^2)^{3/2}}.$$

Using that

$$\partial_\nu g_{\kappa\mu}(x) = g_{\kappa\lambda}(x)\Gamma^{\lambda}_{\mu\nu} + g_{\mu\lambda}(x)\Gamma^{\lambda}_{\kappa\nu}$$

and taking absolute values, we find the following corollary

▶ **Corollary 10.** *If $d_{\mathcal{M}}(p,x) \leq r$, with $R_{max}r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then*

$$|\partial_\nu g_{\kappa\mu}(x)| \leq \frac{20R_{max}r + 10R^{\nabla}_{max}r^2}{(1 - R_{max}d\, r^2)^{3/2}}.$$

We now recall two results from linear algebra:

- Let E be a $d \times d$-matrix, then

$$\frac{1}{\sqrt{d}}\|E\|_\infty \leq \|E\|_2 \leq \sqrt{d}\|E\|_\infty,$$

  see for example (2.3.11) of [19].
- If $G = I + E$, where G and E are $d \times d$-matrices, I denotes the identity matrix and $\|E\|_2 \leq 1$, then

$$\|I - G^{-1}\|_2 \leq \frac{\|E\|_2}{1 - \|E\|_2},$$

  see for example [22, Section 5.8].

If now, $\|E\|_\infty \leq c$, with $\sqrt{d}c < 1$, then

$$\|I - G^{-1}\|_\infty \leq \frac{dc}{1 - \sqrt{d}c}.$$

With this result and Lemma 8, we immediately have the following

▶ **Corollary 11.** *If $d_{\mathcal{M}}(p,x) \leq r$, with $\sqrt{d}R_{max}r^2 \leq 1$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then*

$$|g^{ij} - \delta^{ij}| \leq \frac{dR_{max}r^2}{1 - \sqrt{d}R_{max}r^2}.$$

We'll also make use of the following result [22, Corollary 6.3.4], which we'll formulate as a lemma,

▶ **Lemma 12.** *Let E be an $d \times d$-matrix, and $G = I + E$, with I the identity matrix. If λ is an eigenvalue of G, then $|\lambda - 1| \leq \|E\|_2$.*

We now immediately have

▶ **Corollary 13.** *If $d_{\mathcal{M}}(p,x) \leq r$, with $R_{max}r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, any eigenvalue λ of $g_{ij}(x)$ now satisfies*

$$|\lambda - 1| \leq \sqrt{d}R_{max}r^2.$$

3.3 From bounds on the metrics to bounds on the coordinate transformations

The starting point of this section is the following: We are given a metric in Riemann normal coordinates at two different points. We want to study the coordinate transformation between these coordinates systems, based on our knowledge of the metric in these two coordinate systems.

In fact, we assume we have bounds on the first and second order derivatives of the metric in both coordinate systems. These bounds yield bounds on the first and second derivatives of the coordinates transformation. It is easy to see why this is so, by considering the limit case: Suppose both metrics are the Euclidean metric, then the transformation from one coordinate system to the other is a combination of a rotation and translation.

From bounds on the coordinate transformation, a result of Federer [18] gives bounds on the reach of $\exp_p^{-1}(B)$ where B is a geodesic ball centred at c with radius r, assuming that p and c are not too far from each other. Here we emphasize that the radius of the ball B equals the radius of $\exp_c^{-1}(B)$.

We will consider a coordinate transformation from a coordinate system x to a coordinate system y. Because the emphasis is on coordinate transformations we'll follow a different convention in this section, and only this section, and use Latin indices. We'll use the indices a, b, c, e, f for y-coordinate system and the indices i, j, k, l, m for x and write

$$y^a = T^a_i x^i + Q^a_{ij} x^i x^j + \mathcal{O}(x^3).$$

Here we assumed that the coordinate systems are chosen such that the origins are mapped to one another, which can be done without loss of generality. Q^a_{ij} is symmetric in i and j. We have

$$g_{ij}(x) = g_{ab}(y(x)) \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \qquad g_{ij}(x) = g_{ij}(0) + (\partial_k g_{ij})(0) x^k + \mathcal{O}(x^2)$$

$$g_{ab}(y) = g_{ab}(0) + (\partial_c g_{ab})(0) y^c + \mathcal{O}(y)^2 \qquad y^c = T^c_m x^m + \mathcal{O}(x^2)$$

$$\frac{\partial y^a}{\partial x^i} = T^a_i + Q^a_{ik} x^k + \mathcal{O}(x^2)$$

Combining these gives

$$\begin{aligned} g_{ij}(0) &= g_{ab}(0) T^a_i T^b_j, \\ (\partial_k g_{ij})(0) &= \partial_c g_{ab} T^c_k T^a_i T^b_j + g_{ab}(Q^a_{ik} T^b_j + T^a_i Q^b_{jk}) \end{aligned} \tag{4}$$

3.3.1 Bounds on the transformations

With the concepts and notations developed in the previous section, we can give bounds on the transformation in terms of bounds on the metric and its derivatives. We find the following.

