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ABSTRACT. The paper deals with load balancing in a set of N queues on a line by a local choice policy. Each one-server queue has a Poissonian arrival of customers. When a customer arrives at queue i, he joins the least loaded queue between queues i and $i+1$. When the load tends to zero, we obtain an asymptotic for the steady-state probability that a queue has m customers. It quantifies the difference between this local choice, no choice and the choice between two queues chosen at random.

1. INTRODUCTION

A load balancing policy. The paper deals with the power of choice between two neighbors in a large set of queues. Load balancing is present in a wide literature and includes various techniques as choice or work stealing ([**?**] and [**?**] for example). When the different servers have their own queues, the two-choice policy is a well-known distributed way to improve load balancing. It consists for the arriving customers to choose two queues at random and join the shortest one, ties being solved at random. The paper focuses on the case where, due to geographical constraints, only local choice can be processed. These constraints are present in many applications as for example vehicle-sharing systems or cloud computing.

The model. We present the model called *local choice model*. It consists in the following. Consider a set of N one-server queues with infinite capacity. Customers arrive at each queue independently with rate λ. When a customer arrives at queue i, $1 \leq i \leq N$, he chooses between queues i and $i+1$ the least loaded one and joins it. If queues i and $i+1$ have the same number of customers, he joins one of these queues with probability $1/2$. By convention, queue $N+1$ is queue 1. The service rate at each queue is μ. All inter-arrival and service times are independent with exponential distribution. Let us define $\rho = \lambda/\mu$.

The problem. The main issue addressed in the paper concerns the distribution of the length of a queue at equilibrium for the model with local choice. We investigate the limiting steady-state distribution as the system gets large, N tending to $+\infty$.

More precisely, let X be a random variable distributed as the limiting number of customers in a queue as N tends to infinity in the model. It is well-known, see [11] and [9], that the tail probability $\mathbb{P}(X \geq k)$, $k \geq 0$ when the customers choose two servers at random and join the least loaded queue is doubly exponentially decreasing: $\mathbb{P}(X \geq k) = \rho^{2^k-1}$, $k \geq 0$. Thus it is much smaller than the tail probability $\mathbb{P}(X \geq k) = \rho^k$, $k \geq 0$, when each customer joins his own queue (independent queues). We investigate the behavior of this tail probability for the local choice policy. Moreover, when N tends to infinity, two queues are independent for the choice at random. It is called the propagation of chaos. The problem is also adressed for the local choice policy.

The results. The *local choice model* previously described, a set of N queues where customers join between two neighboring queues the least loaded one, is investigated. The main result of the paper states that, in equilibrium, queue lengths decay geometrically at rate $(\rho/2)^2$ when ρ tends to 0, N fixed. See Proposition 4 for more details. In other words, the local policy changes the exponential rate but does not lead to an improvement comparable to the random choice model.

Related work. The choice between two queues chosen at random among N is well understood for N large until the late 90's with [11] and [9] and knows a great interest in literature. Nevertheless, local choice is a quite challenging open problem in queueing theory. As far as we know, very few papers investigate the problem. For this model where the underlying graph is a line (or a circle), and more generally for a graph $G = (V, E)$, [5] gives an approximation of the steady-state queue length distribution which seems numerically accurate compared to simulations. This approximation, called pair-approximation is obtained from the empirical measure on pairs of neighbors. It is a mean field limit. But this limit, solution of an ODE, is hard to study analytically. In [5], the expression of the ODE is explicitly given but its equilibrium point is mainly investigated by numerical simulations.

Related models. Some papers deal with such types of models, but without departure. In computer science literature, such model is called urn model or graph model, while in statistical physics it is called deposition model or crystal growth model. The problems addressed in both cases are quite different.

THE URN MODEL. Urns are put at vertices of a finite graph $G = (E, V)$ with $|V| = N$. Arrival of balls are associated to edges. For each ball, an edge is choosen at random and the ball is put in the least loaded of the two end-points of the edge. The problem of the maximum number of balls per urn for N balls in N urns is investigated. The conclusion is that the power of choice does not hold for d-regular graphs, d constant, as this maximum is not in $\log \log N$ (see [7], also [2] and references therein). But the main difference with our study is that we deal with the stationary regime. The poorer load balancing result in the urn problem might come from the fact that with N balls in N urns, the equilibrium is not reached.

THE CRYSTAL GROWTH MODEL. In this model, consider N sites $1, \cdots, N$. There is also no departure. Particules arrive at each site, say i, at rate λ. If both neighboring sites $i-1$ and $i+1$ (respectively just one, none) have more particules than site i, the arrival rate at the site i is β_2, respectively β_1 and β_0. See [6, 1, 4] for a presentation and ergodic conditions for the shape process, which is Markov. Our arrival process is a variant of this model in the special case where $\beta_0 = 0$ and $\beta_2 = 2\beta_1$ (see Section 2 for details). Note that if we extend the local choice model, when the customers choose with some probability α and do not choose otherwise, it will still fit in this framework as a variant, but with $\beta_0 \neq 0$.

Outline. The paper is organized as follows. Section 2 gives the model and the notations. Section 3 presents our main result. Section 4 deals with the asymptotic independence issue.

2. Model description and notations

Consider a system of N queues with inifinite capacities, each of them served by one server at rate μ. In all the following, queue $N+1$ means queue 1. The arrival rate at each queue is λ but the arriving customer at queue i joins the least loaded queue between queues i and $i+1$, ties being solved at random. All inter-arrival and service times are independent with exponential distribution. The Bernoulli variables with parameter $1/2$ introduced to solve the ties are independent and independent of the previous random variables. By definition, $\rho = \lambda/\mu$.

2.1. **The state process.** For $1 \leq i \leq N$, let $X_i(t)$ be the number of customers at queue i at time t and $X(t) = (X_i(t))_{1\leq i \leq N}$. The queue length process $(X(t))_{t\geq 0}$ is a Markov process on state space $\mathbb{N}^N$ with jump matrix Q, given for $n = (n_1, \dots, n_N)$ here and in all the following, by its non-negative components, for $1 \leq i \leq N$,

$$\begin{aligned} Q(n, n+e_i) &= \lambda c_i(n) \\ Q(n, n-e_i) &= \mu 1_{n_i > 0} \end{aligned}$$

where $c : \mathbb{N} \times \mathbb{N}^N \longrightarrow \mathbb{R}_+$ is called the *contribution function* and quantifies the amount of arrivals at the different queues.

For our local choice model, this contribution function is called local choice function and is denoted by c^{lc}. Function c^{lc} at queue i, depends only on the state of this queue and the two neighbours $i-1$ and $i+1$ and is defined by

$$c_i^{lc}(n) = d(n_i, n_{i+1}) + d(n_i, n_{i-1}) \text{ where } d(k,l) = \frac{1}{2} 1_{\{k=l\}} + 1_{\{k<l\}} \tag{1}$$

with, by convention, $n_0 = n_N$ and $n_{N+1} = n_1$. Dispatching function d is the basis of our local choice model since it implements the arrival policy: join the least loaded among two neighboring queues.

Remark. The local choice function c^{lc} can also be defined by

$$c_i^{lc}(n) = \omega(\Delta_{i-1} n, -\Delta_i n),$$

in terms of the shape $\Delta n = (\Delta_1 n, \dots, \Delta_N n)$ where $\Delta_j n = n_j - n_{j+1},\ 1 \leq j \leq N$ and the so-called deposition function ω given by

$$\omega(a,b) = \frac{1}{2}\left(1_{\{a=0\}} + 1_{\{b=0\}}\right) + 1_{\{a>0\}} + 1_{\{b>0\}},\ a, b \in \mathbb{Z}. \tag{2}$$

Note that the Gates-Wescott process studied in [4] is the shape process $(\Delta X(t))$ for the model without departure associated to the following deposition function

$$\omega(a,b) = \beta_{1_{\{a>0\}} + 1_{\{b>0\}}},\ a, b \in \mathbb{Z}, \tag{3}$$

with β_0, β_1 and $\beta_2 > 0$.

2.2. **The infinitesimal generator.** It is given by

$$\begin{aligned} Lf(n) &= \sum_{n' \in \mathbb{N}^N} Q(n,n')(f(n') - f(n)) \\ &= \sum_{i=1}^{N} \lambda c_i(n)(f(n+e_i) - f(n)) + 1_{n_i>0}\mu(f(n-e_i) - f(n)), \end{aligned} \tag{4}$$

for $f : \mathbb{N}^N \to \mathbb{R}$ with finite support.

3. Preliminary results

In this section we study, for N fixed, the queue length process $(X(t))$ for this local choice policy. We prove that $(X(t))$ is ergodic for $\rho < 1$ if $c = c^{lc}$. See Proposition 1. Thus, for $\rho < 1$ fixed, $(X(t))$ has a unique invariant measure $y = (y_n,\ n \in \mathbb{N}^N)$ on $\mathbb{N}^N$, solution of the global balance equations

$$\sum_{n' \in \mathbb{N}^N} y(n')Q(n', n) = 0, \quad n \in \mathbb{N}^N. \tag{5}$$

Our aim is not to solve these equations but rather look for an analytical solution for y of the form

$$y_n(\rho) = \sum_{k \geq 0} \alpha_k(n)\rho^k,\ n \in \mathbb{N}^N.$$

Assuming, for ρ in an interval of the type $[0, \varepsilon[$, the existence of such a serie expansion solution of the global balance equations, we prove that each α_k, $k \geq 0$, has a finite support. See Lemma 2. Then we explain the algorithm to obtain by induction the explicit expressions of α_0, α_1, α_2 and so on.

Assuming an analytical solution for y, we give the expansion at order 6 for the marginal stationary distribution of the queue length of a given queue. Then we investigate the accuracy of this expansion, comparing it to simulations.

3.1. Ergodicity for c^{lc}.

For local choice, contribution function c^{lc} is given by equation (1). The following result gives us the necessary and sufficient condition for ergodicity of the Markov state process $(X(t))$. The proof is based on the Foster's criterion.

Proposition 1 (Ergodicity). *The Markov process $(X(t))_{t \geq 0}$ is ergodic if $\rho < 1$ and transient if $\rho > 1$.*

Proof. Assume that $\rho < 1$. We prove ergodicity by Foster's criterion for Markov processes based on a Lyapunov function (see for example [10, Proposition 8.14]). Here the Lyapunov function f is quadratic, given by

$$f(n) = n_1^2 + \cdots + n_N^2, \quad n = (n_1, \dots, n_N).$$

Let us denote $|n| = \sum_{i=1}^N n_i$. From the expression (4) of the infinitesimal generator L, with straightforward algebra, using that, by equation (1),

$$n_1 c_1^{lc}(n) + \cdots + n_N c_N^{lc}(n) \leq |n| \text{ and } c_1^{lc}(n) + \cdots + c_N^{lc}(n) = N, \tag{6}$$

it holds that

$$\begin{aligned} L(f)(n) &= \lambda \sum_{i=1}^N c_i^{lc}(n)((n_i+1)^2 - n_i^2) + \mu \sum_{i=1}^N 1_{n_i > 0}((n_i - 1)^2 - n_i^2) \\ &= 2\lambda \sum_{i=1}^N c_i^{lc}(n) n_i + \lambda \sum_{i=1}^N c_i^{lc}(n) - 2\mu \sum_{i=1}^N n_i + \mu \sum_{i=1}^N 1_{n_i > 0} \\ &\leq (\lambda + \mu)N - 2(\mu - \lambda)|n|. \end{aligned} \tag{7}$$

By the equivalence of norms in $\mathbb{R}^N$, there is a constant $C > 0$ such that, for all n, $\sqrt{f(n)} \leq C^{-1}|n|$ where $|n| = n_1 + \cdots + n_N$. Thus, if $f(n) > K$ then $|n| \geq C\sqrt{K}$. As $\rho = \lambda/\mu < 1$, K can be chosen large enough to get

$$\gamma = -(\lambda + \mu)N + 2(\mu - \lambda)C\sqrt{K} > 0.$$

Thus, by equation (7), if $f(n) > K$ then $L(f)(n) \leq -\gamma$. Moreover the set $F = \{n \in \mathbb{N}^N, f(n) \leq K\}$ is finite and the random variables $\sup_{0\leq s\leq 1} f(X(s))$ and $\int_0^1 L(f)(X(s))ds$ are integrable. Indeed,

$$\sup_{0\leq s\leq 1} f(X(s)) \leq C^{-2} \sup_{0\leq s\leq 1} |X(s)|^2 \leq C^{-2}(\mathcal{N}_{\lambda N}([0,1[)^2$$

where the arrival process in the system is denoted by $\mathcal{N}_{\lambda N}$, Poisson process with intensity $\lambda N ds$, sum of the N independent Poisson processes $\mathcal{N}_\lambda^i$ with parameter λ of arrivals at queue i, $1 \leq i \leq N$. Using again equation (7),

$$\int_0^1 L(f)(X(s))ds \leq (\lambda+1)N.$$

Thus, the Markov process $(X(t))_{t\geq 0}$ is ergodic if $\rho < 1$.

If $\rho > 1$, we apply [10, Theorem 8.10] , a simplified version of Lamperti's result ([8]), to prove the transience of the embedded Markov chain (M_n) at jump times of $(X(t))$. It implies the transience of $(X(t))$. Let g be defined by $g(n) = n_1+\cdots+n_N$. Then, using that $c_1^{lc}(n) + \ldots + c_N^{lc}(n) = N$, see equation (6), for all $n \in \mathbb{N}^N$,

$$\mathbb{E}_n(g(M_1) - g(n)) = Lg(n) = \lambda \sum_{i=1}^N c_i(n) - \mu \sum_{i=1}^N 1_{\{n_i>0\}} \geq (\lambda-\mu)N > 0.$$

Moreover, for all $n \in \mathbb{N}^N$,

$$\begin{aligned}\mathbb{E}_n(|g(M_1) - g(n)|^2) &= \sum_{n'\in\mathbb{N}^N} Q(n,n')|f(n') - f(n)|^2 \\ &= \lambda \sum_{i=1}^N c_i(n) + \mu \sum_{i=1}^N 1_{\{n_i>0\}} \leq (\lambda+\mu)N,\end{aligned}$$

thus $\sup_{n\in\mathbb{N}^N} \mathbb{E}_n(|g(M_1) - g(n)|^2) < \infty$. The sufficient conditions for transience of Theorem 8.10 in [10] hold. It ends the proof. □

Let $\rho < 1$ be fixed. Since $(X(t))$ is ergodic, let $y(\rho) = (y_n(\rho), n \in \mathbb{N}^N)$ be its invariant measure, the unique solution of the global balance equations

$$\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}} + \rho \sum_{i=1}^N c_i(n)\right) y_n(\rho) = \sum_{i=1}^N y_{n+e_i}(\rho) + \rho \sum_{i=1}^N \mathbb{1}_{\{n_i>0\}} c_i(n-e_i) y_{n-e_i}(\rho), \quad n \in \mathbb{N}^N \tag{8}$$

obtained plugging the expression of Q in equation (5).