▶ **Lemma 14.** *Let x and y be two coordinates systems for the same point on the manifold and assume that the metric and its derivatives in these coordinates systems are as follows*

$$g_{ij}(x) = g_{ij}(0) + (\partial_k g_{ij})(0) x^k + \mathcal{O}(x^2)$$
$$g_{ab}(y) = g_{ab}(0) + (\partial_c g_{ab})(0) y^c + \mathcal{O}(y)^2.$$

We assume further that any eigenvalue λ of g_{ij} is bounded by $|\lambda - 1| \leq A$, the any eigenvalue $\tilde{\lambda}$ of g_{ab} is bounded by $|\tilde{\lambda} - 1| \leq B$, and the entries of g^{ef} are bounded from above by $1 + C$. Moreover we assume that for all i, j, k we also have that $|\partial_k g_{ij}| \leq \partial g_{max,x}$, and for all a, b, c, that $|\partial_a g_{bc}| \leq \partial g_{max,y}$.

Now we have that the coordinate transformation between x and y,

$$y^a = T^a_i x^i + Q^a_{ij} x^i x^j + \mathcal{O}(x^3),$$

satisfies the following constraints: The Lipschitz constants, or the metric distortion of the linear approximation T, are bounded by $\sqrt{1+A}/\sqrt{1-B}$ and its inverse by $\sqrt{1+B}/\sqrt{1-A}$, and

$$|Q^a_{ij}| \leq 3d^2 \partial g_{max,x} \frac{(1+C)\sqrt{1+B}}{\sqrt{1-A}} + d^3 \partial g_{max,y} \frac{(1+A)(1+C)}{1-B}.$$

The proof of this statement can be found in the appendix.

3.3.2 Using Federer's estimates: from bounds on the coordinate transformation to bounds on the reach

In this section, we are finally able to give the bounds on the reach by applying Theorem 4.19 of Federer [18]:

▶ **Theorem 15** (Federer). *Let $\mathcal{S}$ be a subset of $\mathbb{R}^d$ with $\mathrm{rch}(\mathcal{S}) > t > 0$, and $s > 0$. If*

$$\tilde{f} : \{x \mid d(x, \mathcal{S}) < s\} \to \mathbb{R}^d$$

is a C^2 diffeomorphism such that

$$|D\tilde{f}| \leq M \qquad |D\tilde{f}^{-1}| \leq N \qquad |D^2\tilde{f}| \leq P,$$

where D denotes the derivative, then

$$\mathrm{rch}(\tilde{f}(\mathcal{S})) \geq \min\{sN^{-1}, (Mt^{-1} + P)^{-1}N^{-2}\}.$$

We can now combine this result with the estimates of the previous sections. We want to investigate how an empty tangent ball to $\mathcal{S}$ transforms under the exponential map. Because a geodesic ball is also an Euclidean ball in the tangent space of its centre (lifted via the exponential map), this is equivalent to giving bounds on the reach of this ball under the map $\exp_p^{-1} \circ \exp_c$.

The bounds on the reach under the map $\exp_p^{-1} \circ \exp_c$ use almost all previous results in this paper: In particular the bounds on the metric and its derivatives are given in Lemma 8, and Corollaries 10, 11, and 13, while Lemma 14 tells us how to go from bounds on the metric to bounds on the coordinate transformation. Federer's result now gives us the reach after the transformation.

Our main result now reads:

▶ **Theorem 16.** *Let $\mathcal{M}$ be a smooth d-dimensional Riemannian manifold whose curvatures are bounded as follows:*

$$|R^{\sigma}_{\mu\nu\lambda}| \leq R_{max} \tag{2}$$

$$|\nabla_\kappa R^{\sigma}_{\mu\nu\lambda}| \leq R^{\nabla}_{max}. \tag{3}$$

Suppose that r_p and r_c are the radii of geodesic balls centred at p and c respectively such that

- *$B(c, 2r_c) \subset B(p, r_p)$, where we have made the centres and radii explicit,*
- *$\sqrt{d}R_{max}r_p^2 \leq 1$ and $r_p \leq \frac{\iota_{\mathcal{M}}}{2}$.*