Suppose also that the following condition holds.

(H_0) There exists $\varepsilon > 0$, such that, for $\rho \in [0,\varepsilon[$ and $n \in \mathbb{N}^N$, $y_n(\rho)$ can be written as a series expansion of the form

$$y_n(\rho) = \sum_{k\geq 0} \alpha_k(n)\rho^k. \tag{9}$$

Remark 1. *According to Proposition 1, for $c = c^{lc}$, as analyticity requires the existence of the stationary measure, thus implicitely the ergodicity of process $(X(t))$, it holds that $\varepsilon \leq 1$. Note that assumption (H_0) could have been written with 1* *instead of ε. We introduce (H_0) of this form because in the following, some results apply for more general c than c^{lc}, where the ergodicity condition can be written $\rho < \varepsilon$.*

Under assumption (H_0), for each $n \in \mathbb{N}^N$, $\rho \mapsto y_n(\rho)$ is C^∞ on $[0, \varepsilon[$ and $\alpha_k(n) = y_n^{(k)}(0)/k!$. Taking the derivative in the global balance equations (8) with respect to ρ, k times, and evaluating it at $\rho = 0$, it holds that, for any $n \in \mathbb{N}^N$ and $k \in \mathbb{N}^*$,

$$\begin{aligned}(10)\quad &\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right)\alpha_k(n) = \\ &\sum_{i=1}^N \alpha_k(n+e_i) + \sum_{i=1}^N \mathbb{1}_{\{n_i>0\}} c_i(n-e_i)\alpha_{k-1}(n-e_i) - \left(\sum_{i=1}^N c_i(n)\right)\alpha_{k-1}(n).\end{aligned}$$

3.2. **Some lemmas.** Equation (10) allows us to prove that for k fixed, α_k has a finite support. It is the purpose of Lemma 2. For that, we need to prove the following technical lemma.

Lemma 1. *Let $\alpha : \mathbb{N}^N \longrightarrow \mathbb{R}$ and $k_0 \in \mathbb{N}^*$ be such that, for $n = (n_1, \ldots, n_N)$ with $|n| = n_1 + \cdots + n_N > k_0$,*

(i) $\alpha(n) \geq 0$,

(ii) the following recurrence equation holds,

$$(11)\qquad \left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right)\alpha(n) = \sum_{i=1}^N \alpha(n+e_i)$$

(iii) $\sum_{n,|n|>k_0} \alpha(n) < \infty$

then, for all n such that $|n| > k_0$, $\alpha(n) = 0$.

Proof. Let $k_0 \in \mathbb{N}^*$ be fixed. Denote for $k \in \mathbb{N}$, $\mathcal{A}_k = \{n \in \mathbb{N}^N, |n| = \sum_{i=1}^N n_i = k\}$. First, let us prove that, for any $k > k_0$,

$$(12)\qquad \sum_{n\in\mathcal{A}_k}\sum_{i=1}^N \alpha(n+e_i) = \sum_{n\in\mathcal{A}_{k+1}}\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right)\alpha(n).$$

Indeed, for $n \in \mathcal{A}_{k+1}$, for i such that $n_i \neq 0$, $\alpha(n)$ can be written as $\alpha(\hat{n} + e_i)$, for a unique $\hat{n} \in \mathcal{A}_k$. The number of elements in $\mathcal{A}_k$ that can generate n when we add them to e_i is exactly equal to the number of non-zero coordinates n_i of n, $1 \leq i \leq N$. Therefore, equation (12) holds.

We replace $\sum_{i=1}^N \alpha(n+e_i)$ in the left-hand side of (12) by the left-hand side of equation (11). It yields, for any $k > k_0$,

$$\sum_{n\in\mathcal{A}_k}\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right)\alpha(n) = \sum_{n\in\mathcal{A}_{k+1}}\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right)\alpha(n).$$

Thus, for any $k > k_0$,

$$(13)\qquad \sum_{n\in\mathcal{A}_k}\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right)\alpha(n) = C$$

where C is positive due to (i) and independent of k. As $\sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}} \leq N$,

$$\sum_{n\in\mathcal{A}_k} N\alpha(n) \geq C.$$

If $C > 0$, $\sum_{k>k_0} \sum_{n\in\mathcal{A}_k} \alpha(n)$ diverges, since $\sum_{n\in\mathcal{A}_k} \alpha(n) \geq \frac{C}{N} > 0$. But, this contradicts the fact that $\sum_{n,|n|>k_0} \alpha(n) < \infty$. Thus $C = 0$. Using the fact that $\alpha(n) \geq 0$ in equation (13), for all n such that $|n| > k_0$, $\alpha(n) = 0$. □

Lemma 2. *Let $k \in \mathbb{N}$. For all n, $|n| > k$, $\alpha_k(n) = 0$.*

Proof. We prove this assertion by induction on k. Take $k = 0$. From equation (9), $y_n(0) = \alpha_0(n)$. As the invariant measure for $\rho = 0$ (no arrival) is $y_n(0) = \delta_{0_N}$, the Dirac mass at $(0, \ldots, 0) \in \mathbb{N}^N$ denoted by 0_N, the assertion is true. Let $k' \in \mathbb{N}$ be fixed. If we suppose that the assertion holds for $k \leq k'$, then Lemma 1, applied to $\alpha = \alpha_{k'+1}$ and $k_0 = k' + 1$, guarantees that the assertion is true for $k = k' + 1$. Indeed let us check assertions (i), (ii) and (iii) for all n with $|n| > k' + 1$. Let such a n be fixed. In equation (10), $\alpha_{(k'+1)-1}(n) = \alpha_{(k'+1)-1}(n - e_i) = 0$ since $|n| > k'$ and $|n - e_i| > k'$ and induction assumption. Therefore equation (10) is rewritten as equation (11), giving (ii). Moreover, by induction assumption, in equation (9), $\alpha_{k'+1}(n)$ represents the first possible non-zero coefficient for $y_n(\rho)$. This coefficient $\alpha_{k'+1}(n) \geq 0$, because otherwise it would exist ρ such that $y_n(\rho) < 0$, which is false as $y(\rho)$ is a probability measure. It gives (i). Eventually, by equation (14), $\sum_{n=(n_1,\ldots,n_N)} \alpha_{k'+1}(n) = 0$ and, as $\sum_{|n|\leq k'+1} \alpha_{k'+1}(n)$ is finite, then $\sum_{|n|>k'+1} \alpha_{k'+1}(n)$ is finite too, which is (iii). □

3.3. **Induction procedure.** Let us recall the definition

$$\mathcal{A}_m = \left\{n \in \mathbb{N}^N,\ m \in \mathbb{N},\ |n| = m\right\}.$$

The algorithm to obtain all the coefficients $\alpha_k(n)$ is an induction procedure on $k \geq 0$. We use that $\alpha_0 = \delta_{0_N}$. Recall the key equation (10)

$$\left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}}\right) \alpha_k(n) =$$
$$\sum_{i=1}^{N} \alpha_k(n + e_i) + \sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}} c_i(n - e_i)\alpha_{k-1}(n - e_i) - \left(\sum_{i=1}^{N} c_i(n)\right) \alpha_{k-1}(n).$$

For $k \geq 1$, assume that we know the coefficients $\alpha_{k-1}(n)$, for all $n \in \mathbb{N}^N$ and find the coefficients $\alpha_k(n)$, $n \in \mathbb{N}^N$. First, by Lemma 2 $\alpha_k(n) = 0$ for $|n| > k$. Second we derive each coefficient $\alpha_k(n)$ for $n \in \mathcal{A}_k$ as the left-hand side of the previous equation. Indeed, in the right-hand side of the same equation, the first term is null due to Lemma 2. The other terms are known as coefficients for $k - 1$. By the same procedure, we compute the $\alpha_k(n)$ for $n \in \mathcal{A}_{k-1}$. Since $n + e_i \in \mathcal{A}_k$, we still know also the first term of the right-hand side of equation (10). Then we determine the coefficients for $n \in \mathcal{A}_{k-2}$, $n \in \mathcal{A}_{k-3}$ and so on, until $n \in \mathcal{A}_1$. It remains to compute the last coefficient $\alpha_k(0_N)$. It is given by the additional equation (14) in the following lemma.

Lemma 3. *The following property holds:*

$$\sum_{n\in\mathbb{N}^N} \alpha_k(n) = 0, \quad k > 0. \tag{14}$$

Proof. Permuting the sums because, due to Lemma 2, the sum over n is finite, it holds that, for $\rho < \varepsilon$,

$$\begin{aligned}\sum_{k\geq 0}\left(\sum_{n\in\mathbb{N}^N}\alpha_k(n)\right)\rho^k &= \sum_{k\geq 0}\left(\sum_{n,|n|\leq k}\alpha_k(n)\right)\rho^k \\ &= \sum_{n\in\mathbb{N}^N}\left(\sum_{k\geq 0}\alpha_k(n)\rho^k\right) = \sum_{n\in\mathbb{N}^N} y_n(\rho) = 1.\end{aligned}$$

as $y(\rho) = (y_n(\rho), n \in \mathbb{N}^N)$ is a probability measure. The left-hand side of this equation is a power series whose all the terms except the first one are null. □

Remark. For Lemma 2 and the previous induction procedure, we do not use the specific expression (1) of contribution function c. We just choose ρ in the domain of analyticity of the $y_n, n \in \mathbb{N}^N$. What follows remains valid in a general framework but needs the following additional assumptions for the contribution function c.

(H_1) For $n \in \mathbb{N}^N,\ c_1(n) + \cdots + c_N(n) = N$.

(H_2) c is invariant by circular permutation or reflection.

More precisely, the second assumption means that, for such a permutation σ on $\{1, 2, ..., N\}$, for $n \in \mathbb{N}^N$ and $1 \leq i \leq N$, $c_{\sigma(i)}(\sigma(n)) = c_i(n)$. These assumptions are obviously true for the local choice function $c = c^{lc}$ defined by equation (1).

3.4. **A third order series expansion.** Let us derive the coefficients until order 3 under (H_0), (H_1) and (H_2). It is given by the following proposition.

Proposition 2. *For* $k = 0$,

$$\alpha_0(n) = \mathbb{1}_{\{n=0_N\}}. \tag{15}$$

For $k = 1$,

$$\begin{cases}\alpha_1(0_N) & = -N,\\ \alpha_1(e_i) & = 1,\ 1 \leq i \leq N\\ \alpha_1(n) & = 0 \text{ otherwise.}\end{cases} \tag{16}$$

For $k = 2$, *for* $i, j \in \{1, 2, ..., N\}$,

$$\begin{cases}\alpha_2(0_N) & = \frac{1}{2}(N^2 - Nc_1(e_1)),\\ \alpha_2(e_i) & = -N,\\ \alpha_2(e_i + e_j) & = c_i(e_j),\\ \alpha_2(n) & = 0 \text{ otherwise.}\end{cases} \tag{17}$$

For $k=3$, for all $i,j,l \in \{1,2,...,N\}$, $i \neq j$, $j \neq l$ and $l \neq i$,

$$
\begin{cases}
\alpha_3(0_N) & = -\sum_{n \neq 0_N} \alpha_3(n) \\
\alpha_3(e_i) & = \frac{1}{2}(N^2 - Nc_1(e_1)) \\
\alpha_3(e_i+e_j) & = \frac{1}{2}\left(\sum_{v=1}^N \alpha_3(e_i+e_j+e_v) - 3Nc_i(e_j)\right), \\
\alpha_3(2e_i) & = \frac{1}{2}\left(\sum_{v=1}^N c_i(e_v)c_i(e_i+e_v) - 3Nc_i(e_i)\right), \\
\alpha_3(e_i+e_j+e_l) & = \frac{1}{3}(c_i(e_j)c_l(e_i+e_j) + c_j(e_l)c_i(e_j+e_l) \\
& \quad +c_l(e_i)c_j(e_l+e_i)), \\
\alpha_3(2e_i+e_j) & = \frac{1}{2}\left(c_i(e_j)c_i(e_i+e_j) + c_i(e_i)c_j(2e_i)\right), \\
\alpha_3(3e_i) & = c_1(e_1)c_1(2e_1) \\
\alpha_3(n) & = 0 \text{ otherwise.}
\end{cases}
\tag{18}
$$

Proof. For $\rho = 0$, the solution is $y_n(0) = \mathbb{1}_{\{n=0_N\}}$, which gives the coefficients for $k=0$. For $k=1,2$ and 3, we use the method described previously and assumptions (H_1) and (H_2). □

It is interesting to notice that, for $k=0$ and 1, the coefficients $\alpha_k(n)$ do not depend on the choice function c. It means that, for ρ sufficiently small, the choice policy does not influence the system. For $k \geq 4$, the expressions become huge.

3.5. **Marginal distribution for one queue.** Our objective is to find the expansion of the probability that queue i, $1 \leq i \leq N$, has $m \in \mathbb{N}$ customers.

As our system is invariant by circular permutation, by assumption(H_2), for $m \in \mathbb{N}$ and $i \in \{1, \cdots, N\}$, the probability that queue i has m customers does not depend on i. This probability, denoted by $\pi_m(\rho)$, is given by

$$\pi_m(\rho) = \sum_{n=(n_1,\dots,n_N), n_1=m} y_n(\rho). \tag{19}$$

Under the assumption (H_0) that $y_n(\rho)$ is analytical n $[0,\varepsilon[$, $\pi_m(\rho)$ has a series expansion, that can be written as

$$\pi_m(\rho) = \sum_{k \geq 0} \phi_k(m)\rho^k, \quad 0 \leq \rho < \varepsilon \tag{20}$$

where $\phi_k(m) = \pi_m^{(k)}(0)/k!$ is given from equation (9) by

$$\phi_k(m) = \mathbb{1}_{\{m \leq k\}} \sum_{n_2+n_3+\ldots+n_N \leq k-m} \alpha_k(m, n_2, n_3, ..., n_N). \tag{21}$$

For $k \leq 3$, from the previous expressions of α_k given by Proposition 2, we can compute ϕ_k. It is summarized by the following result.