Then the reach of $\exp_p^{-1}(B(c, r_c)) \subset T_p\mathcal{M}$ is bounded. Specifically,

$$\mathrm{rch}(\exp_p^{-1}(B(c, r_c))) \geq \min\{r_c N^{-1}, (Mr_c^{-1} + P)^{-1}N^{-2}\},$$

where

$$M = \frac{\sqrt{1+A}}{\sqrt{1-B}} \quad N = \frac{\sqrt{1+B}}{\sqrt{1-A}} \quad P = 12\partial g_{max,x}\frac{(1+C)\sqrt{1+B}}{\sqrt{1-A}} + 8\partial g_{max,y}\frac{(1+A)(1+C)}{1-B},$$

with

$$A = \sqrt{d}R_{max}r_c^2 \qquad B = \sqrt{d}R_{max}r_p^2. \qquad C = \frac{dR_{max}r_p^2}{1 - \sqrt{d}R_{max}r_p^2},$$

and $\partial g_{max,x} = \partial g_{max}(r_c)$, $\partial g_{max,y} = \partial g_{max}(r_p)$ with

$$\partial g_{max}(r) = \frac{20R_{max}r + 10R^{\nabla}_{max}r^2}{(1 - R_{max}d\,r^2)^{3/2}}$$

▶ Remark. Our theorem also gives a bound on the reach (or local feature size, if we want to concentrate on the local case) of a submanifold under $\exp_p^{-1}$ by simply applying the theorem to every tangent ball (or every tangent ball tangent to a given point, for the local feature size). Here we naturally assume that all the tangent balls satisfy the conditions of the theorem.

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A Appendix: Proofs

Proof of Theorem 5. We first note that the fiberwise restriction of f to $B_{N_p\mathcal{S}}(r)$ is a diffeomorphism for each $r < \iota_{\mathcal{M}}$. Because $\mathcal{M}$ is C^∞, f is smooth and we can consider the derivative $T_{(p,0)}f$ at a point $(p, 0) \in N\mathcal{S}$. The tangent space splits as follows $T_{(p,0)}N\mathcal{S} = T_p\mathcal{S} \oplus N_p\mathcal{S}$. $T_{(p,0)}f$ is the identity if restricted to the tangent space of $\mathcal{M}$ as well as to a fiber. This gives that f is an immersion and in particular a local homeomorphism on its image because the codimension is zero. Due to Lemma 3, $f : NB_\epsilon \to \mathcal{M}$ is injective for some sufficiently small $\epsilon > 0$. ◀

Proof of Lemma 14. We write G for the matrix $g_{ij}(0)$, and we assume that the eigenvalues λ_i, are not far from 1, that is $|\lambda_i - 1| \leq A$ for all i and some $A \geq 0$. Similarly, we write $\tilde{G}$ for $g_{ab}(0)$, and assume that its eigenvalues $\tilde{\lambda}_i$ are bounded by $|\tilde{\lambda}_i - 1| \leq B$ for some $B \geq 0$. We'll also write $G = o_A^t D_A o_A$ with o_A the orthogonal matrix that diagonalizes G and D_A the diagonal matrix with the eigenvalues of G on the diagonal, that is $\mathrm{diag}(\lambda_i)$. We let S_A denote the matrix with square roots $\sqrt{\lambda_i}$ on the diagonal, that is $S_A = \mathrm{diag}(\sqrt{\lambda_i})$. And similarly, $\tilde{G} = o_B^t D_B o_B$ and $S_B = \mathrm{diag}(\sqrt{\tilde{\lambda}_i})$.

Now (4) gives

$$\begin{aligned} G &= T^t \tilde{G} T \\ o_A^t S_A I S_A o_A &= T^t o_B^t S_B I S_B o_B T \\ o_A^t S_A^t I S_A o_A &= T^t o_B^t S_B^t I S_B o_B T \\ I &= S_A^{-t} o_A T^t o_B^t S_B^t I S_B o_B T o_A^t S_A^{-1}, \end{aligned}$$

which means that $S_B o_B T o_A^t S_A^{-1} = o$, with o an orthogonal transformation.

This in turn implies that $T = o_B^t S_B^{-1} o S_A o_A$ is close to an orthogonal transformation, if A and B are close to zero. The Lipschitz constant of a composition of function is the product of the Lipschitz constants and thus the Lipschitz constant of T is bounded by $(1-A)(1-B)$ and $(1 + A)(1 + B)$ respectively. Because we have that for any vector $|v^\mu| \leq |v|$, where the first $|\cdot|$ should be read as an absolute value and the second as the norm, and $A_{ij} = e_i^t A e_j$, where the e_i denote basis vectors, the entries of T are bounded by from above by $\sqrt{1 + A}/\sqrt{1 - B}$. By the same argument we have that the entries of T^{-1} are bounded by $\sqrt{1 + B}/\sqrt{1 - A}$.