Lemma 4. *For integer k, $0 \leq k \leq 3$, the coefficients $\phi_k(m)$, $m \in \mathbb{N}$, are given by*

$$
\begin{aligned}
&\phi_0(0) = 1, \text{ and } \phi_0(m) = 0, \ m \geq 1, \\
&\phi_1(0) = -1, \ \phi_1(1) = 1 \text{ and } \phi_1(m) = 0, \ m > 1, \\
&\phi_2(0) = 0, \ \phi_2(1) = -c_1(e_1), \ \phi_2(2) = c_1(e_1) \text{ and } \phi_2(m) = 0, \ m > 2, \\
&\phi_3(0) = 0, \ \phi_3(1) = -\phi_3(2) = Nc_1(e_1) - \sum_{j=1}^N c_1(e_1+e_j)c_1(e_j), \\
&\qquad \phi_3(3) = c_1(e_1)c_1(2e_1) \text{ and } \phi_3(m) = 0, \ m > 3.
\end{aligned}
$$

Additionally, we can prove the following result.

Lemma 5. *For $k \geq 1$, $\phi_k(k) = \alpha_k(ke_1) = \prod_{j=1}^{k-1} c_1(je_1)$.*

Proof. Let $k \in \mathbb{N}^*$ be fixed. By equation (21), $\phi_k(k) = \alpha_k(ke_1)$. Taking $n = ke_1$ in equation (10),

$$\alpha_k(ke_1) = \sum_{i=1}^{N} \alpha_k(ke_1 + e_i) + c_1((k-1)e_1)\alpha_{k-1}((k-1)e_1) - N\alpha_{k-1}(ke_1).$$

By Lemma 2, for any i, $1 \leq i \leq N$, $\alpha_k(ke_1 + e_i) = 0$ and $\alpha_{k-1}(ke_1) = 0$. It gives that

$$\phi_k(k) = c_1((k-1)e_1)\phi_{k-1}(k-1).$$

This recurrence equation in $\phi_k(k)$ leads to the desired result, since $\phi_1(1) = 1$. $\square$

Note that $\phi_k(k)$ is the first possibly non-null coefficient of the expansion of $\pi_k(\rho)$. This follows directly from Lemma 2 and equation (21). The previous results imply the following proposition.

Proposition 3. *If the choice function c satisfies (H_0), (H_1) and (H_2), then*

$$\begin{cases} \pi_0(\rho) = 1 - \rho, \\ \pi_1(\rho) = \rho - c_1(e_1)\rho^2 + \left(Nc_1(e_1) - \sum_{j=1}^{N} c_1(e_1 + e_j)c_1(e_j)\right)\rho^3 + \mathcal{O}(\rho^4), \\ \pi_2(\rho) = c_1(e_1)\rho^2 - \left(Nc_1(e_1) - \sum_{j=2}^{N} c_1(e_1 + e_j)c_1(e_j)\right)\rho^3 + \mathcal{O}(\rho^4), \\ \pi_k(\rho) = \left(\prod_{j=1}^{k-1} c_1(je_1)\right)\rho^k + \mathcal{O}(\rho^{k+1}), \quad k \geq 3 \end{cases}$$

where ρ tends to 0.

Proof. Equation (22) comes straightforwardly from equation (20) and Lemmas 4 and 5. Note that, since at equilibrium, the rates of incoming and exiting customers are the same, i.e., $N\lambda = N\mu(1 - \pi_0)$, it allows to obtain by another way that $\pi_0(\rho) = 1 - \rho$. $\square$

Equation (22) can be rewritten in the case of the local choice function $c = c^{lc}$ defined by equation (1). It gives the following result.

Corollary 1. *For the local choice function c^{lc},*

$$\pi_0(\rho) = 1 - \rho, \qquad \pi_1(\rho) = \rho - \frac{3}{2}\rho^3 + \mathcal{O}(\rho^4)$$

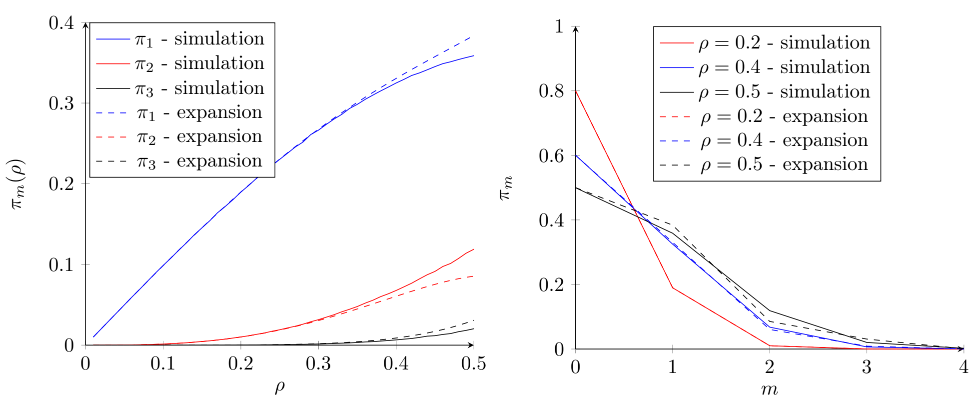
$$\pi_2(\rho) = \frac{3}{2}\rho^3 + \mathcal{O}(\rho^4), \qquad \pi_m(\rho) = \mathcal{O}(\rho^{m+1}), \quad m > 2.$$

3.6. **Further expansions for the local choice function c^{lc}: a numerical approach.** As the amount of cases to analyze grows exponentially with k, it is rather difficult to obtain further series expansions. The following expansions are obtained with help of mathematical software. Moreover we use the following property, which remains to be proved, that for each $k \in \mathbb{N}$, there exists $N_0(k)$ such that if $N > N_0(k)$, $\pi_i^{(k)}$ does not depend on N. Using this, from global balance equations (8) for some N sufficiently large, the following result holds. For $\rho < 1$

tending to 0,

$$\begin{aligned}
\pi_0(\rho) &= 1-\rho \\
\pi_1(\rho) &= \rho - \frac{3}{2}\rho^3 + \frac{11}{8}\rho^4 - \frac{7}{3}\rho^5 + \frac{10727}{2880}\rho^6 + \mathcal{O}(\rho^7) \\
\pi_2(\rho) &= \frac{3}{2}\rho^3 - \frac{11}{8}\rho^4 + \frac{47}{24}\rho^5 - \frac{1583}{320}\rho^6 + \mathcal{O}(\rho^7) \\
\pi_3(\rho) &= \frac{3}{8}\rho^5 + \frac{11}{9}\rho^6 + \mathcal{O}(\rho^7) \\
\pi_i(\rho) &= \mathcal{O}(\rho^7), \quad i > 3.
\end{aligned}$$

We did not go further than this, because the memory required grows as $\mathcal{O}\left(k^k\right)$. In Figure 1a, we investigate numerically the accuracy of the previous expansion. Recall that π is the queue length distribution of any queue in this symmetric system of N queues. We plot π_m for $m = 1$, 2 and 3 as a function of ρ given first by simulation and then by the series expansion at order 7. The conclusion is that the previous series expansion gives a quite good approximation for small values of ρ ($\rho < 0.3$), reasonable for $\rho \leq 0.4$. Figure 1b gives the distribution for different small values of ρ. It indicates that, as ρ increases, the distribution deviates from a geometric distribution. Moreover, the series expansion gives a quite good approximation for $\rho < 0.4$.



(A) For $m = 1$, 2 and 3, as a function of ρ.

(B) For $\rho = 0.2$, 0.4 and 0.5.

FIGURE 1. Invariant distribution for the queue length for N queues with local choice.

Remark. No choice policy: independent queues. For the case where each queue receives independently customers at rate λ and serves them at rate μ, the contribution fuction becomes $c_i(n) = 1$, $n \in \mathbb{N}^N, i \in \mathbb{N}$. We can easily verify that

$$\alpha_k(n) = (-1)^{k-|n|}\binom{N}{k-|n|}\mathbb{1}_{|n|\leq k}$$

satisfies equation (10), where $|n| = n_1 + \ldots + n_N$. Using equation (21), we have for any $r \in \mathbb{N}$, $0 \leq r \leq k$,

$$\begin{aligned}\phi_k(k-r) &= (-1)^r \sum_{i=0}^{r} (-1)^i \binom{N-2+i}{i} \binom{N}{r-i} \\ &= (-1)^r \mathbb{1}_{\{r \leq 1\}}.\end{aligned}$$

The term $\binom{N-2+i}{i}$ comes from the fact that we need to distribute the remaining i customers in the remaining $N-1$ queues. Then, we obtain

$$\pi_m(\rho) = \rho^m - \rho^{m+1}, \qquad m \in \mathbb{N}.$$

This geometric distribution $\pi(\rho) = (\pi_m(\rho), m \in \mathbb{N})$ with parameter ρ is the well-known queue length stationary distribution in a $M/M/1$ queue with arrival-to-service-rate ratio $\rho = \lambda/\mu$.

4. The main result

Proposition 4. *For the local choice function c^{lc} defined by equation* (1)*, for $m \geq 2$, for the stationary probability $\pi_m(\rho)$ that a queue has m custumers,*

$$\pi_m(\rho) = 12 \left(\frac{\rho}{2}\right)^{2m-1} + \mathcal{O}(\rho^{2m})$$

when ρ tends to zero.

To prove Proposition 4, the first step is to obtain that, for a state $n = (n_1, \ldots, n_N)$ existing at order k, which means that $\alpha_k(n) \neq 0$, the maximum possible queue length is $\lceil k/2 \rceil$. Indeed, by Lemma 2, n exists at order k only if $|n| \leq k$. Moreover, we need the following lemma.

Lemma 6. *Let $k \in \mathbb{N}$ and $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$. If $|n| \leq k$ and $n_1 > \lceil k/2 \rceil$ then $\alpha_k(n) = 0$.*

Proof. The following assertion is proved by induction on $p \geq 0$.

($\mathcal{B}_p$) For $k \in \mathbb{N}$ and $n = (n_i)_{1 \leq i \leq N}$, if $|n| = k - p$ and $n_1 > \lceil k/2 \rceil$ then $\alpha_k(n) = 0$.

Let us prove ($\mathcal{B}_0$). Let $k \in \mathbb{N}$ and n such that $|n| = k$ and $n_1 > \lceil k/2 \rceil$. As $|n| = k$, by Lemma 2, for each i, $1 \leq i \leq N$,

$$\alpha_k(n + e_i) = \alpha_{k-1}(n) = 0.$$

Thus equation (10) is rewritten

$$\left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}}\right) \alpha_k(n) = \sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}} c_i^{lc}(n - e_i) \alpha_{k-1}(n - e_i). \tag{22}$$

As $|n| \leq k$ and $n_1 > \lceil k/2 \rceil$,

$$\begin{aligned} n_2 + n_N &\leq k - n_1 \\ &< k - \lceil k/2 \rceil < n_1 \end{aligned}$$

thus

$$n_2 + n_N \leq k - \lceil k/2 \rceil - 1 < n_1 - 1.$$

It means that each neighbouring queue of queue 1 has strictly less than $n_1 - 1$ customers. Thus the contribution on queue 1 for our local choice function c^{lc} defined by equation (1) gives $c_1^{lc}(n - e_1) = 0$ and equation (22) can be rewritten

$$\left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}}\right) \alpha_k(n) = \sum_{i=2}^{N} \mathbb{1}_{\{n_i>0\}} c_i^{lc}(n - e_i)\alpha_{k-1}(n - e_i). \tag{23}$$

Therefore, n exists at order k only if there is $i_1 \neq 1$ such that $\alpha_{k-1}(n - e_{i_1}) \neq 0$. But $|n - e_{i_1}| = k - 1$ and we can repeat the previous arguments for $k - 1$ instead of k and $n - e_{i_1}$ instead of n, with $(n - e_{i_1})_1 > \lceil k/2\rceil \geq \lceil (k-1)/2\rceil$, and so on until we obtain n_1e_1. In conclusion, n exists at order k only if $\alpha_{n_1}(n_1e_1) \neq 0$. It contradicts Lemma 5. Therefore $\alpha_k(n) = 0$.

Assume now, for $p \geq 1$, that $(\mathcal{B}_{p-1})$ is true, and prove $(\mathcal{B}_p)$. For that, let $k \in \mathbb{N}$ and n be such that $|n| = k - p$ and $n_1 > \lceil k/2\rceil$. By induction assumption $(\mathcal{B}_{p-1})$, applied to k and $n + e_i$ as $|n + e_i| = k - (p - 1)$, then to $k - 1$ and n as $|n| = k - 1 - (p - 1)$, it holds that

$$\alpha_k(n + e_i) = \alpha_{k-1}(n) = 0.$$

Then the same arguments as for $(\mathcal{B}_0)$ give that $\alpha_k(n) = 0$. It ends the proof. □

One can then deduces easily the following result.

Lemma 7. *Let m be in $\mathbb{N}^*$. The first possibly non vanishing term of the expansion of the stationary probability $\pi_m(\rho)$ that a queue has m customers is $\phi_{2m-1}(m)\rho^{2m-1}$.*

Proof. For $m \in \mathbb{N}$, by definition, see (20), $\pi_m(\rho) = \sum_{k\geq 0} \phi_k(m)\rho^k$ with

$$\phi_k(m) = \sum_{\substack{n=(m,n_2,\ldots,n_N)\\ |n|\leq k}} \alpha_k(m, n_2, \ldots, n_N).$$

If $k < 2m - 1$ then, for $n = (m, n_2, \ldots, n_N)$ such that $|n| \leq k$, $n_1 = m > \lceil k/2\rceil$. Thus, by Lemma 6, all the $\alpha_k(m, n_2, \ldots, n_N)$ in the right-hand side of the previous equation are null. This ends the proof. □

Moreover the states which exists at order $k = 2m - 1$ with one queue with the maximum value m correspond just to two neighboring queues with m and $j < m$. It is given by the following lemma.

Lemma 8. *If $|n| \leq k = 2m - 1$ (k odd), $n_1 = m$ and there exists $j, l \neq, \neq 1$ such that $n_j > 0$ and $n_l > 0$ then $\alpha_k(n) = 0$.*

Proof. The following assertion is proved by induction on $p \geq 0$

$(\mathcal{B}_p)$ For $k = 2m - 1$, $m \in \mathbb{N}$, for n such that $|n| = k - p$, $n_1 = m$, $n_j > 0$ and $n_l > 0$ with $j, l \neq, \neq 1$ then $\alpha_k(n) = 0$.