We shall now consider the quadratic term. We start with

$$(\partial_k g_{ij})(0) = \partial_c g_{ab} T_k^c T_i^a T_j^b + g_{ab}(Q_{ik}^a T_j^b + T_i^a Q_{jk}^b).$$

Reshuffling and permuting the indices and changing the sign for the last equation gives

$$
\begin{aligned}
(\partial_k g_{ij})(0) - \partial_c g_{ab} T^c_k T^a_i T^b_j &= g_{ab} Q^a_{ik} T^b_j + g_{ab} T^a_i Q^b_{jk} \\
(\partial_i g_{jk})(0) - \partial_c g_{ab} T^c_i T^a_j T^b_k &= g_{ab} Q^a_{ji} T^b_k + g_{ab} T^a_j Q^b_{ki} \\
-(\partial_j g_{ki})(0) + \partial_c g_{ab} T^c_j T^a_k T^b_i &= -g_{ab} Q^a_{kj} T^b_i - g_{ab} T^a_k Q^b_{ij}.
\end{aligned}
$$

Adding the terms yields

$$
\begin{aligned}
(\partial_k g_{ij})(0) + (\partial_i g_{jk})(0) - (\partial_j g_{ki})(0) - \partial_c g_{ab} T^c_k T^a_i T^b_j - \partial_c g_{ab} T^c_i T^a_j T^b_k + \partial_c g_{ab} T^c_j T^a_k T^b_i \\
= g_{ab} Q^a_{ik} T^b_j + g_{ab} T^a_i Q^b_{jk} + g_{ab} Q^a_{ji} T^b_k + g_{ab} T^a_j Q^b_{ki} - g_{ab} Q^a_{kj} T^b_i - g_{ab} T^a_k Q^b_{ij} \\
= 2 g_{ab} Q^a_{ik} T^b_j,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\big((\partial_k g_{ij})(0) + (\partial_i g_{jk})(0) - (\partial_j g_{ki})(0)\big)(T^{-1})^j_e g^{ef} \\
&\qquad - \partial_c g_{ae} T^c_k T^a_i g^{ef} - \partial_c g_{eb} T^c_i T^b_k g^{ef} + \partial_e g_{ab} T^a_k T^b_i g^{ef} \\
&\quad= \big((\partial_k g_{ij})(0) + (\partial_i g_{jk})(0) - (\partial_j g_{ki})(0) \\
&\qquad - \partial_c g_{ab} T^c_k T^a_i T^b_j - \partial_c g_{ab} T^c_i T^a_j T^b_k + \partial_c g_{ab} T^c_j T^a_k T^b_i\big)(T^{-1})^j_e g^{ef} \\
&\quad= 2 g_{ab} Q^a_{ik} T^b_j (T^{-1})^j_e g^{ef} \\
&\quad= Q^f_{ik},
\end{aligned}
$$

The idea now is the following: If we assume that the left hand side of the previous equation is close to zero (this is in line with Corollary 10 because we assume that the derivatives of the metric are not too large if the neighbourhood is not too small), g_{ab} is close to δ_{ab}, and T^b_j is close to a rotation, all entries of Q^a_{ij} have to be close to zero too.

Let us now assume that for all k, i, j we have that

$$
|\partial_k g_{ij}| \leq \partial g_{\max,x},
$$

and for all a, b, c, that

$$
|\partial_a g_{bc}| \leq \partial g_{\max,y}.
$$

We'll also assume that entries of g^{ef} are bounded in absolute value by $1 + C$. We will use that for a tensor $U_{\mu\nu}$, with $|U_{\mu\nu}| \leq U_{\max}$, and the coordinates of vectors v^μ, w^μ are bounded by $|v^\mu| \leq v_{\max}$ and $|w^\mu| \leq w_{\max}$, we have that

$$
|T_{\mu\nu} v^\mu w^\nu| \leq \sum_{\mu,\nu} T_{\max} v_{\max} v_{\max} = d^2 T_{\max} v_{\max} v_{\max} \tag{5}
$$

where we made the summation explicit, as well as its obvious generalization.

Thanks to the triangle inequality we now have

$$
\begin{aligned}
|Q^f_{ik}| =& \big| \left((\partial_k g_{ij})(0) + (\partial_i g_{jk})(0) - (\partial_j g_{ki})(0)\right)(T^{-1})^j_e g^{ef} - \partial_c g_{ae} T^c_k T^a_i g^{ef} \\
&- \partial_c g_{eb} T^c_i T^b_k g^{ef} + \partial_e g_{ab} T^a_k T^b_i g^{ef} \big| \\
\leq& \left| \left((\partial_k g_{ij})(0) + (\partial_i g_{jk})(0) - (\partial_j g_{ki})(0)\right)(T^{-1})^j_e g^{ef} \right| + |\partial_c g_{ae} T^c_k T^a_i g^{ef}| \\
&+ |\partial_c g_{eb} T^c_i T^b_k g^{ef}| + |\partial_e g_{ab} T^a_k T^b_i g^{ef}| \\
\leq& 3 d^2 \partial g_{\max,x} \frac{(1+C)\sqrt{1+B}}{\sqrt{1-A}} + d^3 \partial g_{\max,y} \frac{(1+A)(1+C)}{1-B}
\end{aligned}
$$

◀

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