Let us prove $(\mathcal{B}_0)$. Let $k = 2m - 1$ and n choosen as indicated. As $|n| = k$, by Lemma 2, for each i, $1 \leq i \leq N$,

$$\alpha_k(n + e_i) = \alpha_{k-1}(n) = 0.$$

As before, using Lemma 2, equation (22) holds. By assumption, as in the proof of Lemma 6, it holds that each neighbouring queue of queue 1 has strictly less than $n_1 - 1$ customers, which yields $c_1(n - e_1) = 0$.Thus equation (22) can be rewritten equation (23). We conclude as in the proof of Lemma 6. □

$$n_2 + n_N < k/2 < n_1$$

We can now prove Proposition 4.

Proposition 4. . In Figure 2, the k-th line contains the n at order k that are linked in some way by equation (10) with a state with highest possible queue (1 with a circular permutation) at order odd. In this graph, two states linked by an arrow are related by equation (10). The state n crossed by red is a state that is not linked in any way to a state with highest possible queue at order odd. Figure 2 helps to understand why case 3 is true.

Now, let us analyze the remaining cases: Let $n_2 = (k-1)/2$ and, using case 1, equation (**??**) gives, for $k = 2m-1$ with m integer and $m \geq 2$,

$$2\alpha_{2m-1}(m, m-1, 0, ..., 0) = \frac{1}{2}\alpha_{2(m-1)}(m-1, m-1, 0, ..., 0). \tag{24}$$

Let $k = 2m$, and $n = (m, m, 0, ..., 0)$, then for m integer and $m \geq 2$, as $c_2(m, m-1, 0, ..., 0) = 1$,

$$2\alpha_{2m}(m, m, 0, ..., 0) = 2\alpha_{2m-1}(m, m-1, 0, ..., 0). \tag{25}$$

Combining equations (24) and (25), for $m \geq 2$,

$$\alpha_{2m-1}(m, m-1, 0, ..., 0) = \frac{1}{2^2}\alpha_{2m-3}(m-1, m-2, 0, ..., 0)$$

and then, using equation (18) to show that $\alpha_3(2, 1, 0, ..., 0) = 3/8$, for $m \geq 3$,

$$\alpha_{2m-1}(m, m-1, 0, ..., 0) = \frac{1}{2^{2(m-2)}}\alpha_3(2, 1, 0, ..., 0) = \frac{3}{2^{2m-1}}. \tag{26}$$

Now, the only way to create new states with one queue with m customers at order $2m-1$ is by decrementing the state $n = (m, m-1, 0, ..., 0)$, since, by case 1, the order $2m-2$ does not have states with $n_1 = m$. Then, for $n = (m, i, 0, ..., 0)$, for $0 < i < m-1$, from equation (10),

$$2\alpha_{2m-1}(m, i, 0, ..., 0) = \alpha_{2m-1}(m, i+1, 0, ..., 0).$$

By induction and using equation (26), for $0 < i < m-1$,

$$\alpha_{2m-1}(m, i, 0, ..., 0) = \frac{1}{2^{m-1-i}}\alpha_{2m-1}(m, m-1, 0, ..., 0) = \frac{3}{2^{2m-1}}\frac{1}{2^{m-1-i}}. \tag{27}$$

With similar arguments and then plugging equation (27) for $i = 1$,

$$\begin{aligned} \alpha_{2m-1}(m, 0, 0, ..., 0) &= \alpha_{2m-1}(m, 1, 0, ..., 0) + \alpha_{2m-1}(m, 0, ..., 0, 1) \\ &= \frac{6}{2^{2m-1}}\frac{1}{2^{m-2}}. \end{aligned} \tag{28}$$

Then, from equation (21) and excluding the n's which fall in cases 1, 2 and 3,

$$\begin{aligned} \phi_{2m-1}(m) &= \sum_{i=1}^{m-1}\alpha_{2m-1}(m, i, 0, ...0) + \sum_{i=1}^{m-1}\alpha_{2m-1}(m, 0, ..., 0, i) + \alpha_{2m-1}(m, 0, ..., 0) \\ &= 2\sum_{i=1}^{m-1}\alpha_{2m-1}(m, i, 0, ...0) + \alpha_{2m-1}(m, 0, 0, ...0) \end{aligned}$$

Plugging equations (26), (27) and (28) in the previous one,

$$\begin{aligned}\phi_{2m-1}(m) &= 2\frac{3}{2^{2m-1}} + 2\sum_{i=1}^{m-2}\frac{3}{2^{2m-1}}\frac{1}{2^{m-1-i}} + \frac{6}{2^{2m-1}}\frac{1}{2^{m-2}} \\ &= \frac{6}{2^{2m-1}}\left(\sum_{i=1}^{m-1}\frac{1}{2^{m-1-i}} + \frac{1}{2^{m-2}}\right) = \frac{12}{2^{2m-1}}\end{aligned}$$

Using it in equation (20) gives the result. □

Proposition 4 guarantees that for ρ sufficiently small, the probability of having m customers in the queue follows a geometric decay of parameter $\rho^2/4$ as m grows. The following table illustrates where the choice policy of local choice situates.

Choice policy	$u_m = \sum_{k\geq m}\pi_k$
No-choice	$\sim \rho^m$
Local choice	$\sim (\rho/2)^{2m-1}$
Random choice	$\sim \rho^{2^m-1}$

In this table the equivalent is when ρ tends to 0. For random choice, the user joins between two queues chosen at random the least loaded one. As expected, the performance of *local choice* policy is between the other two policies. However, for low traffic, its behavior is closer to no choice, than to random choice in the following sense. The two first asymptotics are exponential, whilst the third one is double exponential in ρ.

5. No asymptotic independence of queues for local choice

The following result is classical, see [11] for example, and our proof in Appendix.

Proposition 5. *If the choice is at random among two queues, then the invariant measure μ of the queue length process in the system with an infinite number of queues is of product form $\mu = m^{\otimes\mathbb{Z}}$ with m measure on $\mathbb{N}$ given by*

$$m_j = u_j - u_{j+1} \text{ with } u_j = \lambda^{2^j-1},\ j \geq 0. \tag{29}$$

For the crystal growth model with deposition function defined by equation (3), the invariant measure of the shape process has surprisingly a product form if $\beta_1 = (\beta_0+\beta_2)/2$, see [3, Theorem 16 p.68]. Thus the numbers of particules at the different sites at steady-state become independent as the number of sites is large. We will prove in the following proposition that this nice property that the queue lengths are independent does not hold for our system. In general, the introduction of locality in the choice breaks the asymptotic independence. We also checked that the independence is not true for models with no departure with different variants of (3) for ω, for example (2).

Proposition 6. *The queue length process with choice among neighbors given by equation* (1) *has no product form invariant measure.*

Proof. The idea is the following. The stationary measure μ verifies that

$$\int Lf\,d\mu = 0 \tag{30}$$

for all f of the form $f(n) = 1_{\{n_i=a_i,\ldots,n-j=a_j\}}$ where $i < j$, $a_i, \ldots, a_j \in \mathbb{N}$ and L is defined by equation (4). Suppose that $\mu = m^{\otimes\mathbb{Z}}$. We start with $f(n) = 1_{\{n(i)=a_i\}}$ where $a_i \in \mathbb{N}$. If for such function f, equation (30) holds then, by Proposition 5, m is given by equation (29). It can be checked that for this $\mu = m^{\otimes\mathbb{Z}}$, if $f(n) = 1_{\{n_i=a_i,n_{i+1}=a_{i+1}\}}$ where $a_i = a_{i+1} = 0$, equation (30) does not hold. It ends the proof. $\square$

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6. Appendix

6.1. **Proof of Proposition 5.** Let μ be the invariant measure. Thus for any f with finite support,

$$\int Lf d\mu = 0.$$

For f given by $f(n) = 1_{n_i=j}$ with $i \in \{1, \cdots, N\}$ and $n,\ j \in \mathbb{N}$, plugging

$$f(n + e_i) = 1_{n_i=j-1} \text{ and } f(n - e_i) = 1_{n_i=j+1}$$

in equation (4) gives, first for $j = 0$,

$$Lf(n) = -\lambda(1_{n_{i-1}>0} + 1_{n_{i+1}>0} + \frac{1}{2}(1_{n_{i-1}=0} + 1_{n_{i+1}=0}))1_{n_i=0} + (1_{n_i=1} - 1_{n_i=0})1_{n_i>0}.$$

If $\mu = m^{\otimes\mathbb{Z}}$ then

$$\int Lf d\mu = -\lambda(2m([1, +\infty[) + m_0)m_0 + m_1.$$

Then, as $m([1,+\infty[) = 1 - m_0$, $\int L d\mu = 0$ rewrites $m_1 = \lambda(2-m_0)m_0$. If $j > 0$,

$$\begin{aligned} Lf(n) &= \lambda(1_{n_{i-1}>j-1} + 1_{n_{i+1}>j-1} + \frac{1}{2}(1_{n_{i-1}=j-1} + 1_{n_{i+1}=j-1}))1_{n_i=j-1} \\ &\quad - \lambda(1_{n_{i-1}>j-1} + 1_{n_{i+1}>j-1} + \frac{1}{2}(1_{n_{i-1}=j-1} + 1_{n_{i+1}=j-1}))1_{n_i=j} \\ &\quad + (1_{n_i=j+1} - 1_{n_i=j}) \end{aligned}$$

and it gives if $\mu = m^{\otimes\mathbb{Z}}$,

$$\int Lf d\mu = \lambda(2m([j,+\infty[) + m_{j-1})m_{j-1} - \lambda(2m([j+1,+\infty[) + m_j)m_j + m_{j+1} - m_j.$$

Let $u_j = m_j + m_{j+1} + \cdots$. Equation $\int Lf d\mu = 0$ can be rewritten

$$\lambda\left((2u_j + m_{j-1})m_{j-1} - (2u_{j+1} + m_j)m_j\right) + m_{j+1} - m_j = 0$$

Replacing m_l by $u_l - u_{l+1}$ in the previous formula and using that $\lim_{j\to+\infty}$ gives

$$\lambda(u_{j-1}^2 - u_j^2) - (u_j - u_{j+1}) = \lambda(u_j^2 - u_{j+1}^2) - (u_{j+1} - u_{j+2}) = 0, \ \ j > 0.$$

Then with the same argument

$$u_j - \lambda u_{j-1}^2 = u_{j+1} - \lambda u_j^2 = 0, \ \ j > 0$$

which yields $u_j = \lambda u_{j-1}^2, \ \ j > 0$. Equation (29) thus holds. It ends the proof.

If $\lambda > 1$, we apply Lamperti's result [10, Theorem 8.10] to prove the transience of $(X(t))$. Let f be defined by $f(n) = n_1 + \cdots + n_N$. Then, using $(H2)$,

$$Lf(n) = \frac{\lambda}{N(\lambda+1)} \sum_{i=1}^{N} c_i(n) - \sum_{i=1}^{N} 1_{\{n_i>0\}} \geq (\lambda - 1)N.$$

with $\sup_{n\in N^N} Lf^2(n) = \lambda\sum_{i=1}^N c_i(n) + \sum_{i=1}^N 1_{\{n_i>0\}} \leq (\lambda+1)N < \infty$. It ends the proof.

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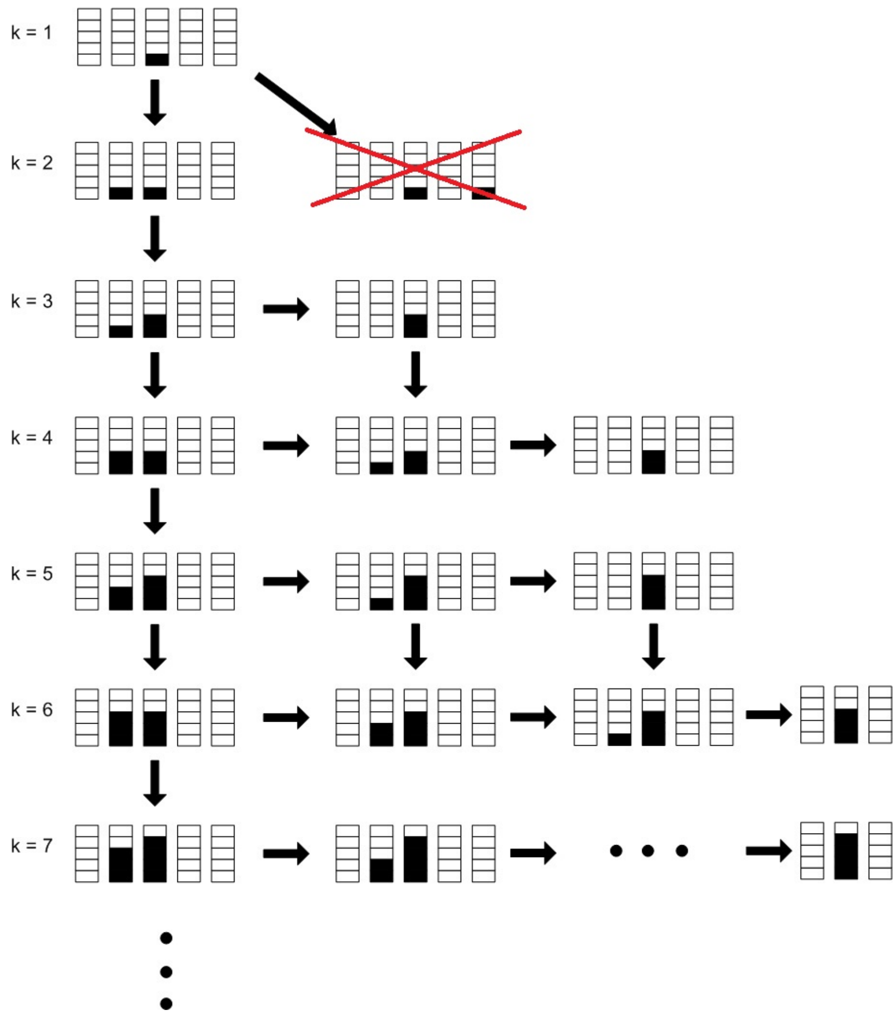

FIGURE 2. Non-zero states that have the greatest possible queue.